1 SOME EXTENSIONS OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION EQUATIONS IN MULTIDIMENSIONS

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ABSTRACT

The local discontinuous Galerkin method has been developed recently by Cockburn and Shu for convection-dominated convection-diffusion equations. In this paper, we extend the method to multidimensional equations with non-periodic boundary conditions, and with a positive semi-definite diffusion coefficient which may depend on space and time. Stability and *a priori* error estimates are derived.

Key words. discontinuous Galerkin method, convection-diffusion equation

1.1 INTRODUCTION

The local discontinuous Galerkin (LDG) method was developed recently by Cockburn and Shu [11] for convection-dominated convection-diffusion equations. This method is a generalization of the method introduced by Bassi and Rebay [2] for the compressible Navier-Stokes equations which is in turn an extension of the so-called Runge-Kutta discontinuous Galerkin methods for conservation laws developed by Cockburn and Shu in a series of papers [6, 10, 9, 12, 8]; see the review on the development of discontinuous Galerkin methods by Cockburn, Karnidakis and Shu [7]. The LDG method is similar in nature to the upwind-mixed or Godunov-mixed methods developed by Dawson *et al* [14, 15, 16, 13]. Both methods can be viewed in an operator splitting context, where a high resolution upwind method is used for advection, combined with some type of finite element method for diffusion. In the case of the LDG method, a discontinuous Galerkin approach is used for diffusion, and in the case of the upwind-mixed method, a mixed finite element method approximates diffusion.

Both of these methods have the nice properties that they are based on conserving mass locally over each element and they approximate sharp fronts accurately and with minimal oscillation; these methods are easily extendible to nonlinear convection-diffusion systems. Moreover, the LDG method can easily be extended to higher-order polynomials, can be defined on any grid including non-conforming or non-matching grids, and also easily allows one to vary the degree of the approximating space from one element to the next.

In [11], the LDG method is described and analyzed for convection-diffusion equations with periodic boundary conditions. It is proven that, for general regular triangulations and approximate solutions using polynomials of degree k in each element, the upper bound of the 'energy norm' is of order h^k for the general convection-diffusion case and of order $h^{k+1/2}$ in the purely hyperbolic case; h stands for the mesh parameter. In the purely parabolic case, for uniform Cartesian grids and approximate solutions using tensor products of polynomials of degree k, the rate of convergence is of order h^{k+1} or h^k for k odd or even, respectively; this result holds for the special choice of numerical fluxes used by Bassi and Rebay [2]. Castillo [5] considered the one-dimensional bounded domain case and identified a special numerical flux for which the improved rate of convergence of order $k+1$ is obtained for arbitrary meshes when the approximate solution uses polynomials of degree k.

In this paper, we extend the work done in [11] in two ways: (i) We consider non-periodic boundary conditions in a multi-dimensional setting, and (ii) we consider the case in which the diffusion/dispersion tensor depends on (x,t). Thus, we focus on the following standard transport equation,

$$\phi\, c_t + \nabla \cdot (uc - D\nabla c) = \phi\, f, \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d = 1$, 2, or 3, and $T > 0$. In porous media applications, see, for example, [3], $c(x,t)$ represents the concentration of some chemical component, $\phi(x)$ is the porosity of the medium and may also include adsorption effects, $u(x,t)$ is the Darcy velocity, $D(x,t)$ is a diffusion/dispersion tensor assumed to be symmetric and at worst positive semi-definite, and $f = f(x,t)$ is a source term. We will assume positive constants ϕ_*, ϕ^* exist such that

$$\phi_* \le \phi(x) \le \phi^*, \tag{1.2}$$

and that the Darcy velocity u satisfies the continuity equation

$$\nabla \cdot u = 0. \tag{1.3}$$

The initial condition for the concentration is the following:

$$c(x,0) = c^0(x), \quad \text{on } \Omega. \tag{1.4}$$

For boundary conditions, let n denote the unit outward normal to $\Gamma \equiv \partial\Omega$. We write $\Gamma = \overline{\Gamma}_I \cup \overline{\Gamma}_O$, where

$$\Gamma_I = \{x \in \partial\Omega : u \cdot n < 0\}, \tag{1.5}$$

and

$$\Gamma_O = \{x \in \partial\Omega : u \cdot n \geq 0\}. \tag{1.6}$$

On these boundaries we assume the following conditions hold:

$$(uc - D\nabla c) \cdot n = uc_I \cdot n, \quad \text{on } \Gamma_I, \tag{1.7}$$

where c_I is specified, and

$$(D\nabla c) \cdot n = 0, \quad \text{on } \Gamma_O. \tag{1.8}$$

We assume the coefficients, initial and boundary data, and domain Ω are sufficiently smooth so that a unique solution c exists for the problem above.

The paper is outlined as follows. In the next section, we define the LDG method in continuous time. In Section 3, we analyze the stability of this method and in Section 4, we derive an *a priori* error estimate. We end in Section 5 with some concluding remarks.

