On the complexity of algebraic power series.

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1 Introduction.

A classical computational model do deal with "computable" power series consists in giving an algorithm to compute their coefficients and to consider suitable truncations in order to perform the required operations. In the case of algebraic power series, i.e. when the series $f(X)$ is given by a polynomial $G(X_1, , X_n, T)$ s.t.$G(X_1, , X_n, f(X)) = 0$, these coefficients can be computed for instance using [K-T]. Since there is in general more than one series vanishing at the origin and satisfying the above identity, one must compute enough terms of the Taylor expansion of f, which, at least, permit to distinguish it from the other roots of G. It is clear that computational problems arise naturally in case the series f should be used for further calculations, e.g. to determine the solution h of a polynomial depending on the X variables and on f.

In order to avoid these problems, in [AMR], we introduced a purely symbolic model of computation, based on the notion of Locally Smooth Systems (LSS), and we showed that these systems have good computational properties: standard bases and normal forms can be calculated in the ring of algebraic power series and it is possible to give effective versions of classical theorems like the Weierstrass Preparation Theorem and the Noether Normalization Lemma.

The aim of this paper is to look for suitable measures for the complexity of algebraic power series. In [R1], [R2] R.Ramanakoraisina defines the complexity of an analytic function f satisfying a polynomial equation to be the degree $c(f)$ of its minimal polynomial; and he shows that this definition satisfies all the required properties of complexities. Since we are interested in the local properties of f (at the origin) we consider here a notion which takes care of

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both the degree and the multiplicity defined as the minimum order $e(f)$ of defining polynomials.

We introduce then a notion of complexity for algebraic series represented in model of computation of LSS's. To do this we introduce suitable costs for Locally Smooth Systems by means of the length $\lambda(f)$(i.e. the number of extra variables) and the degree $\delta(f)$ (i.e. the product of the degrees of the involved polynomials). We define then the complexity of f : $\xi(f) = (\lambda(f), \delta(f))$ as the minimum of the costs of all Locally Smooth Systems representing f .

The main result of this note shows how these costs can be estimated in terms of degree and multiplicity : $\lambda(f) \leq e(f)$ and $\delta(f) \leq c(f)^{\lambda(f)} \leq c(f)^{e(f)}$. Conversely, we have that $o(f) \leq c(f) \leq \delta(f)$, where $o(f)$ is the order of f at the origin.

Then we introduce the complexity ξ' as the minimum cost of Standard Locally Smooth Systems defining f, and we find bounds for ξ' in terms of ξ and of the maximum degree of involved polynomials. Using this we show that the cost of representing the Weierstrass form of a distinguished polynomial of order b can be estimated by $(b(r+1), D^{b(r+1)})$ where r is the lenght of the involved Locally Smooth Sytem and with D depending on the degrees of the data. Finally, using the above estimates, we find a test to check whether an algebraic function is indeed a rational function.

2 Notation and preliminaries

Let K be a subfield of the field of complex numbers, let $X = (X_1, ..., X_n)$ a set of variables and $K[[X]]_{alg}$ the algebraic closure of $K[X]$ in $K[[X]]$, which is the set of algebraic formal power series, and which is also the henselization of the ring of polynomials with respect to the maximal ideal corresponding to the origin.

Let us recall from [AMR] the notion of Locally Smooth Systems. We say that a system of polynomials $\mathbf{F} = (F_1, ..., F_r)$ is a LSS if the F_i's are polynomials in $K[X_1, ..., X_n, Y_1, ..., Y_r]$ vanishing at the origin and s.t. the Jacobian of the F_i's with respect to the Y_j's at the origin is a lower triangular non singular matrix, i.e. we can write:

$$F_i(X, Y_1, ..., Y_r) = \sum_{j=1}^{r} c_{ij} Y_j + H_i(X, Y_1, ..., Y_r)$$

with $H_i \in (X, (Y_1, ..., Y_r)^2)$ and (c_{ij}) a non-singular lower triangular r by r matrix. Under this assumption, by the Implicit Function Theorem, there are unique algebraic series $f_1, ..., f_r \in K[[X]]_{alg}$ s.t. $f_j(0) = 0 \ \forall\, j$, and $F_i(X, f_1, ..., f_r) = 0 \ \forall\, i$. We will also say that $\mathbf{F} = (F_1, ..., F_r)$ is a LSS for the f_i's (or defining the f_i's; or that the f_i's are given by the LSS $\mathbf{F}$ etc.).

