Localizations of transfors

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Abstract

Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be n-dimensional teisi, i.e., higher-dimensional Gray-categorical structures. The following questions can be asked. Does a left q-transfor $\mathbb{C} \to \mathbb{D}$, i.e., a functor $2_q \otimes \mathbb{C} \to \mathbb{D}$, induce a right q-transfor $\mathbb{C} \to \mathbb{D}$, i.e., a functor $\mathbb{C} \otimes 2_q \to \mathbb{D}$? More generally, does a functor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ induce a functor $\mathbb{D} \otimes \mathbb{C} \to \mathbb{E}$? For c, c' elements of $\mathbb{C}$ whose $(k-1)$-sources and $(k-1)$-targets agree, does a q-transfor $\mathbb{C} \to \mathbb{D}$ induce a q-transfor $\mathbb{C}(c, c') \to \mathbb{D}(d, d')$, for appropriate $d, d' \in \mathbb{D}$? For $c, c' \in \mathbb{C}$ and $d, d' \in \mathbb{D}$ whose $(k-1)$-sources and $(k-1)$-targets agree, does a q-transfor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ induce a $(q+k+1)$-transfor $\mathbb{C}(c, c') \otimes \mathbb{D}(d, d') \to \mathbb{E}(e, e')$, for appropriate $e, e' \in \mathbb{E}$? I give answers to these questions in the cases where n-dimensional teisi and their tensor product have been defined, i.e., for $n \leq 3$, and in some cases for n up to 5 which do not need all data and axioms of n-dimensional teisi.

I apply the above to compositions *in* teisi, and in particular to braidings and syllepses. One of the results is that a braiding on a monoidal 2-category induces a pseudo-natural transformation $\widetilde{? \otimes -} \to ? \otimes -$, where $\widetilde{? \otimes -}$ is the "reverse" of $? \otimes -$, and is almost, but not quite, equal to $- \otimes ?$. However, in higher dimensions $\otimes$ need not be reversible, so the previous result does not generalize to higher-dimensional teisi.

1 Introduction

In category theory, a natural transformation α between functors $F, G : \mathbb{C} \to \mathbb{D}$ is a function which assigns to every object C of the category $\mathbb{C}$ an arrow $\alpha_C : F(C) \to G(C)$ in the category $\mathbb{D}$, such that for every arrow $f : C \to C'$ in $\mathbb{C}$ the diagram

$$\begin{tikzcd}
& G(C) \arrow[dr, "G(f)"] & \\
F(C) \arrow[ur, "\alpha_C"] \arrow[dr, "F(f)"'] & & G(C') \\
& F(C') \arrow[ur, "\alpha_{C'}"'] &
\end{tikzcd}$$

commutes [42, p. 16]. Alternatively, α is a function $C_0 \to D_1$ (where C_0 is the set of objects of $\mathbb{C}$ and D_1 is the set of arrows of $\mathbb{D}$ as usual), as in

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$$\begin{tikzcd} C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] & C_0 \arrow[dl, "\alpha"'] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] \\ D_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & D_0 \end{tikzcd} ,$$

such that $G(f) \circ \alpha(C) = \alpha(C') \circ F(f)$. Writing '$+$' for composition and '$\partial$' for $t - s$, and extending the above diagram to

$$\begin{tikzcd} & C_1 \arrow[dl, "\alpha"'] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] & C_0 \arrow[dl, "\alpha"'] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] \\ D_1 \arrow[r, equal] & D_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & D_0 \end{tikzcd} ,$$

where the new $\alpha : C_1 \to D_1$ sends f to either composite in the first diagram above, the naturality condition on α becomes $\partial(\alpha(f)) = G(f) + \alpha(s(f)) - \alpha(t(f)) - F(f) = G(f) - F(f) - \alpha(\partial(f))$. This is precisely the familiar condition $G - F = \partial\alpha + \alpha\partial$ for chain homotopies [48, p. 177].[1] Note that by taking $C_{-1} = \{*\}$ and $\alpha(*) = 0$ this also covers the condition that $\alpha(C)$ has domain $F(C)$ and codomain $G(C)$.

The relationship between topology and category theory is as old as category theory itself. It began with the birth of category theory from homological algebra, with naturality one of the first things being investigated [26], and has continued with and been strenghtened by the development of topos theory [43], and the use of categorical methods in homotopical [46, 27, 31, 7, 12, 16] and homological algebra [41], including K-theory [10, 33, 47], and elsewhere. Recently, the interaction has intensified, with the connection between braids and tangles on the one hand and braided and tortile tensor categories on the other [35, 36], and in the theory of operads [44, 6]. The above connection between categories and chain complexes, and between natural transformations and chain homotopies, is another instance of the interaction between category theory and topology. I will not attempt to make this connection more formal (see Johnson and Wood [34] for that); instead, I will use it in the sequel as a guiding motivation for higher dimensions.

For 2-categories, the evidence from chain complexes suggests that a natural transformation between 2-functors $F, G : \mathbb{C} \to \mathbb{D}$ would consist of functions $\alpha : C_0 \to D_1$ and $\alpha : C_1 \to D_2$, as in

$$\begin{tikzcd} C_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] & C_1 \arrow[dl, "\alpha"'] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] & C_0 \arrow[dl, "\alpha"'] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] \\ D_2 \arrow[r, shift left] \arrow[r, shift right] & D_1 \arrow[r, shift left] \arrow[r, shift right] & D_0 \end{tikzcd}$$

such that $s\alpha = \alpha t + F$ and $t\alpha = G + \alpha s$. Such a concept, up to some signs, i.e., with source and target interchanged in some places, does indeed exist in 2-category theory, but it is known as a *lax-natural transformation*. There are also *pseudo-natural transformations*, which require $\alpha(f)$ (but not $\alpha(C)$) to be invertible, and 2-*natural transformations*, which require $\alpha(f)$ to be an identity [40]. Just as for chain homotopies, lax- and pseudo-natural transformations have the further condition that α is a homomorphism: $\alpha(f + g) = \alpha(f) + \alpha(g)$, with addition on the left hand side being composition

[1] This was first noted explicitly by Johnson [34].

and on the right hand side, in $\mathbb{D}$, being *pasting* along the common boundary, as in

$$
\begin{tikzcd}
 & G(C) \arrow[dr, "G(f)"] & & \\
F(C) \arrow[ur, "\alpha(C)"] \arrow[dr, "F(f)"'] & \Downarrow\alpha(f) & G(C) \arrow[dr, "G(g)"] & \\
 & F(C') \arrow[ur] \arrow[dr, "F(g)"'] & \Downarrow\alpha(f') & G(C'') \\
 & & F(C'') \arrow[ur, "\alpha(C'')"'] &
\end{tikzcd}
\quad .
$$

(As composition is an instance of pasting this does not introduce an asymmetry in the interpretation of addition.)

For 2-categories, there is also room for chain homotopies of degree 2. For two (lax-, pseudo- or 2-) natural transformations $\alpha, \beta : F \to G$ this would consist of a function $\mu : C_0 \to D_2$, as in

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
C_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & C_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, shift right, "\beta"'] \arrow[dl, shift left, "\alpha"] & C_0 \arrow[d, shift left] \arrow[d, shift right] \arrow[dll, "\mu"'] \arrow[dl, shift right, "\beta"'] \arrow[dl, shift left, "\alpha"] \\
D_2 \arrow[r, shift left] \arrow[r, shift right] & D_1 \arrow[r, shift left] \arrow[r, shift right] & D_0
\end{tikzcd}
$$

such that $s\mu = \mu t + \alpha$ and $t\mu = \beta + \mu s$, together with a further naturality condition expressible in terms of a $\mu : C_1 \to D_3$, and a functoriality condition requiring μ to be a homomorphism. The precise interpretation of addition as pasting becomes quite involved, but again this concept does indeed exist in 2-category theory, where it is known as a *modification* [40]. There are no further qualifications, like lax, pseudo, or 2-natural, for a modification because these would indicate properties of $\mu(f)$, which is an identity, i.e., a commutativity condition, anyway.

It is clear from the above that for higher-dimensional categorical structures a natural transformation of degree q will consist of functions $\rho : C_p \to D_{p+q}$, as in

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\cdots \arrow[r, shift left] \arrow[r, shift right] & C_{q+1} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & C_q \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & C_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dlll, "\rho"'] & C_0 \arrow[d, shift left] \arrow[d, shift right] \arrow[dlll, "\rho"'] \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & D_{q+1} \arrow[r, shift left] \arrow[r, shift right] & D_q \arrow[r, shift left] \arrow[r, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & D_1 \arrow[r, shift left] \arrow[r, shift right] & D_0
\end{tikzcd}
$$

such that: for $c \in C_p$ the faces of $\rho(c)$ are given as certain pastings in $\mathbb{D}$, involving ρ of faces of c and $\rho^{\pm}_{q'}(c)$'s, where the $\rho^{\pm}_{q'}$ are natural transformations of lower degree that are the sources and targets of ρ; and for n-composable $c, c' \in \mathbb{C}$, (and denoting n-composition by $\#_n$ as in [24],) $\rho(c' \#_n c)$ is equal to the pasting of $\rho(c')$ and $\rho(c)$ along their common boundary, which will be $(p+n)$-dimensional. This concept does indeed exist in ω-category theory, and is called a *lax-q-transformation* [17, Section 3-9]. One can also consider pseudo-q-transformations and strict q-transformations of course.

Homotopy theory suggests that ω-categories are not the right higher-dimensional categorical structures to consider. In fact, mere categories do suffice to do abstract homotopy as with topological spaces [51], but not in a very nice way, involving the twice iterated subdivision and extension functors. More importantly, categories have only structure up to dimension 1, and hence do not give a separation of homotopy information according to dimension.

Weak ω-groupoids, which are ω-categories in which every element is invertible up to coherent higher-dimensional data, also have the same abstract homotopy theory as topological spaces [38], and do give such a separation of homotopy information according to dimension. However, the invertibility condition on elements involves an infinite amount of extra data, which needs to be carried along, which is hard to check, and which is not even algebraic, being given as the existence of a "quasi-inverse" and not as a specified one.

ω-groupoids, which are ω-categories in which every element has a (strict) inverse, are algebraic structures, because this inverse is then necessarily unique. But the abstract homotopy theory of ω-groupoids [23, 11] is more restricted than that of topological spaces, in particular, it has all Whitehead products trivial [14, p. 114]. This inadequacy of ω-groupoids starts in dimension 3: 2-groupoids do classify homotopy 2-types [45], but 3-groupoids don't classify homotopy 3-types. **Gray**-groupoids, which are like 3-groupoids except that horizontal composition of 2-arrows is not definable in terms of their vertical composition but results in a (specified) (iso-)3-arrow mediating between the two possible ways of doing this, as in

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\delta"] & \bullet \\
\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\gamma"] & \bullet \arrow[r, bend right, "g'"'] & \bullet
\end{tikzcd}
\xrightarrow[3]{\delta\,\#_0\,\gamma}
\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\gamma"] & \bullet \arrow[r, bend left, "g"] & \bullet \\
\bullet \arrow[r, bend right, "f'"'] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\delta"] & \bullet
\end{tikzcd}
,
$$

with further conditions on these 3-arrows, do classify homotopy 3-types [37]. This suggests that **Gray**-categories *are* the right 3-dimensional categorical structures to consider. Further evidence for this comes from the theory of braids [35, 36]: braided (strict) monoidal categories are precisely one-object, one-arrow **Gray**-categories [28].

$(\boldsymbol{\omega}\textbf{-Cat})_{\otimes}$-categories (the explanation of the name will follow later) [17, Section 3-12] are higher-dimensional categorical structures generalizing **Gray**-categories. They are like ω-categories except that 0-composition of a p-arrow and a q-arrow results in a $(p+q-1)$-arrow whose source and target are the two possible ways of 1-composing them, with further conditions on these $(p+q-1)$-arrows. However, the other compositions are exactly as for ω-categories, which is still too strict.

Chain complexes are *dimension invariant*, in the sense that the type of structure at each dimension is independent of the dimension. This makes it possible to *localize*[2] a chain complex $C_\bullet$ at a particular dimension n, namely by looking at the chain complex

$$\cdots \longrightarrow C_{n+2} \longrightarrow C_{n+1} \longrightarrow \ker(\partial) \quad .$$

ω-categories don't have a zero and no addition in each dimension, but they too can be localized, in a slightly more elaborate way: for two n-arrows c and c' of an ω-category $\mathbb{C}$ whose $(n-1)$-sources

[2] This use of the term 'localization' here seems to be unrelated to its use in algebra, where one localizes at prime ideals [1].

and $(n-1)$-targets agree, the collection of elements of $\mathbb{C}$ which have c and c' as n-source and n-target respectively is itself an ω-category, the *localization of* $\mathbb{C}$ *at* c *and* c'. $(\boldsymbol{\omega}\textbf{-Cat})_\otimes$-categories do have localization, but not of the right kind: 0-composition is dimension raising, but n-composition for $n > 0$ is not, which implies that localization always gives an ω-category, and never a more general $(\boldsymbol{\omega}\textbf{-Cat})_\otimes$-category.

My conjecture is that *higher-dimensional categorical structures that have dimension raising compositions and have localization* are the right ones to consider. The reason for this conjecture is that n-composition of a p-arrow and a q-arrow resulting in a $(p+q-n-1)$-arrow is exactly like the Whitehead product $\pi_p \times \pi_q \to \pi_{p+q-n-1}$ [52, p. 472]. Such dimension-raising operations also occur in hypercrossed complexes of groups, which classify all (connected) homotopy types [15].

Apart from the conditions on faces of a composite, there are two further restrictions I will impose on such higher-dimensional categorical structures. Firstly, composition (and identity) should be the *only* dimension raising operation, in particular, there should be functoriality and associativity axioms stating the equality of certain composites. This implies that these structures will be algebraic, which makes — potentially at least — methods from universal algebra [9, 8] available. Secondly, the result of a dimension raising composition should be invertible. This latter condition is not by choice but by necessity, as calculations show that both directions for such a composite appear in faces of lower-dimensional composites [21, p. 8].

There are two main tests for the above conjecture. The first one is that such structures in which moreover all elements are invertible should classify all homotopy types. The second one is that, just as every tricategory is triequivalent to a **Gray**-category [28], these structures should feature in a coherence theorem for weak n-categories [6, 4, 32]. Even the failure of these test-cases would be interesting, as that would give an abstract homotopy theory which is richer than for ω-groupoids but still not as rich as for topological spaces, and it would give a basis for the study of the weaker structures that then *are* weakly equivalent to weak n-categories.

The current and recent terminology for higher-dimensional categorical structures is quite impractical and confusing. What I call ω-categories have also been called '∞-categories' [2], whereas 'ω-category' has also been used without the condition that every element has a (finite) dimension [50]. The use of '∞-categories' was based on the use of '∞-groupoids' for ω-groupoids, because 'ω-groupoid' was already used for cubical sets with extra groupoid structures [13]. Then there are 'weak n-groupoids', referred to above, which weaken the strictness of the groupoid condition [38]. Another use of 'weak' occurs in 'weak n-categories', which have weakened axioms for composition [32], and which some people prefer to call 'n-categories' [3, 49]. And then there is the term '**Gray**-category', which doesn't give any indication that it is 3-dimensional, and for which the boldface font is a bit tiresome.

The reason for all these problems is basically that categorical terminology was never made for higher dimensions. Therefore I have stepped outside the categorical framework, and have baptized the — hypothetical — higher-dimensional categorical structures that have dimension raising compositions, that are dimension invariant and that satisfy the two further conditions above ω-*teisi*. 'Teisi' (pronounced TAY-see) is Welsh, plural of 'tas', which means 'stack'. I have chosen this term because of Grothendieck's programme "Pursuing stacks" [30], in which he advocates the study of weak n-categories, which he calls 'n-stacks', and because of my visit in 1993 to Bangor, Wales. I call finite-dimensional such structures *n-dimensional teisi*, or *nD teisi* for short, reserving the term 'n-teisi' for ω-teisi which behave like ω-categories above dimension $n-1$. I will use 'teisi' as a generic term for ω- and nD teisi.

With this terminology, a category is a 1D tas, a 2-category is a 2D tas, and a **Gray**-category is a 3D tas. So far, I have defined 4D teisi, and 5D and 6D teisi in some special cases, viz. trivial in low dimensions, and I have given some indications for higher-dimensional teisi [21, Section 2].

For teisi, then, natural transformations of degree q should be similar to lax-q-transformations between ω-categories, but with the dimension raising of composition in teisi taken into account. Thus, such natural transformations should still consist of functions $\rho : C_p \to D_{p+q}$, and should satisfy a certain condition on the faces of $\rho(c)$'s in terms of ρ of faces of c and $\rho^{\pm}_{q'}(c)$'s, and a certain condition expressing $\rho(c' \#_n c)$ as a pasting of $\rho(c')$ and $\rho(c)$.

The terminology for natural transformations is another example of the inappropriateness of categorical terminology in higher dimensions, because, from the chain homotopy viewpoint, there is nothing "weak" or "unreal" about lax- and pseudo-naturality respectively. Another objection to the terminology is that it does record the condition on faces but not the condition on composition. Because of this, I have christened the maps of degree q between teisi that satisfy these conditions *q-transfors*. I call the conditions on faces and composition *naturality* and *functoriality*[3] respectively, or *transforiality*[4] for short. I will use 'transfors' as a generic term for q-transfors between teisi.

With this terminology, a natural transformation between categories is a 1-transfor, as is a lax-natural transformation between 2-categories, and a modification between 2-categories is a 2-transfor. So far, I have defined transfors between nD teisi for $n \leq 3$ [19, Section 5], and I have given some indications for transfors between higher-dimensional teisi [18, Section 2.5.1].

Transfors and composition in a tas are both dimension raising operations that are natural and functorial. There is actually a precise connection between them, which is best expressed using enriched category theoretical terminology [39].

Given two higher- (n-)dimensional categorical structures $\mathbb{C}$ and $\mathbb{D}$, one can consider a structure $\mathbb{C} \otimes \mathbb{D}$ which has as generators expressions $c \otimes d$ of dimension $p+q$ for $c \in C_p$ and $d \in D_q$ (for $p+q \leq n$), and where, denoting source and target by d^- and d^+ respectively, the source (target) of such a generator is a pasting of $d^\alpha(c) \otimes d$ and $c \otimes d^\beta(d)$ with $\alpha = -\ (+)$ and $\beta = (-)^{p+1}\ ((-)^p)$. Writing '$+$' for pasting and '$\partial$' for $t-s$, as before, this condition becomes $\partial(c \otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d$, which is precisely the familiar condition for the tensor product of chain complexes [48, p. 321]. Also, for composable c and $c' \in \mathbb{C}$ and $d \in \mathbb{D}$, the generator $(c' \# c) \otimes d$ is required to be equal to the pasting of $c \otimes d$ and $c' \otimes d$, and similarly for $c \in \mathbb{C}$ and composable d and $d' \in \mathbb{D}$. In terms of '$+$' and '∂', this condition becomes $(c + c') \otimes d = c \otimes d + c' \otimes d$, and similarly in the second variable, which is just the condition that the tensor product is a bihomomorphism.

For 2-categories, this construction is known as Gray's tensor product [29]. It being coherently associative and having a unit, namely the trivial 2-category, makes the category of (small) 2-categories (and 2-functors between them) into a monoidal category, usually denoted by **Gray**.[5] For ω-categories, this construction makes the category of ω-categories (and ω-functors between them) into a monoidal category [17, Section 3-7 and 3-8], denoted by $(\boldsymbol{\omega}\textbf{-Cat})_\otimes$. For **Gray**-categories, this construction, with the extra requirement of "functoriality in both variables at the same time", makes the category of **Gray**-categories into a monoidal category [19], denoted 3D-**Teisi**. For higher-dimensional teisi, it should still be possible to make $\mathbb{C} \otimes \mathbb{D}$ into a tas, but the tensor product thus defined will no longer be associative, as can be seen from the diagram on page 46 of [19].

Denoting the ω-tas (which is also the ω-category) generated by one element in dimension q by 2_q, and calling maps between teisi that preserve all the structure *functors*, one sees that *a q-transfor*

[3]Note that in category theory 'functoriality' usually refers to both the *existence* of a map on the level of arrows and this map preserving composition.

[4]The ultimate reason for 'transfor' was to be able to use 'transforial'!

[5]The tensor product of **Gray** is in fact the *iso*-version of Gray's tensor product.

$\mathbb{C} \to \mathbb{D}$ *is precisely a functor* $\mathbb{C} \otimes 2_q \to \mathbb{D}$. Thus, there are also *left q*-transfors, which are precisely functors $2_q \otimes \mathbb{C} \to \mathbb{D}$.

In section 3 I show that for *iso-transfors*, i.e., transfors ρ for which $\rho(c)$ is invertible for c of dimension greater than 0, between 3D teisi there is a correspondence between left transfors ρ and (right) transfors $\widetilde{\rho}$, the *righting* of ρ. This is a special case of the iso-version of the tensor product of 3D teisi being symmetric, with a functor $\chi : \mathbb{C} \otimes_{\text{iso}} \mathbb{D} \to \mathbb{E}$ corresponding to a functor $\widetilde{\chi} : \mathbb{D} \otimes_{\text{iso}} \mathbb{C} \to \mathbb{E}$, the *reversal* of χ. Inspection of the proofs shows that the iso-version of a tensor product of higher-dimensional teisi will not be symmetric.

Enriching with respect to a monoidal category $\mathcal{V}$ of higher- (n-)dimensional structures gives a new kind of higher- ($(n+1)$-)dimensional structure called $\mathcal{V}$-categories, where for each pair of objects C, D of a $\mathcal{V}$-category $\mathbb{C}$ the collection of elements of $\mathbb{C}$ which have C and D as source and target respectively is an object $\mathbb{C}(C,D)$ of $\mathcal{V}$, with 0-composition in $\mathbb{C}$ being given by a collection of arrows $\mathbb{C}(C,D) \otimes \mathbb{C}(D,E) \to \mathbb{C}(C,E)$ in $\mathcal{V}$. In particular, for $\mathcal{V} = \boldsymbol{\omega}$**-Cat** with the cartesian product as tensor product this gives ω-categories again, for $\mathcal{V} =$ **Gray** this gives **Gray**-categories, for $\mathcal{V} = (\boldsymbol{\omega}\textbf{-Cat})_\otimes$ this gives $(\boldsymbol{\omega}\textbf{-Cat})_\otimes$-categories, and for $\mathcal{V} =$ 3D-**Teisi** this gives 4D teisi.

Even though the tensor product of higher-dimensional teisi will not give rise to a monoidal category, it should still be possible to enrich with respect to it: the associativity of the tensor product on $\mathcal{V}$ is only used to formulate associativity of 0-composition in $\mathcal{V}$-categories, and the tensor product of teisi should be sufficiently close to being coherently associative to be able to do that. Carrying out this enrichment should give that $\boldsymbol{\omega}$**-Teisi**-categories are ω-teisi, which is precisely the localizability of ω-teisi. It also implies that locally, *n-composition in a tas* $\mathbb{C}$ *should be a functor* $\mathbb{C}(c,c') \otimes \mathbb{C}(c',c'') \to \mathbb{C}(c,c'')$.

A q-dimensional element of a tas $\mathbb{D}$ gives, by freeness of the tas 2_q, a functor $2_q \to \mathbb{D}$. The tensor product should itself be natural and functorial in both variables, i.e., it should be a functor $\boldsymbol{\omega}$**-Teisi** $\times$ $\boldsymbol{\omega}$**-Teisi** $\to$ $\boldsymbol{\omega}$**-Teisi**. So, a functor $\chi : \mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ and a q-dimensional element of $\mathbb{D}$ should give rise to a q-transfor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$. Applying this to the functor $\#_n : \mathbb{C}(c,c') \otimes \mathbb{C}(c',c'') \to \mathbb{C}(c,c'')$ gives that locally, *right n-composition with a q-dimensional element in a tas* $\mathbb{C}$ *should be a* $(q-n-1)$*-transfor* $\mathbb{C}(c,c') \to \mathbb{C}(c,c'')$. There is a similar statement for left composition and left transfors.