1.2 METHOD FORMULATION

The weak formulation. Let us motivate the weak formulation we will use to define the LDG method. We start by rewriting (1.1) in the following mixed form:

$$\phi\, c_t + \nabla \cdot (uc + z) = \phi\, f, \tag{1.9}$$

$$\tilde{z} = -\nabla c, \tag{1.10}$$

and

$$z = D\tilde{z}. \tag{1.11}$$

and by rewriting the boundary conditions accordingly, that is,

$$(uc + z) \cdot n = (uc_I) \cdot n, \ \text{ on } \Gamma_I \tag{1.12}$$

and

$$z \cdot n = 0, \ \text{ on } \Gamma_O. \tag{1.13}$$

Note that the introduction of the auxiliary variables $\tilde{z}$ and z allows us to treat tensors D that are not invertible; this is not possible to do in the standard mixed approach. This notion was introduced in [1] for the mixed finite element method applied to elliptic equations.

Next, to obtain our weak formulation, we simply multiply the above equations by test functions and integrate on each finite element. To describe this procedure, we need to introduce some notation. Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of finite element partitions of Ω such that no element Ω_e crosses the boundaries of Γ_I or Γ_O, where h_e is the element diameter and h the maximal element diameter; each element Ω_e has a Lipschitz boundary $\partial\Omega_e$. For any function w in $W_e = H^1(\Omega_e)$, we denote its trace on $\partial\Omega_e$ by w^-. Finally, we denote the $L^2(\tilde{\Omega})$ inner product by $(\cdot,\cdot)_{\tilde{\Omega}}$, where we omit $\tilde{\Omega}$ if $\tilde{\Omega} = \Omega$. To distinguish integration over domains $\tilde{\Omega} \in \mathbb{R}^{d-1}$, e.g., surfaces or lines, we will use the notation $\langle\cdot,\cdot\rangle_{\tilde{\Omega}}$; we denote by n_e the unit outward normal vector to $\partial\Omega_e$, with $n_e = n$ on Γ. With this notation, and assuming that c, $\tilde{z}$, and z are smooth enough, we obtain the following weak formulation of equations (1.9)-(1.13):

$$
\begin{aligned}
&(\phi\, c_t, w)_{\Omega_e} - (u\, c + z, \nabla w)_{\Omega_e} + \langle (c\, u + z)\cdot n_e, w^-\rangle_{\partial\Omega_e/\Gamma} \\
&+\langle c\, u\cdot n, w^-\rangle_{\partial\Omega_e\cap\Gamma_O} = -\langle c_I\, u\cdot n, w^-\rangle_{\partial\Omega_e\cap\Gamma_I} + (\phi\, f, w)_{\Omega_e}, \quad w \in W_e,
\end{aligned}
\tag{1.14}
$$

$$
(\tilde{z}, v)_{\Omega_e} - (c, \nabla\cdot v)_{\Omega_e} + \langle c, v^-\cdot n_e\rangle_{\partial\Omega_e} = 0, \quad v \in (W_e)^d,
\tag{1.15}
$$

and

$$
(z, \tilde{v})_{\Omega_e} = (D\tilde{z}, \tilde{v})_{\Omega_e}, \quad \tilde{v} \in (W_e)^d.
\tag{1.16}
$$

The LDG method. Let $W_{h,e} \subset W_e$ denote the set of all polynomials of degree at most k_e defined on Ω_e. On Ω_e, we approximate $c(\cdot,t)$ by $C(\cdot,t) \in W_{h,e}$, $z(\cdot,t)$ by $Z(\cdot,t) \in (W_{h,e})^d$, and $\tilde{z}(\cdot,t)$ by $\tilde{Z}(\cdot,t) \in (W_{h,e})^d$. To define C, Z, and $\tilde{Z}$, we simply have to use the weak formulation (1.14)-(1.16). However, we see that we have terms involving c and z on $\partial\Omega_e$. Since C and Z are discontinuous across these boundaries, we must define how we approximate these terms. For $x \in \partial\Omega_e$ we define

$$
w^-(x) = \lim_{s\to 0^-} w(x + sn_e), \qquad w^+(x) = \lim_{s\to 0^+} w(x + sn_e), \qquad \overline{w} = \frac{1}{2}(w^+ + w^-).
$$

We approximate the value of C on $\partial\Omega_e/\Gamma$ by the so-called "upwind value" of C which is defined as follows:

$$
C^u = \begin{cases} C^-, & u\cdot n_e \geq 0, \\ C^+, & u\cdot n_e < 0. \end{cases}
\tag{1.17}
$$

The value of Z on $\partial\Omega_e/\Gamma$ is approximated by $\overline{Z}$. Finally, the value of C on $\partial\Omega_e \cap \Gamma_O$ is simply approximated by C^-. This takes care of the integrals of the equation (1.14). The approximation of the value of C on $\partial\Omega_e$ in equation (1.15) is taken to be equal to $\overline{C}$ on $\partial\Omega_e/\Gamma$ and equal to C^- on $\partial\Omega_e \cap \Gamma$.

