Hermitian structures and harmonic morphisms in higher dimensional Euclidean spaces

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Abstract

We construct new complex-valued harmonic morphisms from Euclidean spaces from functions which are holomorphic with respect to Hermitian structures. In particular, we give the first global examples of complex-valued harmonic morphisms from $\mathbf{R}^n$ for each $n > 4$ which do not arise from a Kähler structure; it is known that such examples do not exist for $n \leq 4$.

1 Introduction

A *harmonic map* $\phi : U \to \mathbf{C}$ from an open subset U of Euclidean n-space $\mathbf{R}^n$ is a solution of Laplace's equation

$$\Delta\phi \equiv \sum_{i=1}^{n} \frac{\partial^2\phi}{(\partial x^i)^2} = 0, \quad (x^1, \dots x^n) \in U; \tag{1}$$

such a map is a *harmonic morphism* if it additionally satisfies the condition of *horizontal weak conformality*:

$$\sum_{i=1}^{n} \left(\frac{\partial\phi}{\partial x^i}\right)^2 = 0, \quad (x^1, \dots x^n) \in U. \tag{2}$$

(Equivalently, and in more generality, a harmonic morphism is a map between Riemannian manifolds which pulls back germs of harmonic functions to germs of harmonic functions — see [7, 12].)

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Suppose that n is even, say $n = 2m$. Then it is well-known that a map $\phi : U \to \mathbf{C}$ which is holomorphic with respect to the standard complex structure J on $\mathbf{R}^{2m}$ (given, on identifying the latter with $\mathbf{C}^m$, by multiplication by i) is a harmonic morphism; indeed ϕ is holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x^{2j}} = \mathrm{i}\frac{\partial \phi}{\partial x^{2j-1}}, \quad (j = 1, \dots m)$$

which are easily seen to imply (1) and (2). More generally, if ϕ is holomorphic with respect to any Kähler structure (i.e. orthogonal complex structure) on U, it is a harmonic morphism — this is an example of the general result that holomorphic maps from a Kähler manifold to $\mathbf{C}$ (or to a Riemann surface) are harmonic morphisms [7].

Let us now give U a non-Kähler (integrable) Hermitian structure (see §2). Then, unless J is cosymplectic (i.e. has zero divergence, see equation (11) below), a holomorphic map $(U, J) \to \mathbf{C}$ still satisfies (2), however it is no longer automatically harmonic. (Note that it is, however, always a Hermitian harmonic map in the sense of Jost and Yau [13].) In fact it is harmonic if and only if the divergence of J lies in the kernel of the differential of ϕ (see, for example, [11]). It is not easy to see how to find maps satisfying this condition in general. However, in [17] with some further development in [1], strong relationships were shown between Hermitian structures and harmonic maps from $\mathbf{R}^4$, (or, more generally, any 4-dimensional anti-self-dual Einstein manifold) to $\mathbf{C}$ or to a Riemann surface; in particular, this provided a way of finding all (submersive) harmonic morphisms from open subsets of $\mathbf{R}^4$ to $\mathbf{C}$. In this paper we explore this relationship in higher dimensions, and in particular give a new way of finding many interesting locally and globally defined harmonic morphisms from open subsets of $\mathbf{R}^{2m}$ which are holomorphic with respect to non-Kähler structures.

We find many interesting differences between the 4-dimensional case ($m = 2$) and the higher dimensional case ($m > 2$). Our main results are as follows:

(i) In [17] it is shown that in the 4-dimensional case any harmonic morphism $\phi : \mathbf{R}^4 \supset U \to \mathbf{C}$ is holomorphic with respect to precisely one Hermitian structure — unless it has totally geodesic fibres in which case it is holomorphic with respect to precisely two Hermitian structures of opposite orientations. Further, in either case, the fibres are superminimal with respect to these Hermitian structures, i.e. the Hermitian structures are parallel (constant) along the fibres of ϕ. In the higher dimensional case we construct harmonic morphisms $\mathbf{R}^{2m} \supset U \to \mathbf{C}$ which are holomorphic with respect to Hermitian structures, some with fibres which are superminimal and others with fibres which are not superminimal with respect to any Hermitian structure. The examples can be chosen with J cosymplectic or not, see Theorem 4.15. None of these examples is holomorphic with respect to any Kähler structure and they are *full*, (see Definition 4.1). We give another example which is holomorphic with respect to a family of Hermitian structures and which has superminimal fibres with respect to some of the family but not all.

(ii) In the 4-dimensional case it is shown in [17] that a globally defined harmonic morphism $\mathbf{R}^4 \to \mathbf{C}$ is holomorphic with respect to a Kähler structure at least if it is submersive. In contrast, in the higher dimensional case we construct globally defined harmonic morphisms which are not holomorphic with respect to any Kähler structure, and in fact have fibres which are not superminimal with respect to any almost complex structure, see Theorem 4.17. These can be chosen to be full or to factor to full globally defined harmonic morphisms $\mathbf{R}^{2m-1} \to \mathbf{C}$, see Theorem 4.20. They can be chosen to be submersive or not.

(iii) Thus, for all $n > 4$, we have full globally defined harmonic morphisms $\mathbf{R}^n \to \mathbf{C}$ which do not arise (see Definition 4.8) from Kähler structures (Corollary 4.21).

Since, by [2], the fibres of a submersive harmonic morphism to $\mathbf{C}$ (or to a Riemann surface) form a conformal foliation by minimal submanifolds of codimension 2, we obtain many interesting such foliations, in particular (Theorem 4.22), global foliations of $\mathbf{R}^n$ for any $n > 4$ which do not arise from a Kähler structure.

Note that, for simplicity, our codomain is taken to be $\mathbf{C}$. However, the methods of this paper apply replacing this by any Riemann surface. In particular, in the last section, we discuss those harmonic morphisms which factor to give harmonic morphisms from a $2n$-torus to a 2-torus and show, that, if $n > 2$, as well as being holomophic with respect to infinitely many Kähler structures, any such map is also holomorphic with respect to infinitely many non-Kähler Hermitian structures.

In a sequel to this paper, we shall study which harmonic morphisms constructed by our method factor to spheres, real, complex and quaternionic projective spaces, giving new examples on these spaces. We shall also consider harmonic morphisms which are holomorphic with respect to more general CR structures on odd dimensional spaces.

We now describe the idea of our construction of harmonic morphisms.

(1) Suppose that $\phi : \mathbf{R}^{2m} \supset U \to \mathbf{C}^k$ is holomorphic with respect to an almost Hermitian structure J. Then its Laplacian, normally a second order partial differential operator (1) can be expressed just in terms of first derivatives of ϕ and the matrix valued function $M : U \to \mathbf{C}$ representing J, see Equation (9).

(2) Now suppose that $k = m$ and $z : \mathbf{R}^{2m} \supset U \to \mathbf{C}^k$ is holomorphic with respect to a *Hermitian* (i.e. integrable almost Hermitian) structure J and has independent components. Then these components give local complex coordinates for (U, J) and so M is a function of them, holomorphic by integrability of J. More generally, this remains true for any k and any holomorphic map z if we suppose that J is constant along the components of the fibres of z. (In the case $k = 1$ this means that $z : U \to \mathbf{C}$ has superminimal fibres.) In this case

the expression for the Laplacian of z becomes (13).

(3) Now, any holomorphic z must be related to the standard local complex coordinates $w = q - M(z)\bar{q}$ for (U, J) by an equation of the form (18) where f is holomorphic. In the case when $k = m$ we can generally write f in the simpler form (25). In either case, we can find a formula for the Laplacian of z just in terms of the holmorphic data (f, M) (or (h, M)). Our idea is to choose this holomorphic data such that one or more components z^a of z has zero Laplacian and so is a harmonic mrophism. Examples 4.9 – 4.12 are found this way.

(4) If two or more components $z^{a_1}, \dots, z^{a_\ell}$ of z are harmonic, we may compose the function $(z^{a_1}, \dots, z^{a_\ell}) : U \to V \subset \mathbf{C}^l$ with a holomorphic map $\psi : V \to \mathbf{C}$ to obtain (see §3.2) more harmonic morphisms; this device is used to find our global Examples 4.13, 4.14.

(5) We check that all examples $\psi \circ z$ are full and not holomorphic with respect to a Kähler structure using tests in §4.1.

(6) All harmonic morphisms $U \to \mathbf{C}$ which are holomorphic with respect to a Hermitian structure are constructed, at least locally, by our method, see Proposition 3.18.

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2 Parametrization of Hermitian structures on $\mathbf{R}^{2m}$.

Throughout this paper we shall adopt the double summation convention.

We shall consider the Euclidean space $\mathbf{R}^{2m}$ together with its standard orientation. Let $x \in \mathbf{R}^{2m}$. An *almost Hermitian structure at* x or *on* $T_x\mathbf{R}^{2m}$ is an orthogonal transformation $J_x : T_x\mathbf{R}^{2m} \to T_x\mathbf{R}^{2m}$, with the property that $J_x^2 = -I$ where I is the identity transformation. We say that J_x is *compatible with the orientation* or *positive* if there exists an oriented orthonormal basis of $T_x\mathbf{R}^{2m}$ of the form $\{e_1, e_2, \dots, e_{2m}\}$ with $e_{2j} = J_x e_{2j-1}, (j = 1, \dots, m)$. Otherwise, we say that J_x is *incompatible with the orientation* or *negative*. By an *almost Hermitian structure* J on an open subset U of $\mathbf{R}^{2m}$ we mean the choice of an almost Hermitian structure at each point x of U which varies smoothly with x.

By a *complex chart* $\phi : (V, J) \to \mathbf{C}^m$ on an open subset V of U we mean a diffeomorphism of V onto an open subset of $\mathbf{C}^m$ such that $d\phi(JX) = \mathrm{i}\, d\phi(X)$ for all $X \in T_x\mathbf{R}^{2m}$, $x \in V$. We say that J is *integrable on* U if we can find such complex charts in the neighbourhood of each point of U. Note that this gives

U the structure of an m-dimensional complex manifold. An integrable almost Hermitian structure on U is also called a *Hermitian structure on* U. It is called a *Kähler structure* or an *orthogonal complex structure* if it is parallel, i.e. does not vary from point to point. The standard Hermitian (in fact Kähler) structure on $\mathbf{R}^{2m}$ is that given by the complex chart $\mathbf{R}^{2m} \to \mathbf{C}^m$, $(x^1, \ldots, x^{2m}) \mapsto (x^1 + \mathrm{i}x^2, \ldots, x^{2m-1} + \mathrm{i}x^{2m})$.