Another example of a dimension raising operation on categorical structures is given by braidings. These, too, can be fitted into the current framework, namely by considering teisi that are trivial up to a certain dimension.

(Strict) monoidal categories are precisely one-object 2-categories. However, going from one viewpoint to the other involves a *shift in dimension*: the objects of the monoidal category $\mathbb{C}$ become the arrows of the one-object 2-category. Because this process is similar to *delooping* in the theory of loop spaces [44] I denote this latter 2-category by $\Sigma(\mathbb{C})$. As the converse is similar to *looping*, I denote the monoidal category corresponding to a one-object 2-category $\mathbb{C}$ by $\Omega(\mathbb{C})$. The tensor on a monoidal category $\mathbb{C}$ corresponds to 0-composition in $\Sigma(\mathbb{C})$. This suggests that one should be able to *define* a *monoidal (ω- or nD) tas* to be a one-object (ω- or $(n-1)$D) tas, with formal delooping and looping operations to relate these two viewpoints.

As already observed before, braided (strict) monoidal categories are precisely one-object one-arrow 3D teisi. This time, the relation involves a double shift in dimension: the objects of the braided monoidal category $\mathbb{C}$ become the 2-arrows of the one-object one-arrow 3D tas, the *double delooping* of $\mathbb{C}$, denoted by $\Sigma^2(\mathbb{C})$. Conversely, there is a *double looping* of a one-object one-arrow 3D tas $\mathbb{C}$, denoted by $\Omega^2(\mathbb{C})$. Define a 2-*monoidal (ω- or nD) tas* to be a one-object one-arrow (ω- or $(n-2)$D) tas, these two viewpoints being related via Σ^2 and Ω^2. Note that delooping a 2-

monoidal (ω- or nD) tas $\mathbb{C}$ *once* means viewing it as a one-object (ω- or $(n-1)$D) tas with extra structure, and then *looping* it means that it is a monoidal (ω- or nD)tas. The extra structure on this monoidal tas corresponds to 0-composition in $\Sigma^2(\mathbb{C})$, which has been shifted down two dimensions. Hence, such structure is called (-2)-*composition*, or, alternatively, 2-*tensor*, in analogy with (-1)-composition being called tensor, or even *braiding*, as that is what it gives for $n=1$, and also for $n=2$ [21, Section 5].

This process continues, of course. Define a *k-monoidal (ω- or nD) tas* to be a one-object, $\ldots$, one-$(k-1)$-arrow (ω- or $(n-k)$D) tas, these viewpoints being related via *k-fold delooping*, denoted by Σ^k, and *k-fold looping*, denoted by Ω^k. By considering m-fold deloopings of a k-monoidal (ω- or nD) tas for $0 \leq m < k$, one sees that a k-monoidal tas is also m-monoidal. For a $(k-1)$-monoidal tas, being k-monoidal gives an extra structure corresponding to 0-composition in $\Sigma^k(\mathbb{C})$, which has been shifted down k dimensions, and hence is called $(-k)$-*composition*, or k-*tensor*. 3-tensor is also called *syllepsis* [25], as that is what it gives for $n=2$ [21, Section 6].

One can even define an *ω-monoidal (ω- or nD) tas* to be a(n ω- or nD) tas that is k-monoidal for all k. One readily sees that the k-tensor of two objects of a k-monoidal nD tas should result in a $(k-1)$-dimensional element. So if $k = n+2$ the k-tensor only gives an identity, i.e., an equality condition, on n-dimensional elements. If $k > n+2$ the k-tensor gives an identity between identities, hence adds nothing new. Thus, this "proves" the Stabilization Hypothesis for teisi, that *k-monoidal nD teisi are ω-monoidal for $k \geq n+2$* [21, Theorem 3.8]. ω-monoidal is also called *symmetric*, as that is what it gives for $n=1$ and for $n=2$ [21, Section 7].

For a k-monoidal tas $\mathbb{C}$, and $0 < m \leq k$, localization of $\Sigma^k(\mathbb{C})$ at the unique $(m-1)$-arrow of $\Sigma^k(\mathbb{C})$ is precisely m-fold looping, and results in $\Sigma^{k-m}(\mathbb{C})$. Thus, composition in a tas being locally a functor from a tensor product gives, after careful shifting of dimensions, that *m-tensor on a k-monoidal tas $\mathbb{C}$ should be a functor $\Sigma^{m-1}(\mathbb{C}) \otimes \Sigma^{m-1}(\mathbb{C}) \to \Sigma^{m-1}(\mathbb{C})$*. In particular, the tensor of a monoidal tas $\mathbb{C}$ should be a functor $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, as expected. Also, the braiding of a braided tas $\mathbb{C}$ should be a functor $\Sigma(\mathbb{C}) \otimes \Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$. This is not as expected, as it is not *a priori* a 1-transfor $A \otimes B \to B \otimes A$, which is how braidings are usually defined. I will come back to this shortly.

Composition with an element of a tas being locally a transfor gives, after careful shifting of dimensions, that *right m-tensor with a q-dimensional element in a k-monoidal tas $\mathbb{C}$ should be a $(q-m-1)$-transfor $\Sigma^{m-1}(\mathbb{C}) \to \Sigma^{m-1}(\mathbb{C})$*. In particular, tensoring with a q-dimensional element should be a q-transfor $\mathbb{C} \to \mathbb{C}$. Also, braiding with an object A should be a 1-transfor $\Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$, which is not *a priori* a 1-transfor $A \otimes - \to - \otimes A$. I will come back to this shortly too.

Chain homotopies localize along with the localization of the chain complexes. First, a chain map maps $\ker(\partial)$ into $\ker(\partial)$, hence restricts to a chain map between the localized chain complexes. And then a chain homotopy $\rho : C_\bullet \to D_\bullet$ of degree q restricts to a map of degree q as in

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{n+q+1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl] & C_{n+q} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dll] & \cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] \arrow[dll, "\rho"'] & \ker(\partial) \arrow[d, shift left] \arrow[d, shift right] \arrow[dll, "\rho"'] \\
\cdots \arrow[r] & D_{n+q+1} \arrow[r] & D_{n+q} \arrow[r] & \cdots \arrow[r] & D_{n+1} \arrow[r] & \ker(\partial)
\end{tikzcd}
$$

which is trivially a chain homotopy of degree q again.

For higher-dimensional categorical structures, localization of structure preserving maps is equally trivial: a functor $F : \mathbb{C} \to \mathbb{D}$ induces a functor $F : \mathbb{C}(c, c') \to \mathbb{D}(F(c), F(c'))$ because structure is of the same type in all dimensions. The question of localization of transformations is more problematic. Viewing a q-transfor (or lax-q-transformation) $\rho : \mathbb{C} \to \mathbb{D}$ as a functor $\rho : \mathbb{C} \otimes 2_q \to \mathbb{D}$, it localizes to a functor $(\mathbb{C} \otimes 2_q)(x, x') \to \mathbb{D}(\rho(x), \rho(x'))$, for x and x' ℓ-arrows of $\mathbb{C} \otimes 2_q$ whose $(\ell-1)$-sources and $(\ell-1)$-targets agree. The first observation to make is that the localization of ρ *as a transfor* must have as domain $\mathbb{C}(c, c')$ for c and c' k-arrows of $\mathbb{C}$ whose $(k-1)$-sources and $(k-1)$-targets agree. To get to that, one would want to take $x = c \otimes 2_q$ and $x' = c' \otimes 2_q$, but these have non-agreeing sources and targets. The second observation to make is that the localization of ρ as a transfor must have as codomain $\mathbb{D}(d, d')$ for appropriate k-arrows d and d' of $\mathbb{D}$ whose $(k-1)$-sources and $(k-1)$-targets agree. But $\rho(c)$ and $\rho(c')$, besides having non-agreeing sources and targets, are $(k+q)$-dimensional. And furthermore, for $\gamma \in \mathbb{C}(c, c')$, $\rho(\gamma)$ is not in $\mathbb{D}(\rho(c), \rho(c'))$ (which does not even exist), because its faces also involve $\rho^{\beta'}_{q'}(c)$ and $\rho^{\beta'}_{q'}(c')$'s for $q' < q$ and $\beta' = \pm$.

For k-arrows c and c' of $\mathbb{C}$ whose $(k-1)$-sources and $(k-1)$-targets agree, it *is* the case that the $(k-1)$-sources and $(k-1)$-targets of $s_k(\rho(c))$ and $t_k(\rho(c'))$ agree, because these both only involve $\rho^{\beta'}_{q'}(d^{\alpha'}_{p'}(c))$ for $p' \leq p' + q' \leq k-1$. Thus, it is possible to localize $\mathbb{D}$ at these elements. Also, for $\gamma \in \mathbb{C}(c, c')$, $\rho(\gamma)$ is indeed in $\mathbb{D}(s_k(\rho(c)), t_k(\rho(c')))$. Thus, ρ induces a map of degree q $\widehat{\rho} : \mathbb{C}(c, c') \to \mathbb{D}(s_k(\rho(c)), t_k(\rho(c')))$.

This does not make $\widehat{\rho}$ into a q-transfor (or lax-q-transformation) yet. For this, one also needs maps of degree q' $\widehat{\rho}^{\pm}_{q'} : \mathbb{C}(c, c') \to \mathbb{D}(s_k(\rho(c)), t_k(\rho(c')))$ for all $q' < q$ which are themselves q'-transfors (or lax-q'-transformations) such that $\widehat{\rho}(\gamma)$ has the appropriate composites as faces. Now $\rho^{\beta'}_{q'}(\gamma)$ is not in $\mathbb{D}(s_k(\rho(c)), t_k(\rho(c')))$, so the idea is to define $\widehat{\rho}^{\beta'}_{q'}(\gamma)$ by composing $\rho^{\beta'}_{q'}(\gamma)$ with elements in the faces of $\rho(c)$ and $\rho(c')$.

In section 4 I show that for a lax-q-transformation $\rho : \mathbb{C} \to \mathbb{D}$ between ω-categories $\widehat{\rho}$ thus defined is indeed a lax-q-transformation $\mathbb{C}(c, c') \to \mathbb{D}(s_k(\rho(c)), t_k(\rho(c')))$. The reason this works is that composition in an ω-category is a functor $\mathbb{C}(c, c') \times \mathbb{C}(c', c'') \to \mathbb{C}(c, c'')$, and thus that composition with an element is a *functor* $\mathbb{C}(c, c') \to \mathbb{C}(c, c'')$. $\widehat{\rho}^{\beta'}_{q'}$ can now be seen as the composite (*as a lax-q'-transformation* [17, Sections 3-10 and 3-12]) of the lax-q'-transformation $\rho^{\beta'}_{q'}$ with composition functors, which do not spoil transforiality.

For a q-transfor $\rho : \mathbb{C} \to \mathbb{D}$ between teisi, this will not work. This is partly because composition with an element is a *transfor* $\mathbb{C}(c, c') \to \mathbb{C}(c, c'')$, composition (as a transfor) with which can and does spoil transforiality [19, Section 9], but also because in this case $\widehat{\rho}^{\beta'}_{q'}$ can and will involve dimension raising composites of elements in the faces of $\rho(c)$ and $\rho(c')$, which spoils transforiality too.

In section 5 I show that for $n \leq 4$ and $k \geq n-3$, a q-transfor ρ between nD teisi can be localized at k-dimensional elements to a q-transfor $\widehat{\rho}$. Inspection of the proofs shows that this works only because the localized tas is a 2-category: from the next dimension, i.e., higher n and/or lower k, the obstructions mentioned above do indeed occur. Thus, generally speaking, *transfors are not localizable*.

Closely related to localization of transfors is the question of localization of functors from a tensor product. Given a functor $\chi : \mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$, it localizes to a functor $(\mathbb{C} \otimes \mathbb{D})(x, x') \to \mathbb{E}(\chi(x), \chi(x'))$, for x and x' ℓ-arrows of $\mathbb{C} \otimes \mathbb{D}$ whose $(\ell-1)$-sources and $(\ell-1)$-targets agree. The first observation to make is that the localization of χ *as a functor from a tensor product* must have as domain $\mathbb{C}(c, c') \otimes \mathbb{D}(d, d')$ for c and c' k-arrows of $\mathbb{C}$ whose $(k-1)$-sources and $(k-1)$-targets agree and d and d' k-arrows (same k) of $\mathbb{D}$ whose $(k-1)$-sources and $(k-1)$-targets agree. One would want to take $x = c \otimes d$ and $x' = c' \otimes d'$, but, as before, these have non-agreeing sources and targets. The second observation to make is that the localization of χ as a functor from a tensor product must have as codomain $\mathbb{E}(e, e')$ for appropriate k-arrows e and e' of $\mathbb{E}$ whose $(k-1)$-sources and $(k-1)$-targets agree. But, as before, $\chi(c\otimes d)$ and $\chi(c'\otimes d')$, besides having non-agreeing sources and targets, are $(k+k)$-dimensional. And furthermore, although this can be overcome by taking $e = s_k(\chi(c\otimes d))$ and $e' = t_k(\chi(c'\otimes d'))$, as before, it is the case that for $\gamma\in\mathbb{C}(c,c')_i$ and $\delta\in\mathbb{D}(d,d')_j$, $\chi(\gamma\otimes\delta)$ is of dimension $i+k+1+j+k+1$ in $\mathbb{E}$, hence $\in \mathbb{E}(s_k(\chi(c\otimes d),t_k(\chi(c'\otimes d')))_{i+j+k+1}$.

In section 6 I show that for $n\leq 4$ and $k\geq n-3$, a functor χ between nD teisi from a tensor product can be localized at k-dimensional elements to a $(k+1)$-transfor $\widehat{\chi}$ from the localized tensor product. Again, inspection of the proofs shows that this works only because the localized tas is a 2-category: from the next dimension, i.e., higher n and/or lower k, obstructions as mentioned above for localization of transfors do indeed occur in this case too. Thus, generally speaking, *functors from tensor products are not localizable*.

Throughout the paper, I apply all this to composition *in* teisi, and in particular to k-monoidal structures *on* teisi. The conclusions on braidings can be summarized as follows:

- for an object A of a braided 2D tas $\mathbb{C}$, $\widehat{R_{A,-}}$ is a 1-transfor $A\otimes - \to -\otimes A$,
- for an arrow $f: A\to A'$ of a braided 2D tas $\mathbb{C}$, $\widehat{R_{f,-}}$ is a 2-transfor $\widehat{R_{A',-}}\,\#_0\,(f\otimes -) \to (\widetilde{-\otimes f})\,\#_0\,\widehat{R_{A,-}} : A\otimes - \to -\otimes A'$,
- for $R:\Sigma(\mathbb{C})\otimes\Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$ the braiding of a braided 2D tas $\mathbb{C}$, $\widehat{R_{-,?}}$ is a 1-transfor $\widetilde{?\otimes -} \to ?\otimes -$.

This answers the earlier concerns about braidings: *for 2D teisi,* $R_{A,-}$ *is* a posteriori a 1-transfor $A\otimes - \to -\otimes A$, and the braiding is *a posteriori* a 1-transfor $\widetilde{?\otimes -} \to ?\otimes -$, *but not a 1-transfor* $-\otimes ? \to ?\otimes -$, as $\widetilde{?\otimes -} \neq -\otimes ?$. For braidings on higher-dimensional teisi, one needs to impose strong extra conditions to ensure that these results still go through.

The conclusions on syllepses can be summarized as follows:

- for an object A of a sylleptic 2D tas $\mathbb{C}$, $\widehat{v_{A,-}}$ is a 2-transfor $\widehat{R_{A,-}} \to \widehat{\widetilde{R_{A,-}}} : A\otimes - \to -\otimes A$,
- for $v:\Sigma^2(\mathbb{C})\otimes\Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$ the syllepsis of a sylleptic 2D tas $\mathbb{C}$, $\widehat{v_{-,?}}$ is a 2-transfor $\widehat{R_{-,?}} \to \widehat{\widetilde{R_{-,?}}} : \widetilde{?\otimes -} \to ?\otimes -$.

Again, for syllepses in higher-dimensional teisi, one needs to impose strong extra conditions to ensure that these results still go through.

I have limited myself to looking at transfors between nD teisi for $n\leq 4$. I could have defined 5D teisi, and would have obtained slightly more general results than I do here. But the limited setting gives enough interesting results already, enough pointers to the higher-dimensional case, and makes the calculations easier.

One of the obstructions to localization resulting in a transfor has to do with functoriality. This suggests that this obstruction might be dealt with by requiring transformations to be only weakly functorial. If one wanted to apply these results to braidings etc., one would then also need to have higher-dimensional categorical structures in which composition is only weakly functorial. Perhaps the development of the theory of weak n-categories will eventually lead to this kind of structures.

In this paper I have taken a formulaic approach, rather than a diagrammatic one. This is in preparation for the development of the general higher-dimensional theory of teisi, which, at some point, will have to involve formulae.

2 Teisi and transfors

This section contains preliminaries on teisi and transfors. I will only give details insofar as is necessary for this paper; for other details the reader may consult the referred papers.

2.1 Teisi

The basic ingredient of teisi is that they have dimension-raising compositions, with n-composition of a p-arrow and a q-arrow resulting in a $(p+q-n-1)$-arrow. These compositions are to satisfy certain axioms on faces, called *naturality*, certain axioms relating different compositions, called *functoriality* and *interchange*, and axioms on multiple composition, called *associativity*. n-composition is written $\#_n$, in "evaluation order", e.g., $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ is denoted by $g \#_0 f$.

The precise definition is based on Gray's tensor product of 2-categories [29] and on the tensor product of **Gray**-categories [19]. Denote the monoidal category of 2-categories with the iso-version of Gray's tensor product by **Gray**, and the monoidal category of **Gray**-categories with the iso-version of the tensor product of **Gray**-categories by **Gray-Cat**. Recall that a **Gray**-category is a category enriched in **Gray**.

Definition 2.1 A 0-*dimensional tas* is a set. A 1-*dimensional tas* is a category. A 2-*dimensional tas* is a 2-category. A 3-*dimensional tas* is a **Gray**-category. A 4-*dimensional tas* is a category enriched in **Gray-Cat**.

For $0 \leq n \leq 4$, a *functor* between n-dimensional teisi is a function, functor, 2-functor, **Gray**-functor or **Gray-Cat**-functor respectively. $\diamond$

n-dimensional teisi are also called nD teisi for short. An elementary description of 4D teisi is given in [21, Lemma 2.5]. A functor is just a map that preserves all compositions "on the nose".

It is possible to define a similar tensor product of 4D teisi. It is not associative, but it should be possible to generalize the theory of enrichment to cover this case, and to define 5D teisi as categories enriched (in this extended sense) with respect to this tensor product. In any case, it is possible to define 5D teisi explicitly [20]. Besides the classes of axioms mentioned above, 5D teisi have further axioms which can be called *associa-functoriality* and *functorio-functoriality*. These further axioms play no rôle in this paper, though.

In the rest of this paper, when I refer to nD teisi, it will be assumed that $n \leq 5$.

Denote the category of nD teisi and functors by nD-**Teisi**.

An immediate consequence of the fact that teisi are repeatedly-enriched categories is that for two k-arrows c and c' of an nD tas $\mathbb{C}$ satisfying $d^{\alpha}_{k-1}(c) = d^{\alpha}_{k-1}(c')$ for $\alpha = \pm$, the collection of elements of $\mathbb{C}$ which have c and c' as k-source and k-target respectively is an $(n-k)$D tas, which is called the *localization of* $\mathbb{C}$ *at* c *and* c', and is denoted by $\mathbb{C}(c,c')$.

Proposition 2.2 *Let c, c' and c'' be k-arrows in an n-dimensional tas $\mathbb{C}$ satisfying $d^{\alpha}_{k-1}(c) = d^{\alpha}_{k-1}(c') = d^{\alpha}_{k-1}(c'')$ for $\alpha = \pm$. Then k-composition is a functor $\mathbb{C}(c,c') \otimes \mathbb{C}(c',c'') \to \mathbb{C}(c,c'')$.*

2.2 k-monoidal teisi

This subsection recalls the necessary parts of [21, Section 3].

Note that if an nD tas $\mathbb{C}$ has only one k-arrow, it also has only one k'-arrow for every $k' < k$, and for every $0 < k' \leq k$, the unique k'-arrow of $\mathbb{C}$ is an identity on the unique $(k'-1)$-arrow of $\mathbb{C}$.

Definition 2.3 For $n+k \leq 5$ or $n \leq 2$, a *k-monoidal n-dimensional tas* is an $(n+k)$-dimensional tas having one $(k-1)$-arrow. A *k-monoidal functor* between k-monoidal n-dimensional teisi is a functor between the corresponding $(n+k)$-dimensional teisi.

1-monoidal is also called *monoidal*, 2-monoidal is also called *braided*, and 3-monoidal is also called *sylleptic*. For $n=2$ and $k \geq 4$ this definition mentions teisi of dimension > 5. These haven't been defined yet, so the result of [21, Section 7] will be taken to define 4-monoidal 2D teisi as symmetric 2D teisi, and the Stabilization Theorem [21, Theorem 3.8] implies that for $k > 4$, k-monoidal 2D teisi are just symmetric 2D teisi.

k-monoidal nD teisi are considered nD teisi with extra structure. To emphasize this interpretation, the dimensions are renumbered: if $\mathbb{C}$ is an nD tas with one $(k-1)$-arrow, the i-arrows of $\mathbb{C}$ are seen as the $(i-k)$-arrows of the corresponding k-monoidal nD tas, and m-composition of $\mathbb{C}$ is seen as $(m-k)$-*compositions* of the k-monoidal nD tas. $(-m)$-composition is also called *m-tensor*, and is also denoted by $\otimes_m$. 1-tensor is also called *tensor*, 2-tensor is also called *braiding*, and 3-tensor is also called *syllepsis*. For historical reasons, $\otimes$ is not written in evaluation order, $-\otimes_2?$ is also written $R_{-,?}$, and $-\otimes_3?$ is also written $v_{-,?}$.

Denote the category of k-monoidal nD teisi and k-monoidal functors by $\bigotimes^k$-nD-**Teisi**.

There are inclusion functors

$$\Sigma^m : \bigotimes^k\text{-}n\mathrm{D}\text{-}\mathbf{Teisi} \hookrightarrow \bigotimes^{k-m}\text{-}(n+m)\mathrm{D}\text{-}\mathbf{Teisi}$$

for every k and $0 \leq m \leq k$. For an nD tas $\mathbb{C}$ with a k-monoidal structure, $\Sigma^m(\mathbb{C})$ is the same data considered as an $(n+m)$D tas which happens to have one m-arrow, with a $(k-m)$-monoidal structure. $\Sigma^m(\mathbb{C})$ is called the *m-th delooping of* $\mathbb{C}$.

There are also functors

$$U : \bigotimes^k\text{-}n\mathrm{D}\text{-}\mathbf{Teisi} \longrightarrow \bigotimes^j\text{-}n\mathrm{D}\text{-}\mathbf{Teisi}$$

for $j < k$ which simply forget $\otimes_m$ for all $m > j$.

The converse of delooping can be defined more generally.