The key point of our approach in [AMR] was to look for results in $K[[X]]_{alg}$ by working with suitable, and computable, extensions of $K[X]$. Namely, given a LSS **F** , we then consider the rings $K[X_1,,X_n,f_1,....,f_r]$:= K[X,**F**] and $K[X_1,,X_n,f_1,....,f_r]_{loc} := K[X,\boldsymbol{F}]_{loc}$ viewed as a subring of $K[[X]]_{alg}$ and the evaluation map

$$\sigma_F : K[X_1,,X_n,Y_1,....,Y_r]_{loc} \to K[[X]]$$

defined by $\sigma_F(Y_i) = f_i$. We have:

$$K[X,\boldsymbol{F}]_{loc} := K[X,f_1,....,f_r]_{loc} \simeq \frac{K[X,Y_1,....,Y_r]_{loc}}{(F_1,...,F_r)}$$

(Where, for any K-algebra A, with $K[Z] \subset A \subset K[[Z]]$, we denote: $A_{loc} = \{ \frac{a}{1+b}$, a, $b \in A$, $b(0) = 0$ }).

The classical approach to compute with algebraic series (cf [K-T]) consists in representing them as solutions of polynomial equations, i.e. a series $f(X) \in K[[X]]_{alg}$ is given by a polynomial $G(X_1,,X_n,T)$ s.t. $G(X,f(X)) = 0$, and, since there is in general (also in case G is irreducible) more than one series vanishing at the origin and satisfying G, by an algorithm which computes the Taylor expansion of f up to order d, $\forall$ d, or, at least, enough terms in order to distinguish f from the other roots of G. In this paper we introduce suitable measures of the complexities for algebraic power series in both representations and we will compare them. We first recall from [AMR] how the two computational models are compatible (cf [AMR] Appendix and Proposition 2.3).

Theorem 1 *(a)(Artin-Mazur) Let* $f \in K[[X]]_{alg}$, $G \in K[X,T]$ *such that* $G(X,f(X)) = 0$ *and assume that an algorithm to compute the Taylor expansion of f up to order d,* $\forall d$*, is given . Then it is possible to compute a locally smooth system* $(F_1,...,F_r)$ *defining algebraic series* $f_1,...,f_r$*, with* $f_1 = f$*.*

(b) Conversely, let $\mathbf{F} = (F_1,...,F_r)$ *be a LSS in* $K[X,Y_1,...,Y_r]$ *defining the series* $f_1,...,f_r \in K[[X]_{alg}$*, and let* $d_i = deg(F_i)$ *and* $d = \prod_{i=1}^r d_i$*.*
Then: given $h(X) = H(X,f_1(X),..,f_r(X)) \in K[X,\boldsymbol{F}]_{loc}$ *represented by* $H \in K[X,Y_1,...,Y_r]_{loc}$ *with* $H = \frac{H_0}{1+H_1}$ *and* $deg(H_0)$ *and* $deg(H_1)$ *bounded by m, there exist a polynomial* $Q \in K[X,T]$ *with* $deg(Q) \le (m+1)d$ *s.t.* $Q(X,h(X)) = 0$*. (Note:* $h = \sigma_F(H) \in K[X,\boldsymbol{F}]_{loc}$ *).*

Remark 1 *In our computational model all the ring operations with series turn out almost automatically, since our computational tool are the "rings"* $[X,\boldsymbol{F}]_{loc}$*. Suppose that f and g are given by distinct LSS's (respectively* **F** *and* **G** *then it is enough to merge them. More precisely, let* $f = \sigma_F(F) \in [X,\boldsymbol{F}]_{loc}$ *with* $F \in K[X,Y_1,...,Y_r]_{loc}$ *and* $g = \sigma_G(G) \in [X,\boldsymbol{G}]_{loc}$ *with* $G \in K[X,Y_1,...,Y_s]_{loc}$ *then any rational function* $h = H(f,g) \in K[X]_{alg}$ *with* $H = \frac{H_0}{1+H_1}$ *,* $H_0(0) = H_1(0) = 0$ *can be represented by the LSS* $\mathbf{H} := (\mathbf{F},\ \mathbf{G}) = (F_1,...,F_r,F_{r+1},...,F_{r+s}) \subset K[X,Y_1,...,Y_r,Y_{r+1},...,Y_{r+s}]$ *with* $F_{r+i} = G_i(Y_{r+1},...,Y_{r+s})$ *via the evaluation* $h = \sigma_H(H(F,G)) \in [X,\boldsymbol{F}]_{loc}$ *.*

Of course, in many cases it will be not necessary to add so many extra variables: we will propose a test for this in the last section.

