Dodecagonal Tilings as Maximal Cluster Coverings

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Abstract

It is shown that the Socolar tiling, which is quasiperiodic and 12-fold symmetric, can be characterized as the unique tiling which is *maximally covered* by a suitably pair of clusters. Analogous results can be obtained also for other dodecagonal tilings, among them the shield tiling.

1. Introduction

Certain quasiperiodic tilings can be characterized as being the unique tilings which are maximally covered by a single, well chosen cluster. Notable examples are the decagonal Penrose tiling[1,2] and the octagonal Ammann-Beenker tiling.[3,4] While in the case of the Penrose tiling, every tiling that is covered by the cluster is a perfect Penrose tiling, the *maximal* covering property is really necessary in the octagonal case, where only the maximally covered tilings have octagonal symmetry. More details can be found in a recent review[5] of such cluster models. The maximal covering property suggests an explanation for the formation and stability of quasicrystals based on these tilings. If one assumes that the covering cluster is an energetically favourable local configuration, the maximally covered structures appear to be the most favourable global configurations.

In this paper, it will be shown that popular dodecagonal tilings, in particular the Socolar tiling and the shield tiling, can be characterized in a similar way as the octagonal Ammann-Beenker tiling, even though two covering clusters are needed in this case. As in the octagonal case, a two-step procedure has to be used. In a first step, it is shown that all tilings that are covered by the two clusters satisfy the so-called alternation condition. The set of these tilings comprises a one-parameter family of perfectly ordered tilings with at least hexagonal symmetry, among them the desired dodecagonal tiling. In a second step, it is then shown that the dodecagonal tiling is the unique tiling in this family with the highest cluster density.

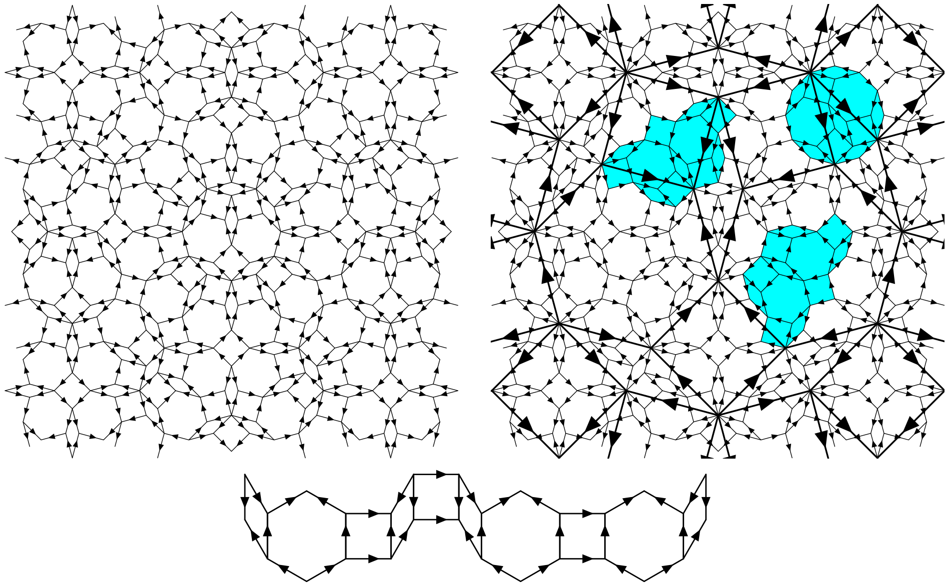

Figure 1: The Socolar tiling (top left), with its inflation superimposed (top right). The arrowing ensures that the alternation condition is satisfied: along any lane of tiles the two types of rhombi must alternate (bottom).

2. The Socolar Tiling

The dodecagonal Socolar tiling[6] (Fig. 1) can be obtained by the projection method, or as the dual of two superimposed triangular grids. It satisfies the alternation condition, which is enforced by the arrowing of the tile edges (Fig. 1). Besides the alternation condition, there is a further local constraint to be satisfied. This becomes apparent if an inflated tiling is superimposed (Fig. 1). The edges of the inflated tiles have an asymmetric environment. There is a dodecagon cluster which is either on the right or on the left of the edge. We may say that square edges buckle inwards, rhombus edges buckle outwards, and the hexagon has opposite edges of different type. In order that an arrowed tiling can be inflated, it must satisfy the local constraint that outward-buckling edges match only inward-buckling edges, and vice versa. Note that the edge types break the single mirror line of the arrowed hexagon, so that there are now left hexagons and right hexagons. Taking the buckling into account, arrowed hexagons are thus completely asymmetric. They the have exactly the same asymmetry as inflated hexagons. A tiling satisfying both the alternation condition and the local constraint can be inflated infinitely often.

The octagonal Ammann-Beenker tiling satisfies a very similar alternation condition. Katz[8] has shown that every square-rhombus tiling satisfying this octagonal alternation condition is perfectly ordered (quasiperiodic or periodic), and has at least four-fold symmetry. A closer inspection of the proof by Katz shows[9] that it can be transferred to the dodecagonal case: every square-hexagon-rhombus tiling satisfying the local constraint and the alternation condition is the dual of two superimposed triangular grids, which are rotated by 90° with respect to each other. These triangular grids may have different scales, so that the symmetry of the tiling is only hexagonal. If the scale of the two grids is the same, the dodecagonal Socolar tiling is obtained. This implies that all tilings admitted by the alternation condition and the local constraint are quasiperiodic (or periodic) and at least hexagonally symmetric. The simplest such tiling is the one with only hexagons.