Definition 2.4 Let $k > 0$, $\mathbb{C}$ an nD tas, and C an object of $\mathbb{C}$. $\Omega^k(\mathbb{C}, C)$ is the subtas of $\mathbb{C}$ having as only $(k-1)$-arrow id_C^{k-1}. ◇

By restricting to appropriate subcategories, there are functors

$$\Omega^m : \bigotimes^k\text{-}n\mathrm{D}\text{-}\mathbf{Teisi} \longrightarrow \bigotimes^{k+m}\text{-}(n-m)\mathrm{D}\text{-}\mathbf{Teisi}$$

for every k and $0 \leq m \leq n$. $\Omega^m(\mathbb{C})$ is called the *m-th looping of* $\mathbb{C}$. $\Omega^m\Sigma^m = \mathrm{id}_{\bigotimes^k\text{-}n\mathrm{D}\text{-}\mathbf{Teisi}}$, and $\Sigma^m\Omega^m(\mathbb{C})$ is a subtas of $\mathbb{C}$ which is equal to $\mathbb{C}$ precisely when $\mathbb{C}$ has one $(m-1)$-arrow. The connection between looping and localization is given by $U(\Omega^k(\mathbb{C}, C)) = \mathbb{C}(\mathrm{id}_C^{k-1}, \mathrm{id}_C^{k-1})$ [21, Lemma 3.6].

By applying proposition 2.2, one sees that tensor in a monoidal tas $\mathbb{C}$ is a functor $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, braiding in a braided 2D (or 3D) tas $\mathbb{C}$ is a functor $\Sigma(\mathbb{C}) \otimes \Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$, and syllepsis in a sylleptic 2D tas $\mathbb{C}$ is a functor $\Sigma^2(\mathbb{C}) \otimes \Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$.

2.3 Transfors

This subsection recalls [19, Section 5.1].

For $q \leq n$, let 2_q be the nD tas generated by one element in dimension q.

Definition 2.5 Let $q \leq n \leq 4$, and let $\mathbb{C}$ and $\mathbb{D}$ be n-dimensional teisi. A (right) q-transfor $\mathbb{C} \to \mathbb{D}$ is a functor $\mathbb{C} \otimes 2_q \to \mathbb{D}$. $\diamond$

Because I will use transfors extensively, I will repeat the explicit description of transfors between 3D teisi $\mathbb{C}$ and $\mathbb{D}$:

Let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors. A *(right)* 1-*transfor*, or *lax-natural transformation*, $\rho : \rho_0^- \to \rho_0^+$ consists of the following data:

- for every object C of $\mathbb{C}$ an arrow $\rho(C) : \rho_0^-(C) \to \rho_0^+(C)$ in $\mathbb{D}$,

- for every arrow $f : C \to C'$ in $\mathbb{C}$ a 2-arrow

$$\begin{tikzcd}
& \rho_0^+(C) \arrow[dr, "\rho_0^+(f)"] & \\
\rho_0^-(C) \arrow[ur, "\rho(C)"] \arrow[dr, "\rho_0^-(f)"'] \arrow[rr, phantom, "\Downarrow\rho(f)"] & & \rho_0^+(C') \\
& \rho_0^-(C') \arrow[ur, "\rho(C')"'] &
\end{tikzcd}$$

  in $\mathbb{D}$,

- for every 2-arrow $\gamma\colon \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in $\mathbb{C}$ a 3-arrow

$$\begin{tikzcd}
& \rho_0^+(C) \arrow[dr, bend left] & & & & \rho_0^+(C) \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Swarrow\rho_0^+(\gamma)"] & \\
\rho_0^-(C) \arrow[ur, "\rho(C)"] \arrow[dr, bend left] \arrow[dr, bend right] \arrow[dr, phantom, "\Swarrow\rho_0^-(\gamma)"'] \arrow[rr, phantom, "\Downarrow\rho(f)"] & & \rho_0^+(C') \arrow[rr, "\rho(\gamma)", "3"'] & & \rho_0^-(C) \arrow[ur, "\rho(C)"] \arrow[dr, bend right] \arrow[rr, phantom, "\rho(f')\Downarrow"] & & \rho_0^+(C') \\
& \rho_0^-(C') \arrow[ur, "\rho(C')"'] & & & & \rho_0^-(C') \arrow[ur, "\rho(C')"'] &
\end{tikzcd}$$

  in $\mathbb{D}$,

satisfying the following conditions:

- (naturality) for every $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, "\gamma' \Lleftarrow_3 \gamma"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\begin{array}{c} \big(\rho(f') \,\#_1\, (\rho_0^+(\varphi) \,\#_0\, \rho(C))\big) \\ \#_2 \\ \rho(\gamma) \end{array} \quad = \quad \begin{array}{c} \rho(\gamma') \\ \#_2 \\ \big((\rho(C') \,\#_0\, \rho_0^-(\varphi)) \,\#_1\, \rho(f)\big) \end{array} ,$$

- (functoriality with respect to 0-composition of arrows) for every $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$,

$$\rho(f' \,\#_0\, f) = \big(\rho(f') \,\#_0\, \rho_0^-(f)\big) \,\#_1\, \big(\rho_0^+(f') \,\#_0\, \rho(f)\big) ,$$

- (functoriality with respect to 1-composition of 2-arrows) for every $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma'"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\rho(\gamma' \,\#_1\, \gamma) = \begin{array}{c} \big(\rho(\gamma') \,\#_1\, (\rho_0^+(\gamma) \,\#_0\, \rho(C))\big) \\ \#_2 \\ \big((\rho(C') \,\#_0\, \rho_0^-(\gamma')) \,\#_1\, \rho(\gamma)\big) \end{array} ,$$

- (functoriality with respect to 0-composition of a 2-arrow with an arrow) for every $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\gamma"] & C' \arrow[r, "f''"] & C'' \end{tikzcd}$ in $\mathbb{C}$,

$$\rho(f'' \,\#_0\, \gamma) = \begin{array}{c} \big((\rho(f'') \,\#_0\, \rho_0^-(f')) \,\#_1\, (\rho_0^+(f'') \,\#_0\, \rho(\gamma))\big) \\ \#_2 \\ \big((\rho(f'') \,\#_0\, \rho_0^-(\gamma)) \,\#_1\, (\rho_0^+(f'') \,\#_0\, \rho(f))\big) \end{array} ,$$

- (functoriality with respect to 0-composition of an arrow with a 2-arrow) for every $\begin{tikzcd} C \arrow[r, "f"] & C' \arrow[r, bend left, "f'"] \arrow[r, bend right, "f''"'] \arrow[r, phantom, "\Downarrow\gamma'"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\rho(\gamma' \,\#_0\, f) = \begin{array}{c} \big((\rho(f'') \,\#_0\, \rho_0^-(f)) \,\#_1\, (\rho_0^-(\gamma') \,\#_0\, \rho(f))^{-1}\big) \\ \#_2 \\ \big((\rho(\gamma) \,\#_0\, \rho_0^-(f)) \,\#_1\, (\rho_0^+(f') \,\#_0\, \rho(f))\big) \end{array} ,$$

- (functoriality with respect to identities) for every C in $\mathbb{C}$, $\rho_{\mathrm{id}_C} = \mathrm{id}_{\rho(C)}$, and for every $f : C \to C'$ in $\mathbb{C}$, $\rho(\mathrm{id}_f) = \mathrm{id}_{\rho(f)}$.

Let $\rho_1^-, \rho_1^+ : \rho_0^- \to \rho_0^+$ be 1-transfors. A *(right) 2-transfor*, or *lax-modification*, $\rho : \rho_1^- \to \rho_1^+$ consists of the following data:

- for every object C of $\mathbb{C}$ a 2-arrow $\rho(C):\ \rho_0^-(C) \underset{\rho_1^+(C)}{\overset{\rho_1^-(C)}{\Downarrow}} \rho_0^+(C)$ in $\mathbb{D}$,

- for every arrow $f : C \to C'$ in $\mathbb{C}$ a 3-arrow

$$
\begin{tikzcd}
 & \rho_0^+(C) \arrow[dr] & \\
\rho_0^-(C) \arrow[ur, bend left, "\rho_1^-(C)"] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow\rho(C)"] \arrow[dr] & \Downarrow\rho_1^+(f) & \rho_0^+(C') \\
 & \rho_0^-(C') \arrow[ur, bend right, "\rho_1^+(C')"'] &
\end{tikzcd}
\xrightarrow[3]{\rho(f)}
\begin{tikzcd}
 & \rho_0^+(C) \arrow[dr] & \\
\rho_0^-(C) \arrow[ur, bend left, "\rho_1^-(C)"] \arrow[dr] & \rho_1^-(f)\Downarrow & \rho_0^+(C') \\
 & \rho_0^-(C') \arrow[ur, bend left] \arrow[ur, bend right, "\rho_1^+(C')"'] \arrow[ur, phantom, "\Downarrow\rho(C')"] &
\end{tikzcd}
.
$$

  in $\mathbb{D}$,

satisfying the following conditions:

- (naturality) for every $\gamma :\ C \underset{f'}{\overset{f}{\Downarrow}} C'$ in $\mathbb{C}$,

$$
\begin{array}{c}
\big((\rho(C')\,\#_0\,\rho_0^-(f'))\,\#_1\,\rho_1^-(\gamma)\big) \\
\#_2 \\
\big((\rho(C')\,\#_0\,\rho_0^-(\gamma))\,\#_1\,\rho_1^-(f)\big) \\
\#_2 \\
\big((\rho_1^+(C')\,\#_0\,\rho_0^-(\gamma))\,\#_1\,\rho(f)\big)
\end{array}
=
\begin{array}{c}
\big(\rho(f')\,\#_1\,(\rho_0^+(\gamma)\,\#_0\,\rho_1^+(C))\big) \\
\#_2 \\
\big(\rho_1^+(f')\,\#_1\,(\rho_0^+(\gamma)\,\#_0\,\rho(C))^{-1}\big) \\
\#_2 \\
\big(\rho_1^+(\gamma)\,\#_1\,(\rho_0^+(f)\,\#_0\,\rho(C))\big)
\end{array}
,
$$

  (functoriality with respect to 0-composition of arrows) for every $C \xrightarrow{f} C' \xrightarrow{f'} C''$ in $\mathbb{C}$,

$$
\rho(f'\,\#_0\,f) =
\begin{array}{c}
\big((\rho(f')\,\#_0\,\rho_0^-(f))\,\#_1\,(\rho_0^+(f')\,\#_0\,\rho_1^-(f))\big) \\
\#_2 \\
\big((\rho_1^+(f')\,\#_0\,\rho_0^-(f))\,\#_1\,(\rho_0^+(f')\,\#_0\,\rho(f))\big)
\end{array}
,
$$

- (functoriality with respect to identities) for every C in $\mathbb{C}$, $\rho(\mathrm{id}_C) = \mathrm{id}_{\rho(C)}$.

Let $\rho_2^-, \rho_2^+ : \rho_1^- \to \rho_1^+$ be 2-transfors. A *(right)* 3-*transfor*, or *perturbation*, $\rho : \rho_2^- \to \rho_2^+$ consists of the following data:

- for every object C of $\mathbb{C}$ a 3-arrow $\rho(C) : \rho_2^-(C) \to \rho_2^+(C)$ in $\mathbb{D}$,

satisfying the following condition:

- (naturality) for every $f : C \to C'$ in $\mathbb{C}$,

$$
\begin{array}{c}
\rho_2^+(f) \\
\#_2 \\
\big(\rho_1^+(f)\,\#_1\,(\rho_0^+(f)\,\#_0\,\rho(C))\big)
\end{array}
=
\begin{array}{c}
\big((\rho(C')\,\#_0\,\rho_0^-(f))\,\#_1\,\rho_1^-(f)\big) \\
\#_2 \\
\rho_2^-(f)
\end{array}
.
$$

Transfors between lower-dimensional teisi are obtained by replacing high-dimensional $\rho(c)$'s by naturality conditions. Transfors between 4-dimensional teisi are obtained by replacing naturality conditions by $\rho(c)$'s and adding naturality and functoriality conditions for them. These further conditions play no rôle in this paper, though.

To see composition in a tas as a transfor, let c, c' and c'' be k-arrows in an nD tas $\mathbb{C}$ satisfying $d^{\alpha}_{k-1}(c) = d^{\alpha}_{k-1}(c') = d^{\alpha}_{k-1}(c'')$ for $\alpha = \pm$, and fix an $(i-k-1)$-arrow a in $\mathbb{C}(c',c'')$, so a is an i-arrow in $\mathbb{C}$. Then k-composition on the right with a is given by

$$\mathbb{C}(c,c') \otimes 2_{i-k-1} \longrightarrow \mathbb{C}(c,c') \otimes \mathbb{C}(c',c'') \longrightarrow \mathbb{C}(c,c''),$$

so it is an $(i-k-1)$-transfor $\mathbb{C}(c,c') \to \mathbb{C}(c,c'')$. Its domain and codomain *as a transfor* are composition with the domain and codomain of a respectively, and composition of such transfors [19, Section 9] corresponds precisely to composition of the a's in $\mathbb{C}$. Analogously, composition on the left is a left transfor.

Applying this to tensor, braiding and syllepsis with an i-arrow of a k-monoidal nD tas $\mathbb{C}$, one gets that tensor with an i-arrow is an i-transfor $\mathbb{C} \to \mathbb{C}$, braiding with an i-arrow is an $(i+1)$-transfor $\Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$, and syllepsis with an i-arrow is an $(i+2)$-transfor $\Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$.

3 From left to right

Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be n-dimensional teisi. Does a left q-transfor $\mathbb{C} \to \mathbb{D}$, i.e., a functor $2_q \otimes \mathbb{C} \to \mathbb{D}$, induce a right q-transfor $\mathbb{C} \to \mathbb{D}$, i.e., a functor $\mathbb{C} \otimes 2_q \to \mathbb{D}$? More generally, does a functor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ induce a functor $\mathbb{D} \otimes \mathbb{C} \to \mathbb{E}$? In this section I answer these questions for $n = 3$ and $n = 4$, and apply the results to compositions in 4D teisi, and in particular to braidings and syllepses.

3.1 Righting left 1-transfors

First note that there is no difference between left and right functors, even as seen as 0-transfors, because $2_0 \otimes \mathbb{C} \cong \mathbb{C} \cong \mathbb{C} \otimes 2_0$.

To go from left to right transfors they need to be *iso-transfors*, i.e., transfors $\rho : \mathbb{C} \to \mathbb{D}$ such that $\rho(c)$ is invertible for every $c \in \mathbb{C}$ that is not 0-dimensional. Left composition with an element of an nD tas is an example of a left iso-transfor.

Let $\mathbb{C}$ and $\mathbb{D}$ be 3D teisi, let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, and let $\rho : \rho_0^- \to \rho_0^+$ be a left 1-transfor. Define a map of degree 1 $\widetilde{\rho} : \mathbb{C} \to \mathbb{D}$ by:

$$\begin{array}{rcl} \widetilde{\rho}(C) & = & \rho(C) \\ \widetilde{\rho}(f) & = & \rho(f)^{-1} \\ \widetilde{\rho}(\gamma) & = & \rho(f')^{-1} \#_1 \rho(\gamma) \#_1 \rho(f)^{-1}. \end{array}$$

Proposition 3.1 *$\widetilde{\rho}$ is a (right) 1-transfor $\rho_0^- \to \rho_0^+$.*

Proof. $\widetilde{\rho}(f)$ and $\widetilde{\rho}(\gamma)$ indeed have the right faces.

For $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f'"'] \arrow[r, phantom, "\gamma' \Lleftarrow_{3} \gamma"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$
\begin{aligned}
&\left(\widetilde{\rho}(f')\,\#_1\,(\widetilde{\rho_0^+}(\varphi)\,\#_0\,\widetilde{\rho}(C))\right)\#_2\,\widetilde{\rho}(\gamma) = \\
&= \quad \left(\rho(f')^{-1}\,\#_1\,(\rho_0^+(\varphi)\,\#_0\,\rho(C))\right)\#_2\left(\rho(f')^{-1}\,\#_1\,\rho(\gamma)\,\#_1\,\rho(f)^{-1}\right) \\
&= \quad \rho(f')^{-1}\,\#_1\left(((\rho_0^+(\varphi)\,\#_0\,\rho(C))\,\#_1\,\rho(f))\,\#_2\,\rho(\gamma)\right)\#_1\,\rho(f)^{-1} \\
&= \quad \rho(f')^{-1}\,\#_1\left(\rho(\gamma')\,\#_2\,(\rho(f')\,\#_1\,(\rho(C')\,\#_0\,\rho_0^-(\varphi)))\right)\#_1\,\rho(f)^{-1} && \text{by naturality of } \rho \\
&= \quad \left(\rho(f')^{-1}\,\#_1\,\rho(\gamma')\,\#_1\,\rho(f)^{-1}\right)\#_2\left((\rho(C')\,\#_0\,\rho_0^-(\varphi))\,\#_1\,\rho(f)^{-1}\right) \\
&= \quad \widetilde{\rho}(\gamma')\,\#_2\left((\widetilde{\rho}(C')\,\#_0\,\widetilde{\rho_0^-}(\varphi))\,\#_1\,\widetilde{\rho}(f)\right)
\end{aligned}
$$

which proves naturality of $\widetilde{\rho}$.

Functoriality of $\widetilde{\rho}$ with respect to 0-composition of arrows is straightforward.

For
$$
\begin{tikzcd}
C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\gamma'"] & C'
\end{tikzcd}
$$
in $\mathbb{C}$,

$$
\begin{aligned}
&\widetilde{\rho}(\gamma'\,\#_1\,\gamma) = \\
&= \quad \rho(f'')^{-1}\,\#_1\,\rho(\gamma'\,\#_1\,\gamma)\,\#_1\,\rho(f)^{-1} \\
&= \quad \rho(f'')^{-1}\,\#_1\left(((\rho_0^+(\gamma')\,\#_0\,\rho(C))\,\#_1\,\rho(\gamma))\,\#_2\right. \\
&\qquad \left.(\rho(\gamma')\,\#_1\,(\rho(C')\,\#_0\,\rho_0^-(\gamma)))\right)\#_1\,\rho(f)^{-1} && \text{by functoriality of } \rho \\
&= \quad \rho(f'')^{-1}\,\#_1\left(((\rho_0^+(\gamma')\,\#_0\,\rho(C))\,\#_1\,(\rho(f')\,\#_1\,\rho(f')^{-1}\,\#_1\,\rho(\gamma))\,\#_2\right. \\
&\qquad \left.(\rho(\gamma')\,\#_1\,\rho(f')^{-1}\,\#_1\,\rho(f')\,\#_1\,(\rho(C')\,\#_0\,\rho_0^-(\gamma)))\right)\#_1\,\rho(f)^{-1} \\
&= \quad \rho(f'')^{-1}\,\#_1\left((\rho(\gamma')\,\#_1\,\rho(f')^{-1}\,\#_1\,(\rho_0^+(\gamma)\,\#_0\,\rho(C))\,\#_1\,\rho(f))\,\#_2\right. \\
&\qquad \left.(\rho(f'')\,\#_1\,(\rho(C')\,\#_0\,\rho_0^-(\gamma'))\,\#_1\,\rho(f')^{-1}\,\#_1\,\rho(\gamma))\right)\#_1\,\rho(f)^{-1} && \text{by naturality in } \mathbb{D} \\
&= \quad \left(\rho(f'')^{-1}\,\#_1\,\rho(\gamma')\,\#_1\,\rho(f')^{-1}\,\#_1\,(\rho_0^+(\gamma)\,\#_0\,\rho(C))\right)\#_2 \\
&\qquad \left((\rho(C')\,\#_0\,\rho_0^-(\gamma'))\,\#_1\,\rho(f')^{-1}\,\#_1\,\rho(\gamma)\,\#_1\,\rho(f)^{-1}\right) \\
&= \quad \left(\widetilde{\rho}(\gamma')\,\#_1\,(\widetilde{\rho_0^+}(\gamma)\,\#_0\,\widetilde{\rho}(C))\right)\#_2\left((\widetilde{\rho}(C')\,\#_0\,\widetilde{\rho_0^-}(\gamma'))\,\#_1\,\widetilde{\rho}(\gamma)\right)
\end{aligned}
$$

which proves functoriality of $\widetilde{\rho}$ with respect to 1-composition of 2-arrows.

The other functoriality axioms for $\widetilde{\rho}$ are left to the reader. □

This will not work for transfors between 4D teisi as the naturality in $\mathbb{D}$ in the first functoriality proof becomes a non-identity arrow. If it does happen to work, including what might go wrong with the higher-dimensional naturality and functoriality conditions, which I didn't mention, call the transfor *rightable*.

As an example, let $\mathbb{C}$ be a 4-dimensional tas, $\gamma\colon$
$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$
a 2-arrow of $\mathbb{C}$, C'' an object of $\mathbb{C}$, and consider the case that ρ is the left 1-transfor $-\,\#_0\,\gamma : \mathbb{C}(C', C'') \to \mathbb{C}(C, C'')$, which has source and target the functors $-\,\#_0\,f$ and $-\,\#_0\,f$ respectively. Righting $-\,\#_0\,\gamma$, one gets the right 1-transfor $\widetilde{-\,\#_0\,\gamma} : \mathbb{C}(C', C'') \to \mathbb{C}(C, C'')$, given by

$$
\begin{aligned}
\widetilde{-\,\#_0\,\gamma}(g) &= g\,\#_0\,\gamma \\
\widetilde{-\,\#_0\,\gamma}(\delta) &= (\delta\,\#_0\,\gamma)^{-1} \\
\widetilde{-\,\#_0\,\gamma}(\psi) &= (\delta'\,\#_0\,\gamma)^{-1}\,\#_1\,(\psi\,\#_0\,\gamma)\,\#_1\,(\delta\,\#_0\,\gamma)^{-1},
\end{aligned}
$$

again with source and target $-\,\#_0\,f$ and $-\,\#_0\,f$ respectively.

A special case of the above example is for a braided 2D tas $\mathbb{C}$: for an object B of $\mathbb{C}$, $R_{-,B}$ is a left 1-transfor $\Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$ with source and target the identity functor. Righting it, one gets the right 1-transfor $\widetilde{R_{-,B}} : \Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$, given by

$$\begin{array}{rcl} \widetilde{R_{-,B}}(A) & = & R_{A,B}{}^{-1} \\ \widetilde{R_{-,B}}(f) & = & R_{A',B}{}^{-1} \#_0 R_{f,B} \#_0 R_{A,B}{}^{-1}. \end{array}$$

3.2 Righting left 2-transfors

Let again $\mathbb{C}$ and $\mathbb{D}$ be 3D teisi, let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, let $\rho_1^-, \rho_1^+ : \rho_0^- \to \rho_0^+$ be left 1-transfors, and let $\rho : \rho_1^- \to \rho_1^+$ be a left 2-transfor. Define a map of degree 2 $\widetilde{\rho} : \mathbb{C} \to \mathbb{D}$ by:

$$\begin{array}{rcl} \widetilde{\rho}(C) & = & \rho(C) \\ \widetilde{\rho}(f) & = & \rho_1^+(f)^{-1} \#_1 \rho(f) \#_1 \rho_1^-(f)^{-1}. \end{array}$$

Proposition 3.2 *$\widetilde{\rho}$ is a right 2-transfor $\widetilde{\rho_1^-} \to \widetilde{\rho_1^+}$.*

Proof. $\widetilde{\rho}(f)$ indeed has the right faces.