3 The complexity of algebraic power series.

We recall a notion of complexity which has been recently introduced for Nash functions by R. Ramanakoraisina (cf [R1] and [R2]). Let $f \in K[[X]]_a lg$; we define the complexity $c(f)$ of f as

$$c(f) = min\{degP, \ where \ P \in K[X,T] \ and \ P(X, f(X)) = 0\}$$

In [R1],[R2] it is shown that: $c(f+g) \leq c(f)c(g)$, $c(fg) \leq 2c(f)c(g)$, $c(f^2) \leq 2c(f)$ and that $c(\frac{\partial f}{\partial X_i}) \leq c(f)^2$

Let us remark that, if P is irreducible, then $c(f) = deg(P)$. Take in fact the Zariski closure in $\mathbf{C}^{n+1}$ of the analytic germ $(X, f(X))$, then $W = \{(x,t) \in \boldsymbol{C^{n+1}} : P(x,t) = 0\}$. If Q is s.t. $Q(X, f(X)) = 0$ then Q vanishes on W, hence $degQ \geq degP$.

Let $o(g)$ denote the order of vanishing of a function g at the origin of the coordinates, defined as the lowest degree for which, in the Taylor expansion of g, there is a non-zero coefficient. We introduce the *representative multiplicity* of $f \in K[[X]]_{alg}$ $(e(f))$ as

$$e(f) = min\{o(P); where P \in K[X,T] and P(X, f(X)) = 0\}.$$

Recall that, for a local ring R, $e(R)$ denotes its multiplicity (cf [Z-S] Ch.VIII 10). Let $P(X,T)$ an irreducible polynomial defining f, then $e(f) = e(\frac{K[X,T]_{loc}}{P(X,T)})$.

Let $\mathbf{F} = (F_1, ..., F_r)$ be a LSS and let $d_i = deg(F_i)$ and d $= \prod_{i=1}^r d_i$ we will say that d is the degree of $\mathbf{F}$, deg($\mathbf{F}$) = d and that r, i.e. the number of new variables, is its lenght, l($\mathbf{F}$) = r. Moreover we introduce the cost of $\mathbf{F}$ as: cost($\mathbf{F}$) = (l($\mathbf{F}$), deg($\mathbf{F}$)), and we order costs lexicographically:

$cost(\boldsymbol{F}) < cost(\boldsymbol{G}) \Leftrightarrow l(\boldsymbol{F}) < l(\boldsymbol{G})$ or $l(\boldsymbol{F}) = l(\boldsymbol{G})$ and $deg(\boldsymbol{F}) < deg(\boldsymbol{G})$.

Moreover we will denote by $V_F \subset \boldsymbol{C^{n+r}}$ the algebraic variety defined by the ideal $(F_1, ..., F_r)$. By definition V_F is non-singular at the origin of the coordinates. Nevertheless V_F can be reducible, therefore we will consider also its irreducible component W_F through the origin. We also remark that, in general, W_F may be not definable by r polynomial equations; i.e. it may be not a complete intersection.

Example 1 *Let*

- $F_1 = 3Y_1 - 4X + Y_2^2 - 2XY_2 + 2Y_1^2 - 2XY_1 - X_2 - X_2Y_1$
- $F_2 = Y_2 - 2Y_1 + X + XY_2 - Y_1^2$

Then (F_1, F_2) is not a prime ideal in $K[X, Y_1, Y_2]$ and the associated prime at the origin is

$$\wp = (F_1, F_2, X^3 + 3X^2 - Y_1Y_2 + 3X - Y1 - Y2)$$

On the other hand we have:

$c(f_1) = 4$ defined by: $X^4 + 4X^3 - T^3 + 6X^2 - 3T^2 + 4X - 3T$.

$c(f_2) = 5$ defined by: $X^5 + 5X^4 + 10X^3 - 3T^2 + 5X - 3T$.

Moreover we have that if $W = V(\wp)$ then $degW = 5$, but for every defining LSS $\mathbf{G}$, $deg(\mathbf{G}) \geq 6$.

We are going now to introduce complexities of algebraic power series in the LSS model; it turns out convenient to introduce this notion for a *set* of such functions.

We say that f is defined *via* the LSS $\mathbf{H}$ if $f \in \{h_1, ..., h_r\}$, where the $h_i's$ are the algebraic power series defined by $\mathbf{H}$. Similarly, we say that $\{f_1, ..., f_s\}$ are defined via $\mathbf{H}$ if $\{f_1, ..., f_s\} \subset \{h_1, ..., h_r\}$ as a set.