We are now ready to formulate the LDG method. At $t = 0$ we define $C(\cdot,0) \equiv C^0 \in W_{h,e}$ by

$$
(\phi(C^0 - c^0), w)_{\Omega_e} = 0, \quad w \in W_{h,e}.
\tag{1.18}
$$

For each $t > 0$, $C(\cdot,t)$, $Z(\cdot,t)$ and $\tilde{Z}(\cdot,t)$ are determined by the following equations:

$$(\phi\, C_t, w)_{\Omega_e} - (uC + Z, \nabla w)_{\Omega_e} + \langle (C^u\, u + \overline{Z}) \cdot n_e, w^- \rangle_{\partial\Omega_e/\Gamma} \\ + \langle C^- u \cdot n, w^- \rangle_{\partial\Omega_e \cap \Gamma_O} = -\langle c_I u \cdot n, w^- \rangle_{\partial\Omega_e \cap \Gamma_I} + (\phi\, f, w)_{\Omega_e}, \quad w \in W_{h,e}, \tag{1.19}$$

$$(\tilde{Z}, v)_{\Omega_e} - (C, \nabla \cdot v)_{\Omega_e} + \langle \overline{C}, v^- \cdot n_e \rangle_{\partial\Omega_e/\Gamma} + \langle C^-, v^- \cdot n \rangle_{\partial\Omega_e \cap \Gamma} = 0 \quad v \in (W_{h,e})^d, \tag{1.20}$$

and

$$(D\tilde{Z}, \tilde{v})_{\Omega_e} - (Z, \tilde{v})_{\Omega_e} = 0, \quad \tilde{v} \in (W_{h,e})^d. \tag{1.21}$$

Note that by (1.20), $\tilde{Z}$ can be eliminated *locally* in terms of C, and by (1.21), Z can be expressed *locally* in terms of $\tilde{Z}$. These relations can be substituted into (1.19), giving an equation for C alone. The stencil for any element Ω_e involves the neighbors of Ω_e, that is, those elements which share an edge with Ω_e, and the neighbors of the neighbors. Thus, on a conforming triangular mesh, for example, the unknowns in Ω_e are related to unknowns in at most nine neighboring elements.

1.3 STABILITY ANALYSIS

To develop an *a priori* error estimate for the scheme derived above, we need to examine its stability. The stability proof below also demonstrates the existence and uniqueness of the numerical solution. We proceed in three steps.

Step 1. First, we rewrite the LDG method (1.18)–(1.21) in compact form. To do that, we add all the left hand sides of equations (1.19)–(1.21), add over all the elements of the partition of Ω, integrate in time from 0 to T and call the result $\mathcal{B}(C, Z, \tilde{Z}; w, v, \tilde{v})$. Then equations (1.19)–(1.21) can be rewritten in compact form as follows:

$$\mathcal{B}(C, Z, \tilde{Z}; w, v, \tilde{v}) = -\int_0^T \langle c_I\, u \cdot n_e, w^- \rangle_{\Gamma_I}\, dt + \int_0^T (\phi\, f, w)\, dt, \tag{1.22}$$

and they have to hold for each test function $(w, v, \tilde{v})$ in $\mathcal{C}^0(0, T; W_h)$, where, of course, $W_h = \{(w, v, \tilde{v}) : \text{On each element } \Omega_e \in \mathcal{T}_h, (w, v, \tilde{v}) \in W_{h,e} \times (W_{h,e})^d \times (W_{h,e})^d\}$.

Now, to obtain our stability result we simply have to set $w = C$, $v = Z$, and $\tilde{v} = \tilde{Z}$ and perform some simple manipulations.

Step 2. By construction, we have that $B(C, Z, \tilde{Z}; C, Z, \tilde{Z}) = \Theta_1 + \Theta_2 + \Theta_3$, where

$$\Theta_1 = \int_0^T \sum_e \left[(\phi\, C_t, C)_{\Omega_e} + (D\tilde{Z}, \tilde{Z})_{\Omega_e} \right] dt,$$

$$\Theta_2 = \int_0^T \sum_e \left[-(C\, u, \nabla C)_{\Omega_e} + \langle C^u\, u \cdot n_e, C^- \rangle_{\partial\Omega_e/\Gamma} + \langle C^-\, u \cdot n_e, C^- \rangle_{\partial\Omega_e \cap \Gamma_O} \right] dt,$$

and

$$\Theta_3 = \int_0^T \sum_e \Big[-(Z, \nabla C)_{\Omega_e} - (C, \nabla \cdot Z)_{\Omega_e} + \langle \overline{Z} \cdot n_e, C^- \rangle_{\partial\Omega_e/\Gamma} \\ + \langle \overline{C}, Z^- \cdot n_e \rangle_{\partial\Omega_e/\Gamma} + \langle C^-, Z^- \cdot n_e \rangle_{\partial\Omega_e \cap \Gamma} \Big] dt.$$

It is very simple to realize that

$$\Theta_1 = \frac{1}{2}\| \phi^{1/2}\, C(T) \|^2 + \int_0^T \| D^{1/2}\, \tilde{Z} \|^2 \, dt - \frac{1}{2}\| \phi^{1/2}\, C(0) \|^2 \tag{1.23}$$

where $\| f \|^2 = \int_\Omega f^2(x)\, dx$.