For $\gamma :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & C' \end{tikzcd}$ in $\mathbb{C}$,

$$\begin{array}{l} \left((\widetilde{\rho}(C') \#_0 \rho_0^-(f')) \#_1 \widetilde{\rho_1^-}(\gamma) \right) \#_2 \\ \left((\widetilde{\rho}(C') \#_0 \rho_0^-(\gamma)) \#_1 \widetilde{\rho_1^-}(f) \right) \#_2 \\ \left((\widetilde{\rho_1^+}(C') \#_0 \rho_0^-(\gamma)) \#_1 \widetilde{\rho}(f) \right) = \end{array}$$

$$\begin{array}{rl} = & \left((\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f')^{-1} \#_1 \rho_1^-(\gamma) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 \rho(f') \#_1 (\rho_1^-(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 \rho(f')^{-1} \#_1 (\rho_1^-(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 \rho_1^+(f') \#_1 (\rho(C') \#_0 \rho_0^-(\gamma))^{-1-1} \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 \rho_1^+(\gamma)^{-1} \#_1 (\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 \rho_1^+(\gamma) \#_1 (\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^+(f)^{-1} \#_1 \rho(f) \#_1 \rho_1^-(f)^{-1} \right) \\ & \hfill \text{by naturality in } \mathbb{D} \text{ twice and naturality of } \rho \\ = & \left(\rho_1^+(f')^{-1} \#_1 \rho(f') \#_1 \rho_1^-(f')^{-1} \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^+(C)) \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 (\rho_0^+(f) \#_0 \rho(C)) \#_1 \rho_1^-(\gamma) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 (\rho_0^+(f) \#_0 \rho(C)) \#_1 \rho_1^-(\gamma)^{-1} \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 (\rho_0^+(\gamma) \#_0 \rho(C))^{-1} \#_1 \rho_1^-(f) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C)) \#_1 \rho(f)^{-1} \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C)) \#_1 \rho(f) \#_1 \rho_1^-(f)^{-1} \right) \#_2 \\ & \left(\rho_1^+(f')^{-1} \#_1 \rho_1^+(\gamma) \#_1 \rho_1^+(f)^{-1} \#_1 (\rho_0^+(f) \#_0 \rho(C)) \right) \\ = & \left(\widetilde{\rho}(f') \#_1 (\rho_0^+(\gamma) \#_0 \widetilde{\rho_1^+}(C)) \right) \#_2 \\ & \left(\widetilde{\rho_1^+}(f') \#_1 (\rho_0^+(\gamma) \#_0 \widetilde{\rho}(C))^{-1} \right) \#_2 \\ & \left(\widetilde{\rho_1^+}(\gamma) \#_1 (\rho_0^+(f) \#_0 \widetilde{\rho}(C)) \right) \end{array}$$

which proves naturality of $\widetilde{\rho}$.

Functoriality of $\widetilde{\rho}$ with respect to 0-composition of arrows is left to the reader. □

For transfors between 4D teisi $\widetilde{\rho_1^-}$ and $\widetilde{\rho_1^+}$ need not be transfors, and it is clear that in higher dimensions there will also be non-identity arrows in proofs of functoriality, so that $\widetilde{\rho}$ need not be a transfor either. If $\widetilde{\rho_1^-}$, $\widetilde{\rho_1^+}$ and $\widetilde{\rho}$ are transfors, call ρ *rightable*.

As an example, let $\mathbb{C}$ be a 4-dimensional tas, $\varphi :$ $\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\gamma' \overset{3}{\Lleftarrow} \gamma"] & C' \end{tikzcd}$ a 3-arrow of $\mathbb{C}$, C'' an object of $\mathbb{C}$, and consider the case that ρ is the left 2-transfor $-\#_0\,\varphi : \mathbb{C}(C',C'') \to \mathbb{C}(C,C'')$, which has source and target the left 1-transfors $-\#_0\,\gamma$ and $-\#_0\,\gamma'$ respectively. Righting $-\#_0\,\varphi$, one gets the right 2-transfor $\widetilde{-\#_0\,\varphi} : \mathbb{C}(C',C'') \to \mathbb{C}(C,C'')$, given by:

$$
\begin{aligned}
\widetilde{-\#_0\,\varphi}(g) &= g\,\#_0\,\varphi \\
\widetilde{-\#_0\,\varphi}(\delta) &= (\delta\,\#_0\,\gamma')^{-1}\,\#_1\,(\delta\,\#_0\,\varphi)\,\#_1\,(\delta\,\#_0\,\gamma)^{-1},
\end{aligned}
$$

with source and target $\widetilde{-\#_0\,\gamma}$ and $\widetilde{-\#_0\,\gamma'}$ respectively.

A special case of the above example one dimension higher is for a sylleptic 2D tas $\mathbb{C}$: for an object B of $\mathbb{C}$, $v_{-,B}$ is a left 2-transfor $\Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$ with source and target the identity left 1-transfor. $\Sigma^2(\mathbb{C})$ is a 4D tas, so things might go wrong, but in this situation there are many identities around, and the map of degree 2 $\widetilde{v_{-,B}} : \Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$ given by

$$
\widetilde{v_{-,B}}(A) \quad = \quad {R_{A,B}}^{-1}\,\#_0\,v_{-,B}(A)\,\#_0\,R_{B,A}
$$

is in fact a right 1-transfor; the calculations for this use the results on syllepses in [21, Section 6], and are left to the reader. Note that, because these calculations make use of naturality axioms in 2-categories, this result will not hold for syllepses on 3D teisi.

3.3 Righting left 3-transfors

Let again $\mathbb{C}$ and $\mathbb{D}$ be 3D teisi, let $\rho_0^-,\rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, let $\rho_1^-,\rho_1^+ : \rho_0^- \to \rho_0^+$ be left 1-transfors, let $\rho_2^-,\rho_2^+ : \rho_1^- \to \rho_1^+$ be left 2-transfors, and let $\rho : \rho_2^- \to \rho_2^+$ be a left 3-transfor. Define a map of degree 3 $\widetilde{\rho} : \mathbb{C} \to \mathbb{D}$ by:

$$
\widetilde{\rho}(C) \quad = \quad \rho(C).
$$

Proposition 3.3 *$\widetilde{\rho}$ is a right 3-transfor $\widetilde{\rho_2^-} \to \widetilde{\rho_2^+}$.*

Proof. For $f : C \to C'$ in $\mathbb{C}$,

$$
\begin{aligned}
&\widetilde{\rho_2^+}(f)\,\#_2\left(\widetilde{\rho_1^+}(f)\,\#_1\,(\widetilde{\rho_0^+}(f)\,\#_0\,\widetilde{\rho}(C))\right) = \\
&\quad = \quad \left(\rho_1^+(f)^{-1}\,\#_1\,\rho_2^+(f)\,\#_1\,\rho_1^-(f)^{-1}\right)\#_2\left(\rho_1^+(f)^{-1}\,\#_1\,(\rho_0^+(f)\,\#_0\,\rho(C))\right) \\
&\quad = \quad \rho_1^+(f)^{-1}\,\#_1\left(\rho_2^+(f)\,\#_2\,((\rho_0^+(f)\,\#_0\,\rho(C))\,\#_1\,\rho_1^-(f))\right)\#_1\,\rho_1^-(f)^{-1} \\
&\quad = \quad \rho_1^+(f)^{-1}\,\#_1\left((\rho_1^+(f)\,\#_1\,(\rho(C')\,\#_0\,\rho_0^-(f)))\,\#_2\,\rho_2^-(f)\right)\#_1\,\rho_1^-(f)^{-1} && \text{by naturality of } \rho \\
&\quad = \quad \left((\rho(C')\,\#_0\,\rho_0^-(f))\,\#_1\,\rho_1^-(f)^{-1}\right)\#_2\left(\rho_1^+(f)^{-1}\,\#_1\,\rho_2^-(f)\,\#_1\,\rho_1^-(f)^{-1}\right) \\
&\quad = \quad \left((\widetilde{\rho}(C')\,\#_0\,\widetilde{\rho_0^-}(f))\,\#_1\,\widetilde{\rho_1^-}(f)\right)\#_2\,\widetilde{\rho_2^-}(f)
\end{aligned}
$$

which proves naturality of $\widetilde{\rho}$. $\square$

Note that the calculation here is very similar to the calculation in the proof of righting a left 1-transfor.

Also note that the simplicity of the proof is due to the fact that the 3-transfor is between 3D teisi; for 3-transfors between 4D teisi the problems will be similar to the problems for righting of 2-transfors.

3.4 Reversing functors of two variables

Recall that the iso-version of Gray's tensor product of 2-categories is symmetric. I will now show that the previous results imply that this is also the case for **Gray**-categories.

Let $\mathbb{C}$ and $\mathbb{D}$ be 3D teisi, and χ a functor $\mathbb{C} \otimes_{\mathrm{iso}} \mathbb{D} \to \mathbb{E}$. Define a function $\widetilde{\chi} : \mathbb{D} \otimes_{\mathrm{iso}} \mathbb{C} \to \mathbb{E}$ by:

$$\begin{array}{rcl}
\widetilde{\chi}(D \otimes c) & = & \chi(c \otimes D) \\
\widetilde{\chi}(d \otimes C) & = & \chi(C \otimes d) \\
\widetilde{\chi}(g \otimes f) & = & \chi(f \otimes g)^{-1} \\
\widetilde{\chi}(g \otimes \gamma) & = & \chi(f' \otimes g)^{-1} \#_1 \chi(\gamma \otimes g) \#_1 \chi(f \otimes g)^{-1} \\
\widetilde{\chi}(\delta \otimes f) & = & \chi(f \otimes g')^{-1} \#_1 \chi(f \otimes \delta) \#_1 \chi(f \otimes g)^{-1}.
\end{array}$$

Theorem 3.4 *$\widetilde{\chi}$ is a functor $\mathbb{D} \otimes_{\mathrm{iso}} \mathbb{C} \to \mathbb{E}$.*

Proof. Recall [19, Section 5.2] that the conditions on χ being a functor $\mathbb{C} \otimes_{\mathrm{iso}} \mathbb{D} \to \mathbb{E}$ are that $\chi(c \otimes -)$ is a left p-transfor for every $c \in C_p$, $\chi(- \otimes d)$ is a right q-transfor for every $d \in D_q$, and an interchange condition on 1-arrows. So the statement follows from propositions 3.1, 3.2 and 3.3 and their duals, with interchange taking care of itself. □

Hence, because $\widetilde{\widetilde{\chi}} = \chi$, the iso-version of the tensor product of **Gray**-categories [19] is symmetric. However, for the iso-version of the tensor product of 4D teisi, $\widetilde{\chi}$ need not be functorial, so this tensor product is not symmetric. In this higher-dimensional case, if $\widetilde{\chi}$ happens to be a functor $\mathbb{D} \otimes_{\mathrm{iso}} \mathbb{C} \to \mathbb{E}$ say that χ is *reversible*.

As an example, let $\mathbb{C}$ be an n-dimensional tas, and consider the case that χ is the functor $\#_0 : \mathbb{C}(C, C') \otimes \mathbb{C}(C', C'') \to \mathbb{C}(C, C'')$. If $n = 4$ then $\widetilde{\#_0}$ is a functor, but in general 0-composition will not be reversible.

A special case of the above example is for a monoidal 3D tas $\mathbb{C}$: tensor is a functor $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$, which is reversible.

Another special case of the above example is for a braided 2D tas $\mathbb{C}$: braiding is a functor $\Sigma(\mathbb{C}) \otimes \Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$, which is again reversible. Braidings on 3D teisi are not reversible in general, however.

Yet another special case of the above example is for a sylleptic 2D tas $\mathbb{C}$: syllepsis is a functor $\Sigma^2(\mathbb{C}) \otimes \Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$, and it is reversible too. Syllepses on 3D teisi are not reversible in general, however.

3.5 What is symmetry again?

The formulae for rightings of the transfors 'braiding with' and 'syllepsis with', and for reversings of the functors 'braiding' and 'syllepsis', suggest the following

Definition 3.5 Let $n \leq 2$ and $k > 0$, and let $\mathbb{C}$ be a k-monoidal nD tas. $\mathbb{C}$ is *strongly symmetric* if $\otimes_k$ is reversible and $\otimes_k = \widetilde{\otimes_k} : \Sigma^{k-1}\mathbb{C} \otimes \Sigma^{k-1}\mathbb{C} \to \Sigma^{k-1}\mathbb{C}$. ◇

To justify the terminology, I need to show that strongly symmetric implies symmetric.

Consider first the case where $\mathbb{C}$ is a strongly symmetric monoidal 2D tas. Because $\mathbb{C}$ is a 2D tas $\otimes$ is reversible. $\otimes = \widetilde{\otimes}$ amounts to:

$$\begin{array}{rcl} A\otimes B & = & B\otimes A \\ f\otimes B & = & B\otimes f \\ A\otimes g & = & g\otimes A \\ f\otimes g & = & (g\otimes f)^{-1}. \end{array}$$

This gives a braiding on $\mathbb{C}$ by $R_{-,?} = \mathrm{id}$, and because the braiding gives identities this trivially makes $\mathbb{C}$ a symmetric 2D tas.

Consider now the case where $\mathbb{C}$ is a strongly symmetric braided 2D tas. $R_{-,?} = \widetilde{R_{-,?}}$ amounts to:

$$\begin{array}{rcl} R_{A,B} & = & {R_{B,A}}^{-1} \\ R_{f,B} & = & {R_{B,A'}}^{-1} \#_0 R_{B,f} \#_0 {R_{B,A}}^{-1} \\ R_{A,g} & = & {R_{B',A}}^{-1} \#_0 R_{g,A} \#_0 {R_{B,A}}^{-1}. \end{array}$$

This gives a syllepsis on $\mathbb{C}$ by $v_{-,?} = \mathrm{id}$, and because the syllepsis gives identities this trivially makes $\mathbb{C}$ a symmetric 2D tas.

Finally, consider the case where $\mathbb{C}$ is a strongly symmetric sylleptic 2D tas. $v_{-,?} = \widetilde{v_{-,?}}$ amounts to

$$v_{A,B} \quad = \quad {R_{B,A}}^{-1} \#_0 v_{B,A} \#_0 R_{A,B}.$$

This precisely makes $\mathbb{C}$ a symmetric 2D tas.

Summarizing:

Proposition 3.6 *Let $n \leq 2$ and $k > 0$, a strongly symmetric k-monoidal nD tas is a symmetric nD tas.* □

For replacing the identity in the definition of strongly symmetric by a natural isomorphism, see the sequel.

3.6 Reversing 1-transfors of two variables

Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be 2D teisi, let $\chi_0^-, \chi_0^+ : \mathbb{C} \otimes_{\mathrm{iso}} \mathbb{D} \to \mathbb{E}$ be functors, and let $\chi : \chi_0^- \to \chi_0^+$ be a (right) 1-transfor $\mathbb{C} \otimes_{\mathrm{iso}} \mathbb{D} \to \mathbb{E}$. Define a map of degree 1 $\widetilde{\chi} : \mathbb{D} \otimes_{\mathrm{iso}} \mathbb{C} \to \mathbb{E}$ by:

$$\begin{array}{rcl} \widetilde{\chi}(D\otimes c) & = & \chi(c\otimes D) \\ \widetilde{\chi}(d\otimes C) & = & \chi(C\otimes d). \end{array}$$

Proposition 3.7 $\widetilde{\chi}$ *is a right* 1*-transfor* $\widetilde{\chi_0^-} \to \widetilde{\chi_0^+}$.

Proof. First note that χ_0^- and χ_0^+ are reversible by theorem 3.4.

Secondly, note that $\chi_0^-(f\otimes g)$ and $\chi_0^+(f\otimes g)$ are invertible, because $f\otimes g$ is invertible.

For $f: C \to C'$ in $\mathbb{C}$ and $g: D \to D'$ in $\mathbb{D}$,

$$
\begin{aligned}
&(\widetilde{\chi}(D' \otimes C') \#_0 \widetilde{\chi_0^-}(g \otimes f)) \#_1 (\widetilde{\chi}(g \otimes C') \#_0 \widetilde{\chi_0^-}(D \otimes f)) \#_1 (\widetilde{\chi_0^+}(g \otimes C') \#_0 \widetilde{\chi}(D \otimes f)) = \\
&= \quad (\chi(C' \otimes D') \#_0 \chi_0^-(f \otimes g)^{-1}) \#_1 \\
&\qquad (\chi(C' \otimes g) \#_0 \chi_0^-(f \otimes D)) \#_1 (\chi_0^+(C' \otimes g) \#_0 \chi(f \otimes D)) \#_1 \\
&\qquad (\chi_0^+(f \otimes g) \#_0 \chi(C \otimes D)) \#_1 (\chi_0^+(f \otimes g)^{-1} \#_0 \chi(C \otimes D)) \\
&= \quad (\chi(C' \otimes D') \#_0 \chi_0^-(f \otimes g)^{-1}) \#_1 (\chi(C' \otimes D') \#_0 \chi_0^-(f \otimes g)) \#_1 \\
&\qquad (\chi(f \otimes D') \#_0 \chi_0^-(C \otimes g)) \#_1 (\chi_0^+(f \otimes D') \#_0 \chi(C \otimes g)) \#_1 \\
&\qquad (\chi_0^+(f \otimes g)^{-1} \#_0 \chi(C \otimes D)) \qquad \text{by naturality of } \chi \\
&= \quad (\widetilde{\chi}(D' \otimes f) \#_0 \widetilde{\chi_0^-}(g \otimes C)) \#_1 (\widetilde{\chi_0^+}(D' \otimes f) \#_0 \widetilde{\chi}(g \otimes C)) \#_1 (\widetilde{\chi_0^+}(g \otimes f) \#_0 \widetilde{\chi}(D \otimes C))
\end{aligned}
$$

which proves naturality of $\widetilde{\chi}$.

Functoriality of $\widetilde{\chi}$ is immediate from functoriality of χ. □

Already for transfors between 3D teisi things go wrong again: there is a problem with functoriality, which has to do with the diagram on page 46 of [19].

An example of the reversal of a 1-transfor will occur in section 6.1.

4 Localization of lax-q-transformations of ω-categories

Let $\mathbb{C}$ and $\mathbb{D}$ be ω-categories and let c, c' be elements of $\mathbb{C}$ whose $(k-1)$-sources and $(k-1)$-targets agree. In this section I show that a lax-q-transformation $\mathbb{C} \to \mathbb{D}$ induces a lax-q-transformation $\mathbb{C}(c, c') \to \mathbb{D}(d, d')$, for appropriate $d, d' \in \mathbb{D}$.

For c and c' k-arrows of an ω-category $\mathbb{C}$ satisfying $d_{k-1}^{\alpha}(c) = d_{k-1}^{\alpha}(c')$ for $\alpha = \pm$, a functor $F: \mathbb{C} \to \mathbb{D}$ trivially induces a functor $\mathbb{C}(c, c') \to \mathbb{D}(F(c), F(c'))$.

Viewing a lax-q-transformation $\rho: \mathbb{C} \to \mathbb{D}$ as a functor $\rho: \mathbb{C} \otimes 2_q \to \mathbb{D}$, it localizes to a functor $(\mathbb{C} \otimes 2_q)(x, x') \to \mathbb{D}(\rho(x), \rho(x'))$, for x and x' ℓ-arrows of $\mathbb{C} \otimes 2_q$ whose $(\ell-1)$-sources and $(\ell-1)$-targets agree. One would want to take $x = c \otimes 2_q$ and $x' = c' \otimes 2_q$, but these have non-agreeing sources and targets. However, the $(k-1)$-sources and $(k-1)$-targets of $s_k(\rho(c))$ and $t_k(\rho(c'))$ do agree, and ρ induces a map of degree q $\widehat{\rho}: \mathbb{C}(c, c') \to \mathbb{D}(s_k(\rho(c)), t_k(\rho(c')))$. The idea is to define $\widehat{\rho}_{q'}^{\beta'}(\gamma)$ by composing $\rho_{q'}^{\beta'}(\gamma)$ with elements in the faces of $\rho(c)$ and $\rho(c')$.

Formally, define assignments $\widehat{\rho}_q: \mathbb{C}(c, c')_p \to \mathbb{D}(s_k(\rho_q(c)), t_k(\rho_q(c')))_{p+q}$ and $\widehat{\rho}_{q'}^{\beta'}: \mathbb{C}(c, c')_p \to \mathbb{D}(s_k(\rho_q(c)), t_k(\rho_q(c')))_{p+q'}$ for every $q' < q$ and $\beta' = \pm$ by

$$\widehat{\rho}(\gamma) \quad = \quad \rho(\gamma)$$

and

$\widehat{\rho}_{q'}^{\beta'}(\gamma)$ is the composite in $\mathbb{D}$ of $\rho_{q'}^{(-)^{k+1}\beta'}(\gamma)$ and $d_{k+q'+1}^{\beta'}(\rho(\gamma))$ according to the pasting scheme $2_{p+k+1} \otimes 2_{q'}^{(-)^{k+1}\beta'} \cup d_{k+q'+1}^{\beta'}(2_{p+k+1} \otimes 2_q) \subset 2_{p+k+1} \otimes 2_q$.

Note that the second part of this composite only involves $\rho_{q''}^{\beta''}$, for $q'' > q'$, of c or c' or of faces of those, and at most $\rho_{q'}^{(-)^{k+1}\beta'}(d_{k+1}^{\beta'}(\gamma))$, which bounds $\rho_{q'}^{(-)^{k+1}\beta'}(\gamma)$, making the composite legitimate, and which is the reason for the occurrence of the sign $(-)^{k+1}$.

Theorem 4.1 *Let c and c' k-arrows of an ω-category $\mathbb{C}$ satisfying $d^{\alpha}_{k-1}(c) = d^{\alpha}_{k-1}(c')$ for $\alpha = \pm$, and let $\rho : \mathbb{C} \to \mathbb{D}$ be a lax-q-transformation. Then $\widehat{\rho}$ is a q-transfor $\mathbb{C}(c,c') \to \mathbb{D}(s_k(\rho_q(c)), t_k(\rho_q(c')))$.*

Proof. $\widehat{\rho}^{\beta'}_{q'}(\gamma)$ has the correct k-faces: $d^{\alpha}_k(\widehat{\rho}^{\beta'}_{q'}(\gamma)) = d^{\alpha}_k(d^{\beta'}_{k+q'+1}(\rho_q(\gamma))) = d^{\alpha}_k(\rho_q(\gamma)))$ which involves at most $\rho_q(d^{\alpha}_k(\gamma))$ hence is equal to $s_k(\rho_q(c))$ and $t_k(\rho_q(c'))$ respectively.

$\widehat{\rho}$ is natural: if γ is $(k+1)$-dimensional,

$$
\begin{aligned}
d^{\alpha}_{k+q'}(\widehat{\rho}^{\beta'}_{q'}(\gamma)) &= d^{\alpha}_{k+q'}(\text{the composite of } \rho^{(-)^{k+1}\beta'}_{q'}(\gamma) \text{ and } d^{\beta'}_{k+q'+1}(\rho(\gamma))) \\
&= \text{the composite of } \rho^{(-)^{k+1}\alpha}_{q'-1}(\gamma) \text{ and } d^{\alpha}_{k+q'}(\rho(\gamma)) \\
&= \widehat{\rho}^{\alpha}_{q'-1}(\gamma),
\end{aligned}
$$

and if γ is of dimension greater than $k+1$ then because the second part of the composite for $\widehat{\rho}^{\beta'}_{q'}(\gamma)$ is low dimensional enough naturality of $\widehat{\rho}$ follows from naturality of ρ.

Functoriality of $\widehat{\rho}$ follows immediately from the fact that composition with lower-dimensional elements is a functor. $\square$

5 Localization in one variable

Let $\mathbb{C}$ and $\mathbb{D}$ be n-dimensional teisi, and let c, c' be elements of $\mathbb{C}$ whose $(k-1)$-sources and $(k-1)$-targets agree. Does a q-transfor $\mathbb{C} \to \mathbb{D}$ induce a q-transfor $\mathbb{C}(c,c') \to \mathbb{D}(d,d')$, for appropriate $d, d' \in \mathbb{D}$? In this section I answer these questions for $n = 3$ and $n = 4$, and apply the results to compositions in 4D teisi, and in particular to braidings and syllepses.

5.1 Localization of functors

Is trivial.

5.2 Localizing a 1-transfor once

Let $\mathbb{C}$ and $\mathbb{D}$ be 3D teisi, let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, and let $\rho : \rho_0^- \to \rho_0^+$ be a 1-transfor. For two objects C, C' of $\mathbb{C}$, note that $\mathbb{C}(C,C')$ and $\mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ are 2D teisi.