Definition 1 *We call cost ξ of $\{f_1, ..., f_s\}$ the minimum of cost($\mathbf{H}$) where $f_1, ..., f_s$ are defined via $\mathbf{H}$. We write:*

$$\xi(f_1, ..., f_s) = (\lambda(f_1, ..., f_s), \delta(f_1, ..., f_s)) = (length(\boldsymbol{H}*), deg(\boldsymbol{H}*)) = cost(\boldsymbol{H}*)$$

where $\mathbf{H}*$ reaches the minimum. Moreover, we set : $d(f_1, ..., f_s) = deg(Z)$ where Z is the Zariski closure of the germ $(X, f_1(X), ..., f_s(X))$ in $\boldsymbol{C}^{n+s}$.

Remark 2 *Notice that our definition does not strictly satisfy the notion of complexity given by Benedetti and Risler (cf[BR]), while it is straightforward to verify the following formulas, (the proofs come out easily by the merging procedure described in Remark 1):*

1. $\xi(f, g) \leq (\lambda(f) + \lambda(g), \delta(f)\delta(g))$
2. $\xi(f + g) \leq (\lambda(f) + \lambda(g) + 1, \delta(f)\delta(g))$
3. $\xi(fg) \leq (\lambda(f) + \lambda(g) + 1, 2\delta(f)\delta(g))$
4. $\xi(f^2) \leq (\lambda(f) + 1, 2\delta(f))$
5. $\xi(\frac{\partial f}{\partial X_i})) \leq (2\lambda(f), \delta(f)^2)$

Proposition 1 *1) If $\mathbf{F}$ and $\mathbf{G}$ define the same algebraic functions $g_1, ..., g_r$ then $W_F = W_G$*

2) $d(f_1, ..., f_s) \leq deg(W_F)$, where $f_1, ..., f_s$ are defined via $\mathbf{F}$

3) $d(f_1, ..., f_s) \leq \delta(f_1, ..., f_s)$

Proof. 1) It is clear, since W_F and W_G are irreducible algebraic varieties with the same germ at the origin of the coordinates.

2) Let U (resp. Z) be the Zariski closure of $(X, g_1(X), ..., g_r(X))$ in $\mathbf{C}^{n+r}$ (resp of $(X, f_1(X), ..., f_s(X))$ in $\mathbf{C}^{n+s}$. Then $f_1, ..., f_s \subset g_1, ..., g_r$ as sets, and w.l.o.g. assume that $f_1 = g_{r-s+1}, ..., f_s = g_r$. We consider the projection $\pi : \mathbf{C}^{n+r} \longrightarrow \mathbf{C}^{n+s}$ to the first n and the last s factors $(x, y_1, ..., y_{r-s}, y_{r-s+1}, ..., y_r) \mapsto (x, y_{r-s+1}, ..., y_r)$. Then $\pi(U) \subset Z$ and:

$$\begin{array}{llll} \mathbf{C}^{n+r} \supset & \text{U} & \subset & V_F \\ \downarrow & \downarrow & & \\ \mathbf{C}^{n+s} \supset & \text{Z} & & \\ \downarrow & \downarrow & & \\ \mathbf{C}^{n} & \mathbf{C}^{n} & & \end{array}$$

By 1) we obtain that $U = W_F$, moreover Z is the Zariski closure of $\pi(U)$ and then: $deg(U) = deg(Z)deg(\pi)$ and hence $deg(W_F) \geq deg(Z) = d(f_1, ..., f_s)$.

3) is clear.

4 Estimating complexities

The following theorem shows us that the measures of complexity for algebraic power series we have introduced in the two models are compatible, i.e. we can estimate $\xi(f)$ in term of $c(f)$ and $e(f)$.

We will furthermore assume that the base field K is algebraically closed.

Theorem 2 *Let $f \in K[[X]]_{alg}$, then:*
A) $\lambda(f) \leq e(f)$
B) $\delta(f) \leq c(f)^{\lambda(f)} \leq c(f)^{e(f)}$

Proof.A) For a local ring R let $e(R)$ denote its multiplicity.

Let $P(X,T)$ an irreducible polynomial defining f, such that $e(f) = e(\frac{K[X,T]_{loc}}{P(X,T)})$.

