

Planarizing Graphs — A Survey and
Annotated Bibliography
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Annegret Liebers

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Planarizing Graphs — A Survey and Annotated Bibliography

Annegret Liebers*

Universität Konstanz
Fakultät für Mathematik und Informatik
Lehrstuhl Prof. Dr. Dorothea Wagner
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Abstract

Given a finite, undirected, simple graph G , we are concerned with operations on G that transform it into a planar graph. We give a survey of results about such operations and related graph parameters. While there are many algorithmic results about planarization through edge deletion, the results about vertex splitting, thickness, and crossing number are mostly of a structural nature. We also include a brief section on vertex deletion.

We do not consider parallel algorithms, nor do we deal with on-line algorithms.

*Fakultät für Mathematik und Informatik, Universität Konstanz, Germany, email: Annegret.Liebers@uni-konstanz.de, WWW: <http://www.fmi.uni-konstanz.de/~liebers/>

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1 Introduction

Many problems in discrete mathematics and combinatorial optimization can be viewed as graph problems. Graphs immediately come to mind for modeling networks of all kinds, but also seemingly unrelated problems from areas like transportation or warehousing can turn out to be, e.g., network flow problems, and their solution involves algorithms on graphs [AMO93].

Graphs that can be drawn without edge crossings (i.e. *planar* graphs) have a natural advantage for visualization, but also other graph problems can be easier to solve when restricted to this special class of graphs. “Easier” might mean that a special algorithm for planar graphs may have a better asymptotic time complexity than the best known algorithm for general graphs, or even that an intractable problem may become tractable if restricted to planar graphs. The former case applies for example to the Vertex- and Edge-Disjoint Menger Problems [RLWW97, Wei97].

The latter case, however, seems to be relatively rare [Joh85, p. 440]: There are polynomial time algorithms for Max Cut restricted to planar graphs [GJ79, Problem ND16], and Vertex Coloring is NP-complete for general graphs, even for a fixed number $k \geq 3$ of colors [GJ79, Problem GT4], but is trivially solvable for a fixed number $k \geq 4$ for planar graphs by virtue of the Four Color Theorem. See [JT95, Section 2.1.] for a discussion of the original proof by Appel and Haken, and of algorithms for actually finding a coloring of a planar graph, also in light of the new proof [RSST96] of the Four Color Theorem.

When visualizing nonplanar graphs, a natural approach is to draw the graph in a way as close to planarity as possible (for example with as few edge crossings as possible). This is one of the problems of graph drawing, a field that has grown tremendously within the last decade [DETT94, DETT99].

In any case there is great interest in the question of how far from being planar a given graph is. We survey ways of transforming a nonplanar graph into a planar graph and discuss measures for the nonplanarity of a graph. We concentrate on sequential algorithms for the off-line case, i.e. we do not consider parallel or on-line algorithms.

One approach is to look for the largest induced planar subgraph of a nonplanar graph. Finding an induced subgraph is equivalent to deleting vertices from a graph and will be discussed in Section 2. It does not seem to be a very common approach, and there is relatively little literature about it.

Another approach is to look for the largest planar subgraph (without the restriction to induced subgraphs). Since deleting an edge from a graph is a less “drastic” operation than deleting a vertex together with all its incident edges, it is not surprising that finding a planar subgraph of a nonplanar graph (i.e. deleting edges) has been studied much more intensively. There is a large amount of literature about finding a planar subgraph, with an emphasis on algorithmic results. They are the subject of Section 3.

Another technique for planarizing a graph is vertex splitting. There are relatively few algorithmic results about vertex splitting, but it turns out that there are many different structural results involving this operation. Section 4 describes the vertex splitting operation as it relates to graph planarization.

Vertex deletion, edge deletion, and vertex splitting are operations performed on single vertices or edges of the graph in question, i.e. they are local operations. Section 5 discusses partitioning the whole graph into several planar layers, hence following a global approach. The greater the number of layers needed, the further away from planarity the graph is. There seem to be few algorithmic results about finding this *thickness* of a graph, but there are many

structural results about thickness within topological graph theory.

Section 6 discusses the problem of drawing a graph so that there are as few edge crossings as possible in the drawing. Again, most results about the *crossing number* of a graph are of a structural nature. Finally, Section 7 mentions the concept of *coarseness*.

We do not study hierarchical graph models such as presented in [Len89, FCE95], nor do we discuss hypergraphs [Ber73, Ber89] or infinite graphs [Kön90].

The remainder of the introduction gives definitions and terminology concerning graphs in Section 1.1, and then gives a brief introduction to planar graphs in Section 1.2. Section 1.3 lists some generalizations of planarity. For an introduction to algorithms and the definition and use of $O(\dots)$ and $\Omega(\dots)$ for asymptotic bounds, the reader is referred to textbooks on algorithms, for example [CLR94]. The complexity classes P and NP and the concept of NP-completeness are also discussed in [CLR94], but a more thorough treatment can be found in [GJ79] and [Pap94].

1.1 Graphs

There are many textbooks on graph theory.¹ Some of the standard ones are [Har69, BM76, CL86, Tut84]. For a focus on algorithmic graph theory, see for example [Eve79, Gol80, GM84, TS92], and for topological graph theory, see [GT87]. A recent text is also [Wes96].

We will now give some definitions and notation concerning graphs that are used throughout the text.

A *finite, undirected, simple graph* G , denoted $G = (V, E)$, consists of a finite vertex set V and a set of undirected edges $E \subseteq \{\{u, v\} \mid u \in V, v \in V, u \neq v\}$. The *end vertices* of an edge $e = \{u, v\} \in E$, u and v , are said to be *adjacent*. u is said to be a *neighbor* of v and vice versa. Furthermore, u and v are said to be *incident* to e (and vice versa). For brevity we often write uv instead of $\{u, v\}$. From now on, when we speak of a graph, we always mean a finite, undirected, simple graph.

The number of edges incident to a vertex u is called the *vertex degree* (or simply *degree*) of u . The *minimum (maximum) degree* of a graph G is the minimum (maximum) degree of all vertices of G . The minimum and maximum degrees of a graph are denoted by δ and Δ , respectively. If all vertices of a graph have the same degree d , the graph is called *d-regular* (or just *regular*). A 3-regular graph is also called *cubic*.

A graph is usually visualized by representing each vertex through a point in the plane, and by representing each edge through a curve in the plane, connecting the points corresponding to the end vertices of the edge. We usually do not distinguish between a vertex and the point representing it, or between an edge and the curve representing it. Such a representation is called a *drawing* of the graph if no two vertices are represented by the same point, if the curve representing an edge does not include any point representing a vertex (except that the endpoints of the curve are the points representing the end vertices of the edge), and if two distinct edges have at most one point in common.

Given a graph $G = (V, E)$, a graph $G' = (V', E')$ is called a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq \{uv \mid u \in V', v \in V', \text{ and } uv \in E\}$. If furthermore $V' = V$ then G' is said to be a *spanning* subgraph of G . If $V' \subset V$ or $E' \subset E$ (or both) then G' is said to be a *proper* subgraph of G . A graph $G'' = (V'', E'')$ is called a *vertex induced* (or simply *induced*)

¹The first textbook devoted solely to graph theory was [Kön36] by König. [Kön90] is the first English translation. The history of graph theory is presented in [BLW76, Wil86].

subgraph of G if $V'' \subseteq V$ and $E'' = \{uv \mid u \in V'' \text{ and } v \in V'' \text{ and } uv \in E\}$. In that case we call G'' the *subgraph of G induced by V''* .

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two (not necessarily distinct) subgraphs of a graph $G = (V, E)$, then the subgraph $G' = (V_1 \cup V_2, E_1 \cup E_2)$ of G is called the *union* of G_1 and G_2 .²

Given a graph $G = (V, E)$, a sequence $v_0e_1v_1e_2v_2 \dots e_kv_k$ is called a *path* in G if the $k + 1$ vertices $v_0 \dots v_k$ are elements of V , if they are pairwise distinct except possibly v_0 and v_k , and if v_{i-1} and v_i are the end vertices of e_i for $1 \leq i \leq k$. k is called the *length* of the path. We also say that the path *connects* the vertices v_0 and v_k . If additionally $v_0 = v_k$, the path is called a *cycle*. The length of a shortest cycle in G is called the *girth* of G . If G has no cycles, it is said to be *acyclic* and the girth is undefined. (Note that an acyclic graph is planar.)

We denote with P_n the graph consisting only of a path of length $n - 1$, where the end vertices of the path are not identical. P_n has n vertices and $n - 1$ edges. C_n denotes a graph consisting of a cycle of length n , having n vertices and n edges. If a path in a graph G includes all vertices of G it is called a *Hamilton path*. If additionally this path is a cycle, it is called a *Hamilton cycle*. Observe that in Figure 6, graph 13 contains a Hamilton path, but no Hamilton cycle, whereas graph 14 contains both.

If for every pair of vertices u and v of a graph $G = (V, E)$ there is a path in G connecting u and v then G is said to be *connected*. Otherwise G is said to be *disconnected*. If $V' \subseteq V$ is a vertex set such that the subgraph G' of G induced by V' is connected and such that for every set V'' with $V' \subset V'' \subseteq V$ the subgraph of G induced by V'' is disconnected, then G' is said to be a *connected component* (or simply *component*) of G .

Given a graph $G = (V, E)$ and a vertex $v \in V$ we say that the subgraph G' of G induced by $V \setminus \{v\}$ is obtained by *deleting v from G* . If G' has more connected components than G then v is said to be a *cut vertex* of G . If at least k vertices have to be deleted from G before the resulting graph is disconnected, or before the resulting graph consists of a single vertex, then G is said to be *k -connected*. Observe that if a graph is 1-connected, then it is connected, and that a connected graph with at least 3 vertices and without cut vertices is 2-connected. In Figure 6, graph 7 has two cut vertices. Graph 16 is 2-connected, but it is not 3-connected.

Analogous definitions exist for edges: Given a graph $G = (V, E)$ and an edge $e \in E$ we say that the subgraph $G' = (V, E \setminus \{e\})$ of G is obtained by *deleting e from G* . If G' has more connected components than G then e is said to be a *cut edge* of G . If at least k edges have to be deleted from G before the resulting graph is disconnected, then G is said to be *k -edge-connected*. The graph consisting of a single vertex is defined to be 0-edge-connected.

If for a graph $G = (V, E)$, $V' \subseteq V$ is a vertex set such that the subgraph of G induced by V' is 2-connected and such that for every set V'' with $V' \subset V'' \subseteq V$ the subgraph of G induced by V'' is not 2-connected, then we call the subgraph of G induced by V' a *2-connected block* (or simply a *block*) of G .

If an edge $e = uv$ of a graph $G = (V, E)$ is replaced by a path $ue'v_e e''v$ introducing a new vertex $v_e \notin V$ then we say that the graph $G' = (V \cup \{v_e\}, (E \setminus \{e\}) \cup \{e', e''\})$ is obtained from G by *subdividing* the edge e . If a graph G'' is obtained from G by any number of (possibly zero) subdivisions of edges then G'' is called a *subdivision* of G . It will be clear from the context whether the term subdivision refers to the operation of subdividing an edge or to the resulting graph. For an illustration of subdivisions, see Figure 7.

For a graph $G = (V, E)$ and an edge $e = uv \in E$, the graph G' obtained from G by deleting

²Note that the term *union* is sometimes defined differently (see for example [Har69, p. 21]).

e , identifying u and v and by removing all edges $f \in \{ux \mid x \in V, x \neq u, x \neq v, ux \in E, \text{ and } vx \in E\}$, is said to have been obtained from G by *contracting* the edge e . In other words, contracting an edge means identifying its two end vertices and making the resulting graph simple by deleting loops and multiple edges. A graph obtained from a subgraph of G by any number (including zero) of edge contractions is said to be a *minor* of G . A subgraph of G is always a minor of G , but not vice versa. In Figure 6, the graph G is a minor of graphs 1 through 6 and 9 through 18, but it is not a minor of graphs 7 and 8. For another illustration of graph minors, see Figure 14.

Besides the paths P_n and the cycles C_n , the following special graphs appear throughout the text:

For $n \geq 2$, the *complete graph*, denoted K_n , consists of n vertices together with all possible $\binom{n}{2}$ edges. So in K_n every vertex is adjacent to every other vertex. We define K_1 to be the graph consisting of a single vertex. K_2 is a single edge with its two end vertices, and K_3 is a triangle.

The *complete bipartite graph*, denoted K_{n_1, n_2} , consists of two disjoint vertex sets $V = \{v_1, \dots, v_{n_1}\}$ and $W = \{w_1, \dots, w_{n_2}\}$ and the edge set $E = \{v_i w_j \mid 1 \leq i \leq n_1 \text{ and } 1 \leq j \leq n_2\}$ of all edges between vertices in V and vertices in W . Note that $K_{n_1, n_2} = K_{n_2, n_1}$.

The *hypercube of dimension n* , denoted Q_n , is the graph with 2^n vertices where each vertex has a label consisting of an n -digit binary number between $0 \dots 0$ and $1 \dots 1$ and with an edge connecting two vertices if and only if the labels of the vertices differ in a single digit. Observe that Q_n has $n \cdot 2^{n-1}$ edges, that $Q_1 = K_2$ and that $Q_2 = C_4$. For a survey on the properties of hypercube graphs see [HHW88].

A connected, acyclic graph is called a *tree*. A tree with n vertices has $n - 1$ edges.

1.2 Planar Graphs

The class of planar graphs has been widely studied, and most of the textbooks mentioned above contain chapters about planar graphs [Har69, BM76, CL86, Tut84, GT87, TS92, Wes96]. A wealth of literature studies properties of planar graphs, algorithms for solving problems on planar graphs, and how close other graphs are to planarity. The latter topic results in algorithms that transform a given graph into a planar graph. These results are briefly summarized in Section 4.2 of the annotated graph drawing bibliography by Di Battista et al. [DETT94].

The book by Nishizeki and Chiba [NC88] is a thorough treatment of planar graphs, with an emphasis on algorithms. [Nis90] can be seen as an update of [NC88]. Johnson [Joh85] surveys the algorithmic complexity of problems on graphs, including problems on planar graphs.

A graph G is said to be *planar* if it admits a drawing such that no two edges contain a common point except possibly a common end vertex. Such a drawing of a planar graph is called a *planar embedding* (or simply an *embedding*) of G . Wagner [Wag36], Fáry [Fár48], and Stein [Ste51] independently showed that every planar graph has an embedding in which the edges are straight line segments. This result also follows from Schnyder's characterization of planarity [Sch89].

Given a planar graph G together with an embedding, each connected subset of the plane that is delimited by a closed curve consisting of vertices and edges of G is called a *face* of the embedding. A face is said to be *incident* to the vertices and edges it is delimited by (and vice versa). All faces except one are bounded subsets of the plane. The unbounded face is called the *outer face*.