To deal with Θ_2 is not that simple. Integrating by parts and taking into account the continuity equation (1.3), we get

$$\begin{aligned} \Theta_2 &= \int_0^T \sum_e \Big[-\frac{1}{2}\langle C^-\, u\cdot n_e, C^-\rangle_{\partial\Omega_e/\Gamma} - \frac{1}{2}\langle C^-\, u\cdot n_e, C^-\rangle_{\partial\Omega\cap\Gamma_I} \\ &\qquad + \langle C^u\, u\cdot n_e, C^-\rangle_{\partial\Omega_e/\Gamma} + \frac{1}{2}\langle C^-\, u\cdot n_e, C^-\rangle_{\partial\Omega_e\cap\Gamma_O}\Big]\, dt \\ &= \int_0^T \Big[\frac{1}{2}\langle\, | u\cdot n\,|, (C^-)^2\rangle_\Gamma + \sum_e \langle\, (C^u - \frac{1}{2}C^-)\, u\cdot n_e, C^-\rangle_{\partial\Omega_e/\Gamma} \Big]\, dt. \end{aligned}$$

Next, we would like to rewrite the sum over the elements e in terms of the sum over the set of *interior* edges $\{\, \gamma_l \,\}_l$. To do that, we need to introduce some notation. Let γ_l be an interior edge belonging to $\partial\Omega_e$ then, we set

$$[F] = (-F^+ + F^-)\, n_e.$$

Note that this quantity is well defined independently of the set Ω_e we take as a reference. We can now write that

$$\begin{aligned} \sum_e \langle\, (C^u - \frac{1}{2}C^-)\, u\cdot n_e, C^-\rangle_{\partial\Omega_e/\Gamma} &= \sum_l \langle u\cdot (C^u\,[C] - \frac{1}{2}[C^2]), 1\rangle_{\gamma_l} \\ &= \sum_l \langle u\cdot (C^u\,[C] - \overline{C}\,[C]), 1\rangle_{\gamma_l} \\ &= \sum_l \langle u\cdot [C], C^u - \overline{C}\rangle_{\gamma_l}, \\ &= \frac{1}{2}\sum_l \langle | \, u\cdot n_l\,|, [C]^2\rangle_{\gamma_l}, \end{aligned}$$

since

$$(C^u - \overline{C})\, u\cdot [C] = \frac{1}{2}[C]^2\, |\, u\cdot n_l\,|,$$

where n_l denotes any unit normal to γ_l.

Hence, we get

$$\Theta_2 = \frac{1}{2}\int_0^T \Big[\langle C^-\, |\, u\cdot n\,|, C^-\rangle_\Gamma + \sum_l \langle |\, u\cdot n_l\,|, [C]^2\rangle_{\gamma_l} \Big]\, dt. \tag{1.24}$$

Let us now treat Θ_3. Integrating by parts and taking into account the definition of $\overline{Z}$ and $\overline{C}$, we get

$$\begin{aligned} \Theta_3 &= \int_0^T \sum_e \Big[\langle \overline{Z}\cdot n_e, C^-\rangle_{\partial\Omega_e/\Gamma} + \langle \overline{C}, Z^-\cdot n_e\rangle_{\partial\Omega_e/\Gamma} - \langle C^-, Z^-\cdot n_e\rangle_{\partial\Omega_e/\Gamma} \Big]\, dt \\ &= \int_0^T \sum_e \Big[\langle Z^+\cdot n_e, C^-\rangle_{\partial\Omega_e/\Gamma} + \langle C^+, Z^-\cdot n_e\rangle_{\partial\Omega_e/\Gamma} \Big]\, dt \\ &= 0. \end{aligned} \tag{1.25}$$

Since $B(C,Z,\tilde{Z};C,Z,\tilde{Z}) = \Theta_1 + \Theta_2 + \Theta_3$, using the identities (1.23),(1.24), and(1.25), we get that

$$\begin{aligned} B(C,Z,\tilde{Z};C,Z,\tilde{Z}) &= \frac{1}{2}\| \phi^{1/2}\, C(T) \|^2 + \int_0^T \| D^{1/2}\, \tilde{Z} \|^2 \, dt - \frac{1}{2}\| \phi^{1/2}\, C(0) \|^2 \\ &+ \frac{1}{2}\int_0^T \Big[\langle | \, u\cdot n \, |, (C^-)^2 \rangle_\Gamma + \sum_l \langle | \, u\cdot n_l \, |, [C]^2 \rangle_{\gamma_l} \Big] \, dt. \end{aligned} \tag{1.26}$$

Step 3. Inserting the above identity into (1.22) with $(w, v, \tilde{v}) = (C, Z, \tilde{Z})$ we get, after some simple manipulations,

$$|\!|\!| \, (C, \tilde{Z}) \, |\!|\!|^2 \le \| \phi^{1/2}\, C(0) \|^2 + 2\int_0^T \langle | \, u\cdot n \, |, (c_I)^2 \rangle_{\Gamma_I} \, dt + 2\int_0^T \| \phi^{1/2}\, f \| \, \| \phi^{1/2}\, C \| \, dt.$$

where

$$\begin{aligned} |\!|\!| \, (C, \tilde{Z}) \, |\!|\!|^2 &\equiv \| \phi^{1/2}\, C(T) \|^2 + \int_0^T \| D^{1/2}\, \tilde{Z} \|^2 \, dt \\ &+ \frac{1}{2}\int_0^T \Big[\langle | \, u\cdot n \, |, (C^-)^2 \rangle_\Gamma + \sum_l \langle | \, u\cdot n_l \, |, [C]^2 \rangle_{\gamma_l} \Big] \, dt. \end{aligned} \tag{1.27}$$

Finally, since (1.18) implies that

$$\| \phi^{1/2}\, C(0) \| \le \| \phi^{1/2}\, c^0 \|,$$

we obtain our stability result stated below by a simple application of the following Lemma.