If $e := e(f) = 1$ we have finished.By induction suppose that there exist k variables $Y_1, ..., Y_k$ and an ideal I_k such that $\frac{K[X,Y_1,...,Y_k]}{I_k}$ is an integral extension of $\frac{K[X,T]}{P(X,T)}$ and such that $e_k := e(\frac{K[X,Y_1,...,Y_k]_{loc}}{I_k}) \leq e + 1 - k$. Let $R_k := \frac{K[X,Y_1,...,Y_k]_{loc}}{I_k}$. If $e_k > 1$, since we know by hypothesis that there is an analytic smooth branch through the origin, the local ring R_k is not unibranche (cf. the proof of Artin-Mazur as in [AMR]) so there exists an integer function h in the normalization of it such that h assumes two different values at two distinct branches of it . Let us consider $R' := \frac{K[X,Y_1,...,Y_k]_{loc}}{I_k}[h] = \frac{K[X,Y_1,...,Y_k]_{loc}[Y_{k+1}]}{I_{k+1}}$ and let $R_{k+1} := \frac{K[X,Y_1,...,Y_k,Y_{k+1}]_{loc}}{I_{k+1}}$. By the projection formula ([Z-S]) we have $e(R_k) = e(R_{k+1}) + \sum [R'/\wp_i : K] e(\wp_i)$ and hence $e(R_k) > e(R_{k+1})$. The procedure then halts when $e_k = 1$, then $k = \lambda(f)$ and $1 \leq e - k + 1 = e - \lambda(f) + 1$.

B) Let Z the Zariski closure of $(X, f(X))$ as above and let W_H be the irreducible component through the origin of V_H where **H** is a LSS defining f and such that W_H is dominated by the normalization Z' of Z. Then there exists a dominating morphism $\psi : W_H \longrightarrow K^n$ and the degree of W_H is given by $[K(W_H) : K(X_1, ..., X_n)] = [K(W_H) : K(Z)][K(Z) : K(X_1, ..., X_n)] = [K(Z) : K(X_1, ..., X_n)] = c(f)$. Hence $deg(W_H) = c(f)$. Suppose now that **H** is minimal, as constructed in A), and that $r = \lambda$. By Heintz' results on definability (cf. [H] Prop.3), there exist $n + r + 1 = n + \lambda(f) + 1$ polynomials $g_j's$ of degree bounded by $c(f)$ such that $W_H = g_1 = ... = g_{n+\lambda(f)+1} = 0$ as a set . By means of a generalization of Heintz proof (it is enough to choose projections which distinguish not only points but also the tangents at the origin), we obtain that the $g_j' s$ can be chosen in order that the variety $g_1 = ... = g_{n+\lambda(f)+1} = 0$ is non singular at the origin of coordinates, then W_H is actually its irreducible component through the origin. Computing the Jacobian determinant of the $g_i' s$ w.r.t. the $Y_j' s$ variables we know that it has rank $\lambda(f)$ and therefore we can choose the corresponding polynomials $g_1', ..., g_{\lambda(f)}'$ which give a LSS $\mathbf{H}'$ defining the $f_i' s$ and with $deg(\mathbf{H}') \leq c(f)^{\lambda(f)} \leq c(f)^{e(f)}$.

On the other side we know that the complexity of f can be estimated by the degree of any LSS defining f (cf Theorem 1 b)). We will give an improvement of this result and will also show how to compute $c(f)$ in term of the LSS, for this we need the following lemma.

Lemma 1 *Let $G(X,T), F(X,T) \in K[X,T]$ be polynomials of degree d and m respectively s.t. F is a factor of G and let $h(X) \in K[[X]]_{alg}$ be s.t. $G(X, h(X)) = 0$. Then if the Taylor expansion of $F(X, h(X))$ vanishes up to order dm we have that $F(X, h(X)) = 0$.*

Proof. In fact take an irreducible factor G_1 of G with $G_1(X, h(X)) = 0$, if $\{F = 0\}$ and $\{G_1 = 0\}$ do not have a common component there exists a set of linear forms H_j through the origin such that $\{F = 0\} \cap \{G_1 = 0\} \cap \{H_1 = ... = H_{n-1=0}\}$ is a finite set of points, whose multiplicity at the origin is greater than dm, in contradiction with Bézout theorem.

Proposition 2 *Let* $\mathbf{F}$ *a LSS defining $f_1, ..., f_r$. Then we have:*
a) $c(f_i) \leq deg(F) \; \forall i$
b) $d(f_1, ..., f_r) \leq \prod_i^r c(f_i)$
And there is an algorithm to compute $c(f_i) \; \forall i$.

Proof. Let $V_F = V(F_1, ..., F_r) \subset K^{n+r}$, $W_i = Zar.cl.\{(X, f_i(X)), X \in K^n\} \subset K^{n+1}$, $W = Zar.cl\{(X, f_1(X), ..., f_r(X)), X \in K^n\} \subset K^{n+r}$, and $\pi_i : K^{n+r} \longrightarrow K^{n+1}$ the projection $(X, Y_1, ..., Y_r) \mapsto (X, Y_i)$. Then

$$\begin{array}{ccccc} W & \subset & V_F & \subset & K^{n+r} \\ & & \downarrow & & \downarrow \\ W_i & \subset & \pi_i(V_F) & \subset & K^{n+1} \end{array}$$

So, $c(f_i) \leq deg(\pi_i(V_F) \leq deg(V_F)deg(F)$, by Bézout inequality.