3. Cluster Covering and Cluster Densities

The two clusters shown in Fig. 1 cover the entire Socolar tiling. This is best seen at the level of the superimposed inflated tiling. The dodecagon cluster alone covers already most of the tiling. The only bits that remain uncovered are a little square inside each big square, and a little hexagon and three little squares inside each big hexagon. There is one dodecagon cluster centre near the middle of each inflated tile edge (but not directly on the edge). Every dodecagon cluster can be assigned to a unique inflated tile. There are two dodecagon clusters per inflated rhombus, and three per inflated hexagon. Inflated squares are not assigned any dodecagon clusters. The remaining uncovered bits of the tiling are then covered by the butterfly cluster (Fig. 1). There is one butterfly cluster per inflated square, and one per inflated hexagon. Each of the two clusters occurs always with exactly the same decoration in the tiling. Since they are arrowed, every tiling that is covered must necessarily satisfy the alternation condition, and due to the structure of the clusters the same must be the case also for the local constraint. Using the results of the previous section, we now know that every tiling that is covered by the two clusters is a quasiperiodic inflated tiling of at least hexagonal symmetry.

It remains to prove that among these tilings, the dodecagonal one has the highest cluster density. Recall that each cluster is assigned to a unique inflated tile, and each such tile of a given kind carries the same number of clusters. Moreover, the inflated tiling is the dual of two superimposed triangular grids, so that we can compute the densities of the inflated tiles, which are parametrized by the ratio $1+x$ of the gridline spacings of the two grids. This parametrization is chosen such that the dodecagonal tiling corresponds to $x=0$. From these tile densities, it is then easy to compute the cluster densities. In suitable units, we obtain:

$$\rho_{\rm dod}(x) = \frac{(6\sqrt{3}+48)(1+x)+3\sqrt{3}x^2}{1+x+x^2/6}$$

$$\rho_{\rm bfl}(x) = \frac{(2\sqrt{3}+6)(1+x)+\sqrt{3}x^2}{1+x+x^2/6}$$

$\rho_{\rm dod}(x)$ has a maximum for $x=0$, but $\rho_{\rm bfl}(x)$ has a minimum there. However, the weighted cluster density

$$\rho(x) = w_{\rm dod}\rho_{\rm dod}(x) + w_{\rm bfl}\rho_{\rm bfl}(x)$$

has a maximum at the dodecagonal point $x = 0$, provided the weights satisfy the mild condition $w_{\mathrm{bfl}}/w_{\mathrm{dod}} < 12 + 10\sqrt{3}$.

4. Discussion and Conclusion

We have shown that square-hexagon-rhombus tilings that are completely covered by the dodecagon and the butterfly cluster satisfy the alternation condition and the local constraint. They are thus quasiperiodic and hexagonally symmetric. Among these tilings, the dodecagonal Socolar tiling has the highest weighted cluster density, provided a mild constraint on the cluster weights is satisfied. This does not rigorously rule out the possibility that there are tilings which are not completely covered, but have an even higher weighted cluster density. Such a scenario seems rather unlikely, however.

The present results are very analogous to what has been obtained for the octagonal Ammann-Beenker tiling.[3] There is one key difference, however. For the dodecagonal Socolar tiling we need two covering clusters, whereas for the octagonal tiling (as well as for the Penrose tiling) one cluster was sufficient. By taking one larger cluster, it is possible reduce the fraction of the uncovered area, even as far as one wants,[10] but it does not seem possible to cover the whole tiling with one cluster of finite size. Whether such an almost covering can enforce an ordered tiling is unkown, however.

These results can be transferred to other dodecagonal tilings which are mutually locally derivable[11] (or locally equivalent) with the Socolar tiling. There are then direct analogues of the dodecagon cluster and the butterfly cluster. Such a pair of cluster has been found for the shield tiling. The only non-trivial part is to find a decoration that enforces the analogue of the alternation condition in the shield tiling.

References

1. Gummelt, P., in *Proc. 5th Int. Conf. on Quasicrystals* eds. Janot, C., and Mosseri, R., World Scientific, Singapore, p. 84 (1995).
2. Gummelt, P., Geometriae Dedicata **62**, 1 (1996).
3. Gähler, F., in *Proc. 6th Int. Conf. on Quasicrystals*, eds. Takeuchi, S., and Fujiwara, T., World Scientific, Singapore, p. 95 (1998).
4. Ben-Abraham, S.I., and Gähler, F., Phys. Rev. B**60**, 860 (1999).
5. Gähler, F., to appear in Mat. Sci. Eng. A (2000).
6. Socolar, J.E.S., Phys. Rev. B**39** 10519 (1989).
7. Socolar, J.E.S., Commun. Math. Phys. **129** 599 (1990).
8. Katz, A., in *Beyond Quasicrystals*, eds. Axel, F., and Gratias, D., Les Editions de Physique and Springer Verlag, Les Ulis, Berlin, Heidelberg, p. 141 (1995).
9. Gähler, F., unpublished.
10. Ben-Abraham, S.I., Gummelt, P., Lück, R., and Gähler, F., *Dodecagonal Tilings Almost Covered by a Single Cluster*, this volume.
11. M. Baake, M. Schlottmann, and P.D. Jarvis, J. Phys. A: Math. Gen. **24** (1991) 4637.