Lemma1 . *Suppose that for all $T > 0$ we have*

$$\chi^2(T) + R(T) \le A(T) + 2\int_0^T B(t)\,\chi(t)\,dt,$$

where R, A, and B are nonnegative functions. Then

$$\sqrt{\chi^2(T) + R(T)} \le \sup_{0\le t\le T} A^{1/2}(t) + \int_0^T B(t)\,dt.$$

Proposition 2. [Stability of the LDG method] *The scheme (1.18)-(1.21) satisfies*

$$|\!|\!| \, (C, \tilde{Z}) \, |\!|\!| \le \Big\{ \|\phi^{1/2} c^0\|^2 + 2\int_0^T \|c_I |u\cdot n|^{1/2}\|^2_{\Gamma_I} dt \Big\}^{1/2} + \int_0^T \|\phi^{1/2} f \, \|_{\Omega_e} \, dt.$$

Note that from the above result, the existence and uniqueness of the approximate solution (C, Z) follows. Since the problem is linear and finite dimensional, existence and uniqueness are equivalent. If we assume two solutions (C_1, Z_1) and (C_2, Z_2) exist and subtract them, then the difference satisfies the inequality above with $c_I = f = c^0 = 0$. Thus $(C_1, Z_1) = (C_2, Z_2)$.

1.4 AN *A PRIORI* ERROR ESTIMATE

We now develop an *a priori* error estimate for the scheme. As in the previous section, we proceed in several steps.

Step 1. We start by introducing some notation. Let $\Pi c(\cdot,t) \in W_{h,e}$ denote the $L^2(\Omega_e)$ projection of $c(\cdot,t)$:

$$(\Pi c - c, w)_{\Omega_e} = 0, \quad w \in W_{h,e}. \tag{1.28}$$

Similarly, let $\Pi\tilde{z}(\cdot,t) \in (W_{h,e})^d$ denote the L^2 projection of $\tilde{z}$ and $\Pi z(\cdot,t) \in (W_{h,e})^d$ the L^2 projection of z. Define Πc^u analogous to C^u. Set $\psi_c = C - \Pi c$, $\tilde{\psi}_z = \tilde{Z} - \Pi\tilde{z}$, $\psi_z = Z - \pi z$, $\theta_c = c - \Pi c$, $\tilde{\theta}_z = \tilde{z} - \Pi\tilde{z}$ and $\theta_z = z - \Pi z$. Our goal is to obtain an estimate of $(\psi_c, \psi_z, \tilde{\psi}_z)$ in terms of $(\theta_c, \theta_z, \tilde{\theta}_z)$.

To do that, we note that by the construction of the form B (1.22), we have

$$B(C, Z, \tilde{Z}; w, v, \tilde{v}) = B(c, z, \tilde{z}; w, v, \tilde{v}),$$

where $(c, z, \tilde{z})$ is the exact solution. This implies that

$$B(\psi_c, \psi_z, \tilde{\psi}_z; w, v, \tilde{v}) = B(\theta_c, \theta_z, \tilde{\theta}_z; w, v, \tilde{v}),$$

and, for $(w, v, \tilde{v}) = (\psi_c, \psi_z, \tilde{\psi}_z)$,

$$B(\psi_c, \psi_z, \tilde{\psi}_z; \psi_c, \psi_z, \tilde{\psi}_z) = B(\theta_c, \theta_z, \tilde{\theta}_z; \psi_c, \psi_z, \tilde{\psi}_z). \tag{1.29}$$

Since the left hand side of this equality is given by (1.26), we only need to obtain an upper bound of its right hand side to obtain the desired error estimate.

Step 2. Let us get the estimate of the right hand side of (1.29). By the definition of the form B, we get

$$\begin{aligned}
&B(\theta_c, \theta_z, \tilde{\theta}_z; \psi_c, \psi_z, \tilde{\psi}_z) \\
&\quad = \int_0^T \Big\{ \sum_e \left[(\phi\,(\theta_c)_t, \psi_c)_{\Omega_e} - (u\theta_c, \nabla\psi_c)_{\Omega_e} - (D\tilde{\theta}_z, \tilde{\psi}_z)_{\Omega_e} \right] \\
&\quad - \sum_l \left[\langle (c - \Pi c^u) u \cdot n_l, [\psi_c] \rangle_{\gamma_l} + \langle (z - \overline{\Pi z}) \cdot n_l, [\psi_c] \rangle_{\gamma_l} + \langle (c - \overline{\Pi c}, [\psi_z] \cdot n_l \rangle_{\gamma_l} \right] \\
&\qquad + \langle (c - \Pi c^-) u \cdot n, \psi_c^- \rangle_{\Gamma_O} + \langle c - \Pi c^-, \psi_z^- \cdot n \rangle_{\Gamma} \Big\}. \\
&\quad = T_1 + \ldots + T_8.
\end{aligned} \tag{1.30}$$