Let $x \in \mathbf{R}^{2m}$ and suppose that J_x is a positive almost Hermitian structure at x. Consider the complex linear extension $J_x^{\mathbf{c}} : T_x^{\mathbf{c}}\mathbf{R}^{2m} \to T_x^{\mathbf{c}}\mathbf{R}^{2m}$ to the complexified tangent space. Since $(J_x^{\mathbf{c}})^2 = -I$, the space $T_x^{\mathbf{c}}\mathbf{R}^{2m}$ splits into the sum of two eigenspaces $T_x^{(1,0)}\mathbf{R}^{2m}$ and $T_x^{(0,1)}\mathbf{R}^{2m}$ corresponding to the eigenvalues $+\mathrm{i}$ and $-\mathrm{i}$ of J_x respectively; these are called the $(1,0)$- and $(0,1)$-tangent spaces of J_x. Then $T_x^{(1,0)}\mathbf{R}^{2m}$ is spanned by vectors of the form $e - \mathrm{i}J_x e$ ($e \in T_x\mathbf{R}^{2m}$). Such a vector is *isotropic* in the sense that $\langle e - \mathrm{i}J_x e,\ e - \mathrm{i}J_x e\rangle^{\mathbf{c}} = 0$. Here $\langle\ ,\ \rangle^{\mathbf{c}}$ denotes the standard symmetric bilinear inner product on $\mathbf{C}^n$ given by $\langle z, w\rangle^{\mathbf{c}} = z^1 w^1 + \ldots + z^n w^n$.

A complex subspace $V \subset \mathbf{C}^{2m}$ is called *isotropic* if $\langle v, v\rangle^{\mathbf{c}} = 0$ for each $v \in V$. (This implies that $\langle v, w\rangle^{\mathbf{c}} = 0$ for all $v, w \in V$.) Such a subspace is called *maximal (isotropic)* (or *Lagrangian*) if it is not contained in a larger isotropic subspace; note that an isotropic subspace is maximal if and only if it has complex dimension m. Suppose that V is maximal isotropic. Then call V *positive* if it has a basis of the form $\{e_1 - \mathrm{i}e_2, \ldots, e_{2m-1} - \mathrm{i}e_{2m}\}$ where $\{e_1, \ldots, e_{2m}\}$ is a positive orthonormal basis of $\mathbf{R}^{2m}$.

For any $x \in \mathbf{R}^{2m}$ we may identify in a canonical way $T_x\mathbf{R}^{2m}$ with $\mathbf{R}^{2m}$ and $T_x^{\mathbf{c}}\mathbf{R}^{2m}$ with $\mathbf{C}^{2m}$. Then there is a one-to-one correspondence between the set $\mathcal{J}_x^+(\mathbf{R}^{2m})$ of positive almost Hermitian structures at x and the set $\mathrm{Iso}^+(\mathbf{C}^m)$ of positive maximal isotropic subspaces of $T_x^{\mathbf{c}}\mathbf{R}^{2m}$ given by sending $J_x \in \mathcal{J}_x^+(\mathbf{R}^{2m})$ to its $(1,0)$-tangent space. Further, both spaces can be identified with the Hermitian symmetric space $SO(2m)/U(m)$. Indeed let $A \in SO(2m)$. The columns $\{e_1, \ldots e_{2m}\}$ of A define a positive orthonormal basis. Then the coset of $SO(2m)/U(m)$ defined by A corresponds to the positive almost Hermitian structure J_x at x with $J_x(e_{2j-1}) = e_{2j}$ $(j = 1, \ldots m)$; this in turn corresponds to the maximal isotropic subspace of $\mathbf{C}^{2m}$ with basis $\{e_1 - \mathrm{i}e_2, \ldots, e_{2m-1} - \mathrm{i}e_{2m}\}$.

We next define the *twistor bundle* $\pi : Z^+ \to \mathbf{R}^{2m}$ as the bundle whose fibre at each point $x \in \mathbf{R}^{2m}$ consists of all positive Hermitian structures at x. As a real fibre bundle, $Z^+ = \mathbf{R}^{2m} \times SO(2m)/U(m)$. The *vertical tangent bundle* $T^V Z^+$ is the subbundle of TZ^+ consisting of tangents to the fibres of Z^+, equivalently, $T^V Z^+$ is the kernel of the differential $d\pi : TZ^+ \to T\mathbf{R}^{2m}$; the *horizontal tangent bundle* $T^H Z^+$ is its orthogonal complement with respect to the product metric on Z^+. We give Z^+ a complex structure in a standard way as follows: For any $J \in Z^+$, we give $T_J^V Z^+$ the almost Hermitian structure induced from that of $SO(2m)/U(m)$. For $T_J^H Z^+$, note that each $J \in Z^+$ defines an almost complex structure on $T_{\pi(J)}\mathbf{R}^{2m}$; we pull this back to $T_J^H Z$ via the isomorphism $d\pi_J$. Finally, we define the almost complex structure $\mathcal{J}$ on Z^+ by

$\mathcal{J} = \mathcal{J}^H \oplus \mathcal{J}^V$. As is well-known, this structure is integrable. Next note that an almost complex structure J on an open subset U of R^{2m} can be considered as a map $J : U \to SO(2m)/U(m)$ or as a section $\sigma_J : U \to Z^+$; then we have:

Lemma 2.1 *[5, 4] Let J be a positive almost Hermitian structure on $U \overset{open}{\subset} \mathbf{R}^{2m}$. Then the following are equivalent:*

1. *J is integrable;*
2. *the corresponding map $J : U \to SO(2m)/U(m)$ is holomorphic;*
3. *the corresponding section $\sigma_J : U \to Z^+$ is holomorphic;*
4. *the image of the section σ_J is a complex submanifold of $(Z^+, \mathcal{J})$.*

We must now interpret our development in terms of the *co*-tangent space. Note that the natural linear isomorphism $\bar{\flat} : T_x^{\mathbf{c}}\mathbf{R}^{2m} \to T_x^{\star\mathbf{c}}\mathbf{R}^{2m}$ given by $X^{\bar{\flat}}(Y) = \langle \bar{X}, Y \rangle^{\mathbf{c}}$, $(X, Y \in T_x^{\mathbf{c}}\mathbf{R}^{2m})$ gives a one-to-one correspondence between (positive) isotropic subspaces of $T_x^{\mathbf{c}}\mathbf{R}^{2m}$ and those of $T_x^{\star\mathbf{c}}\mathbf{R}^{2m}$; the subspace of $T_x^{\star\mathbf{c}}\mathbf{R}^{2m}$ corresponding to the $(1,0)$-tangent space (resp. $(0,1)$-tangent space) of a positive almost Hermitian structure being called its $(1,0)$-*cotangent space*, $T_x^{\star(1,0)}\mathbf{R}^{2m}$ (resp. its $(0,1)$-*cotangent space*, $T_x^{\star(0,1)}\mathbf{R}^{2m}$).

Next, suppose that $V_0 \in \mathrm{Iso}^+(\mathbf{C}^{2m})$ is the $(1,0)$-cotangent space of a positive almost Hermitian structure $J_0 \in \mathcal{J}_x^+(\mathbf{R}^{2m})$. Note that $\mathbf{C}^{2m} = V_0 \oplus \bar{V}_0$. Then it is easily seen that the graph of a linear map $M : V_0 \to \bar{V}_0$ belongs to $\mathrm{Iso}^+(\mathbf{C}^{2m})$ if and only if M is skew-symmetric. This correspondence provides a chart for $\mathrm{Iso}^+(\mathbf{C}^{2m})$ centred on V_0 or, equivalently, for $\mathcal{J}_x^+(\mathbf{R}^{2m})$ centred on J_0. Explicitly, let $\{q^j = x^{2j-1} + \mathrm{i}x^{2j} : j = 1 \dots m\}$ denote complex coordinates for $(\mathbf{R}^{2m}, J_0)$, then the $(1,0)$- and $(0,1)$- cotangent spaces of J_0 have bases $\{dq^j : j = 1, \dots m\}$ and $\{dq^{\bar{j}} : j = 1, \dots m\}$ respectively. The above chart centred on J_0 can be described as follows:

Define a map

$$\iota : \mathbf{C}^{m(m-1)/2} \overset{\cong}{\to} so(m, \mathbf{C}) \to Iso^+(\mathbf{C}^{2m}) \to \mathcal{J}_x^+(\mathbf{R}^{2m}) \tag{3}$$

by

$$\mu = (\mu_1, \dots, \mu_{m(m-1)/2}) \mapsto M \mapsto V \mapsto J \tag{4}$$

where $M = M(\mu)$ is the $m \times m$ antisymmetric matrix given by

$$\left(M_{\bar{j}}^i(\mu)\right) = \begin{pmatrix} 0 & \mu_1 & \mu_2 & \cdots & \mu_{m-1} \\ -\mu_1 & 0 & \mu_m & \cdots & \mu_{2m-3} \\ -\mu_2 & -\mu_m & 0 & \cdots & \mu_{3m-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mu_{m-1} & -\mu_{2m-3} & -\mu_{3m-6} & \cdots & 0 \end{pmatrix},$$

$V = V(M)$ is the positive maximal isotropic subspace given by

$$V = \text{span}\{e^i = dq^i - M^i_{\bar{j}} dq^{\bar{j}} : i = 1, \ldots, m\}$$

and $J = J(V)$ is the positive Hermitian structure at x with $(1,0)$-cotangent space V. Note that these mappings are well-defined since:

(i) we have

$$\begin{aligned} \langle e^i, e^j \rangle^{\mathbf{c}} &= \langle dq^i - M^i_{\bar{k}} dq^{\bar{k}}, dq^j - M^j_{\bar{\ell}} dq^{\bar{\ell}} \rangle^{\mathbf{c}} \\ &= 2(M^i_{\bar{j}} + M^j_{\bar{i}}) \\ &= 0 \end{aligned}$$

since M is skew-symmetric and so $V = V(M)$ is isotropic, further the e^i form a positively oriented basis and so V is maximal isotropic and positive;

(ii) any positive maximal isotropic subspace V is the $(1,0)$-cotangent space of a unique positive almost Hermitian structure $J = J(V)$ at the point $(q^1, \ldots, q^m) \in \mathbf{C}^m = \mathbf{R}^{2m}$.

Note that the map (3) gives a chart for a dense subset of $\text{Iso}^+(\mathbf{C}^{2m}) = SO(2m)/U(m)$ called *a large cell* — in fact it is essentially the exponential map.

For simplicity of notation we shall frequently indicate the various compositions of maps in (4) by giving the dependent and independent variables, e.g. $J(M) = J(V(M))$ and $J(\mu) = J(V(M(\mu)))$, (i.e. ι), and their inverse maps by $M(J)$, $\mu(J)$ etc.

For computations, the following is useful (cf. [16]):

Lemma 2.2 *The $(0,1)$-tangent space of J_x has basis $\{e_{\bar{i}} : i = 1, \ldots, m\}$ where*

$$e_{\bar{i}} = \frac{\partial}{\partial q^{\bar{i}}} + M^j_{\bar{i}} \frac{\partial}{\partial q^j}.$$

Proof It suffices to note that, under the natural pairing of $T^{\star\mathbf{c}}_x(\mathbf{R}^{2m})$ and $T^{\mathbf{c}}_x(\mathbf{R}^{2m})$ the vectors e^i annihilate the vectors $e_{\bar{j}}$.