Figure 1 shows the nonplanar graph G as well as two planar graphs G_1 and G_2 . The drawing for G_1 is not an embedding, but the drawing for G_2 is. In Figure 2, the graphs G_1 , G_2 , and G_3 are planar, and the drawing given for each of them is an embedding. The embedding for G_1 contains three faces, one incident to four vertices, another incident to five vertices, and a third one (the outer face) incident to seven vertices.

A planar graph together with an embedding is also called a *plane* graph. For a connected plane graph G with n vertices, m edges and f faces, Euler found the following formula:

$$n - m + f = 2 \quad (\text{Euler 1750}) \quad (1)$$

This can be shown by an induction over m (see for example [NC88]). Note that if a planar graph with $n \geq 3$ vertices has as many edges as possible, then each face is incident to exactly three vertices (for otherwise an additional edge could be added, dividing a face that is incident to more than three vertices into two faces, without violating planarity). Euler's formula together with this observation yields the following well known corollary:

$$m \leq 3n - 6 \quad (\text{for } n \geq 3) \quad (2)$$

We now turn our attention to the question of deciding whether a given graph is planar. We first note that we can restrict ourselves to 2-connected graphs as stated by Kelmans [Kel93]: Clearly a graph is planar if and only if each of its connected components is planar. Furthermore, a connected graph is planar if and only if each of its 2-connected blocks is planar. [Kel93] goes on to show that we may even restrict our attention to 3-connected graphs.

First we will give some of the known characterizations of planar graphs. We start with Steinitz's Theorem, relating planar graphs to 3-dimensional convex polytopes. Given a 3-dimensional polytope P , its *edge graph* $G_P = (V_P, E_P)$ is formed as follows. Let V_P be the set of 0-dimensional faces³ of P (i.e. the so-called vertices of P) and let E_P be the set of 1-dimensional faces of P (the so-called edges of P). Recalling that a polytope is convex by definition and that all graphs considered here are simple, Steinitz's Theorem [SR34] can be stated as follows [Whi84, p. 53],[RZ95] :

Theorem 3 (Steinitz 1922) *A graph G is the edge graph of a 3-dimensional polytope if and only if G is planar and 3-connected.*

For a proof, see [Grü67, Chapter 13]. As an example, observe that K_4 is the edge graph of a tetrahedron. The most well known characterization of planar graphs is probably the one by Kuratowski [Kur30, KJ83]:

Theorem 4 (Kuratowski [Kur30]) *A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

The graphs K_5 and $K_{3,3}$ are the complete graph on 5 vertices and the complete bipartite graph on two times three vertices as defined above. A subdivision of K_5 or $K_{3,3}$ that is contained as a subgraph in some graph G is called a *Kuratowski subgraph* of G . A proof of Kuratowski's Theorem can be found in [NC88], for example. The theorem was strengthened by Wagner [Wag37b], and, independently, by Hall [Hal43]. Kelmans [Kel93] states the stronger version as follows:

³Note the difference between the face of a plane graph and the face of a polytope.

Theorem 5 (Wagner [Wag37b], Hall [Hal43]) *A 3-connected graph G distinct from K_5 is planar if and only if it does not contain a subdivision of $K_{3,3}$ as a subgraph.*

Wagner [Wag37a], and, independently, Harary and Tutte [HT65] give another characterization that can be stated in the following way:

Theorem 6 (Wagner [Wag37a], Harary and Tutte [HT65]) *A graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

For further characterizations of planar graphs see for example [Whi33, Mac37], [Sch89, dFdM96] [NC88, BS93, Kel93, ABL95], [dV90, dV93, Sch97], and [TT97].

An algorithm for determining whether a given graph is planar was first developed by Auslander and Parter [AP61] and Goldstein [Gol63]. Hopcroft and Tarjan [HT74] improved it to run in linear time. [Wil80] and [Mut94, p. 39] discuss the development of this result and give additional references. The algorithm tests the planarity of a given graph for each of its 2-connected blocks using the following idea recursively: Let $G = (V, E)$ be 2-connected. Let $T = (V, E')$ be a depth first search tree⁴ of G with root v , and let C be a cycle containing v and consisting of edges from E' plus one edge from $E \setminus E'$. For each edge e of G that is not part of C but that has at least one end vertex in C , consider a certain subgraph G_e of G and test (recursively) whether it can be embedded in the plane with certain edges bordering the outer face. After this has been done for each edge e emanating from C , test whether the embeddings of the different subgraphs G_e can be merged to embed G in the plane. [DETT99, Section 3.3] describes this algorithm in detail, and [Meh84, Section IV.10] additionally shows that it can be implemented in linear time.

This algorithm by Hopcroft and Tarjan tests whether a given graph is planar, but it is not obvious how to extract an embedding for the graph from it, if the graph is planar. Mutzel et al. [MMN93, MM96] modified the planarity testing algorithm to then also yield a *combinatorial embedding* of the graph in linear time, i.e. for each vertex a cyclic list of the incident edges so that the graph can be embedded in the plane obeying these edge sequences. Given a combinatorial embedding of a planar graph G with n vertices, de Fraysseix et al. construct a straight line embedding of G on a grid of size $2n - 4$ by $n - 2$ in time $\mathcal{O}(n \log n)$ [dFPP90]. This result was improved to a linear time algorithm finding a straight line embedding on a grid of size $n - 2$ by $n - 2$ by Schnyder [Sch90a]. See [DETT94, Section 5][DETT99, Chapter 4] for further discussions on drawing planar graphs.

Another linear time planarity testing algorithm was developed by Lempel, Even, and Cederbaum [LEC67]. They define an *st-numbering* as follows: Let $G = (V, E)$ be a 2-connected graph, and let $\{s, t\} \in E$ be an edge of G . An *st-numbering* is a bijection $f : V \rightarrow \{1, 2, \dots, |V|\}$ such that $f(s) = 1$, $f(t) = |V|$, and such that for every $v \in V \setminus \{s, t\}$ there are vertices u and w in V with $\{u, v\} \in E$, $\{v, w\} \in E$, and $f(u) < f(v) < f(w)$.

[LEC67] show that an *st-numbering* always exists. The idea of the planarity testing algorithm is this: For a 2-connected graph G , compute an *st-numbering*, and then try to build up a planar graph by starting with the vertex with *st-number* 1 and by adding the vertices of G together with their incident edges one by one according to their ascending *st-numbers*.

Even and Tarjan [ET76] showed that an *st-numbering* can be computed in linear time using depth first search. Using this result, and introducing a data structure called *PQ-*

⁴For a description of depth first search, see for example [Meh84, Sections IV.4 and IV.5] or [TS92, Chapter 11.7].

trees, Booth and Lueker [BL76] improved Lempel, Even, and Cederbaum’s planarity testing algorithm to run in linear time.

The algorithm was modified to also yield a combinatorial embedding for the graph if it is planar by Chiba et al. [CNAO85]. [Eve79, Section 8.4] and [TS92, Section 11.11] describe the original algorithm [LEC67], and [Kan93, Section 2.2.2] describes the implementation [BL76] using PQ -trees.

1.3 Generalizations of Planarity

Just as planar graphs are graphs embeddable in the 2-dimensional plane, we can consider graphs embeddable in other surfaces. By *surface* we mean a topological space that is a compact 2-manifold. A surface is characterized by its property of being either *orientable* or *nonorientable*, and by its *genus* g . The *sphere* is the most simple orientable surface. It has genus 0. Informally speaking, the orientable surface S_g of genus $g \geq 0$ is the sphere with g handles attached to it. So S_0 denotes the sphere itself, whereas S_1 is also known as the *torus*. For the orientable surface S_g , the *Euler characteristic* of S_g is defined to be $E(S_g) = 2 - 2g$. See [WB78] and [Whi84, Chapters 5 and 6] for precise definitions and further explanations, in particular for the nonorientable case.

Note that the 2-dimensional plane is not compact, so it is not a surface in the above sense. But embedding a graph in the plane is equivalent to embedding it in the sphere (see [Whi84, Chapter 5] or [NC88, Section 1.3], for example).

The orientable (nonorientable) genus g of a graph G is defined to be the smallest g so that G can be embedded in an orientable (nonorientable) surface of genus g . It is NP-hard to determine the genus of a given graph [Tho89]. [DR91] provides an algorithm to determine the orientable genus of a graph. The running time of the algorithm is superexponential in the genus. Given an arbitrary but fixed surface S , [Moh96] presents a linear time algorithm that, for a given graph G , either finds an embedding of G in S , or finds a minimal forbidden subgraph H of G that cannot be embedded in S .

Besides considering different surfaces in which to embed a graph, further generalizations of planarity result when weaker forms of embedding a graph in a surface are considered. Graphs that can be drawn in a surface S so that each edge is involved in at most k edge crossings are called *k -embeddable in S* . So planar graphs are precisely the 0-embeddable graphs in the plane. [Sch90b] and [PT97] study 2-embeddable and k -embeddable graphs in the plane, respectively.

Considering graphs that can be drawn in the plane so that there are no k pairwise crossing edges, we get the planar graphs for $k = 2$. [AAP⁺96] shows that for graphs with no three pairwise crossing edges and n vertices, the number of edges is in $O(n)$, and calls such graphs *quasi-planar*. For general k , see also [PSS96] and [Val97, Val98] for recent work and further references.

Chen et al. [CGP98] study intersection graphs of planar regions with disjoint interiors and call them *planar map graphs*. This generalizes planar graphs since planar graphs may be defined as the intersection graphs of planar regions with disjoint interiors such that no four regions meet at a point.

Yet another way of generalizing the concept of planarity is to weaken the characterizations of planarity that involve the Kuratowski graphs, (subdivisions of) K_5 and $K_{3,3}$, as subgraphs or minors of a graph. The result are four classes of graphs: Graphs that do not contain K_5 as a minor (or that do not contain a subdivision of K_5 as a subgraph) have been studied, and

similarly for $K_{3,3}$ (see for example [Bar83, Khu90, KM92, NP94, MP95, Che96, JMOS98]).

2 Vertex Deletion

Given a graph $G = (V, E)$, we can transform it into a planar graph $G' = (V', E')$ in a trivial way by deleting all but four vertices of V from G together with all their incident edges. G' is then a tetrahedron (K_4) or a subgraph thereof, and hence planar. But we would hope to retain more than four vertices of the original graph and still obtain a planar subgraph. This section investigates the question of deleting as few vertices as possible (together with their incident edges) from a given graph G to make it planar. It seems that deleting vertices is too drastic an operation on a given graph to be useful in practice. The author is only aware of few results investigating vertex deletion for planarization.

Definition 7 (maximum induced planar subgraph) *If a graph $G' = (V', E')$ is an induced planar subgraph of a graph $G = (V, E)$ such that there is no induced planar subgraph $G'' = (V'', E'')$ of G with $|V''| > |V'|$, then G' is called a maximum induced planar subgraph of G .*

So the problem of deleting as few vertices as possible from a graph so that the resulting graph is planar means to find, for a given graph G , a maximum induced planar subgraph of G .

Problem 8 (Maximum Induced Planar Subgraph [GJ79, Problem GT21]) *Given a graph $G = (V, E)$ and a positive integer $K \leq |V|$, is there a subset $V' \subseteq V$ with $|V'| \geq K$ such that the subgraph of G induced by V' is planar?*

Lewis and Yannakakis [LY80] showed that this problem is NP-complete. [LY80] is based on independent work by the two authors and actually shows a far more general result:

Theorem 9 [LY80] *If Π is a graph property satisfying the following conditions*

1. *There are infinitely many graphs for which Π holds.*
2. *There are infinitely many graphs for which Π does not hold.*
3. *If Π holds for a graph G and if G' is an induced subgraph of G , then Π holds for G' .*

then the following problem is NP-complete: Given a graph $G = (V, E)$ and a positive integer $K \leq |V|$, is there a subset $V' \subseteq V$ with $|V'| \geq K$ such that Π holds for the subgraph of G induced by V' ?

Note that the graph property of being planar satisfies the three conditions of Theorem 9. Independently from Lewis and Yannakakis, Krishnamoorthy and Deo [KD79] also showed the NP-completeness of a whole range of vertex deletion problems including the maximum induced planar subgraph problem.

Djidjev and Venkatesan [DV95] show that for a graph G with n vertices and with orientable genus g , there exists a set of $4\sqrt{gn}$ vertices whose removal planarizes G , and that the size of this planarizing vertex set is optimal up to a constant factor. The proof is constructive and can be transformed into an $\mathcal{O}(n+g)$ time algorithm to find such a planarizing vertex set if the graph G is given together with an embedding on an orientable surface of genus g . But recall that it is NP-hard to determine the genus of a given graph [Tho89]. [DV95] goes on to show that

if no such embedding of the graph is given, a planarizing vertex set of size $\mathcal{O}(\sqrt{gn \log(2g)})$ can be found in time $\mathcal{O}(n \log(2g))$. This algorithm recursively uses a graph partitioning algorithm also by Djidjev [Dji85]. However, no indications of computational studies or existing implementations are given. [DV95] improves results of [Dji84] and [HM87, Hut89], and also considers the nonorientable case.

Since Maximum Induced Planar Subgraph is an NP-complete problem, we also consider an easier problem:

Definition 10 (maximal induced planar subgraph) *If a graph $G' = (V', E')$ is an induced planar subgraph of a graph $G = (V, E)$ such that every subgraph of G induced by a vertex set $V'' = V' \cup \{v\}$ with $v \in V \setminus V'$ is nonplanar, then G' is called a maximal induced planar subgraph of G .*

For a given graph G we want to find a maximal induced planar subgraph. Note that every maximum induced planar subgraph is also a maximal induced planar subgraph, but not vice versa. Observe that a maximal induced planar subgraph is maximal with respect to inclusion of its vertex set, whereas a maximum induced planar subgraph is maximal with respect to the cardinality of its vertex set. Analogous definitions concerning the edge set will be used in Section 3. Figure 1 illustrates maximal and maximum induced planar subgraphs.

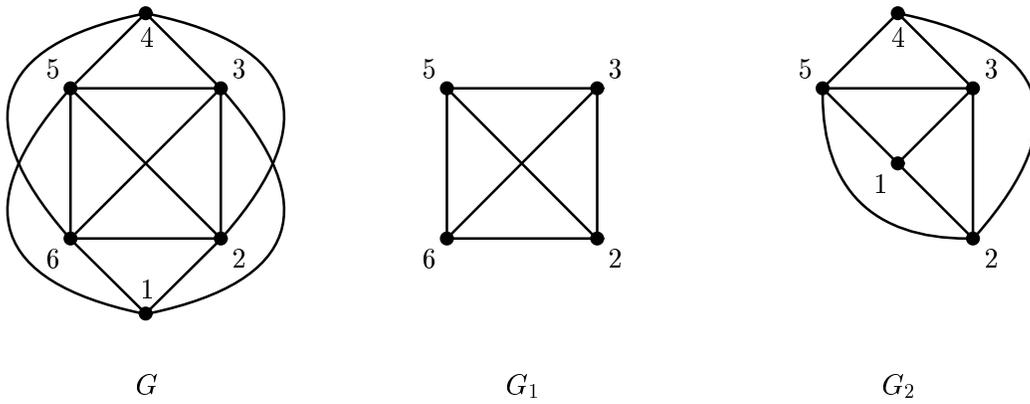


Figure 1: G is a nonplanar graph (note that G contains K_5 as a subgraph). G_1 is a maximal induced planar subgraph of G . G_2 is also a maximum induced planar subgraph of G .