We now examine the terms T_1 through T_8. In the following, K will denote a generic positive constant and $\Pi_0 g$ will denote the projection of $g \in L^2(\Omega)$ into piecewise constants, that is,

$$\sum_e \int_{\Omega_e} (g - \Pi_0 g) dx = 0. \tag{1.31}$$

Consider

$$T_1 \;=\; \int_0^T \sum_e (\phi\,(\theta_c)_t, \psi_c)_{\Omega_e}.$$

Note that if ϕ were constant, this term would be zero, since Πc is the L^2 projection of c into W_h. Thus

$$\begin{aligned}
(\phi\,(\theta_c)_t, \psi_c)_{\Omega_e} &= ((\phi - \Pi_0\phi)(\theta_c)_t, \psi_c)_{\Omega_e} \\
&= \left((\phi - \Pi_0\phi)\phi^{-1/2}(\theta_c)_t, \phi^{1/2}\,\psi_c\right)_{\Omega_e} \\
&\le (\phi_*)^{-1/2}\|\phi - \Pi_0\phi\|_{L^\infty(\Omega_e)}\|\,(\theta_c)_t\|_{\Omega_e}\|\phi^{1/2}\,\psi_c\|_{\Omega_e} \\
&\le (\phi_*)^{-1/2}\,h_e\|\phi\|_{W^1_\infty(\Omega)}\|\,(\theta_c)_t\|_{\Omega_e}\,\|\phi^{1/2}\,\psi_c\|_{\Omega_e},
\end{aligned}$$

for $\phi \in W^1_\infty(\Omega)$. Hence, after a simple application of Cauchy-Schwarz inequality, we get

$$\int_0^T \sum_e (\phi\,(\theta_c)_t, \psi_c)_{\Omega_e} dt \le K_1 \int_0^T \left\{\sum_e h_e^2\,\|(\theta_c)_t\|^2_{\Omega_e}\right\}^{1/2} \|\phi^{1/2}\,\psi_c\|\,dt, \tag{1.32}$$

where

$$K_1 = (\phi_*)^{-1/2}\|\phi\|_{W^1_\infty(\Omega)}. \tag{1.33}$$

Next, consider

$$\begin{aligned}
T_2 &= -\int_0^T \sum_e (u\theta_c, \nabla\psi_c)_{\Omega_e}\,dt \\
&= \int_0^T \sum_e \left((u - \Pi_0 u)\theta_c, \nabla\psi_c\right)_{\Omega_e}\,dt,
\end{aligned}$$

since each component of $\nabla\psi_c \in W_h$, where $\Pi_0 u$ represents the projection of each component of u into piecewise constants. Thus, for $u \in (W^1_\infty(\Omega))^d$,

$$T_2 \le \|\,u\,\|_{(W^1_\infty(\Omega))^d} \int_0^T \sum_e h_e\|\theta_c\|_{\Omega_e}\,\|\nabla\psi_c\|_{\Omega_e}\,dt.$$

Assuming h_e is sufficiently small, and using an inverse estimate ([4],Lemma 4.5.3), we find

$$T_2 \le K_i\,\|\,u\,\|_{(W^1_\infty(\Omega))^d} \int_0^T \sum_e \|\theta_c\|_{\Omega_e}\,\|\psi_c\|_{\Omega_e}\,dt.$$

Finally, using (1.2), we get

$$T_2 \le K_2 \int_0^T \left\{\sum_e \|\theta_c\|^2_{\Omega_e} dt\right\}^{1/2} \|\phi^{1/2}\psi_c\|\,dt \tag{1.34}$$

where

$$K_2 = K_i\,\|u\|_{(W^1_\infty)^d}\,(\phi_*)^{-1/2}. \tag{1.35}$$

The estimates of T_3, T_4, and T_7 are straightforward:

$$\begin{aligned}
T_3 &= -\int_0^T \sum_e (D\tilde{\theta}_z, \tilde{\psi}_z)_{\Omega_e}\,dt \\
&\le 2\int_0^T \sum_e \|D^{1/2}\tilde{\theta}_z\|^2_{\Omega_e}\,dt + \frac{1}{8}\int_0^T \sum_e \|D^{1/2}\tilde{\psi}_z\|^2_{\Omega_e}\,dt,
\end{aligned} \tag{1.36}$$

$$
\begin{aligned}
T_4 &= -\int_0^T \sum_l \langle (u\cdot n_l)(c-\Pi c^u), [\psi_c]\rangle_{\gamma_l}\, dt \\
&\le \int_0^T \sum_l \|\, |u\cdot n_l|^{1/2}(c-\Pi c^u)\|_{\gamma_l}^2\, dt + \frac{1}{4}\int_0^T \sum_l \|\, |u\cdot n_l|^{1/2}[\psi_c]\|_{\gamma_l}^2\, dt, \qquad (1.37)
\end{aligned}
$$

and

$$
\begin{aligned}
T_7 &= \int_0^T \langle (c-\Pi c^-)u\cdot n, \psi_c^-\rangle_{\Gamma_O}\, dt \\
&\le \|\, |u\cdot n|^{1/2}(c-\Pi c^-)\|_{\Gamma_O}^2 + \frac{1}{4}\int_0^T \|\, |u\cdot n|^{1/2}\psi_c^-\|_{\Gamma_O}^2\, dt. \qquad (1.38)
\end{aligned}
$$