Example 2.3 Let $m = 2$. Then

$$M(\mu) = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

and the almost Hermitian structure $J(\mu)$ corresponding to $\mu \in \mathbf{C}$ has $(1,0)$-cotangent space spanned by $\{dq^1 - \mu dq^{\bar{2}},\ dq^2 + \mu dq^{\bar{1}}\}$. This extends to a diffeomorphism $\mathbf{C}P^1 \to \text{Iso}^+(\mathbf{R}^4) \cong SO(4)/U(2)$ given by

$$[\mu_0, \mu_1] \mapsto \text{span}\{\mu_0 dq^1 - \mu_1 dq^{\bar{2}}, \mu_0 dq^2 + \mu_1 dq^{\bar{1}}\}.$$

Example 2.4 Let $m = 3$. Then, attempting to extend the map (4) to $\mathbf{C}P^3$ by introducing μ_0 as in Example 2.3 we find that the corresponding covectors $\mu_0 dq^1 - \mu_1 dq^{\bar{2}} - \mu_2 dq^{\bar{3}}$, $\mu_0 dq^2 + \mu_1 dq^{\bar{1}} - \mu_3 dq^{\bar{3}}$, $\mu_0 dq^3 + \mu_2 dq^{\bar{1}} + \mu_3 dq^{\bar{2}}$ are linearly dependent if $\mu_0 = 0$. However, consider the fourth covector: $\mu_3 dq^1 - \mu_2 dq^2 + \mu_1 dq^1$. If $\mu_0 \neq 0$, this is a linear combination of the above covectors, if $\mu_0 = 0$, it isn't; in both cases the *four* covectors span a positive isotropic subspace of $\mathbf{C}^6$ of dimension *three* and so define a diffeomorphism $\mathbf{C}P^3 \to \mathrm{Iso}^+(\mathbf{C}^6) \cong SO(6)/U(3)$ extending (4).

Note that in higher dimensions, no such extension of (4) is possible since then $\mathrm{Iso}^+(\mathbf{R}^{2m})$ is not homeomorphic to any complex projective space.

We next construct complex coordinates on the twistor bundle $Z^+ = \mathbf{C}^m \times \mathcal{J}^+(\mathbf{R}^{2m})$. Note that this has *product coordinates* given by the formula $(q, J) \mapsto (q^1, \ldots, q^m, \mu_1(J), \ldots, \mu_{m(m-1)/2}(J))$ but these are not complex coordinates for the complex structure $\mathcal{J}$ on Z^+ since the q^i are not holomorphic functions on Z^+. However we have

Proposition 2.5 *The functions* $(q, \mu) \mapsto w^i = q^i - M^i_{\bar{j}}(\mu) q^{\bar{j}}$, $(i = 1, \ldots, m)$ *and* $(q, \mu) \mapsto \mu^i$, $(i = 1, \ldots, m(m-1)/2)$ *define complex coordinates on a dense open subset of the twistor space* $Z^+ = \mathbf{R}^{2m} \times \mathcal{J}^+(\mathbf{R}^{2m})$.

Proof It is clear that the μ^i are holomorphic on Z^+ (indeed they give complex coordinates for each fibre), whilst

$$\begin{aligned} dw^i &= d(q^i - M^i_{\bar{j}}(\mu) q^{\bar{j}}) \\ &= dq^i - M^i_{\bar{j}}(\mu) dq^{\bar{j}} + q^{\bar{j}} dM^i_{\bar{j}}(\mu) \end{aligned}$$

is the sum of covectors of type $(1, 0)$, since $M^i_{\bar{j}}(\mu)$ is holomorphic in μ. Thus the w^i are also holomorphic. (Alternatively we calculate $e_{\bar{j}}(w^i) = 0$.) Clearly the set of differentials $\{dw^i : i = 1, \ldots m\} \cup \{dq^j : j = 1, \ldots m(m-1)/2\}$ is linearly dependent on a dense set

We can also interpret the w^i as complex coordinates on $\mathbf{R}^{2m}$ as follows:

Corollary 2.6 *Let* $J = J(M(\mu(q)))$ *be a Hermitian structure on an open set* U *of* $\mathbf{R}^{2m}$. *Then the functions* $w^i : U \to \mathbf{C}$, $q \mapsto w^i = q^i - M^i_{\bar{j}}(\mu(q)) q^{\bar{j}}$ *are complex coordinates for* (U, J).

Proof The functions $q \mapsto w^i$ are compositions of the holomorphic coordinate functions $(q, \mu) \mapsto w^i$ on Z^+ and the holomorphic section σ_J corresponding to J (see Lemma 2.1).

We now turn to the construction of maps which are holomorphic with respect to a Hermitian structure. In [17] harmonic morphisms ϕ from open sets U of

$\mathbf{R}^4$ to $\mathbf{C}$ (or a Riemann surface) are constructed which are holomorphic with respect to a Hermitian structure J which is constant along the fibres of ϕ and is thus given by a composition of the form $J : U \xrightarrow{\phi} V \xrightarrow{\mu} \mathbf{C}$ where V is open in $\mathbf{C}$ and μ is holomorphic. In the next section we generalise this to consider maps $\phi : \mathbf{R}^{2m} \supset U \to \mathbf{C}^k$ which are holomorphic with respect to a Hermitian structure J that is constant along the fibres of ϕ and thus is given by the composition of ϕ with a holomorphic map

$$\mathbf{C}^k \overset{open}{\supset} V \to so(m, \mathbf{C}), \quad z \mapsto M(z). \tag{5}$$

Suppose that $\phi : \mathbf{R}^{2m} \overset{open}{\supset} U \to V \subset \mathbf{C}^k$, $z = \phi(q)$, is given implicitly in terms of the complex coordinates w for (U, J) by an equation

$$f(w, z) = 0 \tag{6}$$

where $f : \mathbf{C}^m \times \mathbf{C}^k \overset{open}{\supset} W \to \mathbf{C}^k$ is holomorphic. Then, in terms of standard coordinates $q = (q^1, \ldots, q^m)$ on $\mathbf{C}^m = \mathbf{R}^{2m}$, the map $z = \phi(q)$ is a solution to an equation of the form

$$f(q - M(z)\bar{q}, z) = 0 \tag{7}$$

where $M : \mathbf{C} \overset{open}{\supset} V \to so(m, \mathbf{C}) = \mathbf{C}^{m(m-1)/2}$ is a given holomorphic function. In the next section we consider when the components of a solution of such an equation are harmonic and thus harmonic morphisms.

3 Harmonic holomorphic mappings

In this section, we calculate the Laplacian of a map from an open subset U of $\mathbf{R}^{2m}$ which is holomorphic with respect to a Hermitian structure on U, firstly (§3.1) in general, then specializing first to Hermitian structures constant along the fibres (§3.3) then (§3.4) to holomorphic maps defined implicitly by Equation (7). Then in Section 4 we use these formulae to find examples of harmonic holomorphic maps. On the way, we make some general remarks about the space of all harmonic holomorphic maps on U (§3.2).

3.1 Formula for the Laplacian of a map which is holomorphic with respect to some Hermitian structure

Let U be an open subset of $\mathbf{R}^{2m}$ and let $J : U \to \mathcal{J}^+(\mathbf{R}^{2m})$, $J = J(M(q))$ be a positive almost Hermitian structure.

Let $\phi : U \to \mathbf{C}^k$ be a smooth map. We wish to calculate the Laplacian of ϕ. Now ϕ is holomorphic with respect to J if and only if

$$\frac{\partial \phi}{\partial q^{\bar{i}}} + M_{\bar{i}}^{j}(q) \frac{\partial \phi}{\partial q^j} = 0 \quad \text{for } i = 1, \ldots, m. \tag{8}$$

Assuming that this is the case, differentiating (8) with respect to q^i and summing over i gives

$$\frac{1}{4}\Delta\phi \equiv \sum_{i=1}^{m} \frac{\partial^2\phi}{\partial q^i \partial q^{\bar{i}}} = -\sum_{i,j=1}^{m} M_{\bar{i}}^{j} \frac{\partial^2\phi}{\partial q^i \partial q^j} - \sum_{i,j=1}^{m} \frac{\partial M_{\bar{i}}^{j}}{\partial q^i} \frac{\partial\phi}{\partial q^j}.$$

Now the first term is zero by the antisymmetry of $M_{\bar{i}}^{j}$ and the symmetry of $(\partial^2\phi/\partial q^i \partial q^j)$, hence we get

Lemma 3.1 *Let $\phi : U \to \mathbf{C}^k$ be holomorphic with respect to an almost Hermitian structure $J = J(M(q))$ on V. Then*

$$\frac{1}{4}\Delta\phi = -\sum_{i,j=1}^{m} \frac{\partial M_{\bar{j}}^{i}}{\partial q^j} \frac{\partial\phi}{\partial q^i}. \tag{9}$$

Remarks 3.2 1. We do not need J to be integrable for this lemma.

2. More generally, for any oriented Riemannian manifold M^{2m} and map $\phi : M^{2m} \to \mathbf{C}^k$ holomorphic with respect to an almost Hermitian structure J on M^m we have [11]:

$$\Delta\phi = -d\phi(J\delta J). \tag{10}$$

3. Formula (9) breaks down where $J(q)$ does not belong to a large cell for all $q \in V$ and so cannot be written as $J(M(q))$; but unless J is constant, such points are nowhere dense and, if J is integrable, they are isolated since then $J : V \to \mathcal{J}^+(\mathbf{R}^{2m})$ is holomorphic. We can thus replace V by a slightly smaller open set V' such that $J(V')$ does belong to a large cell.

3.2 The space of all harmonic holomorphic maps and cosymplectic manifolds

Let U be an open subset of $\mathbf{R}^{2m}$, $(m \in \mathbf{N})$ and let J be an almost Hermitian structure on U.

Proposition 3.3 *The set* $\mathrm{Haho}(U) = \{\phi : U \to \mathbf{C}^p : p \in \mathbf{N},\ \phi$ is harmonic and holomorphic with respect to $J\}$ *is closed under postcomposition with holomorphic maps $f : \mathbf{C}^p \to \mathbf{C}^q$, i.e. if $\phi \in$* Haho(U) *and $f : V \to \mathbf{C}^q$ is holomorphic, where V is an open subset of $\mathbf{C}^p$ containing $\phi(U)$ then the composition $f \circ \phi$ is in* Haho(U).

Proof By the chain rule, the Laplacian of such a composition $f \circ \phi$ is given by

$$\begin{aligned} \Delta(f \circ \phi) &= \frac{\partial^2 f}{\partial z^j \partial z^k} \langle \nabla\phi^j, \nabla\phi^k \rangle^{\mathbf{c}} + \frac{\partial f}{\partial z^l} \Delta\phi^l \\ &= 0 \end{aligned}$$

where $\nabla\phi^j = (\partial\phi^j/\partial x^1, \ldots, \partial\phi^j/\partial x^{2m})$ for $(x^1, \ldots, x^{2m}) \in \mathbf{R}^{2m}$. Now, as in [9], $\nabla\phi^j$, $\nabla\phi^k$ and $\nabla\phi^j + \nabla\phi^k$, being the gradients of holomorphic functions, are isotropic, so by the polarization identity, $\langle\nabla\phi^j, \nabla\phi^k\rangle^{\mathbf{c}} = 0$.