A straightforward way of finding, for a given graph G with n vertices and m edges, a maximal induced planar subgraph is the Greedy Algorithm: The input is a graph $G = (V, E)$ with n vertices and m edges. The output is a maximal induced planar subgraph $G' = (V', E')$ of G . We start with G' as the empty graph (so $V' = \emptyset$ and $E' = \emptyset$). One vertex of V after the other is taken and either added to V' (if the subgraph of G induced by V' remains planar) or discarded, until every vertex of V has been considered. The order in which the vertices of V are considered is arbitrary. Considering the worst case time complexity of this algorithm, we have to perform a planarity test and to update V' and E' in each of the n iterations. Each planarity test takes $\mathcal{O}(n + m)$ time in the worst case. Each update of V' takes $\mathcal{O}(1)$ time. All updating operations for E' together take $\mathcal{O}(m)$ time. Thus the overall time complexity of the Greedy Algorithm is in $\mathcal{O}(n \cdot m)$ (assuming that G is connected, so that $m \in \Omega(n)$). The

resulting vertex set V' usually depends on the order in which the vertices of V are considered. However, the author is not aware of work investigating the impact of different vertex orderings on the resulting maximal induced planar subgraph.

3 Edge Deletion and Skewness

If a graph $G = (V, E)$ with an edge $e \in E$ is transformed into a graph $G' = (V, E \setminus \{e\})$ then we say that G' was obtained from G by *edge deletion*. By repeatedly deleting edges from a given nonplanar graph G , G can be transformed into a planar graph G' . In this section, we are interested in planarizing G by deleting as few edges as possible.

Deleting edges from a given graph G in order to transform G into a graph G' with a particular property is a common approach (see for example [SC89, Sen90]). We will only discuss edge deletion with the purpose of planarization, a topic that has been studied intensively.

Definition 11 (maximum planar subgraph, skewness) *If a graph $G' = (V, E')$ is a planar subgraph of a graph $G = (V, E)$ such that there is no planar subgraph $G'' = (V, E'')$ of G with $|E''| > |E'|$, then G' is called a maximum planar subgraph of G , and the number of deleted edges, $|E| - |E'|$, is called the skewness of G .*

So the skewness of a graph G is 0 if and only if G is planar. The problem of finding, for a given graph G , a maximum planar subgraph is NP-hard [LG79]. It will be discussed in Section 3.1. For some graph classes, the skewness is known: The complete graph K_n has $n(n-1)/2$ edges. For $n \geq 3$, it has a planar subgraph with $3n-6$ edges. Since a planar graph with $n \geq 3$ vertices cannot have more than $3n-6$ edges (Equation 2), the skewness of the complete graph K_n is $n(n-1)/2 - (3n-6) = (n-3)(n-4)/2$ for $n \geq 3$. A similar argument shows that the skewness of the complete bipartite graph K_{n_1, n_2} is $n_1 \cdot n_2 - 2(n_1 + n_2) + 4$ for $n_1 \geq 2$ and $n_2 \geq 2$. The skewness of the hypercube of dimension n , Q_n , is $2^n(n-2) - n \cdot 2^{n-1} + 4$ [Cim92].

Definition 12 (maximal planar subgraph) *If a graph $G' = (V, E')$ is a planar subgraph of a graph $G = (V, E)$ such that every graph $G'' \in \{(V, E' \cup \{e\}) \mid e \in E \setminus E'\}$ is nonplanar, then G' is called a maximal planar subgraph of G .*

In other words a maximal planar subgraph is maximal with respect to inclusion of its edge set, whereas a maximum planar subgraph is maximal with respect to the cardinality of its edge set. Observe that every maximum planar subgraph is also a maximal planar subgraph, but not vice versa. Also note the analogy with Definitions 7 and 10 concerning the vertex set of a graph. Figure 2 illustrates maximal and maximum planar subgraphs.

Finding a maximum planar subgraph is an NP-hard problem, and Section 3.1 discusses this result. But a maximal planar subgraph can be found in polynomial time, as will be seen in Section 3.2. Finally, Section 3.3 discusses approximative and heuristic approaches for finding a large planar subgraph. It also considers the weighted version, where edges are assigned nonnegative edge weights, and the goal is to find a planar subgraph with total edge weight as large as possible.

For another survey of algorithms for planarization through edge deletion, see Mutzel [Mut94, Chapter 5].

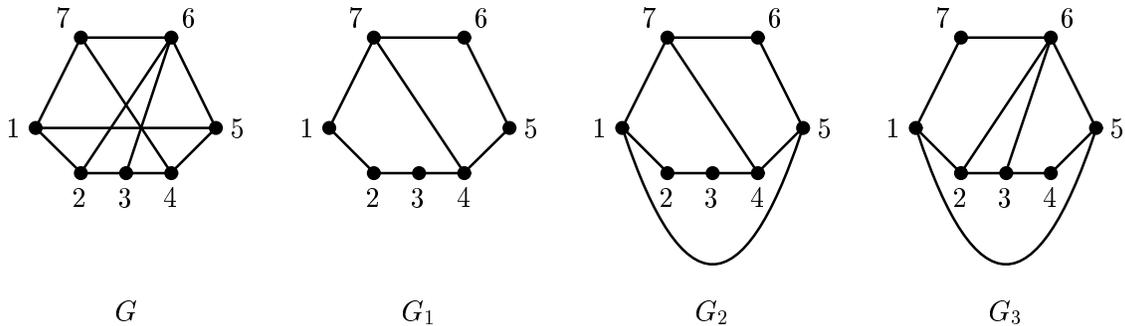


Figure 2: G is a nonplanar graph. Note that G contains $K_{3,3}$ as a minor (contract edge $\{2,3\}$). G_1 is a planar subgraph of G , but it is not a maximal planar subgraph: Edge $\{1,5\}$ can be added to G_1 without destroying planarity. The result is G_2 . Another maximal planar subgraph of G is G_3 . G_3 is also a maximum planar subgraph.

3.1 Finding a Maximum Planar Subgraph

In this section, we study the following problem:

Problem 13 (Maximum Planar Subgraph [GJ79, Problem GT27]) *Given a graph $G = (V, E)$ and a positive integer $K \leq |E|$, is there a subset $E' \subseteq E$ with $|E'| \geq K$ such that the graph $G' = (V, E')$ is planar?*

Liu and Goldmacher [LG79], and, independently, Yannakakis [Yan78]⁵, and, also independently, Watanabe et al. [WAN83],⁶ showed that this problem is NP-complete. The proof of Liu and Goldmacher is a two step reduction using the following problems:

Problem 14 (Vertex Cover [GJ79, Problem GT1])⁷ *Given a graph $G = (V, E)$ and a positive integer $K \leq |V|$, is there a vertex cover of size K or less for G , i.e. is there a subset $V' \subseteq V$ of vertices with $|V'| \leq K$ such that for each edge $uv \in E$ at least one of its end vertices u and v belongs to V' ?*

Problem 15 (Hamilton Path in Graphs Without Triangles) *Given a graph $G = (V, E)$ that does not contain a cycle of length 3, and given two vertices $u \in V$ and $v \in V$, does G contain a Hamilton path from u to v ?*

Karp [Kar72] shows Vertex Cover to be NP-complete. [LG79] first reduces Vertex Cover to Hamilton Path in Graphs Without Triangles, and then reduces this problem to Maximum Planar Subgraph. Recently, Faria, Figueiredo, and Mendonça [FFM98a] have shown that Maximum Planar Subgraph is even NP-complete for cubic graphs (see Section 4.2).

Djidjev and Venkatesan [DV95] show that for a graph G with m edges, maximum vertex degree Δ , and orientable genus g , there exists a set of $4\sqrt{\Delta gm}$ edges whose removal planarizes G , and that the size of this planarizing edge set is optimal up to a constant factor. If G is

⁵[Yan78] shows the NP-completeness of several edge deletion problems. The journal version [Yan81] does not contain the NP-completeness proof for the maximum planar subgraph problem anymore but refers to [LG79], which uses similar ideas as presented in [Yan78, Yan81].

⁶Watanabe et al. [WAN83] show the NP-completeness for a whole class of edge deletion problems.

⁷Liu and Goldmacher call this problem Vertex Edge-Cover.

connected and has n vertices, then there exists a set of $4\sqrt{2\Delta g(n+2g-2)}$ edges whose removal planarizes G . The proofs are constructive and can be transformed into $\mathcal{O}(n+g)$ time algorithms to find such planarizing edge sets if the graph G is given together with an embedding on an orientable surface of genus g . But recall that it is NP-hard to determine the genus of a given graph [Tho89]. [DV95] states that if no such embedding of the graph is given, a planarizing edge set of size $\mathcal{O}(\sqrt{\Delta gm \log(2g)})$ can be found in time $\mathcal{O}(m \log(2g))$.

[DV95] improves results in [Dji84]. No indications of computational studies or existing implementations for these algorithmic results are given though. For the corresponding results concerning planarizing vertex sets see Section 2.

3.2 Finding a Maximal Planar Subgraph

The problem of finding a maximal planar subgraph for a given graph G with n vertices and m edges is solvable in polynomial time. A straightforward way of finding a maximal planar subgraph is the Greedy Algorithm: The input is a graph $G = (V, E)$ with n vertices and m edges. The output is a maximal planar subgraph $G' = (V, E')$ of G . We start with $G' = (V, \emptyset)$ and build up E' by considering one edge e of E after the other. For each $e \in E$, e is added to E' if $G' = (V, E')$ remains planar, and discarded otherwise. We stop either after all edges of E have been considered, or when $|E'|$ becomes equal to $3n - 6$ (since a planar graph cannot have more than $3n - 6$ edges). For each edge of E that is considered we need to perform a planarity test for a graph with n vertices and at most $3n - 6$ edges. Each planarity test takes linear time, i.e. $\mathcal{O}(n)$ in the worst case. The remaining operations like updating E' take $\mathcal{O}(1)$ time per edge. Thus the worst case time complexity is in $\mathcal{O}(n \cdot m)$.

The standard algorithms for planarity testing [HT74, BL76] are rather complicated to implement. Therefore, algorithms for finding a maximal planar subgraph are sought that not only have a better worst case time complexity than the algorithm described above, but that are also less involved.

T. Chiba, Nishioka, and Shirakawa [CNS79] propose an algorithm based on the planarity testing algorithm [HT74]. They achieve a worst case time complexity of $\mathcal{O}(n \cdot m)$, the same as that of the Greedy Algorithm.

A whole series of results about better polynomial time algorithms for finding a maximal planar subgraph starts with [OT81], which also claims to give an $\mathcal{O}(n \cdot m)$ algorithm. In contrast to [CNS79], [OT81] is based on the planarity testing algorithm [LEC67, BL76] using PQ -trees. The algorithm starts with one vertex as the initial planar subgraph and then adds one vertex (together with as many of its incident edges as possible) at a time. But [TJS86] points out that the subgraph generated by this algorithm is not always a maximal planar subgraph, and that it is not even always a spanning subgraph. [JTS89, JST89] claim to amend the problem and give two $\mathcal{O}(n^2)$ algorithms, one to find a spanning planar subgraph of a 2-connected graph G , and one to find a maximal planar subgraph by augmenting the previously found spanning planar subgraph. The latter algorithm is shown to be incorrect by [Kan92, Kan93], claiming to show how to correct the algorithm. [Lei94, JLM97] in turn point out that the result in [Kan92] is not correct either and discuss the difficulties of using PQ -trees.

Di Battista and Tamassia [DT89, DT90] define and use $SPQR$ -trees to describe the recursive decomposition of a 2-connected graph into its 3-connected components. [DT89] obtains an $\mathcal{O}(m \log n)$ time algorithm for finding a maximal planar subgraph as a byproduct of an algorithm for *incremental planarity testing*. An incremental (or *dynamic*) planarity testing

algorithm maintains a data structure representing a planar graph $G = (V, E)$ and can handle requests of the following types: a) For two vertices v_1 and v_2 in G with $v_1v_2 \notin E$, determine whether G stays planar if the edge v_1v_2 is added to G . b) If $v_1 \in V$, $v_2 \in V$, $v_1v_2 \notin E$, add the edge v_1v_2 to G (assuming the corresponding request of type a) yields a positive answer). c) Add a new vertex to G .

Independently, Cai, Han, and Tarjan [CHT93] developed an $O(m \log n)$ algorithm to find a maximal planar subgraph of a graph G with n vertices and m edges. Their algorithm is based on a new version of the Hopcroft and Tarjan planarity testing algorithm [HT74].

La Poutré [La 94] presents algorithms for incremental planarity testing that yield an $O(n + m \cdot \alpha(m, n))$ time algorithm for the maximal planar subgraph problem (where $\alpha(m, n)$ is the functional inverse of the Ackermann function). Djidjev [Dji95], and, independently, Hsu [Hsu95] give linear time algorithms to find a maximal planar subgraph.

This result was recently improved to linear time complexity by Djidjev [Dji95], and, independently, by Hsu [Hsu95].

Given a graph $G = (V, E)$, Djidjev [Dji95] first computes a depth first search tree of G . This spanning tree of G is the initial planar subgraph $G' = (V, E')$ of G . Then for each edge $e \in E \setminus E'$ it is determined whether the graph $(V, E' \cup \{e\})$ is still planar. If so, e is added to E' . The order in which the edges in $E \setminus E'$ are considered is chosen in a sophisticated way so that, with $O(1)$ amortized time per test and insert operation for each edge $e \in E \setminus E'$, the overall time complexity is linear. Many intricate data structures are needed to achieve the $O(1)$ amortized time per test and insert operation. Two of them are *BC*-trees to describe the decomposition of a connected planar graph into its 2-connected components⁸ and *SPQR*-trees to describe the decomposition of a 2-connected graph into its 3-connected components [DT89]. Djidjev's algorithm is linear and therefore asymptotically best possible. However, it is so involved that a linear implementation seems difficult to achieve.

[Hsu95] also starts with a depth first search tree of the given graph $G = (V, E)$, and then determines a postordering of the vertices of G . The postordering is a labeling $l : V \rightarrow \{1, \dots, n\}$ so that if u is an ancestor of v in the depth first search tree, then $l(u) > l(v)$. The initial planar subgraph G' of G is empty, and the vertices are added in ascending order of their labels. So in step i of the algorithm, the vertex with label i (and the edges incident to it) are added to G' . Note that G' is not necessarily connected at all times. According to [Hsu95], the way in which the vertices are added and in which for each edge it is decided whether the edge can be added to G' without destroying planarity ensures the construction of a maximal planar subgraph in linear time. So the algorithm also achieves the asymptotically best possible time complexity, and it appears to be less complicated than that of Djidjev. However, the conference version [Hsu95] does not contain the details of the algorithm and of the proof of its correctness.