Define functions $\widehat{\rho}_0^-, \widehat{\rho}_0^+ : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$
\begin{aligned}
\widehat{\rho}_0^-(c) &= \rho_0^+(c) \#_0 \rho(C) \\
\widehat{\rho}_0^+(c) &= \rho(C') \#_0 \rho_0^-(c).
\end{aligned}
$$

ρ_0^- and ρ_0^+ induce functors $\mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^-(C'))$ and $\mathbb{C}(C,C') \to \mathbb{D}(\rho_0^+(C), \rho_0^+(C'))$, again called ρ_0^- and ρ_0^+ respectively. $\rho(C) : \rho_0^-(C) \to \rho_0^+(C)$ is a 1-arrow in $\mathbb{D}$, so (right) composition by $\rho(C)$ induces a $(1-1)$-transfor, i.e., a functor $\mathbb{D}(\rho_0^+(C), \rho_0^+(C')) \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$. $\widehat{\rho}_0^-$ is precisely the composite of ρ_0^+ and $\rho(C) \#_0 -$, hence is a functor too. One could have also seen this easily directly, of course.

Define a map of degree 1 $\widehat{\rho} : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$
\widehat{\rho}(c) = \rho(c).
$$

Proposition 5.1 $\widehat{\rho}$ *is a* 1*-transfor* $\widehat{\rho}_0^- \to \widehat{\rho}_0^+$.

Proof. I have to show that $\widehat{\rho}(c)$ has the right faces, and that it is transforial.

For an object f of $\mathbb{C}(C,C')$,
$s_0(\widehat{\rho}(f)) = s_1(\rho(f)) = \rho_0^+(f)\,\#_0\,\rho(C) = \widehat{\rho}_0^-(f)$,
and similarly $t_0(\widehat{\rho}(f)) = \widehat{\rho}_0^+(f)$, so $\widehat{\rho}(f)$ is indeed an arrow $\widehat{\rho}_0^-(f) \to \widehat{\rho}_0^+(f)$ in $\mathbb{D}(\rho_0^-(C),\rho_0^+(C'))$.

For an arrow $\gamma\colon f \to f'$ in $\mathbb{C}(C,C')$,

$$\begin{aligned}
s_1(\widehat{\rho}(\gamma)) &= s_2(\rho(\gamma)) \\
&= (\rho(C')\,\#_0\,\rho_0^-(\gamma))\,\#_1\,\rho(f) \\
&= \widehat{\rho}_0^+(\gamma)\,\#_0\,\widehat{\rho}(f),
\end{aligned}$$

and similarly $t_1(\widehat{\rho}(\gamma)) = (\widehat{\rho}(f')\,\#_0\,\widehat{\rho}_0^-(\gamma)$, so $\widehat{\rho}(\gamma)$ is indeed a 2-arrow

$$\begin{tikzcd}
& \widehat{\rho}_0^+(f) \arrow[dr, "\widehat{\rho}_0^+(\gamma)"] \arrow[dd, phantom, "\Downarrow\widehat{\rho}(\gamma)"] & \\
\widehat{\rho}_0^-(f) \arrow[ur, "\widehat{\rho}(f)"] \arrow[dr, "\widehat{\rho}_0^-(\gamma)"'] & & \widehat{\rho}_0^+(f') \\
& \widehat{\rho}_0^-(f') \arrow[ur, "\widehat{\rho}(f')"'] &
\end{tikzcd}$$

in $\mathbb{D}(\rho_0^-(C),\rho_0^+(C'))$.

For a 2-arrow $\varphi\colon \begin{tikzcd} f \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow"] & f' \end{tikzcd}$ in $\mathbb{C}(C,C')$,

$$\begin{aligned}
(\widehat{\rho}(f')\,\#_0\,\widehat{\rho}_0^-(\varphi))\,\#_1\,\widehat{\rho}(\gamma) &= \big(\rho(f')\,\#_1\,(\rho_0^+(\varphi)\,\#_0\,\rho(C))\big)\,\#_2\,\rho(\gamma) \\
&= \rho(\gamma')\,\#_2\,\big((\rho(C')\,\#_0\,\rho_0^-(\varphi))\,\#_1\,\rho(f)\big) && \text{by naturality of } \rho \\
&= \widehat{\rho}(\gamma')\,\#_1\,(\widehat{\rho}_0^+(\varphi)\,\#_0\,\widehat{\rho}(f))
\end{aligned}$$

which proves naturality of $\widehat{\rho}$.

For $f \xrightarrow{\gamma} f' \xrightarrow{\gamma'} f''$ in $\mathbb{C}(C,C')$,

$$\begin{aligned}
\widehat{\rho}(\gamma'\,\#_0\,\gamma) &= \rho(\gamma'\,\#_0\,\gamma) \\
&= \big(\rho(\gamma')\,\#_1\,(\rho_0^+(\gamma)\,\#_0\,\rho(C))\big)\,\#_2\,\big((\rho(C')\,\#_0\,\rho_0^-(\gamma'))\,\#_1\,\rho(\gamma)\big) \\
&\qquad \text{by functoriality of } \rho \text{ with respect to 1-composition of 2-arrows} \\
&= \big(\widehat{\rho}(\gamma')\,\#_0\,\widehat{\rho}_0^-(\gamma)\big)\,\#_1\,\big(\widehat{\rho}_0^+(\gamma')\,\#_0\,\widehat{\rho}(\gamma)\big)
\end{aligned}$$

which proves functoriality of $\widehat{\rho}$.

For f in $\mathbb{C}(C,C')$, $\widehat{\rho}(\mathrm{id}_f) = \rho(\mathrm{id}_f) = \mathrm{id}_{\rho(f)} = \mathrm{id}_{\widehat{\rho}(f)}$. $\square$

This actually also works or $\mathbb{C}$ and $\mathbb{D}$ 4D teisi. What happens here is the reindexation of the data and some of the axioms, namely the ones relevant to $\mathbb{C}(C,C')$. So far, this works almost exactly as for ω-categories, although there is already a difference for transfors between 4D teisi, because functoriality with respect to 0-composition of a 2-arrow with an arrow involves a horizontal composite, unlike for lax-1-transformations.

I have given a somewhat detailed proof here to illustrate the things that are relevant; I will be more succinct in the rest of paper.

As an example, let $\mathbb{C}$ be a 4-dimensional tas, $\delta\colon \begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ a 2-arrow of $\mathbb{C}$, and consider the case that ρ is the 1-transfor $\delta\,\#_0 - \colon \mathbb{C}(C,C') \to \mathbb{C}(C,C'')$, which has source and target the functors

$g\,\#_0 -$ and $g'\,\#_0 -$ respectively. Localizing $\delta\,\#_0 -$ at the objects f and f' of $\mathbb{C}(C,C')$, one gets the 1-transfor $\widehat{\delta\,\#_0 -} : \mathbb{C}(f,f') \to \mathbb{C}(g\,\#_0\, f, g'\,\#_0\, f')$, with source and target $(g'\,\#_0 -)\,\#_1\,(\delta\,\#_0\, f)$ and $(\delta\,\#_0\, f')\,\#_1\,(g\,\#_0 -)$ respectively.

A special case of the above example is for a braided 2D tas $\mathbb{C}$: for an object A of $\mathbb{C}$, $R_{A,-}$ is a 1-transfor $\Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$ with source and target the identity functor. Localizing $R_{A,-}$ at $(*,*)$ gives the 1-transfor $\widehat{R_{A,-}} : \Sigma(\mathbb{C})(*,*) \to \Sigma(\mathbb{C})(*,*)$ with source and target $R_{A,*} \otimes -$ and $- \otimes R_{A,*}$. I.e., $R_{A,-}$ induces a pseudo-natural transformation $A \otimes - \to - \otimes A : \mathbb{C} \to \mathbb{C}$.

5.3 Localizing a 2-transfor once

Let again $\mathbb{C}$ and $\mathbb{D}$ be 3D teisi, let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, let $\rho_1^-, \rho_1^+ : \rho_0^- \to \rho_0^+$ be 1-transfors, and let $\rho : \rho_1^- \to \rho_1^+$ be a 2-transfor.

Define functions $\widehat{\rho}_0^-, \widehat{\rho}_0^+ : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$\begin{array}{rcl} \widehat{\rho}_0^-(c) & = & \rho_0^+(c)\,\#_0\,\rho_1^-(C) \\ \widehat{\rho}_0^+(c) & = & \rho_1^+(C')\,\#_0\,\rho_0^-(c). \end{array}$$

These are very similar to the situation above, and can be seen as instances of the above. So $\widehat{\rho}_0^-$ and $\widehat{\rho}_0^+$ are functors.

Define maps of degree 1 $\widehat{\rho}_1^-, \widehat{\rho}_1^+ : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$\begin{array}{rcl} \widehat{\rho}_1^-(f) & = & \rho_1^+(f)\,\#_1\,\big(\rho_0^+(f)\,\#_0\,\rho(C)\big) \\ \widehat{\rho}_1^+(f) & = & \big(\rho(C')\,\#_0\,\rho_0^-(f)\big)\,\#_1\,\rho_1^-(f) \\ \widehat{\rho}_1^-(\gamma) & = & \big(\rho_1^+(f')\,\#_1\,(\rho_0^+(\gamma)\,\#_0\,\rho(C))^{-1}\big)\,\#_2\,\big(\rho_1^+(\gamma)\,\#_1\,(\rho_0^+(f)\,\#_0\,\rho(C))\big) \\ \widehat{\rho}_1^+(\gamma) & = & \big((\rho(C')\,\#_0\,\rho_0^-(f'))\,\#_1\,\rho_1^-(\gamma)\big)\,\#_2\,\big((\rho(C')\,\#_0\,\rho_0^-(\gamma))\,\#_1\,\rho_1^-(f)\big). \end{array}$$

Proposition 5.2 *$\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ are 1-transfors $\widehat{\rho}_0^- \to \widehat{\rho}_0^+$.*

Proof. I have to show that $\widehat{\rho}_1^-(c)$ has the right faces, and that it is transforial.

For an object f of $\mathbb{C}(C,C')$,

$$\begin{array}{rcl} s_0(\widehat{\rho}_1^-(f)) & = & s_1\big(\rho_1^+(f)\,\#_1\,(\rho_0^+(f)\,\#_0\,\rho(C))\big) \\ & = & \rho_0^+(f)\,\#_0\,s_1(\rho(C)) \\ & = & \rho_0^+(f)\,\#_0\,\rho_1^-(C) \\ & = & \widehat{\rho}_0^-(f), \end{array}$$

and similarly $t_0(\widehat{\rho}_1^-(f)) = \widehat{\rho}_0^+(f)$, so $\widehat{\rho}_1^-(f)$ is indeed an arrow $\widehat{\rho}_0^-(f) \to \widehat{\rho}_0^+(f)$ in $\mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$.

For an arrow $\gamma : f \to f'$ of $\mathbb{C}(C,C')$,

$$\begin{array}{rcl} s_1(\widehat{\rho}_1^-(\gamma)) & = & s_2\big((\rho_1^+(f')\,\#_1\,(\rho_0^+(\gamma)\,\#_0\,\rho(C))^{-1})\,\#_2\,(\rho_1^+(\gamma)\,\#_1\,(\rho_0^+(f)\,\#_0\,\rho(C)))\big) \\ & = & s_2\big(\rho_1^+(\gamma)\,\#_1\,(\rho_0^+(f)\,\#_0\,\rho(C))\big) \\ & = & s_2(\rho_1^+(\gamma))\,\#_1\,\big(\rho_0^+(f)\,\#_0\,\rho(C)\big) \\ & = & \big(\rho_1^+(C')\,\#_0\,\rho_0^-(\gamma)\big)\,\#_1\,\rho_1^+(f)\,\#_1\,\big(\rho_0^+(f)\,\#_0\,\rho(C)\big) \\ & = & \widehat{\rho}_0^+(\gamma)\,\#_0\,\widehat{\rho}_1^-(f), \end{array}$$

and similarly $t_1(\widehat{\rho}_1^-(\gamma)) = \widehat{\rho}_1^-(f')\,\#_0\,\widehat{\rho}_0^-(\gamma)$, so $\widehat{\rho}_1^-(\gamma)$ is indeed a 2-arrow

$$
\begin{tikzcd}
& \widehat{\rho}_0^+(f) \arrow[dr, "\widehat{\rho}_0^+(\gamma)"] & \\
\widehat{\rho}_0^-(f) \arrow[ur, "\widehat{\rho}_1^-(f)"] \arrow[dr, "\widehat{\rho}_0^-(\gamma)"'] \arrow[rr, phantom, "\Downarrow\widehat{\rho}_1^-(\gamma)"] & & \widehat{\rho}_0^+(f') \\
& \widehat{\rho}_0^-(f') \arrow[ur, "\widehat{\rho}_1^-(f')"'] &
\end{tikzcd}
$$

in $\mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$.

For a 2-arrow $\varphi : \begin{tikzcd} f \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] \arrow[r, phantom, "\Downarrow"] & f' \end{tikzcd}$ in $\mathbb{C}(C, C')$,

$$
\begin{array}{ll}
\multicolumn{2}{l}{\big(\widehat{\rho}_1^-(f') \#_0 \widehat{\rho}_0^-(\varphi)\big) \#_1 \widehat{\rho}_1^-(\gamma) =} \\
= & \big(\rho_1^+(f') \#_1 (\rho_0^+(f') \#_0 \rho(C)) \#_1 (\rho_0^+(\varphi) \#_0 \rho_1^-(C))\big) \#_2 \\
& \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho(C))^{-1}\big) \#_2 \\
& \big(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \\
= & \big(\rho_1^+(f') \#_1 (((\rho_0^+(f') \#_0 \rho(C)) \#_1 (\rho_0^+(\varphi) \#_0 \rho_1^-(C))) \#_2 (\rho_0^+(\gamma) \#_0 \rho(C))^{-1})\big) \#_2 \\
& \big(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \\
= & \big(\rho_1^+(f') \#_1 ((\rho_0^+(\gamma') \#_0 \rho(C))^{-1} \#_2 ((\rho_0^+(\varphi) \#_0 \rho_1^+(C)) \#_1 (\rho_0^+(f) \#_0 \rho(C))))\big) \#_2 \\
& \big(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \qquad \text{by naturality in } \mathbb{D} \\
= & \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho(C))^{-1}\big) \#_2 \\
& \big(((\rho_1^+(f) \#_1 (\rho_0^+(\varphi) \#_0 \rho_1^+(C))) \#_2 \rho_1^+(\gamma)) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \\
= & \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho(C))^{-1}\big) \#_2 \\
& \big((\rho_1^+(\gamma') \#_2 (\rho_1^+(f) \#_1 (\rho_1^+(C') \#_0 \rho_0^-(\varphi)))) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \qquad \text{by naturality of } \rho \\
= & \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho(C))^{-1}\big) \#_2 \\
& \big(\rho_1^+(\gamma') \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \#_2 \\
& \big(\rho_1^+(f) \#_1 (\rho_1^+(C') \#_0 \rho_0^-(\varphi)) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \\
= & \widehat{\rho}_1^-(\gamma') \#_1 \big(\widehat{\rho}_0^+(\varphi) \#_0 \widehat{\rho}_1^-(f)\big)
\end{array}
$$

which proves naturality of $\widehat{\rho}_1^-$.

For $f \xrightarrow{\gamma} f' \xrightarrow{\gamma'} f''$ in $\mathbb{C}(C, C')$,

$$
\begin{array}{lll}
\widehat{\rho}_1^-(\gamma' \#_0 \gamma) & = & \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma' \#_1 \gamma) \#_0 \rho(C))^{-1}\big) \#_2 \big(\rho_1^-(\gamma' \#_1 \gamma) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \\
& = & \big(\rho_1^+(f') \#_1 ((\rho_0^+(\gamma') \#_1 \rho_0^+(\gamma)) \#_0 \rho(C))^{-1}\big) \#_2 \\
& & \big(\rho_1^+(\gamma') \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^+(C)) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \#_2 \\
& & \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma')) \#_1 \rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \\
& & \qquad \text{by functoriality of } \rho_1^- \text{ and of } \rho_0^+ \\
& = & \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho(C))^{-1} \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C))\big) \#_2 \\
& & \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho_1^+(C)) \#_1 (\rho_0^+(\gamma) \#_0 \rho(C))^{-1}\big) \#_2 \\
& & \big(\rho_1^+(\gamma') \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^+(C)) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \#_2 \\
& & \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma')) \#_1 \rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \qquad \text{by functoriality in } \mathbb{D} \\
& = & \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho(C))^{-1} \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C))\big) \#_2 \\
& & \big(\rho_1^+(\gamma') \#_1 (\rho_0^+(f') \#_0 \rho(C)) \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C))\big) \#_2 \\
& & \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma')) \#_1 \rho_1^+(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho(C))^{-1}\big) \#_2 \\
& & \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma')) \#_1 \rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \qquad \text{by naturality in } \mathbb{D} \\
& = & \big(\widehat{\rho}_1^-(\gamma') \#_0 \widehat{\rho}_0^-(\gamma)\big) \#_1 \big(\widehat{\rho}_0^+(\gamma') \#_0 \widehat{\rho}_1^-(\gamma)\big)
\end{array}
$$

which proves functoriality of $\widehat{\rho}_1^-$.

For f in $\mathbb{C}(C,C')$, $\widehat{\rho}_1^-(\mathrm{id}_f) = \rho_1^-(\mathrm{id}_f) = \mathrm{id}_{\rho_1^-(f)} = \mathrm{id}_{\widehat{\rho}_1^-(f)}$.
$\widehat{\rho}_1^+$ goes similar. $\square$

For transfors between 4D teisi, in naturality of $\widehat{\rho}_1^-$ the non-identity arrow resulting from naturality in $\mathbb{D}$ can be combined with the non-identity arrow resulting from naturality of ρ_1^-. However, in functoriality of $\widehat{\rho}_1^-$ the non-identity arrow resulting from naturality in $\mathbb{D}$ implies that this result will not hold for for transfors between 4D teisi.

An alternative way to look at $\widehat{\rho}_1^-$ is as follows. First note that $- \#_0 \rho(C)$ is a left 1-transfor $\mathbb{D}(\rho_0^+(C), \rho_0^+(C')) \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$. Righting it (if that is possible) gives a right 1-transfor which can be composed with the functor $\rho_0^+ : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^+(C), \rho_0^+(C'))$, to give a 1-transfor $\mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$. $\widehat{\rho_1^+}$ (as opposed to $\widehat{\rho}_1^+$) is also a 1-transfor $\mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$. 0-composing these *as transfors* [19, Section 9] gives $\widehat{\rho}_1^-$, which is a 1-transfor if $(\widetilde{- \#_0 \rho(C)}) \circ \rho_0^+$ and $\widehat{\rho_1^+}$ are truly composable.

For $\widehat{\rho}_1^+$, $\widehat{\rho_1^-}$ and $(\rho(C') \#_0 -) \circ \rho_0^-$ need to be truly composable.

Define a map of degree 2 $\widehat{\rho} : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$\widehat{\rho}(c) \quad = \quad \rho(c).$$

Proposition 5.3 *$\widehat{\rho}$ is a 2-transfor $\widehat{\rho}_1^- \to \widehat{\rho}_1^+$.*

Proof. I have to show that $\widehat{\rho}(c)$ has the right faces, and that it is transforial. But the definition of $\widehat{\rho}_0^-$, $\widehat{\rho}_0^+$, $\widehat{\rho}_1^-$, and $\widehat{\rho}_1^+$ was made such that $\widehat{\rho}$ has the right faces and is natural. $\square$

Functoriality is not an issue for 2-transfors between 2D teisi, but it is one dimension higher. A long calculation, or a nice big diagram (well, not too big, actually), shows that if $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ are 1-transfors then functoriality of $\widehat{\rho}$ follows from functoriality of ρ. If this is indeed the case, call ρ *localizable*.

As an example, let $\mathbb{C}$ be a 4-dimensional tas, $\psi :$ $\begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\delta' \overset{3}{\Lleftarrow} \delta"] & C'' \end{tikzcd}$ a 3-arrow of $\mathbb{C}$, and consider the case that ρ is the 2-transfor $\psi \#_0 - : \mathbb{C}(C,C') \to \mathbb{C}(C,C'')$, which has source and target the 1-transfors $\delta \#_0 -$ and $\delta' \#_0 -$ respectively. Localizing $\psi \#_0 -$ at the objects f and f' of $\mathbb{C}(C,C')$, one gets the 2-transfor $\widehat{\psi \#_0 -} : \mathbb{C}(f, f') \to \mathbb{C}(g \#_0 f, g' \#_0 f')$, with sources and targets given by:

$$\begin{array}{rcl}
\widehat{\psi \#_0 -}_0^- & = & (g' \#_0 -) \#_1 (\delta \#_0 f) \\
\widehat{\psi \#_0 -}_0^+ & = & (\delta' \#_0 f') \#_1 (g \#_0 -) \\
\widehat{\psi \#_0 -}_1^-(\gamma) & = & (\delta' \#_0 \gamma) \#_2 ((g' \#_0 \gamma) \#_1 (\psi \#_0 f)) \\
\widehat{\psi \#_0 -}_1^+(\gamma) & = & ((\psi \#_0 f') \#_1 (g \#_0 \gamma)) \#_2 (\delta \#_0 \gamma) \\
\widehat{\psi \#_0 -}_1^-(\varphi) & = & \big((\delta' \#_0 \gamma') \#_2 ((g' \#_0 \varphi) \#_1 (\psi \#_0 f))^{-1}\big) \#_3 ((\delta' \#_0 \varphi) \#_2 ((g' \#_0 \gamma) \#_1 (\psi \#_0 f))) \\
\widehat{\psi \#_0 -}_1^+(\varphi) & = & (((\psi \#_0 f') \#_1 (g \#_0 \gamma')) \#_2 (\delta \#_0 \varphi)) \#_3 (((\psi \#_0 f') \#_1 (g \#_0 \varphi)) \#_2 (\delta \#_0 \gamma)) .
\end{array}$$

A special case of the above example is for a braided 2D tas $\mathbb{C}$: for an arrow $f : A \to A'$ of $\mathbb{C}$, $R_{f,-}$ is a 2-transfor $\Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$ with source and target $R_{A,-}$ and $R_{A',-}$ respectively. Localizing

$R_{f,-}$ at $(*,*)$ gives the 2-transfor $\widehat{R_{f,-}} : \mathbb{C} \to \mathbb{C}$ with sources and targets given by:

$$\begin{array}{rcl} \widehat{R_{f,-}}_0^- & = & A \otimes - \\ \widehat{R_{f,-}}_0^+ & = & - \otimes A' \\ \widehat{R_{f,-}}_1^-(B) & = & R_{A',B} \#_0 (f \otimes B) \\ \widehat{R_{f,-}}_1^+(B) & = & (B \otimes f) \#_0 R_{A,B} \\ \widehat{R_{f,-}}_1^-(g) & = & (R_{A',B'} \#_0 (f \otimes g)^{-1}) \#_1 (R_{A',g} \#_0 (f \otimes B)) \\ \widehat{R_{f,-}}_1^+(g) & = & ((B' \otimes f) \#_0 R_{A,g}) \#_1 ((g \otimes f) \#_0 R_{A,B}). \end{array}$$

I.e., $R_{f,-}$ induces a modification $R_{A',-} \#_0 (f \otimes -) \to (- \otimes f) \#_0 R_{A,-}$, provided its source and target are interpreted as pseudo-natural transformations in the above – correct – way. Note, again, that this source and target can be seen as a composite of transfors.

5.4 Localizing a 3-transfor once

Let again $\mathbb{C}$ and $\mathbb{D}$ be 3D teisi, let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, let $\rho_1^-, \rho_1^+ : \rho_0^- \to \rho_0^+$ be 1-transfors, let $\rho_2^-, \rho_2^+ : \rho_1^- \to \rho_1^+$ be 2-transfors, and let $\rho : \rho_2^- \to \rho_2^+$ be a 3-transfor.