There is one special case when *all* holomorphic maps $\phi : U \to \mathbf{C}^p$ are harmonic, namely when J is *cosymplectic*, i.e. its divergence δJ is zero, δJ being defined by

$$\delta J = \sum (\partial_{e_i} J)(e_i) \tag{11}$$

where $\{e_i\}$ is any orthonormal basis of the tangent space (see [14, 11]).

Precisely we have

Proposition 3.4 *Let U be an open subset of $\mathbf{R}^{2m}$ and let $J = J(M)$ be a Hermitian structure on it. Then the following are equivalent:*

(1) for any point $x \in U$ there are m holomorphic harmonic maps $U' \to \mathbf{C}$ defined on a neighbourhood U' of p which are independent*, i.e. their gradients are linearly independent over $\mathbf{C}$;*

(2) all holomorphic maps $U' \to \mathbf{C}$ defined on an open subset U' of U are harmonic;

(3) the complex coordinates $w^i = q^i - M^i_{\bar{j}}(q)q^{\bar{j}}$ (see Corollary 2.6) are harmonic;

(4) the matrix M satisfies

$$\sum_{j=1}^{m} \frac{\partial M^i_{\bar{j}}}{\partial q^j} = 0 \quad \textit{for all} \quad i = 1 \ldots m;$$

(5) The Hermitian structure J is cosymplectic.

Proof That (5) implies (2) follows from the formula (10), and is, in any case, well-known (see [14]). For the converse, apply this formula to m independent holomorphic harmonic functions to see that all the components of the divergence of J vanish.

That (1) implies (2) follows by noting that, if $\phi_1, \ldots, \phi_m : U' \to \mathbf{C}$ are independent holomorphic functions, then any holomorphic function $\phi : U' \to \mathbf{C}$ is, locally a function of them, and so, by Proposition 3.3 , is harmonic. The converse is trivial and the equivalence of (2) and (3) is similar. That (3) implies (4) follows from Formula (9) applied to $\phi = w^k$. Similarly, that (4) implies (5) follows from Formula (10) applied to $\phi = w^k$.

3.3 The Laplacian of a holomorphic map when the Hermitian structure is constant along the fibres

Let $z : \mathbf{R}^{2m} \supset U \to \mathbf{C},\ q \mapsto z(q)$ be a map which is holomorphic with respect to a Hermitian structure $J = J(M(q))$ on U. Suppose that J is constant along the components of the fibres of $z(q)$. (Note that this is automatically true if

$k = m$ and $z(q)$ has independent components. For then these components give local complex coordinates for (U, J) and so M is a funtion of them.) In this case J is of the form:

$$J : \mathbf{R}^{2m} \supset U \to V \to so(m, \mathbf{C}) \to \mathcal{J}^+(\mathbf{R}^{2m}) \ , \quad q \mapsto z \mapsto M \tag{12}$$

for some map $\mathbf{C}^k \overset{open}{\supset} V \to so(m, \mathbf{C})$, $z \mapsto M(z)$ which is holomorphic if J is integrable. Assuming that is the case, by the chain rule Formula (9) becomes

$$\frac{1}{4}\Delta z^a = -\sum_{i,j=1}^{m} \sum_{b=1}^{k} A_i^a A_j^b \frac{\partial M_{\bar{j}}^{i}}{\partial z^b} \ , \quad (a = 1, ..., k) \ , \tag{13}$$

where

$$A_i^a = \partial z^a / \partial q^i \ , \quad (a = 1, ...k, \ i = 1, ...m), \tag{14}$$

which we may write in the form

$$\frac{1}{4}\Delta z^a = -\sum_{1 \le i < j \le m} \sum_{b=1}^{k} C_{ij}^{ab} \frac{\partial M_{\bar{j}}^{i}}{\partial z^b} \ , \quad (a = 1, \dots k) \tag{15}$$

where

$$C_{ij}^{ab} = A_i^a A_j^b - A_i^b A_j^a, \ (i, j = 1, \dots, m, \ , \quad a, b = 1, \dots, k). \tag{16}$$

(Note that $C_{ij}^{ab} = -C_{ji}^{ab} = -C_{ij}^{ba}$, in particular $C_{ij}^{aa} = C_{ii}^{ab} = 0$.) We thus have proved the

Lemma 3.5 *Suppose that $z : U \to \mathbf{C}^k$, $q \mapsto z(q)$ is holomorphic with respect to a Hermitian structure J given by a composition of the form (12), thus $J(q) = J(M(z(q)))$ for $q \in U$. Then the Laplacian of $z(q)$ is given by equations (13, 14), or equivalently by equations (15, 16).*

Our first example involves the concept of superminimality:

Definition 3.6 Let S be an even dimensional submanifold of an open subset U of $\mathbf{R}^{2m}$. We say that an almost complex structure J on U is *adapted* to S if S is a complex submanifold with respect to J, i.e. TS is closed under J. If S is adapted, we say that it is *superminimal with respect to* J if J is parallel (constant) along the submanifold.

If J is an almost Hermitian structure, superminimality implies minimality, see for example, [11].

Example 3.7 Suppose that $k = 1$, that is, suppose that $z : \mathbf{R}^{2m} \supset U \to \mathbf{C}$, $q \mapsto z(q)$ is holomorphic with respect to a Hermitian structure J that is constant along the fibres of $z(q)$; this implies that the regular fibres of $z(q)$ are superminimal. Then (15) reduces to

$$\Delta z = 0.$$

This is clear since $C^{11}_{ij}=0$ is the only coefficient. Of course we know this since superminimal fibres are minimal so that z, being horizontally weakly conformal, is a harmonic morphism by [2].

Remark 3.8 It can be seen directly that, for a conformal J-closed foliation of codimension 2 of a Hermitian manifold (M^{2m},J), the condition $\nabla_V J=0$ for all V tangent to the leaves implies that δJ is tangent to the leaves (see [11]).

3.4 The Laplacian of an implicitly defined holomorphic map

As at the end of §2, we consider holomorphic mappings $z:(V,J)\to\mathbf{C}^k$, $z=z(q)$ given implicitly by an equation of the form (7). To be precise our hypotheses are as follows:

Hypotheses 3.9 We suppose that $f:\mathbf{C}^m\times\mathbf{C}^k \overset{open}{\supset} W\to\mathbf{C}^k$ and $M:\mathbf{C}^k \overset{open}{\supset} V\to so(m,\mathbf{C})$ are given holomorphic mappings.

Defining $F:\mathbf{R}^{2m}\times\mathbf{C}^k \overset{open}{\supset} A\to\mathbf{C}^k$ by $F(q,z)=f(q-M(z)\bar{q},z)$ on the open set $A=\{(q,z): z\in V,\ (q-M(z)\bar{q},z)\in W\}$, supposed non-empty, we suppose that the Jacobian matrix

$$K=(\partial F^a/\partial z^b)_{a,b=1,\cdots,m} \tag{17}$$

is non-singular on a non-empty open subset $A'\subset A$.

Lastly we suppose that $z:\mathbf{R}^{2m} \overset{open}{\supset} U\to\mathbf{C}^k,\ z=z(q)$ is a smooth local solution to the equation

$$F(q,z)\equiv f(q-M(z)\bar{q},z)=0 \tag{18}$$

(i.e. $z(U)\subset V$, $(q,z(q))\in A'$ for all $q\in U$, and

$$F(q,z(q))=0 \tag{19}$$

for all $q\in U$.)

We can now give a formula for the Laplacian of such a solution to equation (18):

Proposition 3.10 *Suppose that Hypotheses 3.9 hold, in particular, suppose that $z:U\to\mathbf{C}^k, z=z(q)$ is a smooth local solution to equation (18). Then*

(i) $z:U\to\mathbf{C}^k$, $q\mapsto z(q)$ is holomorphic with respect to the Hermitian structure J on U given by $J(q)=J(M(z(q)))$ for $q\in U$;

(ii) this Hermitian structure is parallel (i.e. constant) along the fibres of $z(q)$;

(iii) the Laplacian of each component of $z(q)$ is given by (13), or equivalently (15, 16), where

$$A^a_i=-(K^{-1})^a_b\frac{\partial f^b}{\partial w^i}. \tag{20}$$

Remarks 3.11 1. That (18) has smooth local solutions $z(q)$ is guaranteed by the Implicit Function Theorem; indeed given any $(q_0, z_0) \in A$, (18) has a unique local smooth solution $z : U \to \mathbf{C}^k,\ z = z(q)$, on some neighbourhood U of q_0, such that $z(q_0) = z_0$.

2. That J is parallel along the fibres of $z(q)$ does not of course imply that J is parallel along the fibres of each component z^a of $z(q)$.

Proof (i) The map $z(q)$ is of the form

$$\mathbf{R}^{2m} \supset U \to \mathbf{C}^m \xrightarrow{\psi} \mathbf{C}^k, \quad q \mapsto w \mapsto z$$

where, using matrix notation for convenience, $w(q) = q - M(z(q))\bar{q}$, and $z = \psi(w)$ is a local solution to $f(w, z) = 0$, i.e. $f(w, \psi(w)) \equiv 0$. By the Implicit Function Theorem, ψ is holomorphic. Now since, by Corollary 2.6 the components of w give local coordinates, $w(q)$ is holomorphic with respect to the almost Hermitian structure J on U given by the composition

$$J : U \xrightarrow{z} V \subset \mathbf{C}^k \xrightarrow{M} so(m, \mathbf{C}) \cong C^{m(m-1)/2} \xrightarrow{\iota} \mathcal{J}^+(\mathbf{R}^{2m}). \tag{21}$$

Thus $z(q)$ is holomorphic with respect to J.

(Alternatively it can be seen from the chain rule for $F(q, z) = f(w(q, z), z)$ where $w(q, z) = q - M(z)\bar{q}$ that, for $\alpha = 1, \ldots k$,

$$\frac{\partial F^\alpha}{\partial q^i} = \frac{\partial f^\alpha}{\partial w^i} \quad \text{and} \quad \frac{\partial F^\alpha}{\partial q^{\bar{i}}} = -M_{\bar{i}}^j(z)\frac{\partial f^\alpha}{\partial w^j}$$

so that

$$(\frac{\partial}{\partial q^{\bar{i}}} + M_{\bar{i}}^j(z)\frac{\partial}{\partial q^j})F^\alpha = 0, \tag{22}$$

expressing the holomorphicity of $q \mapsto F(q, z)$ with respect to J. Then, differentiating (18) with respect to q^I, $(I = 1, \cdots, \bar{m})$ gives

$$\frac{\partial F^\alpha}{\partial q^I} + \frac{\partial F^\alpha}{\partial z^a}\frac{\partial z^a}{\partial q^I} = 0$$

so that

$$\frac{\partial z^a}{\partial q^I} = -(K^{-1})^a_\alpha \frac{\partial F^\alpha}{\partial q^I},$$

whence from (22) we obtain

$$(\frac{\partial}{\partial q^{\bar{i}}} + M_{\bar{i}}^j(z)\frac{\partial}{\partial q^j})z^a = 0,$$

showing that $z(q)$ is holomorphic with respect to J.)