3.3 Approximations and Heuristics

First consider a trivial approximation for finding a maximum planar subgraph (i.e. for the unweighted case) by observing that for a given graph G with n vertices, any spanning tree of G is a planar subgraph with $n - 1$ edges (assume that G is connected), and that a spanning tree can be found in linear time. Furthermore, a planar subgraph of G cannot have more than $3n - 6$ edges (see Equation 2). So if E' is the edge set of a spanning tree for a given

⁸In [Har69] these trees are called block-cutpoint trees.

graph G , and if E^* is the edge set of a maximum planar subgraph of G , then the ratio $\frac{|E'|}{|E^*|}$ is bounded (see also [Cim92]):

$$\frac{|E'|}{|E^*|} = \frac{n-1}{|E^*|} \geq \frac{n-1}{3n-6} > \frac{1}{3} \quad (16)$$

This bound was improved for the first time by Călinescu et al.: [CFFK96] provide approximations with a lower bound of $\frac{2}{5}$.

Let's turn our attention to the weighted version of the Maximum Planar Subgraph Problem:

Problem 17 (Weighted Maximum Planar Subgraph) *Given a graph $G = (V, E)$ with a nonnegative edge weight $w(e)$ for each edge e , and a positive number K , is there a subset $E' \subseteq E$ with $\sum_{e \in E'} w(e) \geq K$ such that the graph $G' = (V, E')$ is planar?*

Being a generalization of the Maximum Planar Subgraph Problem, Weighted Maximum Planar Subgraph is NP-complete as well.

The Greedy Algorithm of Section 3.2 finds a maximal planar subgraph, which will be at least as good as just taking a spanning tree. [KH78, DFF85, FGG85] use this greedy approach for a heuristic for the Weighted Maximum Planar Subgraph problem: Instead of considering the edges in arbitrary order, they consider them in the order of nonincreasing weight. This Greedy Heuristic does involve repeated planarity testing, and even though planarity testing can be done in linear time, the algorithms are rather complicated. The following heuristics avoid planarity testing.

The Deltahedron Heuristic [FR78, FGG85] starts with a tetrahedron (K_4) as the initial planar subgraph and then adds one vertex at a time, placing each new vertex in one of the faces of the current planar subgraph (see part a) of Figure 5 on page 17 for an illustration). The sequence in which the vertices are added is determined by a vertex weight W that can be defined in various ways, as discussed below. Figure 3 shows the Deltahedron Heuristic in detail. Note that in contrast to the Greedy Heuristic, the Deltahedron Heuristic does not necessarily yield a maximal planar subgraph of the input graph. [FGG85] assigns the vertex weights as the sum of the weights of incident edges: $W(v) = W_{sum}(v) = \sum_{u \in V} w(uv)$. [DFF85] suggests to use the maximum of the vertex weights instead of the sum: $W(v) = W_{max}(v) = \max_{u \in V} \{w(uv)\}$, and also provides a worst case analysis for the performance of the Greedy Heuristic and the two versions of the Deltahedron Heuristic.

Definition 18 (worst case ratio) *Let P be an instance of the Weighted Maximum Planar Subgraph Problem with a graph $G = (V, E)$ and positive edge weights $w(e)$ for $e \in E$. If A is an algorithm that finds a planar subgraph $G' = (V, E')$ of G , and if $E^* \subseteq E$ is an optimal edge set, i.e. if $G^* = (V, E^*)$ is planar, and if $w(E^*) = \sum_{e \in E^*} w(e)$ is as large as possible, then the worst case ratio, denoted ρ_A , is defined as*

$$\rho_A = \inf_P \frac{w(E')}{w(E^*)} .$$

Clearly $\rho_A \leq 1$ for any algorithm A . The closer ρ_A is to 1, the better A performs (in the worst case).

[DFF85] shows that the Deltahedron Heuristic with vertex weights W_{sum} can be arbitrarily bad in the general case but has a performance guarantee in the unweighted case (i.e. if the

Deltahedron Heuristic

Input: A graph $G = (V, E)$ and nonnegative real edge weights $w(e)$ for $e \in E$

Output: A planar subgraph $G' = (V, E')$ of G

- 1 Build a complete graph $G_K = (V, E_K)$ from G by adding an edge $e = uv$ with $w(e) = 0$ for each pair uv of nonadjacent vertices in V .
Let $EE = E_K \setminus E$ be this set of “extra” edges.
- 2 Assign a vertex weight $W(v)$ to each $v \in V$.
- 3 Sort the vertices by vertex weight in nonincreasing order and let a, b, c , and d be the first four vertices in that order.
- 4 Let G' be the tetrahedron on a, b, c , and d ,
i.e. let G' have edge set $E' = \{ab, ac, ad, bc, bd, cd\}$.
Let $T = \{abc, acd, abd, bcd\}$ be the set of faces of G' .
By construction, each face of G' is a triangle.
- 5 As long as there is a vertex in V that has not yet been added to G' :
- 6 Let v be a vertex with largest weight among those not yet in G' .
- 7 Choose a face $xyz \in T$ such that
 $w(xv) + w(yv) + w(zv)$ is as large as possible.
- 8 Add v to the face xyz , i.e. set $E' = E' \cup \{xv, yv, zv\}$,
 and set $T = (T \setminus \{xyz\}) \cup \{xyv, yzv, zxv\}$.
(G' is now a triangulated graph with n vertices and $3n - 6$ edges.)
- 9 If E' contains “extra” edges from EE , eliminate them from E' .

Figure 3: Deltahedron Heuristic for finding a planar subgraph with large edge weights.

edge weights are restricted to 0 and 1). But for the unweighted case, the worst case ratio of the trivial approximation using a spanning tree (Equation 16) is higher anyway.

The Deltahedron Heuristic with vertex weights W_{max} and the Greedy Heuristic both have performance guarantees in the general case. Figure 4 lists the results presented in [DFF85]. They show that the Greedy Heuristic is the best algorithm as far as worst case analysis is concerned. Figure 4 also lists the time complexities of the above algorithms as given in [FGG85].

[FGG85] suggests improving the result of the Deltahedron Heuristic by edge replacement or vertex relocation operations in a postprocessing phase and additionally discusses the wheel expansion approach of [EFG82].

To compare the performance of these heuristics, computational results are reported in detail in [FGG85]. Complete graphs with 10, 20, 30, and 40 vertices and with a normal distribution on the edge weights with mean value 100 and standard deviations in the range from 5 to 30 are generated, and planar subgraphs of these are constructed with the Deltahedron Heuristic, the improved Deltahedron Heuristic, the wheel expansion heuristic (each with vertex weights W_{sum} and each in two versions depending on the choice of the initial K_4), and the Greedy Heuristic). For each one of the altogether 102 planar subgraphs constructed, the performance ratio of the respective algorithm was never below 0.87, where the improved Deltahedron Heuristic with vertex weights W_{sum} ⁹ and the Greedy Heuristic outperformed the other heuristics, and the Greedy Heuristic was better than the improved Deltahedron Heuristic. The performance ratio for the Greedy Heuristic was never below 0.91. The Greedy Heuristic, however, required 5 to 10 times the run time of any of the other heuristics.

algorithm A	worst case ratio ρ_A	worst case time complexity for an input graph with n vertices
Greedy Heuristic	$\frac{1}{3}$	$O(n^3)$
Deltahedron Heuristic with vertex weights W_{sum}	0	$O(n^2)$
Deltahedron Heuristic with vertex weights W_{sum} in the unweighted case	$\frac{1}{6} \leq \rho_A \leq \frac{2}{9}$	
Deltahedron Heuristic with vertex weights W_{max}	$\frac{1}{6}$	

Figure 4: The results of [DFF85] show the worst case performance of three algorithms for finding a planar subgraph with a large sum of edge weights. The time complexity of the algorithms is given in [FGG85].

Besides the worst case analysis mentioned above, [DFF85] also analyses a simplification of

⁹There are no computational results for the Deltahedron Heuristic with vertex weights W_{max} .

the Deltahedron Heuristic: The vertices are considered in arbitrary order instead of in order of nonincreasing vertex weights. In the worst case, this heuristic can be arbitrarily bad, even in the unweighted case. But [DFF85] shows that under the assumption that the edge weights are independent and that they are chosen from a probability density restricted to a bounded interval of the nonnegative reals, as the number n of vertices tends to infinity, the probability that the performance ratio $\frac{w(E')}{w(E^*)}$ is below $1 - n^{-0.1}$ tends to zero.

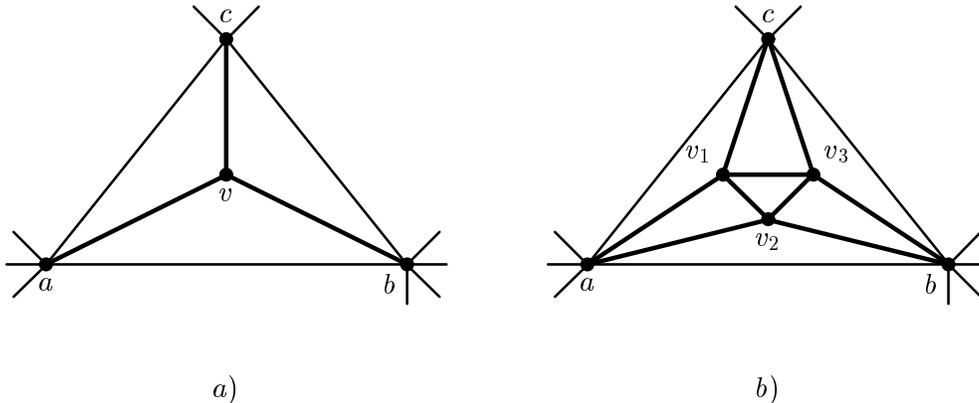


Figure 5: A step in the Deltahedron Heuristic for finding a planar subgraph with large edge weights [FR78, FGG85, DFF85] (operation a), see also Figure 3), or in its generalization [Leu92] (operation a) or b)). In a), vertex v and 3 incident edges are inserted into face abc . In b), vertices v_1, v_2, v_3 and 9 incident edges are inserted into face abc .

Leung [Leu92] generalizes the Deltahedron Heuristic. Starting with a tetrahedron (K_4), a planar subgraph is built such that in each step, the current planar subgraph has only triangular faces. In each step, a single vertex and three incident edges (as in the Deltahedron Heuristic) or a set of three vertices and nine incident edges are placed in one of the faces of the current planar subgraph as illustrated in Figure 5. Unlike in the Deltahedron Heuristic, the vertices to be inserted are not chosen in any predetermined ordering, but in each step the vertex or the set of three vertices, and the face into which to insert them, is determined so that the gain in edge weights per inserted vertex in this step is best possible. The worst case time complexity of this approach is $O(n^4 \log n)$. Computational results are carried out, generating the test base in much the same way as [FGG85]. They suggest that the results of the generalized Deltahedron Heuristic are better than the ones achieved by the original Deltahedron approach discussed in [FR78, FGG85], but there is no comparison with the improved Deltahedron Heuristic of [FGG85] or with the Greedy Heuristic.

A completely different approach is taken by Jünger and Mutzel, who use a heuristic based on polyhedral combinatorics within a branch and cut framework [Mut94, JM96b].

[JM96b] reports computational results where the branch and cut heuristic was applied to various graphs known from the literature with 10 to 100 vertices. In many cases, a provably optimal solution, or at least a solution that is better than the previously known one, could be found. But the running time needed is usually significantly longer than the running time of other algorithms. In fact, Jünger and Mutzel interrupt their algorithm when a time limit of 1000 CPU seconds is reached. They find that the easiest problem instances are sparse graphs

and very dense graphs, and that for weighted graphs the performance of their branch and cut heuristic is much worse than for unweighted graphs.

For the unweighted case (i.e. all edge weights are 1) there are still other approaches. Cimikowski [Cim92] suggests a heuristic based on spanning trees. Suppose a graph G with n vertices and m edges is 2-connected and has two edge disjoint spanning trees whose union is planar. Then this union forms a planar subgraph and has $2n - 2$ edges. If the graph does not have two such spanning trees, some heuristic edge manipulations are performed, so that the output is still a spanning planar subgraph, but without a guaranteed number of edges. If two spanning trees exist, they can be found in $O(m^2)$ [RT85].

Takefuji and Lee [TL89, TLC91] and Goldschmidt and Takvorian [GT94] each propose a two-phase heuristic for finding a planar subgraph with as many edges as possible. In the first phase, a linear ordering of the vertices is determined. The vertices are placed on a line according to that ordering. In the second phase edges are placed above or below the line. The resulting planar subgraph is thus embedded in a book with two pages. The techniques used for each phase are very different in [TL89] and [GT94]. [TL89] places the vertices in a random order in the first phase and uses a neural network technique for the second phase.

[GT94] argues that it is useful to attempt to order the vertices of the input graph $G = (V, E)$ according to a Hamiltonian cycle in the first phase. Given an ordering of the vertices on a line, in the second phase three edge sets $A \subseteq E$, $B \subseteq E$, and $C \subseteq E$ must be determined so that $|A| + |B|$ is as large as possible, and so that no two edges of A (B) intersect if all edges of A (B) are placed above (below) the line of vertices. The edges in C are not part of the planar subgraph. If we imagine the vertices of G to lie on the real line, then each edge $e \in E$ can be regarded as an interval defined by its two end vertices. Let $H = (E, F)$ be a graph such that each edge of G is a vertex of H . Let e_1, e_2 be two edges of G and thus two vertices of H , and let i_1 and i_2 be the intervals corresponding to the edges e_1 and e_2 in G . e_1 and e_2 are connected by an edge in H if and only if the intervals i_1 and i_2 intersect but none is contained in the other. Thus H is an overlap graph (also called circle graph). Finding the sets A , B and C as described above is now equivalent to finding a maximum induced bipartite subgraph of the overlap graph H . Finding a maximum induced bipartite subgraph of an overlap graph is NP-complete [SL89].

[GT94] now uses the following greedy algorithm to construct a maximal induced bipartite subgraph of an overlap graph: Find a maximum independent vertex set in H (the vertices of this set are then the edges in A), delete it from H , and find a maximum independent set in the remaining graph (the vertices of this set are then the edges in B). Since a maximum independent set of an overlap graph can be found in polynomial time [Gav73], this algorithm runs in polynomial time also. [GT94] shows that the number of vertices in the maximal induced bipartite subgraph is at least 0.75 times the number of vertices of a maximum bipartite subgraph.