As for b) let $V_i = W_i \times K^{r-1} \subset K_{n+r}$, then we observe that $deg(V_i) = c(f_i)$ and that W is an irreducible component of $W' = V_1 \cap ... \cap V_r$, and apply Bézout inequality. Moreover: $\pi_i(V_F)$ is defined by the ideal $J = (F_1, ..., F_r) \cap K[X, Y_i]$ which can be calculated by an elimination Groebner basis computation. Now, J is a principal ideal, say $J = (Q)K[X, Y_i]$. Therefore we only have to compute the irreducible factor Q' of Q such that $Q'(X, f_i(X)) = 0$. This can be done using Lemma 4.

Corollary 1 *Let* $\mathbf{F}$ *a LSS defining* $f_1, ..., f_r$ *,* $h = \sigma_F(H)$ *with* $H \in K[X,Y]$ *,* $d = deg(F)$ *and* $m = deg(H)$*, then :*

i) $o(h) \leq c(h) \leq md$

ii) $o(f_i) \leq c(f_i) \leq d \forall i$ *provided* $f_i \neq 0$*.*

Remark 3 *In the Example above, we have:* $e(f_1) = e(f_2) = 1$ $c(f_1) = 4$ *;* $\xi(f_1) = (1,4)$*;* $c(f_2) = 5$ *;* $\xi(f_2) = (1,5)$*;* $d(f_1, f_2) = 5$ *;* $\xi(f_1, f_2) = (2,6)$ *;* $\delta(f_1, f_2) = 6$*.*

Moreover,let us consider: $f(X) = X\sqrt{1-X}$ *;* $g(X) = 1 - \sqrt{1-X}$*. Then* $c(f) = 3$ *,* $e(f) = 2$ *and* $\xi(f) = \xi(f,g) = (2,4)$*.*

5 Standard Locally Smooth Systems

In [AMR] an important role in order to perform computations in the ring $K[[X]]_{alg}$ has been played by the notion of standard locally smooth system (SLSS) We recall (in a simplified version) this definition: $\mathbf{G}$ is a standard locally smooth system (SLSS) if:

1) $\mathbf{G} = (G_1, ..., G_r)$ is a LSS for the functions $f_1, ..., f_r$

2) $f_i \neq 0 \forall i$

3) $G_i = Y_i(1 + Q_i) - R_i$ with Q_i , $R_i \in (X, Y)$,
$R_i \in K[X, Y_1, ..., Y_{i-1}, Y_{i+1}, ..., Y_r]$ and $in(R_i) = in(f_i) \in K[X]$.

The introduced notion turns out to be quite important both for its own sake, since it directly gives explicit information on the $f_i's$, and because it is a computational tool for standard bases, Weierstrass Preparation Algorithm and elimination algorithms (cf [AMR]). In this section we look for bounds for the costs of SLSS's.

To do this, we introduce the complexity ξ' of $\{f_1, ..., f_s\}$ as the minimum of the cost($\mathbf{H}$) where the $f_i's$ are defined via a SLSS $\mathbf{H}$, and we write:

$$\xi'(f_1, ..., f_s) = (\lambda'(f_1, ..., f_s), \delta'(f_1, ..., f_s)) = (length(\boldsymbol{H'}), deg(\boldsymbol{H'})),$$

where $\mathbf{H}'$ reaches the minimum. We further assume that $f_i \neq 0 \forall\, i$ and that $f_1, ..., f_s$ are linearly independent. We also introduce the natural numbers

$\Delta = Sup\{d_i, d_i = deg(F_i)\}$

$\Omega = \prod_{i=1}^{s} o(f_i)$.

Lemma 2 *Let* $\mathbf{F} = (F_1, ..., F_r)$ *be a LSS defining* $f_1, ..., f_r$ *, where* $f_i \neq 0$ *for every i, and let* $\mathbf{G} = (G_1, ..., G_r)$ *be a SLSS obtained by* $\mathbf{F}$ *, then*

$o(f_i) \leq deg(G_i) \leq \Delta + o(f_i) - 1$

Proof. G_i is obtained by the corresponding F_i, as follows: while there is a term t, $deg(t) \leq o(f_i)$ depending on some Y_j, we substitute an occurence of Y_j by $Y_j - F_j$. Each such substitution can at worst introduce terms of degree $o(f_i) + \Delta - 1$.