That J is Hermitian follows from the fact that (21) exhibits J as the composition of holomorphic maps, so it is a holomorphic map $(U, J) \to \mathcal{J}^+(\mathbf{R}^{2m})$. By Lemma 2.1 it is integrable and so Hermitian.

(ii) Obvious from the form of J in (21) .

(iii) By Lemma 3.5 the Laplacian of $z(q)$ is given by equations (13, 14). Now, differentiating (19) totally with respect to q^i, we get, for $i = 1, \ldots, m$,

$$\frac{\partial F^b}{\partial q^i} + \frac{\partial F^b}{\partial z^a}\frac{\partial z^a}{\partial q^i} = 0 \tag{23}$$

whence

$$\frac{\partial z^a}{\partial q^i} = -(K^{-1})^a_b \frac{\partial F^b}{\partial q^i}. \tag{24}$$

$F(q, z) = f(q - M(z)\bar{q}, z)$,

$$\frac{\partial F^b}{\partial q^i} = \frac{\partial f^b}{\partial w^a}\frac{\partial w^a}{\partial q^i} = \frac{\partial f^b}{\partial w^i}$$

giving the formula (20) for the matrix A.

The special case $k = 1$.

In this case the right hand side of (15) is always zero and we get a result:

Proposition 3.12 *Suppose that Hypotheses 3.9 hold with $k = 1$, in particular, suppose that $z : \mathbf{R}^{2m} \supset U \to \mathbf{C}^k$, $q \mapsto z(q)$ is a smooth local solution to equation (18) with $k = 1$. Then*

(i) $z : U \to \mathbf{C}$ is holomorphic with respect to the Hermitian structure J on U given by $J(q) = J(M(z(q)))$ for all $q \in U$;

(ii) this Hermitian structure is parallel along the fibres of $z(q)$, i.e. the fibres of $z(q)$ are superminimal *with respect to J;*

(iii) $z(q)$ is a harmonic morphism;

(iv) $z(q)$ is submersive at points q where $\frac{\partial F}{\partial q}(q, z(q)) \neq 0$.

Remarks 3.13 (i) This result also follows from [10], see[11].

(ii) In the case $m = 2$ this is a slight reformulation of the results of [17]. In that case it gives, locally, *all* submersive harmonic morphisms $\mathbf{R}^4 \overset{open}{\supset} U \to \mathbf{C}$. We shall see below that this is no longer the case when $m > 2$.

Another special case of interest is when $k = m$ and f is of the form

$$f^a(w, z) = w^a - h^a(z) \tag{25}$$

with h holomorphic. We shall call this the *canonical form* of the equations (6). (The reason for this name is that if the holomorphic function f in Hypotheses 3.9 has Jacobian $(\partial f^a/\partial w^b)_{a,b=1,\ldots m}$ non-singular, we can solve the equations (6) locally for $w = h(z)$ and then the equations (6) can be written in the form (25).) In this case we can calculate the C^{ab}_{ij} in (15) by means of the following:

Lemma 3.14 *For any non-singular complex $k \times k$ matrix K, set $A = K^{-1}$ and $C^{ab}_{ij} = A^a_i A^b_j - A^b_i A^a_j$, $(i, j, a, b = 1, \dots, k)$. Then*

$$C^{ab}_{ij} = \frac{1}{\det K}(-1)^{i+j+a+b}\mathrm{sign}(j-i)\mathrm{sign}(b-a)E^{ij}_{ab} \tag{26}$$

where for $a < b$, $i < j$, E^{ij}_{ab} is the determinant of the matrix obtained from K by omitting rows i and j and columns a and b.

Proof Since permuting the rows and columns of a determinant simply multiplies it by the sign of the permutation, it suffices to prove the lemma for C^{12}_{12}. Let $L = (L^a_i)$ denote the $k \times k$-matrix of cofactors of K. Then

$$(\det K)^2 C^{12}_{12} = L^1_1 L^2_2 - L^1_2 L^2_1.$$

Expanding the determinants L^a_i along the top rows, we obtain

$$\begin{aligned}
(\det K)^2 C^{12}_{12} &= \left\{\sum_{r=2}^{k}(-1)^r K^r_2 E^{12}_{1r}\right\}\left\{K^1_1 E^{12}_{12} + \sum_{s=3}^{k}(-1)^s K^s_1 E^{12}_{2s}\right\} \\
&\quad - \left\{\sum_{r=2}^{k}(-1)^r K^r_1 E^{12}_{1r}\right\}\left\{K^1_2 E^{12}_{12} + \sum_{s=3}^{k}(-1)^s K^s_2 E^{12}_{2s}\right\} \\
&= E^{12}_{12}\sum_{r=2}^{k}(-1)^r E^{12}_{1r}\begin{vmatrix} K^1_1 & K^r_1 \\ K^1_2 & K^r_2 \end{vmatrix} + \sum_{r=2}^{k}\sum_{s=3}^{k}(-1)^{r+s}E^{12}_{1r}E^{12}_{2s}\begin{vmatrix} K^s_1 & K^r_1 \\ K^s_2 & K^r_2 \end{vmatrix} \\
&= E^{12}_{12}\sum_{r=2}^{k}(-1)^r E^{12}_{1r}\begin{vmatrix} K^1_1 & K^r_1 \\ K^1_2 & K^r_2 \end{vmatrix} - \sum_{\substack{r,s=2 \\ r<s}}^{k}(-1)^{r+s}(E^{12}_{1r}E^{12}_{2s} - E^{12}_{1s}E^{12}_{2r})\begin{vmatrix} K^r_1 & K^s_1 \\ K^r_2 & K^s_2 \end{vmatrix}.
\end{aligned} \tag{27}$$

The result will follow after we have established:

Claim 3.15

$$E^{12}_{1r}E^{12}_{2s} - E^{12}_{1s}E^{12}_{2r} = E^{12}_{12}\mathrm{sign}(s-r)E^{12}_{rs}.$$

For then the right hand side of (27) equals

$$E^{12}_{12}\left\{\sum_{r=2} +k(-1)^r E^{12}_{1r}\begin{vmatrix} K^1_1 & K^r_1 \\ K^1_2 & K^r_2 \end{vmatrix} - \sum_{\substack{r,s=2 \\ r<s}}^{k}(-1)^{r+s}E^{12}_{rs}\begin{vmatrix} K^r_1 & K^s_1 \\ K^r_2 & K^s_2 \end{vmatrix}\right\}$$

$$= -E^{12}_{12}\left\{\sum_{\substack{r,s=2 \\ r<s}}^{k}(-1)^{r+s}E^{12}_{rs}\begin{vmatrix} K^r_1 & K^s_1 \\ K^r_2 & K^s_2 \end{vmatrix}\right\}.$$

That the expression in brackets is $-\det K$ follows from Laplace's Theorem for determinants [15].

Proof of Claim 3.15 It suffices to prove the case $r < s$. We apply Sylvester's Theorem [15]. For this we need some notation. Denote the $(k-2)\times(k-2)$-determinant obtained from K by omitting rows 1 and 2 and columns r and s by $|12\cdots(r-1)(r+1)\cdots(s-1)(s+1)\cdots k|$. Then

$$E^{12}_{1r}E^{12}_{2s} \equiv |23\cdots(r-1)(r+1)\cdots k|\,|13\cdots(s-1)(s+1)\cdots k|$$

which by Sylvester's Theorem equals

$$\begin{aligned}
&|13\cdots(r-1)(r+1)\cdots k|\,|23\cdots(s-1)(s+1)\cdots k|\\
&+|33\cdots(r-1)(r+1)\cdots k|\,|124\cdots(s-1)(s+1)\cdots k|\\
&+|43\cdots(r-1)(r+1)\cdots k|\,|132\cdots(s-1)(s+1)\cdots k|\\
&\quad\vdots\\
&+|r3\cdots(r-1)(r+1)\cdots k|\,|13\cdots(r-1)2(r+1)\cdots(s-1)(s+1)\cdots k|\\
&\quad\vdots\\
&=|13\cdots(r-1)(r+1)\cdots k|\,|23\cdots(s-1)(s+1)\cdots k|\\
&+|r3\cdots(r-1)(r+1)\cdots k|\,|13\cdots(r-1)2(r+1)\cdots(s-1)(s+1)\cdots k|\,,
\end{aligned}$$

(all other terms vanishing due to their having two identical columns in a determinant). But this is precisely $E^{12}_{2r}E^{12}_{1s} + (-1)^{r+1}(-1)^{r+1}E^{12}_{12}E^{12}_{rs}$, establishing the result.

We can now find a formula for the Laplacian of a solution to (25). Firstly the hypotheses:

Hypotheses 3.16 Suppose that $h : \mathbf{C}^m \overset{open}{\supset} V \to \mathbf{C}^m$ and $M : \mathbf{C}^m \overset{open}{\supset} V \to so(m,\mathbf{C})$ are given holomorphic functions with h of maximal rank.

Set

$$F(q,z) = q - M(z)\bar{q} - h(z) \quad \text{for} \quad (q,z) \in \mathbf{R}^{2m}\times V\ .$$

We suppose that the Jacobian matrix $K = (\partial F^b/\partial z^a)_{a,b=1\cdots,k}$ is non-singular on some non-empty open set $A' \subset \mathbf{R}^{2m}\times V$.

Let $z : \mathbf{R}^{2m} \overset{open}{\supset} U \to V \subset \mathbf{C}^m,\ z = z(q)$ be a smooth local solution to equation (18):

$$F(q,z) \equiv q - M(z)\bar{q} - h(z) = 0 \tag{28}$$

Now the result:

Proposition 3.17 *Suppose that Hypotheses 3.16 hold. Then*

(i) $z : U \to \mathbf{C}^k$ *is holomorphic with respect to the Hermitian structure* J *on* U *given by* $J(q) = J(M(z(q))$ *for* $q \in U$*;*

(ii) this Hermitian structure is parallel (i.e. constant) along the fibres of $z(q)$*;*

(iii) the Laplacian of each component of $z(q)$ *is given by*

$$\frac{1}{4}\Delta z^a = \frac{(-1)^a}{\det K} \sum_{1\le i<j\le m} (-1)^{i+j-1} \begin{vmatrix} \frac{\partial M^i_{\bar{j}}}{\partial z^1} & \dots & \widehat{\frac{\partial M^i_{\bar{j}}}{\partial z^a}} & \dots & \frac{\partial M^i_{\bar{j}}}{\partial z^m} \\ K^{k_1}_1 & \dots & K^{k_1}_a & \dots & K^{k_1}_m \\ K^{k_{m-2}}_1 & \dots & K^{k_{m-2}}_a & \dots & K^{k_{m-2}}_m \end{vmatrix} \tag{29}$$

where $(k_1, ...k_{m-2}) = (1, \dots, \hat{\imath}, \dots, \hat{\jmath}, \dots, m)$*.*

Proof Parts (i) and (ii) follow from Proposition 3.10. For part (iii), by Proposition 3.10 we have that the Laplacian is given by (15) where $A = K^{-1}$. Then the C^{ab}_{ij} are given by (16). To get Formula (29) from (15) and (16) simply note that $(-1)^b \mathrm{sign}(b-a) = \mathrm{sign}\,\sigma$ where

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & \hat{a} & \dots & \dots & \dots & \dots & m \\ b & 1 & \dots & \dots & \hat{a} & \dots & \hat{b} & \dots & m \end{pmatrix}$$

and for $i < j$,

$$\sum_{b=1,\dots,\hat{a},\dots,m} \mathrm{sign}(\sigma) \frac{\partial M^i_{\bar{j}}}{\partial z^b} E^{ij}_{ab}$$

is the above determinant.