Define functions $\widehat{\rho}_0^-, \widehat{\rho}_0^+ : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$\begin{array}{rcl} \widehat{\rho}_0^-(c) & = & \rho_0^+(c) \#_0 \rho_1^-(C) \\ \widehat{\rho}_0^+(c) & = & \rho_1^+(C') \#_0 \rho_0^-(c). \end{array}$$

These are the same as for the localization of a 2-transfor. So $\widehat{\rho}_0^-$ and $\widehat{\rho}_0^+$ are functors.

Define maps of degree 1 $\widehat{\rho}_1^-, \widehat{\rho}_1^+ : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$\begin{array}{rcl} \widehat{\rho}_1^-(f) & = & \rho_1^+(f) \#_1 \left(\rho_0^+(f) \#_0 \rho_2^-(C)\right) \\ \widehat{\rho}_1^+(f) & = & \left(\rho_2^+(C') \#_0 \rho_0^-(f)\right) \#_1 \rho_1^-(f) \\ \widehat{\rho}_1^-(\gamma) & = & \left(\rho_1^+(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_2^-(C))^{-1}\right) \#_2 \left(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho_2^-(C))\right) \\ \widehat{\rho}_1^+(\gamma) & = & \left((\rho_2^+(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(\gamma)\right) \#_2 \left((\rho_2^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^-(f)\right). \end{array}$$

These are very similar to the situation for localization of a 2-transfor above, and can be seen as instances of the above. So $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ are 1-transfors $\widehat{\rho}_0^- \to \widehat{\rho}_0^+$. But not always for ρ a 3-transfor between 4D teisi.

Define maps of degree 2 $\widehat{\rho}_2^-, \widehat{\rho}_2^+ : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$\begin{array}{rcl} \widehat{\rho}_2^-(f) & = & \rho_2^+(f) \#_2 \left(\rho_1^+(f) \#_1 (\rho_0^+(f) \#_0 \rho(C))\right) \\ \widehat{\rho}_2^+(f) & = & \left((\rho(C') \#_0 \rho_0^-(f)) \#_1 \rho_1^-(f)\right) \#_2 \rho_2^-(f). \end{array}$$

Proposition 5.4 *$\widehat{\rho}_2^-$ and $\widehat{\rho}_2^+$ are 2-transfors $\widehat{\rho}_1^- \to \widehat{\rho}_1^+$.*

Proof. $\widehat{\rho}_2^-(c)$ has the right faces more or less by definition of $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$, and a lengthy calculation similar to the ones above shows that $\widehat{\rho}_2^-$ is natural.

$\widehat{\rho}_2^+$ goes similar. □

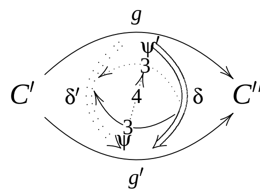
This becomes more interesting one dimension higher, as the proof of naturality turns into a composite of non-identity arrows.

As before, $\widehat{\rho}_2^-$ and $\widehat{\rho}_2^+$ can be seen as composites of transfors, namely $\widehat{\rho}_2^-$ as $\widehat{\rho_2^+} \#_1 \left(\widehat{\rho_1^+} \#_0 ((- \#_0 \rho(C)) \circ \rho_0^+)\right)$, and $\widehat{\rho}_2^+$ similar.

Proposition 5.5 $\widehat{\rho}_2^- = \widehat{\rho}_2^+$.

Proof. Exactly by naturality of ρ. $\square$

This becomes more interesting one dimension higher, of course.

As an example, let $\mathbb{C}$ be a 4-dimensional tas, $\Delta :$  a 4-arrow of $\mathbb{C}$, and consider the case that ρ is the 3-transfor $\Delta \#_0 - : \mathbb{C}(C,C') \to \mathbb{C}(C,C'')$, which has source and target the 2-transfors $\psi \#_0 -$ and $\psi' \#_0 -$ respectively. Localizing $\Delta \#_0 -$ at the objects f and f' of $\mathbb{C}(C,C')$, one gets a 3-transfor $\widehat{\Delta \#_0 -} : \mathbb{C}(f, f') \to \mathbb{C}(g \#_0 f, g' \#_0 f')$, with sources and targets given by:

$$
\begin{array}{lcl}
\widehat{\Delta \#_0 -}_0^- & = & (g' \#_0 -) \#_1 (\delta \#_0 f) \\
\widehat{\Delta \#_0 -}_0^+ & = & (\delta' \#_0 f') \#_1 (g \#_0 -) \\
\widehat{\Delta \#_0 -}_1^- (\gamma) & = & (\delta' \#_0 \gamma) \#_2 ((g' \#_0 \gamma) \#_1 (\psi \#_0 f)) \\
\widehat{\Delta \#_0 -}_1^+ (\gamma) & = & ((\psi' \#_0 f') \#_1 (g \#_0 \gamma)) \#_2 (\delta \#_0 \gamma) \\
\widehat{\Delta \#_0 -}_1^- (\varphi) & = & \big((\delta' \#_0 \gamma') \#_2 ((g' \#_0 \varphi) \#_1 (\psi \#_0 f))^{-1}\big) \#_3 ((\delta' \#_0 \varphi) \#_2 ((g' \#_0 \gamma) \#_1 (\psi \#_0 f))) \\
\widehat{\Delta \#_0 -}_1^+ (\varphi) & = & (((\psi' \#_0 f') \#_1 (g \#_0 \gamma')) \#_2 (\delta \#_0 \varphi)) \#_3 (((\psi' \#_0 f') \#_1 (g \#_0 \varphi)) \#_2 (\delta \#_0 \gamma)) \\
\widehat{\Delta \#_0 -}_2^- (\gamma) & = & (\psi' \#_0 \gamma) \#_3 ((\delta' \#_0 \gamma) \#_2 ((g' \#_0 \gamma) \#_1 (\Delta \#_0 f))) \\
\widehat{\Delta \#_0 -}_2^+ (\gamma) & = & (((\Delta \#_0 f') \#_1 (g \#_0 \gamma)) \#_2 (\delta \#_0 \gamma)) \#_3 (\psi \#_0 \gamma),
\end{array}
$$

i.e., the equality of $\widehat{\Delta \#_0 -}_2^-$ and $\widehat{\Delta \#_0 -}_2^-$.

A special case of the above example is for a braided 2D tas $\mathbb{C}$: for a 2-arrow $\alpha : A \underset{f'}{\overset{f}{\Downarrow}} A'$ of $\mathbb{C}$, $R_{\alpha,-}$ is a 3-transfor $\Sigma(\mathbb{C}) \to \Sigma(\mathbb{C})$ with source and target $R_{f,-}$ and $R_{f',-}$ respectively. Localizing $R_{\alpha,-}$ at $(*,*)$ gives the 3-transfor $\widehat{R_{\alpha,-}} : \mathbb{C} \to \mathbb{C}$ with sources and targets given by:

$$
\begin{array}{lcl}
\widehat{R_{\alpha,-}}_0^- & = & A \otimes - \\
\widehat{R_{\alpha,-}}_0^+ & = & - \otimes A' \\
\widehat{R_{\alpha,-}}_1^- (B) & = & R_{A',B} \#_0 (f \otimes B) \\
\widehat{R_{\alpha,-}}_1^+ (B) & = & (B \otimes f') \#_0 R_{A,B} \\
\widehat{R_{\alpha,-}}_1^- (g) & = & (R_{A',B'} \#_0 (f \otimes g)^{-1}) \#_1 (R_{A',g} \#_0 (f \otimes B)) \\
\widehat{R_{\alpha,-}}_1^+ (g) & = & ((B' \otimes f') \#_0 R_{A,g}) \#_1 ((g \otimes f') \#_0 R_{A,B}) \\
\widehat{R_{\alpha,-}}_2^- (B) & = & R_{f',B} \#_1 (R_{A',B} \#_0 (\alpha \otimes B)) \\
\widehat{R_{\alpha,-}}_2^+ (B) & = & ((B \otimes \alpha) \#_0 R_{A,B}) \#_1 R_{f,B}.
\end{array}
$$

I.e., $\widehat{R_{\alpha,-}}$ says that $\widehat{R_{\alpha,-}}_2^-$ equals $\widehat{R_{\alpha,-}}_2^+$, which is the correct way of stating that $\widehat{R_{f,-}}$ is 2-natural in

α. Note, here, that

$$\begin{array}{rcl}
\widehat{R_{\alpha,-}}_0^{\,-} & = & \widehat{R_{f,-}}_0^{\,-} \\
\widehat{R_{\alpha,-}}_0^{\,+} & = & \widehat{R_{f',-}}_0^{\,+} \\
\widehat{R_{\alpha,-}}_1^{\,-} & = & \widehat{R_{f,-}}_1^{\,-} \\
\widehat{R_{\alpha,-}}_1^{\,+} & = & \widehat{R_{f',-}}_1^{\,+}.
\end{array}$$

Also note that $\widehat{R_{\alpha,-}}_2^{\,-}$ is a composite of transfors, namely $\widehat{R_{f',-}} \#_1 \left(\widehat{R_{A',-}} \#_0 (\alpha \otimes -)\right)$.

5.5 Localizing a 1-transfor twice

Localizing a 1-transfor twice is the same as localizing it once and then once again, of course. But it is useful to express the result in terms of the original 1-transfor.

The situation would be too trivial when starting from transfors between 3D teisi, so from now on I will consider transfors between 4D teisi. As they will be localized twice, I will not need a full definition of such transfors, and the parts which do affect the localizations will become apparent in the sequel.

So let $\mathbb{C}$ and $\mathbb{D}$ be 4D teisi, let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, and let $\rho : \rho_0^- \to \rho_0^+$ be a 1-transfor. For two arrows $f, f' : C \to C'$ of $\mathbb{C}$, note that $\mathbb{C}(f, f')$ and $\mathbb{D}(\rho_0^+(f) \#_0 \rho(C), \rho(C') \#_0 \rho_0^-(f'))$ are 2D teisi. (Note that $\rho_0^+(f) \#_0 \rho(C) = s_1(\rho(f))$ and, $\rho(C') \#_0 \rho_0^-(f') = t_1(\rho(f'))$.)

Define functions $\widehat{\rho}_0^-, \widehat{\rho}_0^+ : \mathbb{C}(f, f') \to \mathbb{D}(s_1(\rho(f)), t_1(\rho(f')))$ by:

$$\begin{array}{rcl}
\widehat{\rho}_0^-(c) & = & (\rho(C') \#_0 \rho_0^-(c)) \#_1 \rho(f) \\
\widehat{\rho}_0^+(c) & = & \rho(f') \#_1 (\rho_0^+(c) \#_0 \rho(C)).
\end{array}$$

An argument similar to for localization once gives that $\widehat{\rho}_0^-$ and $\widehat{\rho}_0^+$ are functors. Alternatively, they are functors because they are the result of repeated localization once, which is known to give functors.

Define a map of degree 1 $\widehat{\rho} : \mathbb{C}(f, f') \to \mathbb{D}(s_1(\rho(f)), t_1(\rho(f')))$ by:

$$\widehat{\rho}(c) \quad = \quad \rho(c).$$

Proposition 5.6 *$\widehat{\rho}$ is a 1-transfor $\widehat{\rho}_0^- \to \widehat{\rho}_0^+$.*

Proof. That $\widehat{\rho}(c)$ has the right faces, and that $\widehat{\rho}$ is natural is almost immediate. Functoriality of $\widehat{\rho}$ with respect to 0-composition of arrows follows from functoriality of ρ with respect to 2-composition of 3-arrows. $\square$

Alternatively, $\widehat{\rho}$ is a repeated localization once, and use the remark following proposition 5.1.

As an example, let $\mathbb{C}$ be a 5-dimensional tas, $\delta : \begin{tikzcd} C' \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & C'' \end{tikzcd}$ a 2-arrow of $\mathbb{C}$, and consider the case that ρ is the 1-transfor $\delta \#_0 - : \mathbb{C}(C, C') \to \mathbb{C}(C, C'')$, which has source and target the functors $g \#_0 -$ and $g' \#_0 -$ respectively. Localizing $\delta \#_0 -$ at the arrows γ and $\gamma' : f \to f'$ of $\mathbb{C}(C, C')$, one gets the 1-transfor $\widehat{\delta \#_0 -} : \mathbb{C}(\gamma, \gamma') \to \mathbb{C}(s_1(g \#_0 \gamma), t_1(g' \#_0 \gamma'))$, with source and target $((\delta \#_0 f') \#_1 (g \#_0 -)) \#_2 (\delta \#_0 \gamma)$ and $(\delta \#_0 \gamma') \#_2 ((g' \#_0 -) \#_1 (\delta \#_0 f))$ respectively.

5.6 Localizing a 2-transfor twice

Let again $\mathbb{C}$ and $\mathbb{D}$ be 4D teisi, let $\rho_0^-,\rho_0^+:\mathbb{C}\to\mathbb{D}$ be functors, let $\rho_1^-,\rho_1^+:\rho_0^-\to\rho_0^+$ be 1-transfors, and let $\rho:\rho_1^-\to\rho_1^+$ be a 2-transfor.

Define functions $\widehat{\rho}_0^-,\widehat{\rho}_0^+:\mathbb{C}(f,f')\to\mathbb{D}(s_1(\rho(f)),t_1(\rho(f')))$ by:

$$\begin{aligned}
\widehat{\rho}_0^-(c) &= (\rho_1^+(C')\#_0\rho_0^-(c))\#_1\rho_1^+(f)\#_1(\rho_0^+(f)\#_0\rho(C))\\
\widehat{\rho}_0^+(c) &= (\rho(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(f')\#_1(\rho_0^+(c)\#_0\rho_1^-(C)).
\end{aligned}$$

By the same argument as before, $\widehat{\rho}_0^-$ and $\widehat{\rho}_0^+$ are functors.

Define maps of degree 1 $\widehat{\rho}_1^-,\widehat{\rho}_1^+:\mathbb{C}(f,f')\to\mathbb{D}(s_1(\rho(f)),t_1(\rho(f')))$ by:

$$\begin{aligned}
\widehat{\rho}_1^-(\gamma) = {} & \big((\rho(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(\gamma)\big)\#_2\\
& \big((\rho(C')\#_0\rho_0^-(\gamma))\#_1\rho_1^-(f)\big)\#_2\\
& \big((\rho_1^+(C')\#_0\rho_0^-(\gamma))\#_1\rho(f)\big)\\
\widehat{\rho}_1^+(\gamma) = {} & \big(\rho(f')\#_1(\rho_0^+(\gamma)\#_0\rho_1^+(C))\big)\#_2\\
& \big(\rho_1^+(f')\#_1(\rho_0^+(\gamma)\#_0\rho(C))^{-1}\big)\#_2\\
& \big(\rho_1^+(\gamma)\#_1(\rho_0^+(f)\#_0\rho(C))\big)\\
\widehat{\rho}_1^-(\varphi) = {} & \big(\big((\rho(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(\gamma')\big)\#_2\\
& \big((\rho(C')\#_0\rho_0^-(\gamma'))\#_1\rho_1^-(f)\big)\#_2\\
& \big((\rho_1^+(C')\#_0\rho_0^-(\varphi))\#_1\rho(f)\big)^{-1}\Big)\#_3\\
& \big(\big((\rho(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(\gamma')\big)\#_2\\
& \big((\rho(C')\#_0\rho_0^-(\varphi))\#_1\rho_1^-(f)\big)\#_2\\
& \big((\rho_1^+(C')\#_0\rho_0^-(\gamma))\#_1\rho(f)\big)\big)\#_3\\
& \big(\big((\rho(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(\varphi)\big)\#_2\\
& \big((\rho(C')\#_0\rho_0^-(\gamma))\#_1\rho_1^-(f)\big)\#_2\\
& \big((\rho_1^+(C')\#_0\rho_0^-(\gamma))\#_1\rho(f)\big)\big)\\
\widehat{\rho}_1^+(\varphi) = {} & \big(\big(\rho(f')\#_1(\rho_0^+(\gamma')\#_0\rho_1^+(C))\big)\#_2\\
& \big(\rho_1^+(f')\#_1(\rho_0^+(\gamma')\#_0\rho(C))^{-1}\big)\#_2\\
& \big(\rho_1^+(\varphi)\#_1(\rho_0^+(f)\#_0\rho(C))\big)\big)\#_3\\
& \big(\big(\rho(f')\#_1(\rho_0^+(\gamma')\#_0\rho_1^+(C))\big)\#_2\\
& \big(\rho_1^+(f')\#_1((\rho_0^+(\gamma')\#_0\rho(C))^{-1}\#_2(\rho_0^+(\varphi)\#_0\rho(C))\#_2(\rho_0^+(\gamma)\#_0\rho(C))^{-1})\big)\#_2\\
& \big(\rho_1^+(\gamma)\#_1(\rho_0^+(f)\#_0\rho(C))\big)\big)\#_3\\
& \big(\big(\rho(f')\#_1(\rho_0^+(\varphi)\#_0\rho_1^+(C))\big)\#_2\\
& \big(\rho_1^+(f')\#_1(\rho_0^+(\gamma)\#_0\rho(C))^{-1}\big)\#_2\\
& \big(\rho_1^+(\gamma)\#_1(\rho_0^+(f)\#_0\rho(C))\big)\big).
\end{aligned}$$

Proposition 5.7 *$\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ are 1-transfors $\widehat{\rho}_0^-\to\widehat{\rho}_0^+$.*

Proof. That $\widehat{\rho}_1^-(c)$ has the right faces, and that $\widehat{\rho}_1^-$ is natural is a long but straightforward calculation. Functoriality of $\widehat{\rho}_1^-$ with respect to 0-composition of arrows follows, again by a long calculation, from functoriality of ρ_1^- with respect to 2-composition of 3-arrows and functoriality of 1-composition in a 4-dimensional tas. □

As before, functoriality uses naturality in $\mathbb{D}$ several times, so will not hold for transfors between higher-dimensional teisi.

As before, $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ can be seen as composites of transfors.

Define a map of degree 2 $\widehat{\rho} : \mathbb{C}(f, f') \to \mathbb{D}(s_1(\rho(f)), t_1(\rho(f')))$ by:

$$\widehat{\rho}(c) \quad = \quad \rho(c).$$

Proposition 5.8 *$\widehat{\rho}$ is a 2-transfor $\widehat{\rho}_1^- \to \widehat{\rho}_1^+$.*

Proof. Again, as in proposition 5.3, the definitions of $\widehat{\rho}_0^-$, $\widehat{\rho}_0^+$, $\widehat{\rho}_1^-$, and $\widehat{\rho}_1^+$ have been made exactly with this purpose; in particular, naturality of $\widehat{\rho}$ follows from naturality of ρ. □

In the higher-dimensional situation, if $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ are 1-transfors call ρ *twice localizable*. Again, this can be expressed in terms of true composability of the transfors making up $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$.

As an example, let $\mathbb{C}$ be a 5-dimensional tas, $\psi :$
$$\begin{tikzcd} C' \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "g'"'] \arrow[r, phantom, "\delta' \overset{3}{\Lleftarrow} \delta"] & C'' \end{tikzcd}$$
a 3-arrow of $\mathbb{C}$, and consider the case that ρ is the 2-transfor $\psi \#_0 - : \mathbb{C}(C, C') \to \mathbb{C}(C, C'')$, which has source and target the 1-transfors $\delta \#_0 -$ and $\delta' \#_0 -$ respectively. Localizing $\psi \#_0 -$ at the arrows γ and $\gamma' : f \to f'$ of $\mathbb{C}(C, C')$, one gets the 2-transfor $\widehat{\psi \#_0 -} : \mathbb{C}(\gamma, \gamma') \to \mathbb{C}(s_1(g \#_0 \gamma), t_1(g' \#_0 \gamma'))$, whose sources and targets can be derived from the formulae for $\widehat{\rho}_0^-$, $\widehat{\rho}_0^+$, $\widehat{\rho}_1^-$, and $\widehat{\rho}_1^+$, as before.

A special case of the above example is for a sylleptic 2D tas $\mathbb{C}$: for an object A of $\mathbb{C}$, $v_{A,-}$ is a 2-transfor $\Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$ with source and target the identity transfor, or, more precisely, $v_{\mathrm{id}_*,-}$ and $v_{\mathrm{id}_*,-}$. Localizing $v_{A,-}$ twice, at $(\mathrm{id}_*, \mathrm{id}_*)$, gives the 2-transfor $\widehat{v_{A,-}} : \mathbb{C} \to \mathbb{C}$ with sources and targets given by:

$$\begin{array}{rcl}
\widehat{v_{A,-}}_0^- & = & A \otimes - \\
\widehat{v_{A,-}}_0^+ & = & - \otimes A \\
\widehat{v_{A,-}}_1^-(B) & = & R_{A,B} \\
\widehat{v_{A,-}}_1^+(B) & = & {R_{B,A}}^{-1} \\
\widehat{v_{A,-}}_1^-(g) & = & R_{A,g} \\
\widehat{v_{A,-}}_1^+(g) & = & {R_{B',A}}^{-1} \#_0 R_{g,A} \#_0 {R_{B,A}}^{-1}.
\end{array}$$

Note that here $\widehat{v_{A,-}}_1^- = \widehat{R_{A,-}}$, but also note that there is more involved in arriving at $R_{A,-}$ as a 1-transfor $\mathbb{C} \to \mathbb{C}$ than just localizing it: it is composed with further transfors which are identities because of unit axioms. $\widehat{v_{A,-}}_1^+ = \widetilde{\widehat{R_{A,-}}}$, also composed with further identity functors. I.e., $v_{A,-}$ induces a modification $R_{A,-} \to R_{-,A}$ but only provided, again, its source and target are interpreted as pseudo-natural transformations in the above – correct – way.

5.7 Localizing a 3-transfor twice

Let again $\mathbb{C}$ and $\mathbb{D}$ be 4D teisi, let $\rho_0^-, \rho_0^+ : \mathbb{C} \to \mathbb{D}$ be functors, let $\rho_1^-, \rho_1^+ : \rho_0^- \to \rho_0^+$ be 1-transfors, let $\rho_2^-, \rho_2^+ : \rho_1^- \to \rho_1^+$ be 2-transfors, and let $\rho : \rho_2^- \to \rho_2^+$ be a 3-transfor.

Define functions $\widehat{\rho}_0^-, \widehat{\rho}_0^+ : \mathbb{C}(f, f') \to \mathbb{D}(s_1(\rho(f)), t_1(\rho(f')))$ by:

$$\begin{array}{rcl}
\widehat{\rho}_0^-(c) & = & (\rho_1^+(C') \#_0 \rho_0^-(c)) \#_1 \rho_1^+(f) \#_1 (\rho_0^+(f) \#_0 \rho_2^-(C)) \\
\widehat{\rho}_0^+(c) & = & (\rho_2^+(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f') \#_1 (\rho_0^+(c) \#_0 \rho_1^-(C)).
\end{array}$$

These are the similar to for the localization twice of a 2-transfor. So $\widehat{\rho}_0^-$ and $\widehat{\rho}_0^+$ are functors.