Proposition 3 *With the above notation, we have:*

a) $\lambda' = \lambda$

b) $Sup\{\Omega, \delta(f_1, ..., f_s)\} \leq \delta'(f_1, ..., f_s) \leq Inf\{2^\lambda \Delta^\lambda, \Delta^\lambda \delta(f_1, ..., f_s)\}$.

Proof. Let us take **F** minimal and **G** obtained from **F** as in the above Lemma. Let $d_i = degG_i$. Then $d_i \leq o(f_i) + \Delta - 1 \leq d_i + \Delta - 1$ and $\prod_{h=1}^{\lambda}(d_h + \Delta - 1) \leq Inf\{2^\lambda \Delta^\lambda, \Delta^\lambda \delta(f_1, ..., f_s)\}$.

6 Weierstrass Preparation Theorem

In [AMR] §5 we gave constructive versions of Weierstrass Preparation and Division theorems, we are going now to bound the complexity of these constructions.

Let $X' = (X_1, ..., X_{n-1})$ so $X = (X', X_n)$ and let g be an algebraic series distinguished in X_n , say $g(0, X_n) = X_n^b +$ *higher degree terms*, then there exist a unit v and series $h_i \in K[[X']]_{alg}$ such that $g = v(X_n^b + \sum_{i<b} h(X')X_n^i)$. Let us represent data and output in our computational model.

DATA: a LSS $\mathbf{F_0}$ defining series $f_{0,i}$ with $cost(F_0) = (r, d_0)$, $deg(F_{0,i}) = d_{0,i}$ and $D_0 = sup\{d_{0,i}\}$; $G_0 \in K[X, Y_1, ..., Y_r]$ such that $\sigma_{F_0}(G) = g$, $deg(G_0) = m$ and $o(g(0, X_n)) = b$

OUTPUT: a LSS $\mathbf{H} :=$ $(H_{1,0}, ..., H_{r,0}, H_0, H_{1,1}, ..., H_{r,1}, H_1, ..., H_{1,b-1}, ..., H_{r,b-1}, H_{b-1})$ in $K[X', U_{1,0}, ..., U_{r,0}, U_0, U_{1,1}, ..., U_{r,1}, U_1, ..., U_{1,b-1}, ..., U_{r,b-1}, U_{b-1}]$ defining algebraic series $h_{1,0}, ..., h_{r,0}, h_0, h_{1,1}, ..., h_{r,1}, h_1, ..., h_{1,b-1}, ..., h_{r,b-1}, h_{b-1} \in K[[X']]_{alg}$ and such that $h_i = \sigma_H(U_i)$ for all i's.

Theorem 3 *Let* $D = sup\{b(m+1), b(2\Delta_0 + 1)\}$,
then $cost(H) \leq (b(r+1), D^{b(r+1)})$.

Proof. Let us recall from [AMR] the lines of the algorithm.

(1) Construct a new LSS $\mathbf{F_2}$ defining series $f_{2,i}$ which differ from the $f_{0,i}$'s only by polynomials of degrees $\leq b$ and such that $T(f_{0,i}) > X_n^b$ and a new $\mathbf{G_2}$ such that $\sigma_{F_2}(G) = g$.

(2) Let **F** the SLSS costructed by $\mathbf{F_2}$ and G such that $\sigma_F(G) = g$.

(3) Let $U = (U_{1,0}, ..., U_{r,0}, U_0, U_{1,1}, ..., U_{r,1}, U_1, ..., U_{1,b-1}, ..., U_{r,b-1}, U_{b-1})$ a new set of variables and let

- $P := X_n^b - \sum_{j=0}^{b-1} U_j X_n^j$
- $P_i := Y_i - \sum_{j=0}^{b-1} U_{ij} X_n^j \ \forall\, i = 1, ..., r.$

(4) Apply Buchberger reduction (with respect to a suitable term ordering) to $G, F_1, ..., F_r$, we obtain polynomials $H_0, ..., H_{d-1}, H_{1,0}, ..., H_{1,b-1}, ..., H_{r,0}, ..., H_{r,b-1} \in (X_1, ..., X_{n-1}, U)K[X_1, ..., X_{n-1}, U] = K[X', U]$ s.t.:

- $G - \sum_{j=0}^{b-1} U_j X_n^j \in (P, P_1, ..., P_r)$
- $F_i - \sum_{j=0}^{b-1} U_{ij} X_n^j \in (P, P_1, ..., P_r) \; \forall \; i$

Let us now examinate the costs of these constructions:

(1) $length(F_2) \leq r$; $deg(F_{2,i}) \leq bd_{0,i}$; $\Delta_2 = sup\{deg(F_{2,i})\} \leq b\Delta_0$ and $deg(G_2) \leq bm$.