We shall see that the data h, M in Hypotheses 3.16 can be chosen such that one or more of the components of $z(q)$ is a harmonic morphism. In fact this construction gives *all* harmonic morphisms locally from open subsets of $\mathbf{R}^{2m}$ to $\mathbf{C}$ which are holomorphic with respect to some Hermitian structure on U. Precisely:

Proposition 3.18 *Let* $\phi : \mathbf{R}^{2m} \overset{open}{\supset} U \to \mathbf{C}$ *be holomophic with respect to some Hermitian structure* J *on* U*. (In particular, if* ϕ *is harmonic, it is a harmonic morphism.)*

Let $q_0 \in U$ *be a regular point of* ϕ*.*

(i) By a permutation of the coordinates $(q^i,\ q^{\bar{i}})$*, we can assume that* $J(q_0)$ *is positive and lies in a large cell of* $\mathcal{J}^+(\mathbf{R}^{2m})$*, so that* $J(q_0) = J(M)$ *for some* $M \in so(m, \mathbf{C})$*.*

(ii) Then in some neighbourhood $U' \subset U$ *of* q_0*, the given map* ϕ *is the first component* z^1 *of a smooth solution* $U' \to \mathbf{C}^m$*,* $q \mapsto z(q)$ *to an equation*

$$F(q, z) = 0$$

with

$$F(q,z) \equiv f(q - M(z)\bar{q}, z)$$

where $f : \mathbf{C}^m \times \mathbf{C}^k \overset{open}{\supset} W \to \mathbf{C}^k$ *and* $M : \mathbf{C}^m \overset{open}{\subset} V \to so(m, \mathbf{C}) = \mathbf{C}^{m(m-1)/2}$ *are holomorphic maps and* k *is an integer,* $1 \leq k \leq m$*. The Laplacian of* $\phi = z^1$ *is then given by (13, 14) or, equivalently, (15, 16).*

(iii) We can choose $k = m$ *and* f *to be in canonical form (25):*

$$f(w,z) = w - h(z)$$

where $h : \mathbf{C}^m \overset{open}{\supset} V \to \mathbf{C}^m$ *is holomorphic and of rank* m *everywhere. The Laplacian of* ϕ *is then given by (29).*

Proof Since, on a neighbourhood of q_0, the map ϕ has maximal rank (one), we can choose local complex coordinates $(z^1, \cdots, z^m)$ on a possibly smaller neighbourhood U' of q such that $z^1 = \phi$. But then these complex coordinates and the standard complex coordinates $(w^1, \cdots, w^m)$ for (U', J) are related by a holomorphic diffeomorphism $w = h(z)$ so that $w - h(z) = 0$ and we are done.

3.5 Low dimensional formulae

To find examples of harmonic morphisms using Proposition (3.17) it is useful write out formula (29) in the cases $m = k = 2$ or 3.

The case $m = k = 2$.
Writing μ for μ^1, the matrix M reads

$$M = M(\mu) = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

so that the standard complex coordinates for the Hermitian structure $J = J(M)$ (see §2) are given by

$$\left.\begin{array}{l} w^1 = q^1 - \mu q^{\bar{2}} \\ w^2 = q^2 + \mu q^{\bar{1}} \end{array}\right\}.$$

Equations (18) for $z : q = (q^1, q^2) \mapsto z = (z^1, z^2)$ read

$$f(q^1 - \mu(z)q^{\bar{2}}, q^2 + \mu(z)q^{\bar{1}}, z) = 0.$$

When f is in canonical form (25) these read

$$\left.\begin{array}{l} F^1(q,z) \equiv q^1 - \mu(z^1,z^2)q^{\bar{2}} - h^1(z^1,z^2) = 0, \\ F^2(q,z) \equiv q^2 + \mu(z^1,z^2)q^{\bar{1}} - h^2(z^1,z^2) = 0. \end{array}\right\} \tag{30}$$

Then (29) simplifies to give:

Proposition 3.19 *Suppose Hypotheses 3.16 hold with $m = 2$. Then the Laplacians of the components of $z(q)$ are given by:*

$$
\begin{aligned}
\frac{1}{4}\Delta z^1 &= \frac{1}{\det K}\frac{\partial \mu}{\partial z^2}\ , \\
\frac{1}{4}\Delta z^2 &= -\frac{1}{\det K}\frac{\partial \mu}{\partial z^1}.
\end{aligned}
$$

Since by Theorem 3.18, any submersive holomorphic map is locally the solution to equations (30), these formulae confirm the result of [17] that a submersive harmonic map $\mathbf{R}^4 \supset U \to \mathbf{C}$ which is holomorphic with respect to a Hermitian structure on U is a harmonic morphism if and only if its fibres are superminimal.

The case $m = k = 3$.
This time the matrix M reads

$$
M = M(\mu_1, \mu_2, \mu_3) = \begin{pmatrix} 0 & \mu_1 & \mu_2 \\ -\mu_1 & 0 & \mu_3 \\ -\mu_2 & -\mu_3 & 0 \end{pmatrix}, \tag{31}
$$

so that the standard complex coordinates for the Hermitian structure $J = J(M)$ are given by

$$
\left.\begin{aligned}
w^1 &= q^1 - \mu_1 q^{\bar 2} - \mu_2 q^{\bar 3} \\
w^2 &= q^2 + \mu_1 q^{\bar 1} - \mu_3 q^{\bar 3} \\
w^3 &= q^3 + \mu_2 q^{\bar 1} + \mu_3 q^{\bar 2}
\end{aligned}\right\}.
$$

With f in canonical form (25) the equations (28) for $z : q = (q^1, q^2, q^3) \mapsto z = (z^1, z^2, z^3)$ read

$$
\left.\begin{aligned}
F^1(q,z) &\equiv q^1 - \mu_1(z) q^{\bar 2} - \mu_2(z) q^{\bar 3} - h^1(z) = 0 \\
F^2(q,z) &\equiv q^2 + \mu_1(z) q^{\bar 1} - \mu_3(z) q^{\bar 3} - h^2(z) = 0 \\
F^3(q,z) &\equiv q^3 + \mu_2(z) q^{\bar 1} + \mu_3(z) q^{\bar 2} - h^3(z) = 0
\end{aligned}\right\} \tag{32}
$$

and from equations (29) we obtain

Proposition 3.20 *Suppose Hypotheses 3.16 hold with $m = 3$. Then the Laplacians of the components of $z(q)$ are given by:*

$$
\begin{aligned}
\frac{1}{4}\Delta z^1 &= \frac{1}{\det K}\left\{ \begin{vmatrix} \frac{\partial \mu_1}{\partial z^2} & \frac{\partial \mu_1}{\partial z^3} \\ K^3_2 & K^3_3 \end{vmatrix} - \begin{vmatrix} \frac{\partial \mu_2}{\partial z^2} & \frac{\partial \mu_2}{\partial z^3} \\ K^2_2 & K^2_3 \end{vmatrix} + \begin{vmatrix} \frac{\partial \mu_3}{\partial z^2} & \frac{\partial \mu_3}{\partial z^3} \\ K^1_2 & K^1_3 \end{vmatrix} \right\} \\
\frac{1}{4}\Delta z^2 &= \frac{1}{\det K}\left\{ \begin{vmatrix} \frac{\partial \mu_1}{\partial z^3} & \frac{\partial \mu_1}{\partial z^1} \\ K^3_3 & K^3_1 \end{vmatrix} - \begin{vmatrix} \frac{\partial \mu_2}{\partial z^3} & \frac{\partial \mu_2}{\partial z^1} \\ K^2_3 & K^2_1 \end{vmatrix} + \begin{vmatrix} \frac{\partial \mu_3}{\partial z^3} & \frac{\partial \mu_3}{\partial z^1} \\ K^1_3 & K^1_1 \end{vmatrix} \right\} \\
\frac{1}{4}\Delta z^3 &= \frac{1}{\det K}\left\{ \begin{vmatrix} \frac{\partial \mu_1}{\partial z^1} & \frac{\partial \mu_1}{\partial z^2} \\ K^3_1 & K^3_2 \end{vmatrix} - \begin{vmatrix} \frac{\partial \mu_2}{\partial z^1} & \frac{\partial \mu_2}{\partial z^2} \\ K^2_1 & K^2_2 \end{vmatrix} + \begin{vmatrix} \frac{\partial \mu_3}{\partial z^1} & \frac{\partial \mu_3}{\partial z^2} \\ K^1_1 & K^1_2 \end{vmatrix} \right\}
\end{aligned} \tag{33}
$$

where

$$
\begin{aligned}
K^1_a = \frac{\partial F^1}{\partial z^a} &= -\frac{\partial \mu_1}{\partial z^a} q^{\bar{2}} - \frac{\partial \mu_2}{\partial z^a} q^{\bar{3}} - \frac{\partial h^1}{\partial z^a} \\
K^2_a = \frac{\partial F^2}{\partial z^a} &= \quad \frac{\partial \mu_1}{\partial z^a} q^{\bar{1}} - \frac{\partial \mu_3}{\partial z^a} q^{\bar{3}} - \frac{\partial h^2}{\partial z^a} \\
K^3_a = \frac{\partial F^3}{\partial z^a} &= \quad \frac{\partial \mu_2}{\partial z^a} q^{\bar{1}} + \frac{\partial \mu_3}{\partial z^a} q^{\bar{2}} - \frac{\partial h^3}{\partial z^a}
\end{aligned}
\tag{34}
$$

4 Examples and results

In this section we shall construct some examples using the theory of the last section together with Proposition 3.3. The idea is to choose the data h and $M = M(\mu)$ in Hypotheses 3.16 and to solve Equation (28) for $z = \phi(q)$. Then the Laplacians of the components of z are found by Equation (29) and the data is chosen judiciously such that one or more of the components is a harmonic morphism with specific properties. Sometimes, z is composed with a futher holomorphic map ϕ to obtain more harmonic holomorphic maps by Proposition 3.3. Before doing this, we must consider, in the first subsection, three important properties of such examples. Readers wishing to get straight to the examples should skip to §4.2.