The estimates of the remaining terms are more delicate. Let us start by considering T_5:

$$
\begin{aligned}
T_5 &= \int_0^T \sum_l \langle (z-\overline{\Pi z})\cdot n_l, [\psi_c]\rangle_{\gamma_l}\, dt \\
&= \int_0^T \left\{\sum_l \rho_l\, \|(z-\overline{\Pi z})\|_{\gamma_l}^2\right\}^{1/2} \left\{\sum_l \rho_l^{-1}\, \|[\psi_c]\|_{\gamma_l}^2\right\}^{1/2} dt, \qquad (1.39)
\end{aligned}
$$

where ρ_l is a positive number to be determined. Let $\{\Omega_{e_l}\}$ denote the set of elements whose boundaries intersect γ_l. Since Ω_e has a Lipschitz boundary, by ([4],Theorem 1.6.6),

$$
\begin{aligned}
\|[\psi_c]\|_{\gamma_l} &\le K_{tr} \sum_{e_l} \|\psi_c\|_{L^2(\Omega_{e_l})}^{1/2} \|\psi_c\|_{H^1(\Omega_{e_l})}^{1/2} \\
&\le K_{tr}\, K_i \sum_{e_l} h_{e_l}^{-1/2} \|\psi_c\|_{L^2(\Omega_{e_l})} \\
&\le K_{tr}\, K_i\, h_l^{-1/2} \sum_{e_l} \|\psi_c\|_{L^2(\Omega_{e_l})} \\
&\le K_{tr}\, K_i\, h_l^{-1/2}\, M_l^{1/2} \left\{\sum_{e_l} \|\psi_c\|_{L^2(\Omega_{e_l})}^2\right\}^{1/2},
\end{aligned}
\qquad (1.40)
$$

where $h_l = \min_{e_l} h_{e_l}$ and M_l is the maximum number of elements whose boundary intersects γ_l (in a conforming mesh, $M_l = 2$). Hence,

$$
\begin{aligned}
\sum_l \rho_l^{-1}\, \|\psi_c\|_{L^2(\Omega_{e_l})}^2 &\le \sum_l \rho_l^{-1}\; K_{tr}^2\, K_i^2\, h_l^{-1}\, M_l \sum_{e_l} \|\psi_c\|_{L^2(\Omega_{e_l})}^2 \\
&\le \sum_l \sum_{e_l} \|\psi_c\|_{L^2(\Omega_{e_l})}^2, \qquad (1.41)
\end{aligned}
$$

if we take $\rho_l^{-1}\; K_{tr}^2\, K_i^2\, h_l^{-1}\, M_l = 1$. Finally,

$$
\sum_l \rho_l^{-1}\, \|\psi_c\|_{L^2(\Omega_{e_l})}^2 \le N\, \|\psi_c\|^2 \le N\, \phi_*^{-1} \|\phi^{1/2}\, \psi_c\|^2, \qquad (1.42)
$$

where N is the maximum number of internal edge segments that any element intersects. Inserting this inequality in the bound for T_5, we get

$$
T_5 \le K_5 \int_0^T \left\{\sum_l h_l^{-1}\, \|(z-\overline{\Pi z})\|_{\gamma_l}^2\right\}^{1/2} \|\phi^{1/2}\, \psi_c\|\, dt, \qquad (1.43)
$$

where

$$K_5 = \{K_{tr}\, K_i\, N\, \phi_*^{-1} \sup_l M_l\}^{1/2}.$$

In a similar manner,

$$\begin{aligned} T_6 &= \int_0^T \sum_l \langle c - \overline{\Pi c}, [\psi_z] \cdot n_l \rangle_{\gamma_l}\, dt \\ &\le K_6 \int_0^T \left\{ \sum_l h_l^{-1} \, \|(c - \overline{\Pi c})\|_{\gamma_l}^2 \right\}^{1/2} \|\, \psi_z \|\, dt, \end{aligned} \tag{1.44}$$

where

$$K_6 = \{K_{tr}\, K_i\, N \sup_l M_l\}^{1/2}.$$

By (1.16) and (1.21),

$$\|\psi_z\|_{\Omega_e}^2 = (D\tilde{\psi}_z, \psi_z)_{\Omega_e} - (D\tilde{\theta}_z, \psi_z)_{\Omega_e},$$

and so,

$$\|\psi_z\|_{\Omega_e} \le K_D \,\|\, D^{1/2} \tilde{\psi}_z \,\|_{\Omega_e} + \|\, D\tilde{\theta}_z \,\|_{\Omega_e},$$

Substituting into (1.44) and performing a few simple manipulations, we get

$$\begin{aligned} T_6 \le\; & K_7 \int_0^T \sum_l h_l^{-1} \, \|(c - \overline{\Pi c})\|_{\gamma_l}^2 \, dt + \frac{1}{8} \int_0^T \|\, D^{1/2} \tilde{\psi}_z \|^2 \, dt \\ & + K_6 \int_0^T \left\{ \sum_l h_l^{-1} \, \|(c - \overline{\Pi c})\|_{\gamma_l}^2 \right\}^{1/2} \left\{ \sum_e \|\, D\tilde{\theta}_z \,\|_{\Omega_e}^2 \right\}^{1/2} dt, \end{aligned} \tag{1.45}$$

where

$$K_7 = 2\, K_6^2\, K_D^2.$$

Finally, we treat the last term

$$T_8 = \int_0^T \langle c - \Pi c^-, \psi_z^- \cdot n \rangle_\Gamma \, dt \tag{1.46}$$

in a similar way. Let $\{\Omega_{e_\Gamma}\}$ denote the set of elements whose boundaries intersect Γ. Then we obtain

$$\begin{aligned} T_8 \le\; & K_7 \int_0^T \sum_{e_\Gamma} h_{e_\Gamma}^{-1} \, \|(c - \overline{\Pi c})\|_{e_\Gamma \cap \Gamma}^2 \, dt + \frac{1}{8} \int_0^T \|\, D^{1/2} \tilde{\psi}_z \|^2 \, dt \\ & + K_6 \int_0^T \left\{ \sum_{e_\Gamma} h_{e_\Gamma}^{-1} \, \|(c - \overline{\Pi c})\|_{e_\Gamma \cap \Gamma}^2 \right\}^{1/2} \left\{ \sum_{e_\Gamma} \|\, D\tilde{\theta}_z \,\|_{\Omega_{e_\Gamma}}^2 \right\}^{1/2} dt, \end{aligned} \tag{1.47}$$

where $\{\Omega_{e_\Gamma}\}$ denotes the set of elements whose boundaries intersect Γ.