Define maps of degree 1 $\widehat{\rho}_1^-, \widehat{\rho}_1^+ : \mathbb{C}(f, f') \to \mathbb{D}(s_1(\rho(f)), t_1(\rho(f')))$ by:

$$
\begin{aligned}
\widehat{\rho}_1^-(\gamma) \;=\;& \big((\rho_2^+(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(\gamma)\big) \#_2 \\
& \big((\rho_2^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^-(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_2^+(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^+(f) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big) \\
\widehat{\rho}_1^+(\gamma) \;=\;& \big((\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C))\big) \#_2 \\
& \big(\rho_2^-(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^+(C))\big) \#_2 \\
& \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_2^-(C))^{-1}\big) \#_2 \\
& \big(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho_2^-(C))\big) \\
\widehat{\rho}_1^-(\varphi) \;=\;& \big(\big((\rho_2^+(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(\gamma')\big) \#_2 \\
& \big((\rho_2^+(C') \#_0 \rho_0^-(\gamma')) \#_1 \rho_1^-(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma')) \#_1 \rho_2^+(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\varphi)) \#_1 \rho_1^+(f) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big)^{-1}\Big) \#_3 \\
& \big(\big((\rho_2^+(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(\gamma')\big) \#_2 \\
& \big((\rho_2^+(C') \#_0 \rho_0^-(\gamma')) \#_1 \rho_1^-(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\varphi)) \#_1 \rho_2^+(f)\big)^{-1} \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^+(f) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big)\big) \#_3 \\
& \big(\big((\rho_2^+(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(\gamma')\big) \#_2 \\
& \big((\rho_2^+(C') \#_0 \rho_0^-(\varphi)) \#_1 \rho_1^-(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_2^+(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^+(f) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big)\big) \#_3 \\
& \big(\big((\rho_2^+(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(\varphi)\big) \#_2 \\
& \big((\rho_2^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^-(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_2^+(f)\big) \#_2 \\
& \big((\rho_1^+(C') \#_0 \rho_0^-(\gamma)) \#_1 \rho_1^+(f) \#_1 (\rho_0^+(f) \#_0 \rho(C))\big)\big) \\
\widehat{\rho}_1^+(\varphi) \;=\;& \big(\big((\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C))\big) \#_2 \\
& \big(\rho_2^-(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho_1^+(C))\big) \#_2 \\
& \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho_2^-(C))^{-1}\big) \#_2 \\
& \big(\rho_1^+(\varphi) \#_1 (\rho_0^+(f) \#_0 \rho_2^-(C))\big)\big) \#_3 \\
& \big(\big((\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C))\big) \#_2 \\
& \big(\rho_2^-(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho_1^+(C))\big) \#_2 \\
& \big(\rho_1^+(f') \#_1 ((\rho_0^+(\gamma') \#_0 \rho_2^-(C))^{-1} \#_2 (\rho_0^+(\varphi) \#_0 \rho_2^-(C)) \#_2 (\rho_0^+(\gamma) \#_0 \rho_2^-(C))^{-1})\big) \#_2 \\
& \big(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho_2^-(C))\big)\big) \#_3 \\
& \big(\big((\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_1^-(C))\big) \#_2 \\
& \big(\rho_2^-(f') \#_1 (\rho_0^+(\varphi) \#_0 \rho_1^+(C))\big) \#_2 \\
& \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_2^-(C))^{-1}\big) \#_2 \\
& \big(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho_2^-(C))\big)\big) \#_3 \\
& \big(\big((\rho(C') \#_0 \rho_0^-(f')) \#_1 \rho_1^-(f') \#_1 (\rho_0^+(\varphi) \#_0 \rho_1^-(C))\big) \#_2 \\
& \big(\rho_2^-(f') \#_1 (\rho_0^+(\gamma') \#_0 \rho_1^+(C))\big) \#_2 \\
& \big(\rho_1^+(f') \#_1 (\rho_0^+(\gamma) \#_0 \rho_2^-(C))^{-1}\big) \#_2 \\
& \big(\rho_1^+(\gamma) \#_1 (\rho_0^+(f) \#_0 \rho_2^-(C))\big)\big)\,.
\end{aligned}
$$

Proposition 5.9 *$\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ are 1-transfors $\widehat{\rho}_0^- \to \widehat{\rho}_0^+$.*

Proof. Only slightly more complicated than the proof of proposition 5.7, and left to the reader. □

As before, functoriality uses naturality in $\mathbb{D}$ several times, so will not hold for transfors between higher-dimensional teisi.

As before, $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ can be seen as composites of transfors.

Define maps of degree 2 $\widehat{\rho} : \mathbb{C}(C,C') \to \mathbb{D}(\rho_0^-(C), \rho_0^+(C'))$ by:

$$
\begin{aligned}
\widehat{\rho}_2^-(\gamma) \quad = \quad & \Big(\big(((\rho(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(f')\#_1(\rho_0^+(\gamma)\#_0\rho_1^-(C))\big)\#_2 \\
& \rho_2^-(\gamma)\big)\#_3 \\
& \Big(\big((\rho(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(\gamma)\big)\#_2 \\
& \big((\rho_2^-(C')\#_0\rho_0^-(\gamma))\#_1\rho_1^-(f)\big)\#_2 \\
& \big((\rho_1^+(C')\#_0\rho_0^-(\gamma))\#_1\rho_2^-(f)\big)\Big)\#_3 \\
& \Big(\big((\rho_2^+(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(\gamma)\big)\#_2 \\
& \big((\rho(C')\#_0\rho_0^-(\gamma))\#_1\rho_1^-(f)\big)\#_2 \\
& \big((\rho_1^+(C')\#_0\rho_0^-(\gamma))\#_1\rho_2^-(f)\big)\Big)\#_3 \\
& \Big(\big((\rho_2^+(C')\#_0\rho_0^-(f'))\#_1\rho_1^-(\gamma)\big)\#_2 \\
& \big((\rho_2^+(C')\#_0\rho_0^-(\gamma))\#_1\rho_1^-(f)\big)\#_2 \\
& \big((\rho_1^+(C')\#_0\rho_0^-(\gamma))\#_1\rho(f)\big)\Big) \\
\widehat{\rho}_2^+(\gamma) \quad = \quad & \Big(\big(\rho_2^+(f')\#_1(\rho_0^+(\gamma)\#_0\rho_1^+(C))\big)\#_2 \\
& \big(\rho_1^+(f')\#_1(\rho_0^+(\gamma)\#_0\rho_2^+(C))^{-1}\big)\#_2 \\
& \big(\rho_1^+(\gamma)\#_1(\rho_0^+(f)\#_0\rho(C))^{-1}\big)\Big)\#_3 \\
& \Big(\big(\rho_2^+(f')\#_1(\rho_0^+(\gamma)\#_0\rho_1^+(C))\big)\#_2 \\
& \big(\rho_1^+(f')\#_1((\rho_0^+(\gamma)\#_0\rho_2^+(C))^{-1}\#_2(\rho_0^+(\gamma)\#_0\rho(C))\#_2(\rho_0^+(\gamma)\#_0\rho_2^-(C))^{-1})\big)\#_2 \\
& \big(\rho_1^+(\gamma)\#_1(\rho_0^+(f)\#_0\rho_2^-(C))\big)\Big)\#_3 \\
& \Big(\big(\rho(f')\#_1(\rho_0^+(\gamma)\#_0\rho_1^+(C))\big)\#_2 \\
& \big(\rho_1^+(f')\#_1(\rho_0^+(\gamma)\#_0\rho_2^-(C))^{-1}\big)\#_2 \\
& \big(\rho_1^+(\gamma)\#_1(\rho_0^+(f)\#_0\rho_2^-(C))\big)\Big)\#_3 \\
& \big(\rho_2^+(\gamma)\#_2 \\
& \big((\rho_1^+(C')\#_0\rho_0^-(\gamma))\#_1\rho_1^+(f)\#_1(\rho_0^+(f)\#_0\rho(C))\big)\big)\,.
\end{aligned}
$$

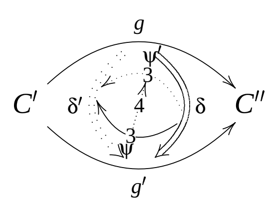
Proposition 5.10 *$\widehat{\rho}_2^-$ and $\widehat{\rho}_2^+$ are 2-transfors $\widehat{\rho}_1^- \to \widehat{\rho}_1^+$.*

Proof. Straightforward, and left to the reader. □

In the higher-dimensional situation, if $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$ are 1-transfors call ρ *twice localizable*. Again, this can be expressed in terms of true composability of the transfors making up $\widehat{\rho}_1^-$ and $\widehat{\rho}_1^+$.

Proposition 5.11 $\widehat{\rho}_2^- = \widehat{\rho}_2^+$.

Proof. Precisely by naturality of ρ. □

As an example, let $\mathbb{C}$ be a 5-dimensional tas, Δ : 

a 4-arrow of $\mathbb{C}$, and consider the case that ρ is the 3-transfor $\Delta\#_0 - : \mathbb{C}(C,C') \to \mathbb{C}(C,C'')$, which has source and target the 2-transfors $\psi\#_0 -$ and $\psi'\#_0 -$ respectively. Localizing $\Delta\#_0 -$ at the arrows γ and $\gamma' : f \to f'$

of $\mathbb{C}(C,C')$, one gets the 3-transfor $\widehat{\Delta\#_0 -} : \mathbb{C}(\gamma,\gamma') \to \mathbb{C}(s_1(g\#_0\gamma), t_1(g'\#_0\gamma'))$, whose sources and targets can be derived from the formulae for $\widehat{\rho}_0^-$, $\widehat{\rho}_0^+$, $\widehat{\rho}_1^-$, and $\widehat{\rho}_1^+$, as before.

A special case of the above example is for a sylleptic 2D tas $\mathbb{C}$: for an arrow $f : A \to A'$ of $\mathbb{C}$, $v_{f,-}$ is a 3-transfor $\Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$ with source and target $v_{A,-}$ and $v_{A',-}$ respectively. Localizing $v_{f,-}$ twice, at $(\mathrm{id}_*, \mathrm{id}_*)$, gives the 3-transfor $\widehat{v_{f,-}} : \mathbb{C} \to \mathbb{C}$ with sources and targets given by:

$$\begin{array}{rcl}
\widehat{v_{f,-}}{}_0^- & = & A \otimes - \\
\widehat{v_{f,-}}{}_0^+ & = & - \otimes A' \\
\widehat{v_{f,-}}{}_1^-(B) & = & R_{A',B}\#_0(f\otimes B) \\
\widehat{v_{f,-}}{}_1^+(B) & = & (B\otimes f)\#_0 R_{B,A}{}^{-1} \\
\widehat{v_{f,-}}{}_1^-(g) & = & (R_{A',B'}\#_0(f\otimes g)^{-1})\#_1(R_{f,B'}\#_0(f\otimes B)) \\
\widehat{v_{f,-}}{}_1^+(g) & = & ((B'\otimes f)\#_0(R_{B,A}{}^{-1}\#_0 R_{B,f}\#_0 R_{B,A'}{}^{-1}))\#_1((g\otimes f)\#_0 R_{B,A}{}^{-1}) \\
\widehat{v_{f,-}}{}_2^-(B) & = & ((B\otimes f)\#_0 v_{A,B})\#_1 R_{f,B} \\
\widehat{v_{f,-}}{}_2^+(B) & = & ((R_{B,A}{}^{-1}\#_0 R_{B,f}\#_0 R_{B,A'}{}^{-1})\#_1(v_{A,B}\#_0(f\otimes B).
\end{array}$$

I.e., $\widehat{v_{f,-}}$ says that $\widehat{v_{f,-}}{}_2^-$ equals $\widehat{v_{f,-}}{}_2^+$, which is the correct way of stating that $\widehat{v_{A,-}}$ is natural in f.

5.8 Localizing a transfor more often

One special case of some interest here is for a symmetric 2D tas $\mathbb{C}$: for an object B of $\mathbb{C}$, $\sigma_{-,B}$ is a 3-transfor $\Sigma^3(\mathbb{C}) \to \Sigma^3(\mathbb{C})$ with source and target the identity transfor. Localizing $\sigma_{-,B}$ thrice, at $(\mathrm{id}_*^2, \mathrm{id}_*^2)$, gives the 3-transfor $\widehat{\sigma_{-,B}} : \mathbb{C} \to \mathbb{C}$ with sources and targets given by:

$$\begin{array}{rcl}
\widehat{\sigma_{-,B}}{}_0^- & = & B\otimes - \\
\widehat{\sigma_{-,B}}{}_0^+ & = & -\otimes B \\
\widehat{\sigma_{-,B}}{}_1^-(A) & = & R_{B,A} \\
\widehat{\sigma_{-,B}}{}_1^+(A) & = & R_{A,B}{}^{-1} \\
\widehat{\sigma_{-,B}}{}_1^-(f) & = & R_{f,B} \\
\widehat{\sigma_{-,B}}{}_1^+(f) & = & R_{A',B}{}^{-1}\#_0 R_{B,f}\#_0 R_{A,B}{}^{-1} \\
\widehat{\sigma_{-,B}}{}_2^-(A) & = & R_{A,B}{}^{-1}\#_0 v_{A,B}\#_0 R_{B,A} \\
\widehat{\sigma_{-,B}}{}_2^+(A) & = & v_{B,A}.
\end{array}$$

I.e., $\widehat{\sigma_{-,B}}$ says that $\widehat{\sigma_{-,B}}{}_2^-$ equals $\widehat{\sigma_{-,B}}{}_2^+$.

This localizing is expected to continue, with ever longer formulas, and with more and more trueness conditions for localizability.

The localization question has a topological interpretation: transfors are “governed by” $2_p \otimes 2_q$ (and functoriality by $(2_p \cup_n 2'_p) \otimes 2_q$), and localization of a transfor is asking for a canonical functor/map $2_{p-k}\otimes 2_q \to \Omega^k(2_p \otimes 2_q)$. This seems to me an interesting question in its own, topological, right.

6 Localization in both variables

Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be n-dimensional teisi, and let $c, c' \in \mathbb{C}$ and $d, d' \in \mathbb{D}$ whose $(k-1)$-sources and $(k-1)$-targets agree, does a functor $\mathbb{C}\otimes\mathbb{D} \to \mathbb{E}$ induce a $(k+1)$-transfor $\mathbb{C}(c,c')\otimes\mathbb{D}(d,d') \to \mathbb{E}(e,e')$, for appropriate $e, e' \in \mathbb{E}$? In this section I answer this question for $n = 3$ and $n = 4$, and apply the results to compositions in 4D teisi, and in particular to braidings and syllepses.

6.1 Localizing a functor of two variables once

Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be 3D teisi, and let χ be a functor $\mathbb{C}\otimes\mathbb{D}\to\mathbb{E}$.

Define functions $\widehat{\chi}_0^-, \widehat{\chi}_0^+ : \mathbb{C}(C,C')\otimes\mathbb{D}(D,D')\to\mathbb{E}(\chi(C\otimes D),\chi(C'\otimes D'))$ by:

$$\begin{array}{rcl}
\widehat{\chi}_0^-(f\otimes g) & = & \chi(f\otimes D')\,\#_0\,\chi(C\otimes g)\\
\widehat{\chi}_0^-(\gamma\otimes g) & = & \chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes g)\\
\widehat{\chi}_0^-(f\otimes \delta) & = & \chi(f\otimes D')\,\#_0\,\chi(C\otimes \delta)\\
\widehat{\chi}_0^-(\varphi\otimes g) & = & \chi(\varphi\otimes D')\,\#_0\,\chi(C\otimes g)\\
\widehat{\chi}_0^-(f\otimes \psi) & = & \chi(f\otimes D')\,\#_0\,\chi(C\otimes \psi)\\
\widehat{\chi}_0^-(\gamma\otimes \delta) & = & (\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes \delta))^{-1}\\
\widehat{\chi}_0^+(c\otimes d) & = & \chi(C'\otimes d)\,\#_0\,\chi(c\otimes D).
\end{array}$$

By definition of functors from a tensor product, $\chi(-\otimes D')$ is a functor $\mathbb{C}\to\mathbb{E}$ and $\chi(C\otimes -)$ is a functor $\mathbb{D}\to\mathbb{E}$, which localized give functors $\mathbb{C}(C,C')\to\mathbb{E}(\chi(C\otimes D'),\chi(C'\otimes D'))$ and $\mathbb{D}(D,D')\to\mathbb{E}(\chi(C\otimes D),\chi(C\otimes D'))$ respectively. Also, 0-composition in $\mathbb{E}$ gives a functor $\mathbb{E}(\chi(C\otimes D),\chi(C\otimes D'))\otimes\mathbb{E}(\chi(C\otimes D'),\chi(C'\otimes D'))\to\mathbb{E}(\chi(C\otimes D),\chi(C'\otimes D'))$. Combining these gives the functor $\#_0\circ(\widehat{\chi(C\otimes -)}\otimes\widehat{\chi(?\otimes D')}) : \mathbb{D}(D,D')\otimes\mathbb{C}(C,C')\to\mathbb{E}(\chi(C\otimes D),\chi(C'\otimes D'))$, and now $\widehat{\chi}_0^- = (\#_0\circ(\widehat{\chi(C\otimes -)}\otimes\widehat{\chi(?\otimes D')}))^{\smile}$. There is a similar story for $\widehat{\chi}_0^+$ but without reversing. So $\widehat{\chi}_0^-$ and $\widehat{\chi}_0^+$ are functors $\mathbb{C}(C,C')\otimes\mathbb{D}(D,D')\to\mathbb{E}(\chi(C\otimes D),\chi(C'\otimes D'))$. But for $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ 4D teisi $\widehat{\chi}_0^-$ is a functor only if the above functor can be reversed.

Define a map of degree 1 $\widehat{\chi} : \mathbb{C}(C,C')\otimes\mathbb{D}(D,D')\to\mathbb{E}(\chi(C\otimes D),\chi(C'\otimes D'))$ by:

$$\begin{array}{rcl}
\widehat{\chi}(f\otimes g) & = & \chi(f\otimes g)\\
\widehat{\chi}(\gamma\otimes g) & = & \chi(\gamma\otimes g)\\
\widehat{\chi}(f\otimes \delta) & = & \chi(f\otimes \delta)^{-1}.
\end{array}$$

Proposition 6.1 *$\widehat{\chi}$ is a 1-transfor $\widehat{\chi}_0^-\to\widehat{\chi}_0^+$.*

Proof. It is natural because I have constructed $\widehat{\chi}$ and $\widehat{\chi}_0^-$ and $\widehat{\chi}_0^+$ such that it follows from naturality of χ, and it is functortorial by functoriality of χ with respect to 1-composition of 2-arrows in each variable. □

Note that interchange is not an issue for functors from a tensor product of 3D teisi. But for 4D teisi, $\widehat{\chi}$ does satisfy interchange because of local interchange of χ.

As an example, let $\mathbb{C}$ be a 4-dimensional tas, and χ the functor $\#_0 : \mathbb{C}(C,C')\otimes\mathbb{C}(C',C'')\to\mathbb{C}(C,C'')$ Localizing χ at the objects f and f' of $\mathbb{C}(C,C')$ and g and g' of $\mathbb{C}(C',C'')$, one gets the 1-transfor $\widehat{\#_0} : \mathbb{C}(f,f')\otimes\mathbb{C}(g,g')\to\mathbb{C}(g\,\#_0\,f, g'\,\#_0\,f')$, whose source and target can be derived from the formulae for $\widehat{\chi}_0^-$ and $\widehat{\chi}_0^+$ above.

A special case of the above example is for a braided 2D tas $\mathbb{C}$, where the braiding is a functor $R:\Sigma(\mathbb{C})\otimes\Sigma(\mathbb{C})\to\Sigma(\mathbb{C})$, where the usual $R_{-,?}$ corresponds to $R(?\otimes -)$, see [21, p. 20]. Localizing

$R_{-,?}$ at $(*,*)$ and $(*,*)$ gives the 1-transfor $\widehat{R_{-,?}} : \mathbb{C}\otimes\mathbb{C} \to \mathbb{C}$ with source and target given by:

$$\begin{array}{rcl}
\widehat{R_{-,?}}_0^-(A,B) &=& A\otimes B \\
\widehat{R_{-,?}}_0^-(f,B) &=& f\otimes B \\
\widehat{R_{-,?}}_0^-(A,g) &=& A\otimes g \\
\widehat{R_{-,?}}_0^-(\alpha,B) &=& \alpha\otimes B \\
\widehat{R_{-,?}}_0^-(A,\beta) &=& A\otimes \beta \\
\widehat{R_{-,?}}_0^-(f,g) &=& (f\otimes g)^{-1} \\
\widehat{R_{-,?}}_0^+(a,b) &=& b\otimes a.
\end{array}$$

I.e., $R_{-,?}$ induces a pseudo-natural transformation $\widetilde{?\otimes -} \to ?\otimes -$. As $\widetilde{?\otimes -} \neq -\otimes ?$, this is not a pseudo-natural transformation $-\otimes ? \to ?\otimes -$!

Note that $\widehat{R_{-,?}}(f,B) = R_{f,B}{}^{-1}$. The different choices for direction of $f\otimes g$, $R_{A,g}$ and $R_{f,B}$ can now be attributed to the use of the braiding as a pseudo-natural transformation $-\otimes ? \to ?\otimes -$ as the guiding motivation [5, Definition 6],[25, Definition 12].

Because in higher dimensions the functor $\otimes$ is not always reversible, there is no hope that braidings on higher-dimensional teisi are localizable.

For later use, in section 6.3, I need to consider the reverse of the 1-transfor (see section 3.6) $\widehat{R_{-,?}} : \widetilde{?\otimes -} \to ?\otimes -$. This results in the 1-transfor $\widetilde{\widehat{R_{-,?}}} : ?\otimes - \to \widetilde{?\otimes -}$ defined by:

$$\begin{array}{rcl}
\widetilde{\widehat{R_{-,?}}}(A,B) &=& \widehat{R_{-,?}}(B,A) = R_{B,A} \\
\widetilde{\widehat{R_{-,?}}}(f,B) &=& \widehat{R_{-,?}}(f,B) = R_{B,f} \\
\widetilde{\widehat{R_{-,?}}}(A,g) &=& \widehat{R_{-,?}}(A,g) = R_{g,A}{}^{-1}.
\end{array}$$

(So far it seems that $\widetilde{\widehat{R_{-,?}}} = \widehat{R_{?,-}}$ but they have different functors as source and target, and differ in their proof of naturality.)

6.2 Localizing a 1-transfor of two variables once

Let again $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be 3D teisi, let $\chi_0^-, \chi_0^+ : \mathbb{C}\otimes\mathbb{D} \to \mathbb{E}$ be functors, and let χ be a 1-transfor $\mathbb{C}\otimes\mathbb{D} \to \mathbb{E}$.

Define functions $\widehat{\chi}_0^-, \widehat{\chi}_0^+ : \mathbb{C}(C,C')\otimes\mathbb{D}(D,D') \to \mathbb{E}(\chi(C\otimes D), \chi(C'\otimes D'))$ by:

$$\begin{array}{rcl}
\widehat{\chi}_0^-(f\otimes g) &=& \chi_0^+(f\otimes D') \#_0 \chi_0^+(C\otimes g) \#_0 \chi(C\otimes D) \\
\widehat{\chi}_0^-(\gamma\otimes g) &=& \chi_0^+(\gamma\otimes D') \#_0 \chi_0^+(C\otimes g) \#_0 \chi(C\otimes D) \\
\widehat{\chi}_0^-(f\otimes \delta) &=& \chi_0^+(f\otimes D') \#_0 \chi_0^+(C\otimes \delta) \#_0 \chi(C\otimes D) \\
\widehat{\chi}_0^-(\varphi\otimes g) &=& \chi_0^+(\varphi\otimes D') \#_0 \chi_0^+(C\otimes g) \#_0 \chi(C\otimes D) \\
\widehat{\chi}_0^-(f\otimes \psi) &=& \chi_0^+(f\otimes D') \#_0 \chi_0^+(C\otimes \psi) \#_0 \chi(C\otimes D) \\
\widehat{\chi}_0^-(\gamma\otimes \delta) &=& (\chi_0^+(\gamma\otimes D') \#_0 \chi_0^+(C\otimes \delta) \#_0 \chi(C\otimes D))^{-1} \\
\widehat{\chi}_0^+(c\otimes d) &=& \chi(C'\otimes D') \#_0 \chi_0^-(C'\otimes d) \#_0 \chi_0^-(c\otimes D).
\end{array}$$

$\chi(C\otimes D)$ is a 1-arrow $\chi_0^-(C\otimes D) \to \chi_0^+(C\otimes D)$ of $\mathbb{E}$, so left composition with it is a left 1-transfor $\mathbb{E}(\chi_0^+(C\otimes D), \chi_0^+(C'\otimes D')) \to \mathbb{E}(\chi_0^-(C\otimes D), \chi_0^+(C'\otimes D'))$. Arguments as before give that $\chi_0^+(-\otimes D')$ and $\chi_0^+(C\otimes -)$ localized give functors $\mathbb{C}(C,C') \to \mathbb{E}(\chi_0^+(C\otimes D'), \chi_0^+(C'\otimes D'))$ and

$\mathbb{D}(D,D') \to \mathbb{E}(\chi_0^+(C\otimes D), \chi_0^+(C\otimes D'))$ respectively. Combining this with 0-composition in $\mathbb{E}$ gives that $\widehat{\chi}_0^- = -\#_0\, \widetilde{\chi(C\otimes D)} \circ \#_0 \circ (\widehat{\chi_0^+(C\otimes -)} \otimes \widehat{\chi_0^+(?\otimes D')})$. There is a similar story for $\widehat{\chi}_0^+$ but without reversing. So $\widehat{\chi}_0^-$ and $\widehat{\chi}_0^+$ are functors $\mathbb{C}(C,C')\otimes\mathbb{D}(D,D') \to \mathbb{E}(\chi_0^-(C\otimes D), \chi_0^+(C'\otimes D'))$. But for $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ 4D teisi $\widehat{\chi}_0^-$ is a functor only if the above functors can be reversed.