4.1 Fullness, superminimality and Kähler structures

1. Fullness and reduction

Definition 4.1 Call a map $\phi : \mathbf{R}^n \overset{open}{\supset} U \to \mathbf{C}$ *full* if we cannot write it as $\phi = \psi \circ \pi_A$ for some orthogonal projection π_A onto a subspace A of $\mathbf{R}^n$ and map $\psi : \pi_A(U) \to \mathbf{C}$. If, on the other hand ϕ does so factor we say that ϕ *reduces* to A and ψ is a *reduction* of ϕ.

Note that if ϕ does so factor, then it is a harmonic map (resp. harmonic morphism) if and only if ψ is a harmonic map (resp. harmonic morphism), see [8].

Similarly, we shall say that a foliation $\mathcal{F}$ on U is *full* if it is not the inverse image $(\pi_A)^{-1}(\mathcal{F}')$ of a foliation $\mathcal{F}'$ on $\pi_A(U)$ for any subspace A.

We now give a test for fullness for holomorphic maps $\phi \circ z$ where $z(q)$ is given by a solution of equation (28). To be precise we assume Hypotheses 3.16 and that $\phi : V \to \mathbf{C}$ is a given holomorphic map. We seek conditions that $\phi \circ z$ factor to $\mathbf{R}^{2m-1}$. As usual, write $K^a_b = \partial F^a / \partial z^b$, $(a, b = 1, \dots m)$ and $(\hat{K}^a_b)$ for the matrix of cofactors of K so that $(K^{-1})^a_b = (1/\det K)\hat{K}^b_a$.

Let A be a $(2m-1)$-dimensional subspace and let $v = a^j \partial/\partial x^j$ be a non-zero vector orthogonal to it. Define $\alpha \in \mathbf{C}^m$ by $\alpha^j = a^{2j-1} - \mathrm{i}a^{2j}$, $(j = 1, \dots m)$.

Proposition 4.2 *Suppose Hypotheses 3.16, and let $\phi : V \to \mathbf{C}$ be a holomorphic function. Then the map $U \to \mathbf{C}$, $q \mapsto \phi(z(q))$ factors to $\pi_A(U) \subset A$ if and only if*

$$\sum_{a=1}^{m} \frac{\partial \phi}{\partial z^a} \begin{vmatrix} K_1^1 & \cdots & K_{a-1}^1 & w^1(\alpha, z(q)) & K_{a+1}^1 & \cdots & K_m^1 \\ K_1^2 & \cdots & K_{a-1}^2 & w^2(\alpha, z(q)) & K_{a+1}^2 & \cdots & K_m^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_1^m & \cdots & K_{a-1}^m & w^m(\alpha, z(q)) & K_{a+1}^m & \cdots & K_m^m \end{vmatrix} = 0 \text{ for all } q \in U \tag{35}$$

where the K_b^a are evaluated at $(q, z(q))$.

Proof A function $\psi : U \to \mathbf{C}$ factors if and only if its directional derivative in direction v is zero, i.e.

$$\alpha^I \partial \psi / \partial q^I = 0 \tag{36}$$

(where we sum over $I = 1, \ldots, m, \bar{1}, \ldots \bar{m}$). Now, setting $\psi = \phi \circ z$, (24) gives

$$\frac{\partial \psi}{\partial q^I} = \frac{\partial \phi}{\partial z^a} \frac{\partial z^a}{\partial q^I} = -\frac{\partial \phi}{\partial z^a} (K^{-1})_b^a \frac{\partial F^b}{\partial q^I} = -\frac{\partial \phi}{\partial z^a} \frac{1}{\det K} \hat{K}_a^b \frac{\partial F^b}{\partial q^I} \tag{37}$$

so that (36) reads

$$\frac{\partial \phi}{\partial z^a} (\alpha^I \frac{\partial F^b}{\partial q^I}) \hat{K}_a^b = 0.$$

Noting that $\alpha^I \partial F^b / \partial q^I = \alpha^I \partial w^b / \partial q^I = w^b(\alpha, z)$ (the last equality using the fact that $w(\alpha, z) = \alpha - M\bar{\alpha}$ is homogeneous of degree one in α), we obtain the formula (35).

Example 4.3 Putting $a = (1, 0, \ldots, 0)$ we see that the function z^1 factors through the natural projection $\mathbf{R}^{2m} \to \mathbf{R}^{2m-1}$, $(x^1, x^2, \ldots, x^{2m}) \mapsto (x^2, \ldots, x^{2m})$, i.e. is independent of x^1, if and only if

$$\begin{vmatrix} 1 & K_2^1 & \ldots & K_m^1 \\ \mu_1 & K_2^2 & \ldots & K_m^2 \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{m-1} & K_2^m & \ldots & K_m^m \end{vmatrix} = 0 \quad \text{for all} \quad q \in U \tag{38}$$

where the K_b^a are evaluated at $(q, z(q))$.

2. Superminimality

The definition of superminimality was given in Definition 3.6. It is easy to show that if an even dimensional submanifold S of an open subset of $\mathbf{R}^{2m}$ equipped with an adapted almost Hermitian structure J is superminimal with respect to J then its Weingarten map A satisfies

$$A_{JX} V = J A_X V \quad \text{for all } x \in S,\ X \in T_x S \text{ and } V \in T_x \mathbf{R}^{2m} \text{ normal to } M\ ; \tag{39}$$

from this it easily follows that an even dimensional submanifold which is superminimal with respect to an almost Hermitian structure is minimal. We further note that if S is of real codimension 2, the property (39) characterizes superminimality with respect to an almost Hermitian structure (see, for example, [11]).

We can characterize this condition geometrically by

Proposition 4.4 *A codimension 2 submanifold S of $\mathbf{R}^{2m}$ is superminimal with respect to some adapted almost Hermitian structure if and only if, for all $x \in S$, $A_Z V$ is isotropic for all $V \in T_x S$ and all isotropic $Z \in T_x \mathbf{R}^{2m}$ normal to S.*

We also give a criterion for superminimality with respect to an almost complex structure:

Proposition 4.5 *Let $\psi : \mathbf{R}^{2m} \overset{open}{\supset} U \to \mathbf{C}$ be a submersive harmonic morphism (or any submersion which satisfies Equation (2)). Then a connected component of a fibre E of ψ is superminimal with respect to some adapted almost complex structure J on U if*

(1) there exists an m-dimensional complex subspace W' of $\mathbf{C}^{2m}$ with $W' \cap \bar{W}' = \{0\}$ and

$$\nabla\psi = (\partial\psi/\partial x^1, \ldots, \partial\psi/\partial x^{2m}) \in W' \tag{40}$$

at all points q of E, or equivalently,

(2) there exists an m-dimensional complex subspace W of $\mathbf{C}^{2m}$ with $W \cap \bar{W} = \{0\}$ such that

$$b^I \frac{\partial\psi}{\partial q^I} = 0 \tag{41}$$

for all $b \in W,\ q \in E$.

Proof For the "only if" part of (1), set W' = the $(1,0)$-tangent space of $\mathbf{R}^{2m}$ with respect to J. This satisfies the desired conditions. For the converse, choose J to have +i- (resp. −i-) eigenspace W' (resp. W) at all points of E. To see the equivalence of (1) and (2), set W = the orthogonal complement of W' in $\mathbf{C}^{2m}$ noting that (41) says that $\nabla\psi$ is orthogonal to W.

For the fibres of a map $\phi \circ z$ where $z(q)$ is a solution to (28) we deduce the following test for superminimality:

Proposition 4.6 *Suppose Hypotheses 3.16. Additionally, let $\phi : V \to \mathbf{C}$ be a holomorphic function. Then a connected component E of a fibre of $q \mapsto \phi(z(q))$ is superminimal with respect to some almost complex structure if and only if there is an m-dimensional linear subspace W of $\mathbf{C}^{2m}$ with $W \cap \bar{W} = 0$ such that*

$$\sum_{a=1}^{m} \frac{\partial\phi}{\partial z^a} \begin{vmatrix} K_1^1 & \cdots & K_{a-1}^1 & w^1(b, z(q)) & K_{a+1}^1 & \cdots & K_m^1 \\ K_1^2 & \cdots & K_{a-1}^2 & w^2(b, z(q)) & K_{a+1}^2 & \cdots & K_m^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_1^m & \cdots & K_{a-1}^m & w^m(b, z(q)) & K_{a+1}^m & \cdots & K_m^m \end{vmatrix} = 0 \tag{42}$$

holds for all $b \in W$ and all $q \in E$.

Proof Apply the last Proposition to $\psi = \phi \circ z$ calculating its gradient as in Proposition 4.2.

3. Kähler structures

Lastly we consider whether a given map $\psi = \phi \circ z : \mathbf{R}^{2m} \overset{open}{\supset} U \to \mathbf{C}$ is holomorphic with respect to a parallel complex structure on $\mathbf{R}^{2m}$, for example a Kähler structure. This is very similar to the last test, only now we insist that the linear subspace does not vary from fibre to fibre.

Proposition 4.7 *Suppose Hypotheses 3.16. Additionally, let $\phi : V \to \mathbf{C}$ be a holomorphic function. Then $q \mapsto \phi(z(q))$ is holomorphic with respect to a parallel complex structure on $\mathbf{R}^{2m}$ if and only if there is an m-dimensional linear subspace W of $\mathbf{C}^{2m}$ with $W \cap W' = 0$ such that Equation (42) holds for all $b \in W$ and $q \in U$.*

To include the odd dimensional case we introduce some terminology:

Definition 4.8 Say that a map $\phi : \mathbf{R}^n \overset{open}{\supset} U \to \mathbf{C}$ *arises from a Kähler structure* if either n is even and it is holomorphic with respect to a Kähler structure on $\mathbf{R}^n$, or n is odd and it is the reduction of a map on an open subset of $\mathbf{R}^{n+1}$ which is holomorphic with respect to a Kähler structure on $\mathbf{R}^{n+1}$.

Note that in the second case (n odd), ϕ further reduces to a map on an even dimensional subspace of $\mathbf{R}^n$ (for it is invariant in the direction $v \in (\mathbf{R}^n)^{\perp} \cap \mathbf{R}^{n+1}$ and so it is also invariant in the direction Jv. Hence it reduces to the orthogonal complement of Jv in $\mathbf{R}^n$.

4.2 Examples of harmonic morphisms on even dimensional Euclidean spaces

In this subsection we construct examples of harmonic morphisms from open subsets of $\mathbf{R}^{2m}$ by using Propositions 3.17 and 3.3.

Examples with superminimal fibres with respect to a Hermitian structure which is not cosymplectic

Suppose that in Proposition 3.17 we choose $\mu_i : \mathbf{C}^m \supset V \to \mathbf{C}$ holomorphic with $\mu_i = \mu_i(z^1), i = 1, \cdots, m(m-1)/2$, depending on z^1 only, with the holomorphic functions h^i chosen arbitrarily. Then by (29), $\Delta z^1 = 0$. In this case the Hermitian structure $J(M(\mu))$ is constant along the fibres of z^1 which are therefore superminimal with respect to it.