Computational results reported by Goldschmidt and Takvorian [GT94] compare their implementation of their heuristic with their implementation of [TL89] on a set of 19 graphs with 10 to 150 vertices and two larger graphs with 300 and 1000 vertices, respectively. For each instance, their heuristic finds at least as good a solution as [TL89]. For the graphs with 50 or more vertices, the solution of [GT94] is even dramatically better than that of [TL89]. But note that the test base is small, that it is unclear how representative it is, and that even the results of [GT94] might still be very far away from an optimal solution.

The approach of [GT94] is further refined by Resende and Ribeiro [RR97]. They apply a greedy randomized adaptive search procedure (GRASP), a metaheuristic for combinatorial

optimization [FR95], to the problem of planarizing a graph through edge deletion. Experimental results using most graphs from the test base in [GT94] as well as graphs with up to 300 vertices collected by Cimikowski are discussed in [RR97, RR98]. They indicate that the GRASP compares favorably with the results of [GT94]. In comparison with the branch and cut heuristic [Mut94, JM96b], however, the situation is not so clear: On some instances the branch and cut heuristic is clearly better, on others the GRASP outperforms the branch and cut heuristic. The latter happens in particular when the time limit set for the branch and cut heuristic is reached so that the computation is halted and the best solution found until then is reported.

Cimikowski also presents results comparing different heuristics [Cim95a].

For algorithmic results, and in particular for approximations and heuristics, computational results are an important performance measure, both regarding the quality of the result of the algorithm and the running time needed. But a fair comparison of algorithms with each other on the basis of computational results is usually difficult, if not impossible, since the implementation of an algorithm and the graphs used for the test strongly influence the computational results. With this in mind, the comparisons of algorithms made in this section have to be considered with caution.

4 Vertex Splitting and Splitting Number

The vertex splitting operation on a graph is the reversal of identifying two vertices:

Definition 19 (vertex splitting) *If $G' = (V', E')$ is a graph with two vertices $v_1 \in V'$ and $v_2 \in V'$, and if $G = (V, E)$ is the graph obtained from G' with*

$$\begin{aligned} V &= (V' \setminus \{v_1, v_2\}) \cup \{v\} \quad \text{and} \\ E &= (E' \setminus \{uv_i \mid u \in V' \text{ and } i \in \{1, 2\} \text{ and } uv_i \in E'\}) \\ &\quad \cup \{uv \mid u \in V \setminus \{v\} \text{ and } (uv_1 \in E' \text{ or } uv_2 \in E')\} \end{aligned}$$

then we say that G' was obtained from G by splitting the vertex v .

If a graph G' has been obtained from a graph G by a (possibly empty) sequence of vertex splitting operations, we call G' a *splitting* of G . Note that even if there is a vertex $x \in V'$ such that $xv_1 \in E'$ and $xv_2 \in E'$, no multiple edges are formed in G by the vertex identification operation. Likewise, no loop vv is formed in G , even if $v_1v_2 \in E'$.

The vertex identification of given vertices v_1 and v_2 in a given graph G' yields a unique graph G . But the opposite is not true: Given a graph G and one of its vertices, v , there are many ways to split this vertex. Given the graph $G = K_3$, for example, and one of its vertices, v , there are six ways to perform a vertex splitting at v such that the resulting graphs are pairwise non-isomorphic (see Figure 6).

One might want to define a vertex splitting in a more general way as the reversal of identifying k vertices of a graph at once, where $k \geq 2$. So a splitting of a vertex v would result in vertices $v_1 \dots v_k$ so that the adjacencies of $v_1 \dots v_k$ cover the adjacencies of v in the original graph. But since splitting a vertex k ways can always be regarded as $(k-1)$ successive vertex splitting operations where each vertex splitting is only a 2-way-splitting, we restrict our definition of vertex splitting to splitting a vertex v into exactly two vertices v_1 and v_2 .

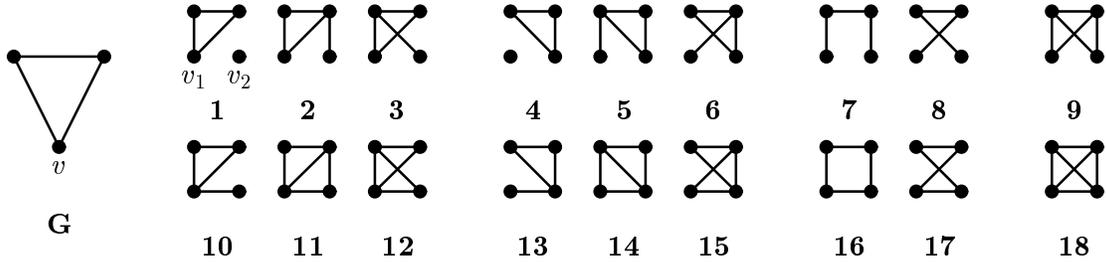


Figure 6: Eighteen possible ways to split $G = K_3$ at v . Essentially, there are only six ways to split v in G : The graphs numbered 1 and 4 are isomorphic. The graphs numbered 2, 3, 5, 6, 10, and 13 are isomorphic. The graphs numbered 7 and 8 are isomorphic. The graphs 9, 11, 12, 14, and 15 are isomorphic. The graphs 16 and 17 are isomorphic.

The vertex splitting operation has appeared in very different contexts. Note, for example, that decomposing a graph into its 2-connected blocks means performing a proper vertex splitting at every cut vertex. Already Steinitz and Rademacher [SR34] observed (as restated in [Sch91]) that every triangulation of the plane can be generated from a planar embedding of K_4 by vertex splitting operations (note that a planar embedding of K_4 is in itself a triangulation of the plane). Similar results hold for other surfaces [Bar82, Bar87, BE88, BE89, Bar90, Bar91, Sch91, MM92, FMN94]. The vertex splitting operation is central to Tutte’s [Tut61, Tut66], Slater’s [Sla74], Chen and Kanevsky’s [CK93], and Gubser’s [Gub93] characterizations of various classes of graphs.

Given a graph $G = (V, E)$ and a function $f : V \rightarrow \mathbb{N}$, Nash-Williams [NW79, NW85a, NW85b, NW87] answers questions of the following type: Can G be transformed into a graph H by a sequence of proper vertex splitting operations such that H has a certain property and such that each $v \in V$ results in $f(v)$ vertices $v_1, v_2, \dots, v_{f(v)}$ in H ? The work in [Arc84], [Yap81, Yap83], and [Sel88] about graph coloring also uses vertex splitting, and Mayer and Ercal [ME93a, ME93b] attack the following NP-hard problem with genetic algorithms: Given a directed, acyclic graph G and a positive number δ , determine a set X of vertices in G with minimum cardinality so that performing a vertex splitting operation on each vertex in X transforms G into a graph where the length of the longest directed path is at most δ .

Eades and Mendonça [Men94, EM96] address the problem of finding a planar embedding for a graph G with edge weights such that for each edge uv , the Euclidean distance between u and v in the layout is proportional to the weight of the edge uv . In general, finding such a layout for a given graph G with given weights is impossible, but by applying proper vertex splitting operations to G , G can be transformed into a graph H that admits a layout with the desired property. Determining the least number of vertex splitting operations required to achieve this is NP-complete [Men94, Section 4.4.1], [ELaMM95]. Heuristics are given to solve the problem.

For the remainder of this chapter, we are concerned with vertex splitting operations as a means to planarize a given graph: Given a graph G , we want to know the smallest number k , so that G can be planarized by k vertex splitting operations. In other words:

Definition 20 (splitting number) *Given a graph G , the splitting number of G , denoted $\sigma(G)$, is the smallest number k , so that G can be obtained from a planar graph G' by k vertex identifications (of 2 vertices each).*

Clearly $\sigma(G) = 0$ if and only if G is planar. If a planar graph G' was obtained from a graph G by vertex splitting operations, we call G' a *planar* splitting of G . If additionally, G' was obtained by $\sigma(G)$ vertex splitting operations, we call G' an *optimal* planar splitting of G . For a general surface S , $\sigma(G, S)$ denotes the smallest number k , so that G can be obtained from a graph G' by k vertex identifications, where G' is embeddable in S .

Investigation of the splitting number seems to have started with the work of Nora Hartsfield, Brad Jackson and Gerhard Ringel in the 1980s about lower bounds for the splitting number and about splitting vertices of complete graphs and of complete bipartite graphs so that the resulting graph is embeddable in a given surface as described in Sections 4.1 and 4.3 [JR85, JR84a, JR84b, HJR85, Har86, Har87]. Section 4.2 describes the work of Eades, Faria, Figueiredo and Mendonça on establishing the NP-hardness of finding the splitting number for a given graph [EM93, Men94, FFM98a].

4.1 Lower Bounds for the Splitting Number

First consider the different ways of vertex splitting as illustrated in Figure 6. The graphs numbered 1 and 7 (and all graphs isomorphic to them) have the same number of edges as the original graph K_3 . The other graphs have more edges than K_3 : In the graphs numbered 10 through 18, v_1v_2 is an additional edge, and in graphs such as the ones numbered 2 or 18, some vertex u that was adjacent to v in K_3 is now adjacent to both v_1 and v_2 . For each u that was adjacent to v and is now adjacent to both v_1 and v_2 , we call one of the edges uv_1 and uv_2 *superfluous*. Likewise, we call an edge v_1v_2 superfluous. We say a vertex splitting is *proper* if it does not create superfluous edges, and if none of the resulting vertices v_1 and v_2 is isolated. Otherwise we call it *improper*.

Now observe that when splitting vertices of a graph G with the goal of planarizing it, we can restrict our attention to proper vertex splittings. For assume we obtain a planar graph G' from G by using improper vertex splittings. Now perform the same sequence of vertex splittings on G again, but in each vertex splitting, leave out all superfluous edges. Also, skip all the vertex splittings that create an isolated vertex. This yields a graph G'' . Since G'' is a subgraph of G' and since G' is planar, G'' is also planar.

The upper bound for the number of edges for planar graphs from Equation 2 immediately yields a lower bound for the splitting number: Let G be a graph with n vertices and m edges, and let $\sigma(G)$ be the splitting number of G . Let G' be a graph obtained from G by $\sigma(G)$ vertex splitting operations so that G' is planar. Then G' has $n' = n + \sigma(G)$ vertices, and by the above argument about superfluous edges, we can construct G' in such a way that $m' = m$. Since G' is planar, Equation 2 says that it has at most $m' \leq 3n' - 6$ edges (for $n' \geq 3$). Since $m = m'$, this implies $n' \geq \frac{1}{3}m + 2$ for $n' \geq 3$. Every graph on $n \leq 4$ vertices is planar, so for $n \leq 4$ we have $n' = n$. For $n \geq 5$, we have $n' \geq n$. Therefore, the condition $n' \geq 3$ is equivalent to the condition $n \geq 3$, and with $\sigma(G) = n' - n$ we obtain the lower bound

$$\sigma(G) \geq \left\lceil \frac{1}{3} \cdot m - n + 2 \right\rceil \quad \text{for } n \geq 3 \quad (21)$$

If we know the girth g of a graph G with n vertices and m edges, a better bound for $\sigma(G)$ can be derived [JR85]: Let again G' be a graph obtained from G by $\sigma(G)$ vertex splitting

operations so that G' is planar. Let n' and m' be the number of vertices and edges of G' , respectively. Let f' be the number of faces of G' in a given planar embedding. Euler's formula for planar graphs (Equation 1) says $n' - m' + f' = 2$. If g' is the girth of G' , then every face of G' is incident to at least g' edges. Furthermore, if m'' edges are incident to exactly two faces, then $f' \cdot g' \leq 2 \cdot m'' \leq 2 \cdot m'$. Combining this inequality with Euler's formula, we have

$$2 + m' - n' = f' \leq \frac{2m'}{g'} .$$

Since $n' = n + \sigma(G)$, $m' = m$, and $g' \geq g$, we have

$$2 + m - n - \sigma(G) \leq \frac{2m}{g'} \leq \frac{2m}{g} .$$

Since $\sigma(G)$ is an integer, we can conclude

$$\sigma(G) \geq \left\lceil m - \frac{2m}{g} - n + 2 \right\rceil \tag{22}$$

Note that if a graph G has cycles, but its girth is not known, combining $g \geq 3$ with Equation 22 yields Equation 21. This is not surprising, since the formula $m' \leq 3n' - 6$ follows from Euler's formula $2 + m' - n' = f'$ with the observation that each of the f' faces is incident to at least 3 edges.

[JR85] provides this lower bound for a general surface S of Euler characteristic $E(S)$:

$$\sigma(G) \geq \left\lceil m - \frac{2m}{g} - n + E(S) \right\rceil \tag{23}$$

[JR85] also shows that this bound is achieved for all complete bipartite graphs on any surface S .

4.2 Finding the Splitting Number of a Graph

It has only recently been shown that determining the splitting number of a given graph G is an NP-hard problem [FFM98a]. The investigation of the complexity status of the splitting number problem begins with Mendonça's definition of the following two problems and his proof that the first one is NP-complete [Men94, Section 4.3.1]:

Problem 24 (Eligible Set Split Planar Graph [Men94, Section 4.3.1]) *Given a graph $G = (V, E)$, a subset of vertices $S \subseteq V$, and a positive integer $K \leq |S|$, can G be transformed into a planar graph G' by K or less vertex splitting operations that involve only vertices in S ? The vertices in S are called eligible vertices¹⁰.*

Problem 25 (Split Planar Graph [Men94, Section 4.3.1]) *Given a graph $G = (V, E)$ and a positive integer $K < |E|$, can G be transformed into a planar graph G' by K or less vertex splitting operations?*

A reduction from the Maximum Planar Subgraph Problem (see also Section 3.1) shows that Eligible Set Split Planar Graph is NP-complete.

¹⁰In [Men94], we actually have " $K < |S|$ "

Problem 26 (Maximum Planar Subgraph [GJ79, Problem GT27]) Given a graph $G = (V, E)$ and a positive integer $K \leq |E|$, is there a subset $E' \subseteq E$ with $|E'| \geq K$ such that the graph $G' = (V, E')$ is planar?

Theorem 27 [Men94, Section 4.3.1] Eligible Set Split Planar Graph is NP-complete.

For the proof, let the graph $G = (V, E)$ and the positive integer $K \leq |E|$ be an arbitrary instance of Maximum Planar Subgraph. Construct an instance of Eligible Set Split Planar Graph as follows: Replace each edge $e = uv \in E$ with a path $ue'v_e e''v$, i.e. subdivide each edge e once. Call the resulting graph H . Let $K' = |E| - K$ (note that $K' \leq |E| = |S|$), and let $S = \{v_e \mid e \in E\}$ be the set of vertices created through the subdivisions. H , S , and K' define an instance of Eligible Set Split Planar Graph. G has a planar subgraph with K or more edges if and only if H can be planarized by K' vertex splitting operations on S . For if G has a planar subgraph $G' = (V, E')$ with $|E'| \geq K$ edges, then for each edge $e \in E \setminus E'$, split the vertex v_e in H so that one of the copies of v_e , v_{e1} , is incident to e' , and the other one, v_{e2} , is incident to e'' , and v_{e1} and v_{e2} are not adjacent. The resulting graph H' is planar and the number of vertex splitting operations was $k' = |E| - |E'| = K' + K - |E'| \leq K'$. On the other hand, if there are $k' \leq K'$ vertex splitting operations on vertices in S that transform H into a planar graph H' , then for each vertex $v_e \in S$ that was involved in a vertex splitting, delete the corresponding edge e from G . The resulting graph G' is planar since H' is planar, and the number of deleted edges is $l \leq k' \leq K'$, so G' has $|E'| = |E| - l \geq |E| - K' = K$ edges. Figure 7 shows the steps of this reduction for $G = K_{3,3}$.