(2) $length(F) \leq r$; $deg(F_i) \leq 2b\Delta_0$;$G = G_2$ and $\Delta := sup\{deg(F_i), deg(G)\} \leq sup\{bm, 2b\Delta_0\}$

(4) $length(H) \leq b(r+1)$, $deg(H_i)$ and $deg(H_{ij})$ are bounded by $\Delta + b$ and therefore $degH) \leq (\Delta + b)^{b(r+1)} \leq D^{b(r+1)}$.

We conclude this section remarking that, by the above result we obtain a single exponential complexity also for the Weierstrass Division Theorem, however the main drawback consists in the large number of extra variables needed to represent the Weierstrass polynomials: by an initial form computation we can cancel those which are zero, in the next section we propose a test to check if they are in fact rational functions.

7 Applications

We apply now the above results, and precisely the Proposition 2 and the Corollary 1, in order to obtain some informations on the rationality of power series. This is clearly of great interest since, as we have remarked in the previous section, the cost grows exponentially on the number of involved functions.

Let f be a power series in X, $f = \sum_{a \in IN^n} f_a X^a$, with $f_a \in K$; then we write $f_{(i)} := \sum_{|a|=i} f_a X^a$, so that $f = \sum_{i=0}^{\infty} f(i)$.

Proposition 4 *Let $h \in K[[X]]_{alg}$ such that $c(h) \leq t$, then: $h \in K[X] \iff h_{(j)} = 0 \; \forall j : t < j \leq t^2$.*

Proof. The same proof of Prop. 2.7 of [AMR] works.

Corollary 2 *It is possible to check whether $h \in K[X]_{alg}$ is a rational function.*

Proof. Let $Q \in K[X, T]$ be an irreducible polynomial such that $Q(X, h(X)) = 0$. Let $s = degT(Q) \leq c(h) = deg(Q) \leq t$ and write

$$Q(X, T) = a_0(X)T^s + a_1(X)T^{s-1} + ... + a_{s-1}(X)T + a_s(X) \in K[X, T]$$

Let us further introduce the following polynomial $Q* \in K[X, T]$:
$Q*(X, T) = a_0^s Q(X, \frac{T}{a_0}) = T^s + a_1 T^{s-1} + a_0 a_2 T^{s-2} + ... + a_0^{s-2} a_{s-1} T + a_0^{s-1} a_s = T^s + \sum_{i=1}^{s} a_0^{i-1} a_i T^{s-i}$ Then we obtain that $deg(Q*) \leq s(t - s + 1) \leq \frac{(t+1)^2}{4}$ .Let us consider $k = ha_0 \in K[[X]]_{alg}$. Now, if $h(X) = \frac{f(X)}{g(X)} \in K[X]_{loc}$, it is easy to see that g is a factor of a_0 and therefore we obtain that $k(X) = \frac{a0(X)}{g(X)} f(X) \in K[X]$. It is straightforward to see that k is a root of $Q*$. Then

$c(k) \leq \frac{(t+1)^2}{4}$. Conversely, suppose that k is a polynomial root of $Q*$, then $h = \frac{k}{a_0} \in K(X) \cap K[[X]]_{alg} = K[X]_{loc}$. In order to apply Proposition 4 , we need to check whether $k(j) = 0$ for $\frac{(t+1)^2}{4} \leq j \leq \frac{(t+1)^4}{16}$, this can be done using suitable linear systems (with the coefficients of a_0 as unknowns), once we know the Taylor expansion of h up to degree $\frac{(t+1)^4}{16}$ and we know that $deg(a_0) \leq t-s \leq t$. More precisely we write a_0 as a polynomial of degree $t-1$ (or if is possible we use a better estimate) with unknown coefficients U_a with $a = (a_1, ..., a_n)$. We then consider starting from $k(b) := \frac{\partial^\beta k}{\partial X^\beta}$ where $b = |b| = [\frac{(t+1)^2}{4}]$ an enough number of equations (at least $\geq max(\begin{pmatrix} t+n-1 \\ n \end{pmatrix}, b^2 - b))$ involving the unknowns U and the Taylor coefficients of h up to degree d^2 . The problem is then reduced to compute whether this system has a non-zero solution, i.e. to compute the vanishing of a suitable determinant

Remark 4 *If we dispose of an efficient factorization modulus, we can test whether an algebraic function is polynomial (resp. rational) in the following way.*

1. *Given f produce a polynomial $Q(X,T)$ over which it vanishes.*
2. *Factorize Q to check whether it has a polynomial factor which is linear in T.*

   Then check whether it is of the form: $T - f(x)$.
3. *If not, construct $Q*$ and factorize it, etc.as in 2).*

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