Define maps of degree 1 $\widehat{\chi}_1^-, \widehat{\chi}_1^+ : \mathbb{C}(C,C')\otimes\mathbb{D}(D,D') \to \mathbb{E}(\chi_0^-(C\otimes D), \chi_0^+(C'\otimes D'))$ by:

$$
\begin{aligned}
\widehat{\chi}_1^-(f\otimes g) &= (\chi(C'\otimes D')\,\#_0\,\chi_0^-(f\otimes g))\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi_0^-(C\otimes g))\,\#_1\,(\chi_0^+(f\otimes D')\,\#_0\,\chi(C\otimes g))\\
\widehat{\chi}_1^+(f\otimes g) &= (\chi(C'\otimes g)\,\#_0\,\chi_0^-(f\otimes D))\,\#_1\,(\chi_0^+(C'\otimes g)\,\#_0\,\chi(f\otimes D))\,\#_1\,(\chi_0^+(f\otimes g)\,\#_0\,\chi(C\otimes D))\\
\widehat{\chi}_1^-(\gamma\otimes g) &= \big((\chi(C'\otimes D')\,\#_0\,\chi_0^-(f'\otimes g))\,\#_1\\
&\quad (\chi(f'\otimes D')\,\#_0\,\chi_0^-(C\otimes g))\,\#_1\\
&\quad (\chi_0^+(\gamma\otimes D')\,\#_0\,\chi(C\otimes g))^{-1}\big)\,\#_2\\
&\quad \big((\chi(C'\otimes D')\,\#_0\,\chi_0^-(f'\otimes g))\,\#_1\\
&\quad (\chi(\gamma\otimes D')\,\#_0\,\chi_0^-(C\otimes g))\,\#_1\\
&\quad (\chi_0^+(f\otimes D')\,\#_0\,\chi(C\otimes g))\big)\,\#_2\\
&\quad \big((\chi(C'\otimes D')\,\#_0\,\chi_0^-(\gamma\otimes g))\,\#_1\\
&\quad (\chi(f\otimes D')\,\#_0\,\chi_0^-(C\otimes g))\,\#_1\\
&\quad (\chi_0^+(f\otimes D')\,\#_0\,\chi(C\otimes g))\big)\\
\widehat{\chi}_1^+(\gamma\otimes g) &= \big((\chi(C'\otimes g)\,\#_0\,\chi_0^-(f'\otimes D))\,\#_1\\
&\quad (\chi_0^+(C'\otimes g)\,\#_0\,\chi(f'\otimes D))\,\#_1\\
&\quad (\chi_0^+(\gamma\otimes g)\,\#_0\,\chi(C\otimes D))\big)\,\#_2\\
&\quad \big((\chi(C'\otimes g)\,\#_0\,\chi_0^-(f'\otimes D))\,\#_1\\
&\quad (\chi_0^+(C'\otimes g)\,\#_0\,\chi(\gamma\otimes D))\,\#_1\\
&\quad (\chi_0^+(f\otimes g)\,\#_0\,\chi(C\otimes D))\big)\,\#_2\\
&\quad \big((\chi(C'\otimes g)\,\#_0\,\chi_0^-(\gamma\otimes D))\,\#_1\\
&\quad (\chi_0^+(C'\otimes g)\,\#_0\,\chi(f\otimes D))\,\#_1\\
&\quad (\chi_0^+(f\otimes g)\,\#_0\,\chi(C\otimes D))\big)\\
\widehat{\chi}_1^-(f\otimes\delta) &= \big((\chi(C'\otimes D')\,\#_0\,\chi_0^-(f\otimes g'))\,\#_1\\
&\quad (\chi(f\otimes D')\,\#_0\,\chi_0^-(C\otimes g'))\,\#_1\\
&\quad (\chi_0^+(f\otimes D')\,\#_0\,\chi(C\otimes\delta))\big)\,\#_2\\
&\quad \big((\chi(C'\otimes D')\,\#_0\,\chi_0^-(f\otimes g'))\,\#_1\\
&\quad (\chi(f\otimes D')\,\#_0\,\chi_0^-(C\otimes\delta))\,\#_1\\
&\quad (\chi_0^+(f\otimes D')\,\#_0\,\chi(C\otimes g))\big)\,\#_2\\
&\quad \big((\chi(C'\otimes D')\,\#_0\,\chi_0^-(f\otimes\delta))\,\#_1\\
&\quad (\chi(f\otimes D')\,\#_0\,\chi_0^-(C\otimes g))\,\#_1\\
&\quad (\chi_0^+(f\otimes D')\,\#_0\,\chi(C\otimes g))\big)\\
\widehat{\chi}_1^+(f\otimes\delta) &= \big((\chi(C'\otimes g')\,\#_0\,\chi_0^-(f\otimes D))\,\#_1\\
&\quad (\chi_0^+(C'\otimes g')\,\#_0\,\chi(f\otimes D))\,\#_1\\
&\quad (\chi_0^+(f\otimes\delta)\,\#_0\,\chi(C\otimes D))\big)\,\#_2\\
&\quad \big((\chi(C'\otimes g')\,\#_0\,\chi_0^-(f\otimes D))\,\#_1\\
&\quad (\chi_0^+(C'\otimes\delta)\,\#_0\,\chi(f\otimes D))^{-1}\,\#_1\\
&\quad (\chi_0^+(f\otimes g)\,\#_0\,\chi(C\otimes D))\big)\,\#_2\\
&\quad \big((\chi(C'\otimes\gamma)\,\#_0\,\chi_0^-(f\otimes D))\,\#_1\\
&\quad (\chi_0^+(C'\otimes g)\,\#_0\,\chi(f\otimes D))\,\#_1\\
&\quad (\chi_0^+(f\otimes g)\,\#_0\,\chi(C\otimes D))\big)\,.
\end{aligned}
$$

Proposition 6.2 *$\widehat{\chi}_1^-$ and $\widehat{\chi}_1^+$ are 1-transfors $\widehat{\chi}_0^- \to \widehat{\chi}_0^+$.*

Proof. Left to the reader. □

Again, note how $\widehat{\chi}_1^-$ and $\widehat{\chi}_1^+$ are composites of 1-transfors and their reversings, and how their being 1-transfors depends on trueness of these compositions.

Define a map of degree 2 $\mathbb{C}(C,C') \otimes \mathbb{D}(D,D') \to \mathbb{E}(\chi_0^-(C \otimes D), \chi_0^+(C' \otimes D'))$ by:

$$\widehat{\chi}(f \otimes g) \quad = \quad \chi(f \otimes g).$$

Proposition 6.3 *$\widehat{\chi}$ is a 2-transfor $\widehat{\chi}_1^- \to \widehat{\chi}_1^+$.*

Proof. Straightforward; there is not much to check. □

Again, in higher dimensions there is the obstruction that $\widehat{\chi}_1^-$ and $\widehat{\chi}_1^+$ might not be 1-transfors. If this and other obstructions do not occur call χ *localizable*.

The example of triple composition in a 4D tas with one factor fixed is left to the reader. A more interesting example is to apply this localization to a transfor obtained from localization, which is the subject of the next subsection.

A specific case of triple composition is a Yang-Baxter operator obtained from a braiding, so this (and other) localizations will be relevant for Yang-Baxter operators, see [22].

6.3 Localizing a functor of two variables twice

Localizing a functor twice could be defined as localizing it once and then once again, of course. But a more direct approach gives a slightly different result, expressed in terms of the original functor: the need to introduce some extra inverses for localization once is not there for localization twice. Also, a functor might be twice localizable even if it is not once localizable as obstructions for once might disappear on localizing further.

The situation would be too trivial when starting from functors between 3D teisi, so from now on I will consider functors between 4D teisi. As they will be localized twice, I will not need a full definition of such functors, and the parts which do affect the localizations will become apparent in the sequel.

Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be 4D teisi, let $\chi_0^-, \chi_0^+ : \mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ be functors, and let χ be a functor $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$.

Define functions $\widehat{\chi}_0^-, \widehat{\chi}_0^+ : \mathbb{C}(f,f') \otimes \mathbb{D}(g,g') \to \mathbb{E}(s_1(\chi(f \otimes g)), t_1(\chi(f' \otimes g'))$ by:

$$\begin{array}{rcl}
\widehat{\chi}_0^-(c \otimes \delta) & = & (\chi(C' \otimes g) \#_0 \chi(c \otimes D)) \#_1 \chi(f \otimes g') \#_1 (\chi(f \otimes D') \#_0 \chi(C \otimes \delta)) \\
\widehat{\chi}_0^-(\gamma \otimes d) & = & (\chi(C' \otimes g) \#_0 \chi(\gamma \otimes D)) \#_1 \chi(f \otimes g') \#_1 (\chi(f \otimes D') \#_0 \chi(C \otimes d)) \\
\widehat{\chi}_0^-(\varphi \otimes \psi) & = & (\chi(C' \otimes g) \#_0 \chi(\varphi \otimes D)) \#_1 \chi(f \otimes g') \#_1 (\chi(f \otimes D') \#_0 \chi(C \otimes \psi))^{-1} \\
\widehat{\chi}_0^+(c \otimes d) & = & (\chi(C' \otimes d) \#_0 \chi(f' \otimes D)) \#_1 \chi(f' \otimes g) \#_1 (\chi(c \otimes D') \#_0 \chi(C \otimes g)).
\end{array}$$

By the same argument as before, $\widehat{\chi}_0^-$ and $\widehat{\chi}_0^+$ are functors.

Define maps of degree 1 $\widehat{\chi}_1^-, \widehat{\chi}_1^+ : \mathbb{C}(f,f') \otimes \mathbb{D}(g,g') \to \mathbb{E}(s_1(\chi(f \otimes g)), t_1(\chi(f' \otimes g'))$ by:

$$
\begin{array}{rcl}
\widehat{\chi}_1^-(\gamma\otimes\delta) & = & ((\chi(C'\otimes\delta)\,\#_0\,\chi(f'\otimes D))\,\#_1\,\chi(\gamma\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes\delta)\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes g')\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes\delta)) \\
\widehat{\chi}_1^+(\gamma\otimes\delta) & = & (\chi(f'\otimes\delta)\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes g')))\,\#_2 \\
 & & \big(\chi(f'\otimes g')\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\delta))^{-1}\big)\,\#_2 \\
 & & (\chi(\gamma\otimes g')\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi(C\otimes\delta))) \\
\widehat{\chi}_1^-(\varphi\otimes\delta) & = & (((\chi(C'\otimes\delta)\,\#_0\,\chi(f'\otimes D))\,\#_1\,\chi(\gamma'\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes\delta)\,\#_0\,\chi(\gamma'\otimes D))\,\#_1\,\chi(f\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes g')\,\#_0\,\chi(\varphi\otimes D))\,\#_1\,\chi(f\otimes\delta))^{-1}\Big)\,\#_3 \\
 & & (((\chi(C'\otimes\delta)\,\#_0\,\chi(f'\otimes D))\,\#_1\,\chi(\gamma'\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes\delta)\,\#_0\,\chi(\varphi\otimes D))\,\#_1\,\chi(f\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes g')\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes\delta)))\,\#_3 \\
 & & (((\chi(C'\otimes\delta)\,\#_0\,\chi(f'\otimes D))\,\#_1\,\chi(\varphi\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes\delta)\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes g')\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes\delta))) \\
\widehat{\chi}_1^+(\varphi\otimes\delta) & = & ((\chi(f'\otimes\delta)\,\#_1\,(\chi(\gamma'\otimes D')\,\#_0\,\chi(C\otimes g')))\,\#_2 \\
 & & \big(\chi(f'\otimes g')\,\#_1\,(\chi(\gamma'\otimes D')\,\#_0\,\chi(C\otimes\delta))^{-1}\big)\,\#_2 \\
 & & (\chi(\varphi\otimes g')\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi(C\otimes\delta))))\,\#_3 \\
 & & ((\chi(f'\otimes\delta)\,\#_1\,(\chi(\gamma'\otimes D')\,\#_0\,\chi(C\otimes g')))\,\#_2 \\
 & & \big(\chi(f'\otimes g')\,\#_1\,((\chi(\gamma'\otimes D')\,\#_0\,\chi(C\otimes\delta))^{-1}\,\#_2\,(\chi(\varphi\otimes D')\,\#_0\,\chi(C\otimes\delta))\#_2 \\
 & & (\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\delta))^{-1})\big)\,\#_2 \\
 & & (\chi(\gamma\otimes g')\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi(C\otimes\delta))))\,\#_3 \\
 & & ((\chi(f'\otimes\delta)\,\#_1\,(\chi(\varphi\otimes D')\,\#_0\,\chi(C\otimes g')))\,\#_2 \\
 & & \big(\chi(f'\otimes g')\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\delta))^{-1}\big)\,\#_2 \\
 & & (\chi(\gamma\otimes g')\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi(C\otimes\delta)))) \\
\widehat{\chi}_1^-(\gamma\otimes\psi) & = & (((\chi(C'\otimes\delta')\,\#_0\,\chi(f'\otimes D))\,\#_1\,\chi(\gamma\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes\delta')\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes g')\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes\psi)))\,\#_3 \\
 & & (((\chi(C'\otimes\delta')\,\#_0\,\chi(f'\otimes D))\,\#_1\,\chi(\gamma\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes\psi)\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes g))^{-1}\,\#_2 \\
 & & ((\chi(C'\otimes g')\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes\delta))^{-1}\Big)\,\#_3 \\
 & & \Big(((\chi(C'\otimes\psi)\,\#_0\,\chi(f'\otimes D))\,\#_1\,\chi(\gamma\otimes g))^{-1}\,\#_2 \\
 & & ((\chi(C'\otimes\delta)\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes g))\,\#_2 \\
 & & ((\chi(C'\otimes g')\,\#_0\,\chi(\gamma\otimes D))\,\#_1\,\chi(f\otimes\delta))) \\
\widehat{\chi}_1^+(\gamma\otimes\psi) & = & ((\chi(f'\otimes\delta')\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes g')))\,\#_2 \\
 & & \big(\chi(f'\otimes g')\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\delta'))^{-1}\big)\,\#_2 \\
 & & (\chi(\gamma\otimes g')\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi(C\otimes\psi))))\,\#_3 \\
 & & ((\chi(f'\otimes\delta')\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes g')))\,\#_2 \\
 & & \big(\chi(f'\otimes g')\,\#_1\,((\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\delta'))^{-1}\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\psi))\#_1 \\
 & & (\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\delta))^{-1})\big)^{-1}\,\#_2 \\
 & & (\chi(\gamma\otimes g')\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi(C\otimes\delta))))\,\#_3 \\
 & & \Big((\chi(f'\otimes\psi)\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes g')))^{-1}\,\#_2 \\
 & & \big(\chi(f'\otimes g')\,\#_1\,(\chi(\gamma\otimes D')\,\#_0\,\chi(C\otimes\delta))^{-1}\big)\,\#_2 \\
 & & (\chi(\gamma\otimes g')\,\#_1\,(\chi(f\otimes D')\,\#_0\,\chi(C\otimes\delta))))\,.
\end{array}
$$

(cf. the formulae for localization of 2-transfors twice and thrice and 3-transfors twice.)

Proposition 6.4 *$\widehat{\chi}_1^-$ and $\widehat{\chi}_1^+$ are 1-transfors $\widehat{\chi}_0^- \to \widehat{\chi}_0^+$.*

Proof. Left to the reader. $\square$

Again, note how $\widehat{\chi}_1^-$ and $\widehat{\chi}_1^+$ are composites of 1-transfors and their reversings, and how their being 1-transfors depends on trueness of these compositions.

Define a map of degree 2 $\widehat{\chi} : \mathbb{C}(f, f') \otimes \mathbb{D}(g, g') \to \mathbb{E}(s_1(\chi(f \otimes g)), t_1(\chi(f' \otimes g'))$ by:

$$\widehat{\chi}(\gamma \otimes \delta) \quad = \quad \chi(\gamma \otimes \delta).$$

Proposition 6.5 *$\widehat{\chi}$ is a 2-transfor $\widehat{\chi}_1^- \to \widehat{\chi}_1^+$.*

Proof. Straightforward; there is not much to check. $\square$

In higher dimensions, $\widehat{\chi}(c,d)$ is not just equal to $\chi(c,d)$, so there are reversibility and trueness issues then. If these do hold call χ *localizable*.

As an example, let $\mathbb{C}$ be a 4-dimensional tas, and χ the functor $\#_0 : \mathbb{C}(C,C') \otimes \mathbb{C}(C',C'') \to \mathbb{C}(C,C'')$ Localizing χ at the arrows γ and γ' of $\mathbb{C}(C,C')$ and δ and δ' of $\mathbb{C}(C',C'')$, one gets the 2-transfor $\widehat{\#_0} : \mathbb{C}(\gamma,\gamma') \otimes \mathbb{C}(\delta,\delta') \to \mathbb{C}(s_2(\delta \#_0 \gamma), t_2(\delta' \#_0 \gamma'))$, with source and target can be derived from the formulae for $\widehat{\chi}_1^-$ and $\widehat{\chi}_1^+$ above.

A special case of the above example is for a sylleptic 2D tas $\mathbb{C}$, where the syllepsis is a functor $v : \Sigma^2(\mathbb{C}) \otimes \Sigma^2(\mathbb{C}) \to \Sigma^2(\mathbb{C})$, where the usual $v_{-,?}$ corresponds to $v(? \otimes -)$, see [21, p. 33]. Localizing $v_{-,?}$ twice, at $(\mathrm{id}_*, \mathrm{id}_*)$ and $(\mathrm{id}_*, \mathrm{id}_*)$ gives the 2-transfor $\widehat{v_{-,?}} : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ with source and target given by:

$$\begin{array}{rcl}
\widehat{v_{-,?}}{}_0^-(A,B) & = & A \otimes B \\
\widehat{v_{-,?}}{}_0^-(f,B) & = & f \otimes B \\
\widehat{v_{-,?}}{}_0^-(A,g) & = & A \otimes g \\
\widehat{v_{-,?}}{}_0^-(\alpha,B) & = & \alpha \otimes B \\
\widehat{v_{-,?}}{}_0^-(A,\beta) & = & A \otimes \beta \\
\widehat{v_{-,?}}{}_0^-(f,g) & = & (f \otimes g)^{-1} \\
\widehat{v_{-,?}}{}_0^+(a,b) & = & b \otimes a \\
\widehat{v_{-,?}}{}_1^-(A,B) & = & R_{A,B} \\
\widehat{v_{-,?}}{}_1^-(f,B) & = & {R_{f,B}}^{-1} \\
\widehat{v_{-,?}}{}_1^-(A,g) & = & R_{A,g} \\
\widehat{v_{-,?}}{}_1^+(A,B) & = & {R_{B,A}}^{-1} \\
\widehat{v_{-,?}}{}_1^+(f,B) & = & {R_{B,A'}}^{-1} \#_0 {R_{B,f}}^{-1} \#_0 {R_{B,A}}^{-1} \\
\widehat{v_{-,?}}{}_1^+(A,g) & = & {R_{B',A}}^{-1} \#_0 R_{g,A} \#_0 {R_{B,A}}^{-1}.
\end{array}$$

Comparing these formulae to ones seen earlier in this paper, we see that $\widehat{v_{-,?}}$ becomes a modification

$$\begin{tikzcd}
\widetilde{? \otimes -} \arrow[r, bend left, "\widehat{R_{-,?}}"] \arrow[r, bend right, "\widetilde{\widehat{R_{-,?}}}"'] \arrow[r, phantom, "\Downarrow"] & ? \otimes -
\end{tikzcd} \quad .$$

Using the final result of section 6.1, there is another way to describe the target of $\widehat{\nu_{-,?}}$, namely as $\widetilde{\widehat{R_{-,?}}}^{-1}$, the inverse *as an invertible transfor*, that is, which also involves the appropriate mates. I do not know whether this is a coincidence, or an instance of the possible more general phenomenon that for a (reversible, localizable) functor $\chi : \mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ it is always the case that $\widehat{\widehat{\chi}} = \widetilde{\widetilde{\widehat{\chi}}}^{-1}$.

As for braidings, there is no hope that generally syllepses on higher dimensional teisi are localizable.

6.4 Localizing functors and transfors of two and more variables more often

One special case of some interest here is for a symmetric 2D tas $\mathbb{C}$, where the symmetry is a functor $\sigma_{-,?} : \Sigma^3(\mathbb{C}) \otimes \Sigma^3(\mathbb{C}) \to \Sigma^3(\mathbb{C})$. Localizing $\sigma_{-,?}$ thrice, at $(\mathrm{id}^2_*, \mathrm{id}^2_*)$ and $(\mathrm{id}^2_*, \mathrm{id}^2_*)$, gives the 3-transfor $\widehat{\sigma_{-,?}} : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ with sources and targets given by:

$$\begin{aligned}
\widehat{\sigma_{-,?}}_0^- &= \widetilde{- \otimes ?} \\
\widehat{\sigma_{-,?}}_0^+ &= - \otimes ? \\
\widehat{\sigma_{-,?}}_1^- &= \widehat{R_{?,-}} \\
\widehat{\sigma_{-,?}}_1^+ &= \widehat{\widetilde{R_{?,-}}} \\
\widehat{\sigma_{-,?}}_2^- &= \widehat{\widetilde{\nu_{?,-}}} \\
\widehat{\sigma_{-,?}}_2^+ &= \widehat{\nu_{?,-}}.
\end{aligned}$$

I.e., $\widehat{\sigma_{-,?}}$ says that $\widehat{\widetilde{\nu_{?,-}}}$ equals $\widehat{\nu_{?,-}}$.

This localizing is expected to continue, with ever longer formulas, and with more and more trueness and reversibility conditions for localizability.

This localization question has a topological interpretation too: functors $\mathbb{C} \otimes \mathbb{D} \to \mathbb{E}$ are "governed by $2_p \otimes 2_q \otimes 2_r$", and localization of such a functor is asking for a canonical functor/map $2_{p-k} \otimes 2_{q-k} \otimes 2_{r+k} \to \Omega^k(2_p \otimes 2_q \otimes 2_r)$. This seems to me an interesting question in its own, topological, right.

For more variables there is a similar question, which will be relevant for Yang-Baxter and Zamolodchikov operators, see [22].

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