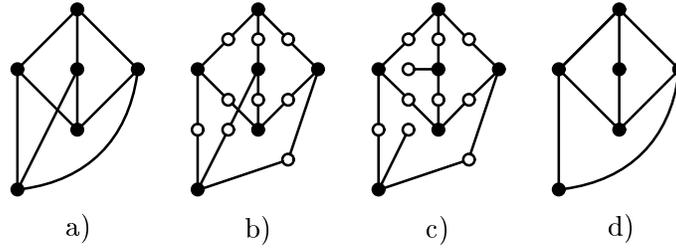


Figure 7: Illustration of the reduction for Eligible Set Split Planar Graph from Maximum Planar Subgraph with $K_{3,3}$. a) $K_{3,3}$. b) Every edge is subdivided by a white vertex. c) One of the white vertices needs to be split to planarize the graph in b). d) Alternatively, the deletion of the edge that was subdivided by the vertex split in c) yields a planar subgraph.

A similar transformation does not seem to work for Split Planar Graph, but Mendonça points out that for the class of graphs with vertex degree not greater than 3, Split Planar Graph and Maximum Planar Subgraph are equivalent [Men94, Section 4.3.1]. If Maximum Planar Subgraph restricted to graphs with vertex degree not greater than 3 were known to be NP-complete, then the following reduction would yield the NP-completeness of Split Planar Graph: A graph $G = (V, E)$ with vertex degrees not greater than 3 has a planar subgraph $G' = (V, E')$ with $|E'| \geq K$ edges if and only if G can be transformed into a planar graph G by less than or equal to $|E| - K$ vertex splitting operations. For assume $E' \subseteq E$ with $|E'| \geq K$ exists so that $G' = (V, E')$ is planar. Then for each edge $e = uv \in E \setminus E'$, perform a proper splitting operation on either u or v in G so that one of the resulting two vertices is only

incident to e . The resulting graph is planar, and the number of vertex splitting operations was $|E| - |E'| \leq |E| - K$. On the other hand, assume that G can be planarized by K' vertex splitting operations. Then each (proper) splitting operation yields at least one vertex v with degree 1. Let E'' be the set of edges incident to those vertices. $|E''| \leq K'$. Then $G' = (V, E \setminus E'')$ is a planar graph with $|E| - |E''| \geq |E| - K'$ edges.

Faria, Figueiredo, and Mendonça have now settled the complexity status of Split Planar Graph:

Theorem 28 [FFM98a] *Split Planar Graph is NP-complete, even when restricted to cubic graphs.*

The proof uses a rather involved reduction from 3-SAT [GJ79, Problem LO2], where for an instance of 3-SAT with n variables and m clauses, a graph of maximum degree 3 with more than $1200 \cdot n^3 \cdot m^2$ vertices is constructed. A variation of the reduction where every vertex has degree exactly 3 is also given, completing the proof of the above theorem. [FFM98a] then observes that the NP-completeness of Split Planar Graph for cubic graphs implies the NP-completeness of Maximum Planar Subgraph when restricted to cubic graphs.

Eades and Mendonça, in their work towards a layout system for diagrams, have developed and implemented a heuristic for planarizing a graph through vertex splitting [EM93] and [Men94, Sections 4.3.2 and 4.3.3]. It is based on Lempel, Even and Cederbaum's planarity testing algorithm and its implementation using PQ -tree algorithms by Booth and Lueker [LEC67, BL76] mentioned in Section 1.2 and uses ideas of [JTS89, JST89]. The vertices of the original graph are considered one at a time. A vertex is added to the graph being constructed if the resulting graph remains planar. Otherwise, the vertex is split and both copies of the vertex are added so that the resulting graph is planar. The running time of the heuristic is in $\mathcal{O}(n^2)$ for graphs with n vertices. There seem to be no computational studies on the performance of this heuristic.

4.3 Results for Particular Classes of Graphs

This section discusses the results about the splitting number of complete bipartite graphs and complete graphs. The splitting number of the hypercube of dimension 4, Q_4 , is 4 [FFM98b], and the splitting number is also known for the Cartesian product of an m -cycle C_m and an n -cycle C_n . The latter result allows the construction of a graph with genus g and splitting number σ , for any integers $\sigma \geq g \geq 1$ [Sch86].

The first class of graphs for which the splitting number was determined was the class of complete bipartite graphs. Note that the complete bipartite graph K_{n_1, n_2} is planar if and only if $n_1 \in \{1, 2\}$ or $n_2 \in \{1, 2\}$. The girth of K_{n_1, n_2} is 4 (for $n_1, n_2 \geq 2$), so the lower bound 22 yields

$$\begin{aligned} \sigma(K_{n_1, n_2}) &\geq \left\lceil n_1 \cdot n_2 - \frac{2n_1 n_2}{4} - n_1 - n_2 + 2 \right\rceil \\ &= \left\lceil \frac{(n_1 - 2)(n_2 - 2)}{2} \right\rceil \end{aligned} \tag{29}$$

In [JR85] and [JR84b], Jackson and Ringel show that this lower bound is also an upper bound. Again, they consider the general case of transforming $G = K_{n_1, n_2}$ into a graph G' that is embeddable in a surface S with Euler characteristic $E(S)$. They show that if S is a closed

orientable or nonorientable surface, then $\sigma(K_{n_1, n_2}, S) = \max\left(\left\lceil \frac{(n_1-2)(n_2-2)}{2} \right\rceil - 2 + E(S), 0\right)$. Recall that embedding a graph in the plane is equivalent to embedding it in the sphere. The sphere is commonly referred to as S_0 , and $E(S_0) = 2$, so we have

Theorem 30 (Jackson, Ringel [JR85, JR84b]) *The splitting number of the complete bipartite graph K_{n_1, n_2} is*

$$\sigma(K_{n_1, n_2}) = \left\lceil \frac{(n_1 - 2)(n_2 - 2)}{2} \right\rceil \quad \text{for } n_1, n_2 \geq 2 .$$

Figure 8 illustrates the idea of the constructive proof given in [JR84b] for the case that n_1 or n_2 is an even number.

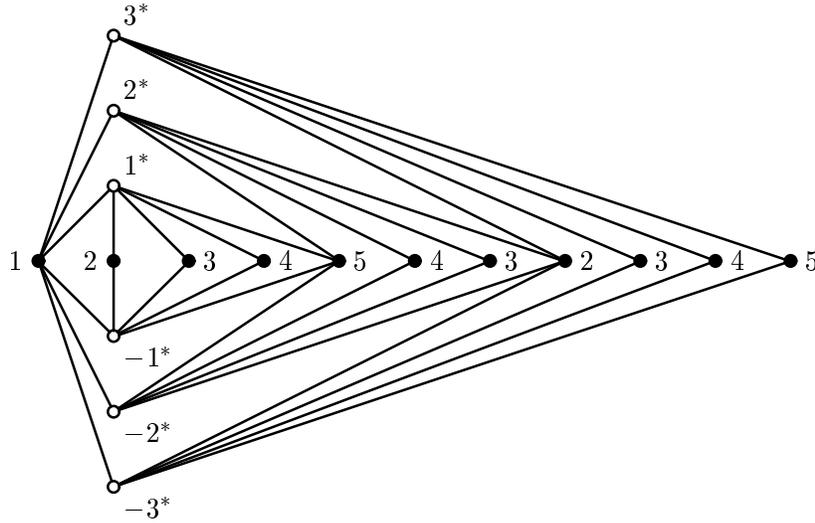


Figure 8: An optimal planar splitting of $K_{6,5}$, as constructed in the proof of Theorem 30. Observe that if we identify, for each j , $2 \leq j \leq 5$, all vertices labeled j , we obtain the original graph $G = K_{6,5}$. Counting the number of such vertex identifications shows that $K_{6,5}$ can be constructed from this graph by $\sigma = \left\lceil \frac{(6-2)(5-2)}{2} \right\rceil$ vertex identifications.

The second class of graphs for which the splitting number was found is the class of complete graphs. This result is much more involved than the one for complete bipartite graphs. First recall that for $n \geq 5$, the complete graph K_n is nonplanar. If less than $(n - 4)$ vertex splitting operations are performed on a graph K_n with $n > 5$, the resulting graph G' contains (at least) 5 vertices that were not involved in splitting operations. These 5 vertices induce the nonplanar graph K_5 , so G' cannot be planar. This yields the trivial lower bound

$$\sigma(K_n) \geq n - 4 \tag{31}$$

With the girth $g = 3$ the lower bounds (21) and (22) both yield

$$\sigma(K_n) \geq \left\lceil \frac{1}{3} \binom{n}{2} - n + 2 \right\rceil = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil \quad \text{for } n \geq 3 \tag{32}$$

The lower bound (31) is only interesting for $n = 6$ and $n = 7$. For $n \geq 8$, the bound (32) is greater than or equal to the bound (31).

Hartsfield, Jackson and Ringel show that except for $n = 6, 7$ or 9 the lower bound (32) is also an upper bound. Unlike the result of Theorem 30 for K_{n_1, n_2} , this result does not extend to general surfaces. In the conference presentations [JR85] and [JR84a], partial results towards finding the splitting number of K_n are announced. [HJR85] then presents a proof for the following theorem:

Theorem 33 (Hartsfield, Jackson, Ringel [HJR85]) *The splitting number of the complete graph K_n is*

$$\sigma(K_n) = \begin{cases} \lceil \frac{(n-3)(n-4)}{6} \rceil & \text{for } n \geq 3 \text{ and } n \notin \{6, 7, 9\} \\ \lceil \frac{(n-3)(n-4)}{6} \rceil + 1 & \text{for } n \in \{6, 7, 9\} \end{cases} .$$

For each n , $3 \leq n \leq 8$, Theorem 33 yields the higher one of the lower bounds (31) and (32). But for $n = 9$, (31) and (32) both yield 5 splitting operations as a lower bound, whereas Theorem 33 yields $\sigma(K_9) = 6$. Figure 9 shows a planar splitting with 6 splitting operations for K_9 . [HJR85] explains that the proof for $\sigma(K_9) = 6$ involves checking many cases and that Mark Jungerman has verified the proof using a computer.

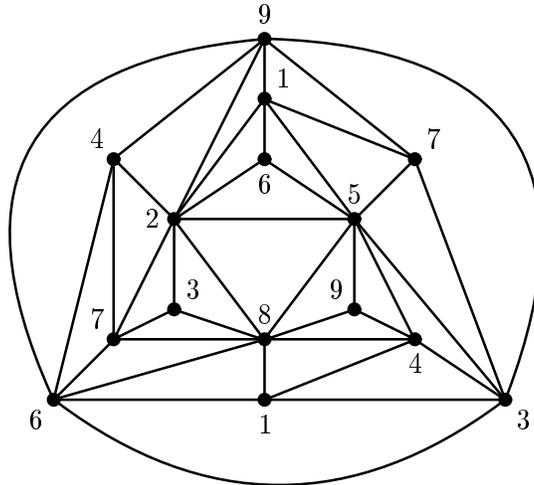


Figure 9: An optimal planar splitting of K_9 as exhibited in [HJR85]. $\sigma(K_9) = 6$. Note that there are superfluous edges connecting vertices labeled 1 and 6, 3 and 7, and 9 and 4, respectively.

The proof for large n is a meticulous case analysis for the congruence classes of n modulo 12. It is actually carried out in a dual formulation of the problem: A planar splitting of K_n is represented as a map where the countries represent the vertices. Countries that correspond to vertices with the same label belong to a common *empire*. Two empires e_i, e_j are *adjacent* if there exist countries c_i and c_j belonging to the empires e_i and e_j , respectively, that share a common border. Countries whose corresponding vertices are adjacent in the planar splitting share a common border in the map.

Finding an optimal planar splitting of K_n is then equivalent to finding a map with n mutually adjacent empires where the overall number of countries is minimum. Figure 10 shows an optimal planar splitting of K_{10} , and Figure 11 shows the corresponding map. This map was actually found by Jungerman's program mentioned above [JR84a, HJR85]. Finding maps of mutually adjacent empires is an old problem: According to [JR84a], Heawood found a map of 12 mutually adjacent empires of 2 countries each in 1890 [Hea90]. Note that indeed $\sigma(K_{12}) = 12$.

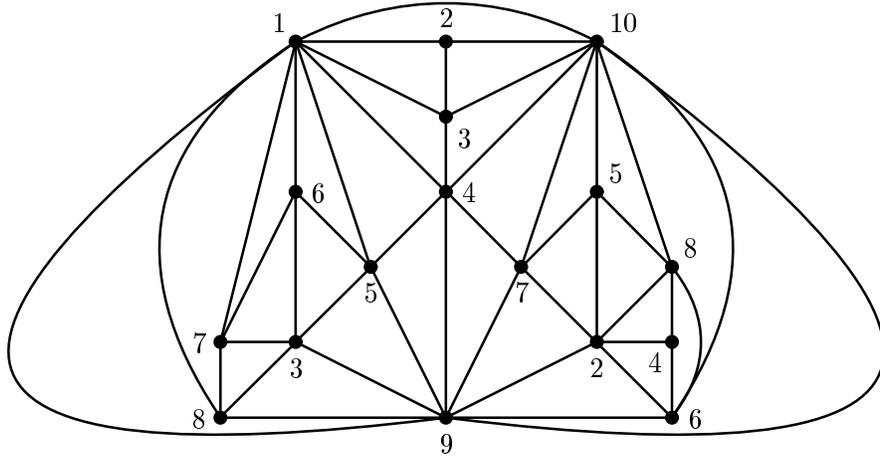


Figure 10: An optimal planar splitting of K_{10} . $\sigma(K_{10}) = 7$.

The splitting number of the complete graph on the torus is given in [Har86], and [HJR85, Har87] give results about the splitting number of the complete graph on two nonorientable surfaces.

5 Thickness

In Sections 2, 3, and 4, we have performed the operations vertex deletion, edge deletion, and vertex splitting on a graph G with the goal of obtaining a new planar graph G' . We now ask for a collection of planar subgraphs of a given graph G , the union of which is G :

Definition 34 (thickness) *The thickness of a graph G , denoted $\theta(G)$, is the minimum number of planar subgraphs of G whose union is G .*

Clearly the thickness of a graph is 1 if and only if the graph is planar.