Thus, substituting the estimates (1.32), (1.34), (1.36), (1.37), (1.38), (1.43), (1.45) and (1.47), in the expression for $B(\theta_c, \theta_z, \tilde{\theta}_z; \psi_c, \psi_z, \tilde{\psi}_z)$, (1.30), we find

$$B(\theta_c, \theta_z, \tilde{\theta}_z; \psi_c, \psi_z, \tilde{\psi}_z) \le \mathcal{T}_1 + \mathcal{T}_2 + \int_0^T \mathcal{T}_3 \,\|\, \phi^{1/2}\, \psi_c \,\|^2 \, dt,$$

where

$$
\begin{aligned}
\mathcal{T}_1 &= 2\int_0^T \sum_e \|D^{1/2}\tilde{\theta}_z\|^2_{\Omega_e}\,dt \\
&+\int_0^T \sum_l \|\,|u\cdot n_l|^{1/2}(c-\Pi c^u)\|^2_{\gamma_l}\,dt + \int_0^T \|\,|u\cdot n|^{1/2}(c-\Pi c^-)\|^2_{\Gamma_O}\,dt \\
&+K_6\int_0^T \left\{\sum_l h_l^{-1}\|c-\overline{\Pi c}\|^2_{\gamma_l}\right\}^{1/2}\left\{\sum_e \|D\tilde{\theta}_z\|^2_{\Omega_e}\right\}^{1/2} dt \\
&+K_6\int_0^T \left\{\sum_{e_\Gamma} h_{e_\Gamma}^{-1}\|c-\overline{\Pi c}\|^2_{e_\Gamma\cap\Gamma}\right\}^{1/2}\left\{\sum_{e_\Gamma} \|D\tilde{\theta}_z\|^2_{\Omega_{e_\Gamma}}\right\}^{1/2} dt, \qquad (1.48)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_2 &= \frac{3}{8}\int_0^T \sum_e \|D^{1/2}\tilde{\psi}_z\|^2_{\Omega_e}\,dt \\
&+\frac{1}{4}\int_0^T \sum_l \|\,|u\cdot n_l|^{1/2}[\psi_c]\|^2_{\gamma_l}\,dt + \frac{1}{4}\int_0^T \|\,|u\cdot n|^{1/2}[\psi_c]\|^2_{\Gamma_O}\,dt,
\end{aligned}
$$

and

$$
\mathcal{T}_3 = K_1\left\{\sum_e h_e^2\|(\theta_c)_t\|^2_{\Omega_e}\right\}^{1/2} + K_2\left\{\sum_e \|\theta_c\|^2_{\Omega_e}\right\}^{1/2} + K_5\left\{\sum_l h_l^{-2}\|z-\overline{\Pi z}\|^2_{\gamma_l}\right\}^{1/2} \qquad (1.49)
$$

Step 3. Now, we insert the above estimates in the identity (1.29) and use the identity (1.26) to obtain, after a few algebraic manipulations,

$$
|\!|\!|\,(\psi_c,\tilde{\psi}_z)\,|\!|\!|^2 \;\le\; A(T) + 2\int_0^T B(t)\,\|\phi^{1/2}\psi_c\|\,dt,
$$

where $|\!|\!|\,(\cdot,\cdot)\,|\!|\!|$ is defined by (1.27), $B=\mathcal{T}_3$, see (1.49), and $A(T)=\|\phi^{1/2}\psi_c(0)\|^2+2\mathcal{T}_1$, see (1.48). Now, a simple application of Lemma 1 gives the following result.

Theorem 3. [First error estimate]. *The scheme (1.18)-(1.21) satisfies the following error estimate*

$$
|\!|\!|\,(\psi_c,\tilde{\psi}_z)\,|\!|\!| \le A^{1/2}(T) + \int_0^T B(t)\,dt.
$$

From this result, by the triangle inequality and some simple approximation results, we get the following estimate.

Theorem 4. [Second error estimate]. *The scheme (1.18)-(1.21) satisfies the following error estimate*

$$
|\!|\!|\,(c-C,\tilde{z}-\tilde{Z})\,|\!|\!| \le |\!|\!|\,(\theta_c,\tilde{\theta}_z)\,|\!|\!| + A^{1/2}(T) + \int_0^T B(t)\,dt \le K\,h^k,
$$

if the exact solution is sufficiently smooth, where $k=\min_e k_e$.

1.5 CONCLUDING REMARKS

The stability estimate in Proposition 2 can be easily be obtained for nonlinear convection-diffusion equations. The error estimates in Theorems 3 and 4 can also be obtained for numerical fluxes more general than the ones presented in this paper, for more general boundary conditions, and for operators Π that are not necessarily the L^2-projection operators.

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