As an example, consider the two planar subgraphs of $K_{3,3}$ whose union is $K_{3,3}$ in Figure 12, and the three planar subgraphs of K_9 whose union is K_9 in Figure 13. Since $K_{3,3}$ is nonplanar, the exhibition of two planar subgraphs of $K_{3,3}$ whose union forms $K_{3,3}$ shows that $\theta(K_{3,3}) = 2$. The thickness of K_9 is not so easily determined: Figure 13 only shows that $\theta(K_9) \leq 3$. [BHK62] shows that indeed $\theta(K_9) = 3$. (Alternative proofs are provided in [Tut63a, Wes86].) See Section 5.3 for further results about the thickness of complete and complete bipartite graphs.

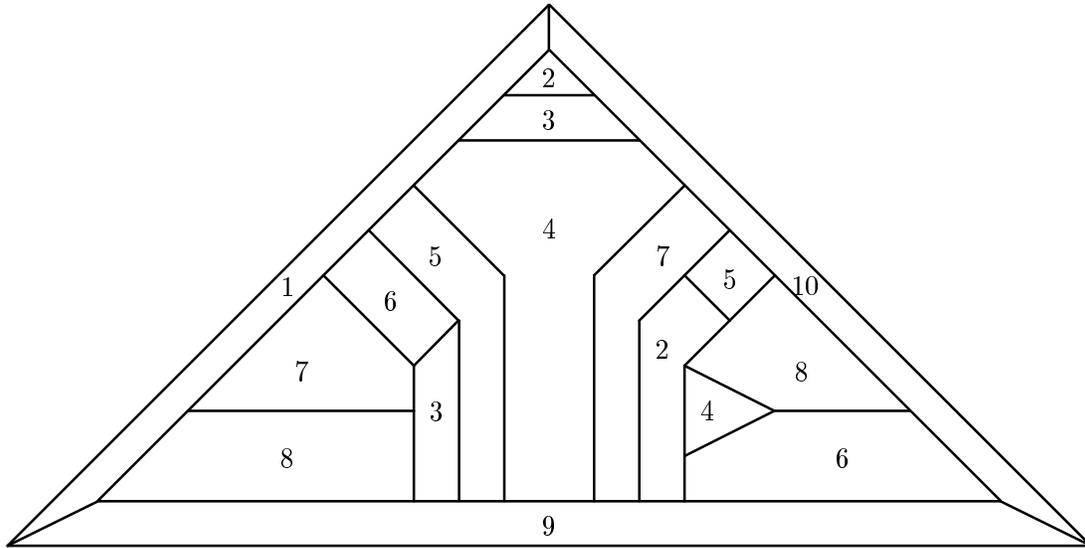


Figure 11: The optimal planar splitting of K_{10} of Figure 10, represented as a map.

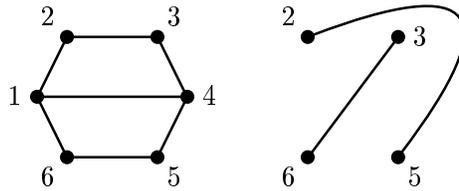


Figure 12: Two planar subgraphs of $K_{3,3}$ whose union is $K_{3,3}$.

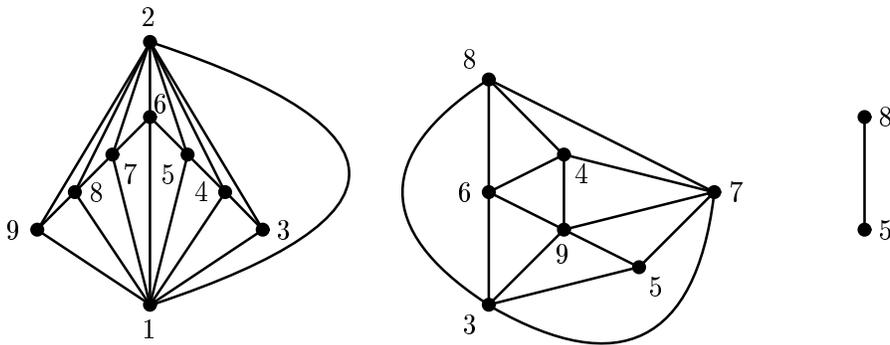


Figure 13: Three planar subgraphs G_1 , G_2 , and G_3 of K_9 whose union is K_9 .

Since each planar subgraph of a given graph G with $n \geq 3$ vertices and m edges can have at most $3n - 6$ edges (Equation 2), we obtain an immediate lower bound for the thickness of G :

$$\theta(G) \geq \left\lceil \frac{m}{3n - 6} \right\rceil \quad \text{for } n \geq 3 \quad (35)$$

For upper bounds, see page 32.

Observe that if the graphs in Figure 12 were printed onto slides, the two planar subgraphs given could actually be placed on top of each other so that each vertex labeled i in the first subgraph lies exactly on top of the vertex labeled i in the second subgraph. So we do not only have two *subgraphs* whose union is $K_{3,3}$, but we have two *embeddings* of two planar graphs so that the union of the embeddings yields a drawing of $K_{3,3}$. Kainen [Kai73] showed that this observation can be generalized:

Theorem 36 [Kai73] *Given a graph G with thickness $\theta(G)$, there exists a drawing of G , and there exist subgraphs $G_1, \dots, G_{\theta(G)}$ whose union is G , such that the drawing of G restricted to G_i is a planar embedding of G_i , for $1 \leq i \leq \theta(G)$.*

Note that the three subgraphs of K_9 in Figure 13 are drawn in a way so that the union of their embeddings does not yield a drawing of K_9 .

Knowing the thickness of a given graph can be helpful in some application problems. [AKS91] proposes two new multilayer grid models for VLSI layout and shows for one of them that a graph with n vertices and thickness 2 can be embedded in two layers in an area of size $O(n^2)$. Furthermore, another algorithm embeds a graph with n vertices and thickness t in t layers in $O(n^3)$ area, respecting some additional constraints. [RL92, RL93] give approximate algorithms for scheduling multihop radio networks. They find a schedule whose length is a function of the thickness of the network.

The thickness of graphs has been widely studied as part of topological graph theory, but few algorithmic results for finding the thickness of a graph seem to be available. Early work about thickness and the introduction of the study of thickness into graph theory is described in detail by Hobbs [Hob69]. In particular, Tutte [Tut63b] establishes many results about the thickness of graphs in one of the earliest papers about this topic. Surveys about thickness are [WB78], [Bei88], and [MOS98].

The following sections give a brief summary of the known results about thickness: Section 5.1 describes the result of Mansfield [Man83] that says that determining the thickness of a graph is NP-hard, and mentions heuristic approaches for finding the thickness. Thickness-minimal graphs are discussed in Section 5.2, and Section 5.3 lists results about the thickness of graphs belonging to particular classes of graphs. Finally, Section 5.4 mentions two variations of the thickness.

5.1 Finding the Thickness of a Graph

Mansfield [Man83] defines the following problem (that was already mentioned in [GJ79, Problem OPEN3]):

Problem 37 (Thickness [Man83]) *Given a graph G and a positive integer K , does the thickness of G satisfy $\theta(G) \leq K$?*

Mansfield shows that this problem is NP-complete for the fixed value $K = 2$, thus establishing the NP-completeness of Thickness. The proof uses a reduction from Planar 3-SAT [GJ79, Problem LO1]. Before we state this problem, recall that given a set $U = \{u_1, \dots, u_m\}$ of Boolean variables, the set $L = \{u_1, \bar{u}_1, \dots, u_m, \bar{u}_m\}$ is the set of *literals* over U . A subset of literals $c \subseteq L$ is called a *clause* over U . A clause c is said to be *satisfied* if the disjunction of the literals in c has the Boolean value “true” (for some truth assignment for U). Given a set U of Boolean variables and a collection C of clauses over U , consider the bipartite graph $G_{U,C} = (U \cup C, E)$ with $E = \{uc \mid (u \in c \text{ or } \bar{u} \in c) \text{ and } u \in U \text{ and } c \in C\}$.

Problem 38 (Planar 3-SAT [GJ79, Problem LO1]) *Given a set U of Boolean variables and a collection C of clauses over U with $|c| \leq 3$ for all $c \in C$, and given that the graph $G_{U,C}$ is planar, is there a truth assignment for U that satisfies all clauses in C simultaneously?*

Lichtenstein [Lic82] showed that Planar 3-SAT is NP-complete. Mansfield first shows that Planar 3-SAT remains NP-complete if each clause contains *exactly* three literals, and then reduces this restricted version of Planar 3-SAT to Thickness with $K = 2$.

So it is unlikely that a polynomial time algorithm that determines the thickness of a given graph will be found. A heuristic approach for finding an upper bound on the thickness of a graph $G = (V, E)$ is to find a planar subgraph $G' = (V, E')$ of G , to form the difference graph $H = (V, E \setminus E')$, to then find a planar subgraph of H and so on until the difference graph itself is planar. This approach is studied in [OS94, MOS98] and, independently, in [Cim95b]. [MOS98] reports on using three different algorithms to find a planar subgraph: The maximal planar subgraph algorithm [CHT93], the maximal planar subgraph algorithm [JTS89, JST89], and the branch and cut heuristic [Mut94, JM96b]. Computational studies are carried out for 19 complete graphs with 10 to 100 vertices, for 9 complete bipartite graphs $K_{n,n}$ with 20 to 100 vertices, and for 14 further graphs with 28 to 680 vertices originating in VLSI design. The two thickness heuristics using the maximal planar subgraph algorithms perform very similarly throughout. For the complete and complete bipartite graphs, their results are reported to be on average 38 and 24 percent, respectively, off the optimal solution, while the heuristic using the branch and cut approach is only off by 20 and 12 percent, respectively. For the other 14 graphs, the thickness is not known. The results of all three heuristics are very similar for these graphs, with a small advantage for the branch and cut heuristic. But as with the maximum planar subgraph heuristics discussed in Section 3.3, the branch and cut heuristic often needs more than 100 times the run time of the heuristics based on maximal planar subgraphs.

[Cim95b] also reports on a thickness heuristic based on extracting maximal planar subgraphs. Heuristic improvements are made to increase the size of the planar subgraphs obtained, and computational studies on complete, complete bipartite, and random graphs with 10 to 115 vertices are carried out. For the complete and complete bipartite graphs that were also used in [MOS98], the performance of the heuristics in [Cim95b] is similar to the performance of the heuristics using maximal planar subgraphs reported in [MOS98].

5.2 θ -Minimal Graphs

The following facts about thickness and the concept of thickness-minimal graphs (also called θ -minimal graphs) are due to Tutte [Tut63b]: If a graph G has thickness $\theta(G) = t$, then every subgraph of G has thickness at most t . Furthermore, if a subgraph G' of G has exactly one edge less than G or exactly one vertex (and all its incident edges) less than G , then either

$\theta(G') = t$ or $\theta(G') = t - 1$. In other words, deleting one edge or deleting one vertex decreases the thickness of a graph by at most one. These facts motivate the following definition:

Definition 39 (thickness-minimal graphs)¹¹ *If a graph G has thickness t and if every proper subgraph of G has thickness less than t , then G is called a thickness-minimal (or θ -minimal) graph. If G is thickness-minimal with $\theta(G) = t$, we also call G t -minimal.*

The 2-minimal graphs are exactly the subdivisions of K_5 and $K_{3,3}$. Note that if a graph G has thickness $t \geq 2$, then there exists a t -minimal subgraph of G . For $t \geq 2$, every t -minimal graph is 2-connected and has minimum vertex degree at least t and maximum vertex degree at least $2t - 1$. Tutte then establishes the following important theorem:

Theorem 40 [Tut63b] *For each integer $t \geq 2$ there exist infinitely many pairwise nonisomorphic t -minimal graphs with maximum vertex degree $2t - 1$, and of girth greater than any specified integer N .*

This theorem establishes the *existence* of infinitely many t -minimal graphs. But given an integer $t \geq 2$, it does not provide an explicit construction of t -minimal graphs. Beineke [Bei67] showed that for any integer $t \geq 2$, the complete bipartite graph $K_{2t-1, 4t^2-10t+7}$ is t -minimal. Hobbs and Grossman [HG68a], and, independently, Bouwer and Broere [BB68] showed that $K_{4t-5, 4t-5}$ is t -minimal for any integer $t \geq 2$. Hobbs and Grossman [HG68b] also showed that any t -minimal graph is t -edge-connected.

Since $\theta(K_9) = 3$ [BHK62, Tut63a, Wes86], K_9 is a candidate for being 3-minimal. Figure 13 displays three subgraphs G_1 , G_2 , and G_3 of K_9 whose union is K_9 , where G_3 consists of a single edge. Thus any proper subgraph of K_9 is the union of a subgraph of G_1 and a subgraph of G_2 , and has therefore thickness at most 2. So K_9 is 3-minimal. K_9 appears to be the only θ -minimal complete graph.

Wessel [Wes83b, Wes89], and, independently, Širáň and Horák [HŠ87] finally give, for each integer $t \geq 2$, an explicit construction of an infinite number of t -minimal graphs. Širáň and Horák show that the bounds established by [Tut63b] and [HG68b] on connectedness and minimum vertex degree are actually tight: Their graphs are 2-connected, but not 3-connected, they are t -edge-connected, but not $(t + 1)$ -edge-connected, and they have minimum vertex degree t .

5.3 Results for Particular Classes of Graphs

There are few classes of graphs for which the thickness is known. For the complete graphs, the thickness was settled in a long process described in detail by White and Beineke [WB78, Section 9]. It is clear that $\theta(K_1) = \theta(K_2) = \theta(K_3) = \theta(K_4) = 1$, and it is easily seen that $\theta(K_5) = \theta(K_6) = \theta(K_7) = \theta(K_8) = 2$. Figure 13 shows that $\theta(K_9) \leq 3$. Battle, Harary, and Kodama [BHK62] were the first to show that indeed $\theta(K_9) = 3$. Alternative proofs were given by Tutte [Tut63a] and Wessel [Wes86]. Beineke and Harary [BH65] showed the formula for $\theta(K_n)$ for most cases, and Alekseev and Goncakov [AG76], and, independently, Vasak [Vas76], completed the result:

$$\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor \quad \text{for } n \geq 1, n \neq 9, n \neq 10$$

¹¹[Bei67, Wes83b, Wes89] use the term *critical* instead of *minimal*.

$$\theta(K_9) = \theta(K_{10}) = 3$$

For the complete bipartite graph, the thickness is still not settled for all cases. Beineke, Harary, and Moon [BHM64] found the following result:

$$\theta(K_{n_1, n_2}) = \left\lceil \frac{n_1 \cdot n_2}{2(n_1 + n_2 - 2)} \right\rceil$$

except possibly when n_1 and n_2 are both odd, and, assuming $n_1 \leq n_2$, there is an integer k such that $n_2 = \left\lfloor \frac{2k(n_1-2)}{n_1-2k} \right\rfloor$. In [Bei67], Beineke gives a more detailed description of the proof than in [BHM64].

The thickness of the hypercube of dimension n , Q_n , is $\theta(Q_n) = \lceil \frac{n+1}{4} \rceil$ [Kle67].

For a graph G and a general surface S , let the *thickness of G on S* , denoted $\theta(G, S)$, be the smallest number of subgraphs of G so that the subgraphs are all embeddable in S and so that their union is G . When S is the torus (also denoted S_1), $\theta(G, S_1)$ is also called the *toroidal thickness* of G . To avoid confusion, the thickness of a graph is sometimes called the *planar thickness*. [WB78] reviews known results about the thickness on other surfaces. [Rin65] and, independently, [Bei69] give the toroidal thickness of the complete graphs. [Bei69] also discusses the complete bipartite graphs, as well as some other surfaces. Further results about the toroidal thickness of graphs are given in [And82b, And82a].

Jünger et al. showed that the thickness of graphs that do not contain K_5 as a minor is at most 2 [JMOS94, Ode94], using a decomposition theorem of Truemper [Tru92, Theorem 10.5.24]. They were able to extend their result to graphs that do not contain the graph G_{12} as a minor [JMOS98] (G_{12} is depicted in Figure 14). Since G_{12} contains K_5 as well as $K_{3,3}$ as a minor, the result for G_{12} implies that the thickness of graphs that do not contain K_5 as a minor and the thickness of graphs that do not contain $K_{3,3}$ as a minor is at most two. For such a graph the thickness can be determined in linear time then, since it can be tested in linear time whether the graph is planar. If it is, its thickness is 1, otherwise it is 2.

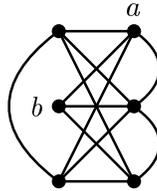


Figure 14: The graph G_{12} . G_{12} clearly has $K_{3,3}$ as a subgraph, so it has $K_{3,3}$ as a minor. To see that it also has K_5 as a minor, contract the edge ab .

For some graph classes, bounds on the thickness are known: A graph of orientable genus 1 (i.e. a nonplanar graph embeddable in the torus) has thickness 2 [Asa87], and a graph of orientable genus 2 has thickness either 2 or 3 [Asa94]. Every graph $G = (V, E)$ has thickness at most $\left\lfloor \sqrt{\frac{|E|}{3}} + \frac{3}{2} \right\rfloor$ [DHS91]. If δ and Δ are the minimum and maximum vertex degree of G , respectively, then $\left\lfloor \frac{\delta+1}{6} \right\rfloor \leq \theta(G) \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$ [Wes84]. Independently, [Hal91] presents similar results about the relation between the minimum and maximum vertex degrees of a graph and its thickness.

5.4 Variations of Thickness

Bernhart and Kainen [BK79, Kai90] introduced the *book thickness* of a graph. A *book* B with $n \geq 0$ pages consists of a line L in 3-dimensional space, called the *spine*, together with n distinct half-planes (called the *pages*) with L as their common boundary. A graph G is embeddable in B if the vertices of G can be placed on L and if each edge can be embedded in at most one page of B . The *book thickness* (also called *pagenumber*) of a graph G is the smallest number n so that G can be embedded in a book with n pages. The book thickness has been studied for several classes of graphs, see for example [CLR87, Hea87, MLW88, HI92, Obr93, Mal94, SGB95].

[BS84] showed that any planar graph can be embedded in a 9-page book. [Hea84] lowered this bound to 7 pages and also gave an $O(n^2)$ algorithm to actually find an embedding. Yannakakis [Yan86, Yan89] showed that any planar graph can be embedded in a book with 4 pages. Yannakakis also gives a linear time algorithm to find such an embedding.

If a graph G has a straight line drawing and two subgraphs G_1 and G_2 whose union is G , and if the straight line drawing of G restricted to G_i is a planar embedding of G_i , for $1 \leq i \leq 2$, then G is called *doubly linear*. Clearly any doubly linear graph has thickness at most two. Hutchinson et al. [HSV96] study doubly linear graphs. They show that a doubly linear graph with n vertices has at most $6n - 18$ edges.

Other variations of thickness are discussed in [Hob69, HŠ82, Hor83, Wes83a, PCK89], for example.

6 Crossing Number

In graph drawing, but also in other application areas such as VLSI layout, we are interested in a drawing of a given graph with as few edge crossings as possible. Here, we do not allow drawings of graphs where a point in the plane belongs to more than two curves representing edges (unless the point represents a common end vertex of those edges).

Definition 41 (crossing number) *The crossing number of a graph G , denoted $\nu(G)$, is the smallest number k so that G can be drawn in the plane with at most k edge crossings.*

Clearly the crossing number of a graph is 0 if and only if the graph is planar, and the crossing number of a graph is bounded from below by the skewness of the graph. Surveys on the crossing number can be found in [SSV95, Sch95].

A graph G with n vertices, m edges, and with crossing number $\nu(G) > 0$ (and a given drawing with $\nu(G)$ edge crossings) can be transformed into a planar graph by introducing $\nu(G)$ new vertices and placing them at the edge crossings of the drawing. The new graph G' has $n + \nu(G)$ vertices and $m + 2 \cdot \nu(G)$ edges. Figure 15 illustrates this process.

This planarization technique is used in graph drawing: Introduce dummy vertices to planarize a graph, then apply a graph drawing algorithm to the planar graph to generate an orthogonal drawing with few bends, and afterwards eliminate the dummy vertices and re-introduce edge crossings into the drawing [DETT99, Section 2.3].

[SSV95] contains a survey on lower bounds for the crossing number, starting with the following observation: Equation 2 for the planar graph G' says that $m + 2 \cdot \nu(G) \leq 3 \cdot (n + \nu(G)) - 6$. This immediately yields the following lower bound for the crossing number of G :

$$\nu(G) \geq m - 3 \cdot n + 6 \tag{42}$$

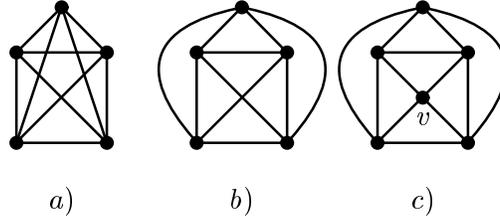


Figure 15: a) K_5 in a drawing with 5 edge crossings. b) K_5 with only one edge crossing. c) Vertex v is introduced instead of the edge crossing to planarize K_5 .

If the girth g of G is known, we have a more precise lower bound [Kai72a]:

$$\nu(G) \geq m - \frac{g}{g-2}(n-2) \quad \text{for } g \geq 3 \quad (43)$$

Note that Equation 42 follows from Equation 43 with $g = 3$. We may assume that $g \geq 3$ since otherwise G has no cycles and is planar. Kainen [Kai72a] actually generalizes Equation 43 to the crossing number of a graph on an orientable surface with given genus, and Kainen and White [KW78] provide the corresponding result for nonorientable surfaces.

Another general lower bound for the crossing number was found by Ajtai et al. [ACNS82] and, independently, by Leighton [Lei83, p. 108]:¹²

Theorem 44 [ACNS82], [Lei83, p. 108] *If G is a graph with n vertices and m edges, and if $m \geq 4n$, then*

$$\nu(G) \geq \frac{m^3}{100n^2} .$$

[SSSV94, SSSV96c] generalize this result to the crossing number of graphs on orientable surfaces with given genus. [SSV95] reports further results on lower bounds for the crossing number, involving the bisection width of the graph ([Lei83, PSS94, SV94] and [SV93a]) as well as embedding a graph into another one ([Lei83, SS92, SS94, SSSV94]). In particular, we have:

Theorem 45 [SV93a] *If G is a graph with n vertices, maximum degree Δ , and with bisection width b , then*

$$\nu(G) \geq \frac{b^2}{75\Delta} - n .$$

The bisection width of a graph $G = (V, E)$ with n vertices is defined to be $b = \min\{|E(V_1, V_2)|\}$ where the minimum is taken over all partitions of V into two sets V_1 and V_2 with $|V_1|, |V_2| \geq n/3$, and where $E(V_1, V_2)$ denotes the set of edges with one endpoint in V_1 and the other in V_2 .

Still another lower bound for the crossing number of a graph was recently given in [PT97].

Section 6.1 deals with the problem of finding the crossing number for a general graph. Section 6.2 introduces the notion of crossing-critical graphs. Section 6.3 lists papers that

¹²[Lei83, p. 108] actually shows that if $m \geq 4n$ then $\nu(G) \geq \frac{m^3}{375n^2}$.

examine the crossing number for particular graph classes. The partial results that are known for complete and complete bipartite graphs are also listed. Finally, Section 6.4 mentions some variations of the crossing number.

6.1 Finding the Crossing Number of a Graph

We are interested in the following problem:

Problem 46 (Crossing Number) *Given a graph G and a positive integer K , is there a drawing of G with K or less edge crossings?*

The complexity status of this problem was mentioned as being open in [GJ79, Problem OPEN3]. Then Garey and Johnson [GJ83] showed that Crossing Number is NP-complete. They use a two-step reduction, starting with the following NP-complete problem [GJS76]:

Problem 47 (Optimal Linear Arrangement [GJ79, Problem GT42]) *Given a graph $G = (V, E)$ and a positive integer K , is there a bijection $f : V \rightarrow \{1, 2, \dots, |V|\}$ such that $\sum_{uv \in E} |f(u) - f(v)| \leq K$?*

First, Optimal Linear Arrangement is reduced to a problem introduced as Bipartite Crossing Number:

Problem 48 (Bipartite Crossing Number [GJ83]) *Given a connected bipartite graph $G = (V_1, V_2, E)$ with multiple edges allowed, and given a positive integer K , can G be drawn in the unit square so that all vertices in V_1 are on the northern boundary, all vertices in V_2 are on the southern boundary, all edges are within the square, and there are at most K edge crossings?*

Then, Bipartite Crossing Number is reduced to Crossing Number.

So it is unlikely that a polynomial time algorithm that determines the crossing number of a given graph will be found. Note that any algorithm for drawing a graph in the plane can trivially be seen as a heuristic for the crossing number: We simply count the edge crossings of the resulting drawing. The author is only aware of [OS94] for a concrete heuristic designed specifically for drawing a graph with few crossings. It is based on finding a maximal planar subgraph. Two variations are tested on complete graphs with 5 to 14 vertices, on complete bipartite graphs $K_{n,n}$ with 6 to 16 vertices, and on 11 other graphs for which the crossing number is known or conjectured. One of the two variants performs rather well on the graphs tested, but it incurs high running times.

6.2 Crossing-Critical Graphs

If a graph G has crossing number k , then clearly any subgraph of G has crossing number at most k . We are interested in the “smallest” graph with crossing number k :

Definition 49 (crossing-critical graphs) *If a graph G has crossing number k and if every proper subgraph of G has crossing number less than k , then G is said to be crossing-critical.*

[Koc87] gives, for any $k \geq 2$, a construction of an infinite family of 3-connected crossing-critical graphs with crossing number k . This improves the result in [Šir84].

Note the analogy of the above definition to thickness-minimal graphs discussed in Section 5.2. But this analogy does not carry over to structural results about the crossing number and crossing-critical graphs. While deleting an edge from a graph G can decrease the thickness of G by at most one, the same is not true for the crossing number:

Theorem 50 [Koc91] *For any positive integer k there is a 3-connected graph $G = (V, E)$ with crossing number $\nu(G) = 4k$ and with an edge $e \in E$ so that $\nu(G' = (V, E \setminus \{e\})) = 3k$, and so that e belongs to no Kuratowski subgraph of G .*

Further results about crossing-critical graphs can be found in [Šir83], [Ric88, MR94] and [RT93].

6.3 Results for Particular Classes of Graphs

There seem to be few results about the exact crossing number of particular graph classes. Some crossing numbers are known exactly, but often, only lower or upper bounds are known.

The crossing number of complete graphs is not known exactly:

Theorem 51

$$\nu(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

and equality holds for $n \leq 10$.

See for example [Guy71, Guy72] or [WB78] or [Wes96, Section 7.3] for a description of the history of this result. [Guy71, Guy72] also contains the proof of equality for $n \leq 10$.

The crossing number of complete bipartite graphs is not known exactly either:

Theorem 52

$$\nu(K_{n_1, n_2}) \leq \left\lfloor \frac{n_1}{2} \right\rfloor \left\lfloor \frac{n_1-1}{2} \right\rfloor \left\lfloor \frac{n_2}{2} \right\rfloor \left\lfloor \frac{n_2-1}{2} \right\rfloor$$

and equality holds for $\min(n_1, n_2) \leq 6$.

The history of this result is described in detail in [CL86, Chapter 4.3]. See also [WB78] or [Wes96, Section 7.3], for example. The most recent paper that contributed to this result is [Kle70]. [Woo93] was able to extend the proof of equality to $K_{7,7}$ and $K_{7,9}$.

The crossing number of the hypercube of dimension n , Q_n , is studied in [Kai72a, Kai72b, Mad91], [SV92, SV93b], and [DR95, Far94, FF]. The crossing numbers of many Cartesian product graphs have been studied [KW78, RB78, JŠ82, PPV86, Kle91, Kle94, Kle95, DR95, KRS96, SSSV96a]. Further classes of graphs were investigated in [GH73, Asa86, Fio86, MR92], [SV92, SV93b], and [RŠ96].

6.4 Variations of Crossing Number

The following restricted version of the crossing number is studied intensely within the field of graph drawing:

Definition 53 (rectilinear crossing number) *The rectilinear crossing number of a graph G , denoted $\overline{\nu}(G)$, is the smallest number k so that G can be drawn in the plane with at most k edge crossings, and so that each edge of G is drawn as a straight-line segment.*

Clearly $\overline{\nu}(G) \geq \nu(G)$. For graphs with bounded degree, the crossing number and the rectilinear crossing number are bounded functions of one another [BD92, SSSV95]. But for every $m > k \geq 4$, there exists a graph G with $\nu(G) = k$ and $\overline{\nu}(G) \geq m$ [BD93]. So the rectilinear crossing number can be arbitrarily large in comparison to the crossing number.

A further restriction of the rectilinear crossing number yields the following problem: The vertices of the graph under consideration are partitioned into $k \geq 2$ classes (usually called *layers* in this context) numbered from 1 to k such that edges only exist between vertices of consecutive layers. All vertices of a particular layer have to be drawn on a horizontal line, while edges still have to be drawn as straight-line segments, with as few edge crossings as possible. See [JM96a, JLMO97, SSSV97a, SSSV97b], for example.

Still further variations of the crossing number are studied in [MNKF90, SSSV95, SSSV96b], for example.

7 Coarseness

Apparently by accident, Paul Erdős introduced the notion of coarseness of a graph [Har69]:

Definition 54 (coarseness) *The coarseness of a graph G , denoted $\xi(G)$, is the largest number of pairwise edge disjoint nonplanar subgraphs contained in G .*

We only mention the coarseness for the sake of completeness, and refer the interested reader to the literature: The coarseness (and variations thereof) of some graph classes have been studied in [GB68, BG69, Kai73, Har79, AP93], for example.

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