Example 4.9 Let $\mu_1 = z^1, \mu_2 = z^1, \mu_3 = 0, h^1 = z^2, h^2 = z^3, h^3 = z^2 z^3$. Then z^1 satisfies the system of equations

$$\left. \begin{array}{llll} q^1 - z^1 q^{\bar{2}} & -z^1 q^{\bar{3}} & -z^2 & = 0 \\ q^2 + z^1 q^{\bar{1}} & & -z^3 & = 0 \\ q^3 + z^1 q^{\bar{1}} & & -z^2 z^3 & = 0 \end{array} \right\} .$$

Then $\det K$ is non-zero almost everywhere and, eliminating z^2 and z^3 we see that z^1 is determined by the quadratic equation

$$(z^1)^2 q^{\bar{1}}(q^{\bar{2}} + q^{\bar{3}}) + z^1 \left[q^{\bar{1}} + q^2(q^{\bar{2}} + q^{\bar{3}}) - |q^1|^2 \right] + q^3 - q^1 q^2 = 0.$$

Neither z^2 nor z^3 is harmonic hence the corresponding complex structure is not cosymplectic. Furthermore, it can be easily checked that z^1 is not holomorphic with respect to any Kähler structure.

This example generalises to all $m > 2$ by setting $h^1 = z^2, h^2 = z^3, \ldots, h^{m-1} = z^m, h^m = z^2 z^3 \cdots z^m$, with the μ_i's appropriately chosen functions of z^1.

Examples with superminimal fibres with respect to a Hermitian structure which is cosymplectic

Note firstly that for $m = 2$, J cosymplectic implies J Kähler; this is no longer the case for $m > 2$. Suppose again that $\mu_i = \mu_i(z^1)$ for each $i = 1, 2, \ldots, m(m-1)/2$, so that z^1 has superminimal fibres with respect to $J = J(M(\mu))$, so that. In particular z^1 is harmonic and by Proposition 3.4, J is cosymplectic if and only if $z^2, z^3, \ldots z^m$ are also harmonic. For simplicity of exposition we consider the case $m = 3$.

By equation (33)

$$\begin{array}{rcl} \dfrac{\det K}{4} \Delta z^2 & = & \dfrac{\partial \mu_1}{\partial z^1} \dfrac{\partial h^3}{\partial z^3} - \dfrac{\partial \mu_2}{\partial z^1} \dfrac{\partial h^2}{\partial z^3} + \dfrac{\partial \mu_3}{\partial z^1} \dfrac{\partial h^1}{\partial z^3} \\ & = & \dfrac{\partial}{\partial z^3} (h^3 \dfrac{\partial \mu_1}{\partial z^1} - h^2 \dfrac{\partial \mu_2}{\partial z^1} + h^1 \dfrac{\partial \mu_3}{\partial z^1}) \end{array}$$

Similarly

$$\frac{\det K}{4} \Delta z^3 = \frac{\partial}{\partial z^2} (h^3 \frac{\partial \mu_1}{\partial z^1} - h^2 \frac{\partial \mu_2}{\partial z^1} + h^1 \frac{\partial \mu_3}{\partial z^1}) \tag{43}$$

so that J is cosymplectic if and only if

$$h^3 \frac{\partial \mu_1}{\partial z^1} - h^2 \frac{\partial \mu_2}{\partial z^1} + h^1 \frac{\partial \mu_3}{\partial z^1} = \alpha(z^1), \tag{44}$$

where α is a holomorphic function of z^1 only. Then z^1 satisfies the single equation

$$\frac{\partial\mu_1}{\partial z^1}(q^3+\mu_2 q^{\bar{1}}+\mu_3 q^{\bar{2}})-\frac{\partial\mu_2}{\partial z^1}(q^2+\mu_1 q^{\bar{1}}-\mu_3 q^{\bar{3}})+\frac{\partial\mu_3}{\partial z^1}(q^1-\mu_1 q^{\bar{2}}-\mu_2 q^{\bar{3}})=\alpha(z^1). \tag{45}$$

Conversely, given μ_1,μ_2,μ_3 as functions of z^1 and a solution z^1 to the above equation, setting $h^1=q^1-\mu_1 q^{\bar{2}}-\mu_2 q^{\bar{3}}, h^2=q^2+\mu_1 q^{\bar{1}}-\mu_3 q^{\bar{3}}, h^3=q^3+\mu_2 q^{\bar{1}}+\mu_3 q^{\bar{2}}$, we see that z^1 is holomorphic with respect to a cosymplectic structure.

We can now easily generate non-trivial solutions. For instance

Example 4.10 Set $\mu_1=z^1,\mu_2=(z^1)^2,\mu_3=(z^1)^3,h^3=2z^1z^2-3(z^1)^2z^3,h^2=z^2,h^1=z^3$. Then the above procedure gives a harmonic morphism z^1 of the type considered given by any local solution to the equation

$$(z^1)^4q^{\bar{3}}+2(z^1)^3q^{\bar{2}}+z^2(q^{\bar{1}}-3q^1)+2z^1q^2-q^3=0.$$

Examples with non-superminimal fibres

Choose $\mu_i:\mathbf{C}^m\supset V\to\mathbf{C}$ holomorphic with $\mu_1=\mu_1(z^1,\dots,z^{m-1}),\mu_a=\mu_a(z^1),\ (a=2,\dots,m(m-1)/2)$. Then

$$\Delta z^1=-\frac{1}{\det K}\begin{vmatrix}\frac{\partial\mu_1}{\partial z^2} & \frac{\partial\mu_1}{\partial z^3} & \cdots & \frac{\partial\mu_1}{\partial z^{m-1}} & 0\\ K^3_2 & K^3_3 & \cdots & K^3_{m-1} & K^3_m\\ K^4_2 & K^4_3 & \cdots & K^4_{m-1} & K^4_m\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ K^m_2 & K^m_3 & \cdots & K^m_{m-1} & K^m_m\end{vmatrix}, \tag{46}$$

with $K^a_b=-\partial h^a/\partial z^b,\ (a,b\geq 2)$.

Then if $h^3,h^4,\dots,h^m$ are independent of z^4 the last column of the above determinant vanishes and so $\Delta z^1=0$. Noting that unless μ_1 depends on z^1 only, the complex structure $J=J(\mu)$ varies with $z^2,\cdots,z^{m-1}$ as well as with z^1, giving local examples with fibres not superminimal with respect to J. However it might still be the case that z^1 is holomorphic with respect to another Hermitian structure with respect to which it has superminimal fibres, as the following example shows.

Example 4.11 Let $\mu_1=z^2,\mu_2=z^1,\mu_3=0,h^1=z^3,h^2=z^3,h^3=z^2$. Then

$$\det K=(q^{\bar{1}})^2+q^{\bar{1}}q^{\bar{2}}+q^{\bar{3}}$$

is non-zero on the complement U of the surface $(q^{\bar{1}})^2+q^{\bar{1}}q^{\bar{2}}+q^{\bar{3}}=0$ and

$$z^1=\frac{-q^3(q^{\bar{1}}+q^{\bar{2}})+q^1-q^2}{(q^{\bar{1}})^2+q^{\bar{1}}q^{\bar{2}}+q^{\bar{3}}}$$

is a harmonic morphism from U to $\mathbf{C}$, holomorphic wih respect to the Hermitian structure represented by

$$M = M(z(q)) = \begin{pmatrix} 0 & z^2 & z^1 \\ -z^2 & 0 & 0 \\ -z^1 & 0 & 0 \end{pmatrix}$$

Furthermore $\Delta z^2 = -4/\det K$ and $\Delta z^3 = -2q^{\bar{1}}/\det K$ which are non-zero, so that J is not cosymplectic. However a simple check, using Theorem 4.5 shows that along each connected fibre component, ∇z^1 is contained in a fixed $\mathbf{C}^3 \subset \mathbf{C}^6$, showing that in fact the fibres of z^1 are superminimal with respect to some complex structure

This illustrates the possibility of a harmonic morphism being holomorphic with respect to more than one Hermitian structure. In fact there are examples which are holomorphic with respect to a family of Hermitian structures:

Example 4.12 Let $m = 3$ and suppose $\mu_1 = \mu_1(z^1, z^2), h^3 = h^3(z^1)$. Then the harmonic morphism defined by equations (32) is determined by the last of these equations:

$$q^3 + \mu_2(z^1)q^{\bar{1}} + \mu_3(z^1)q^{\bar{2}} - h^3(z^1) = 0$$

and provided μ_2, μ_3 and h^3 are chosen judiciously, for example, $h^3 = z^1, \mu_2 = (z^1)^2, \mu_3 = (z^1)^3$, the resulting map is full.

We are now at liberty to choose μ_1, h^1, h^2 as we wish, provided of course that the system gives well-defined solutions. To be specific, if we set $h^1 = z^2, h^2 = z^3, \mu_1 = \mu_1(z^1, z^2)$, then the Jacobian matrix K has determinant

$$\left(\frac{\partial \mu_2}{\partial z^2} q^{\bar{2}} - 1\right)(2z^1 q^{\bar{1}} + 3(z^1)^2 q^{\bar{2}} - 1),$$

which is non-zero in general, giving a family of Hermitian structures parametrised by the function $\mu_1(z^1, z^2)$ with respect to each of which z^1 is holomorphic. Amongst this family are those parametrised by functions μ_1 independent of z^2 and, with respect to such a Hermitian structure, z^1 has superminimal fibres. There are examples of harmonic holomorphic maps which do not have superminimal fibres (with respect to any almost complex structure) which we shall describe shortly when we consider global examples.

The above construction generalizes to higher dimensions. For example, if $m = 4$, set $\mu_2 = z^1, \mu_3 = \mu_4 = 0, \mu_5 = z^1, \mu_6 = (z^1)^3, h^3 = z^1 z^2, h^4 = z^2$, and suppose $\mu_1 = \mu_1(z^1, z^2)$. Then z^1 is harmonic and is determined by the last two equations of the system (28):

$$\left.\begin{array}{rcl} q^1 - \mu_1 q^{\bar{2}} - \mu_2 q^{\bar{3}} - \mu_3 q^{\bar{4}} - h^1 & = & 0 \\ q^2 + \mu_1 q^{\bar{1}} - \mu_4 q^{\bar{3}} - \mu_5 q^{\bar{4}} - h^2 & = & 0 \\ q^3 + \mu_2 q^{\bar{1}} + \mu_4 q^{\bar{2}} - \mu_6 q^{\bar{4}} - h^3 & = & 0 \\ q^4 + \mu_3 q^{\bar{1}} + \mu_5 q^{\bar{2}} + \mu_6 q^{\bar{3}} - h^4 & = & 0. \end{array}\right\}$$

Eliminating z^2 from these gives z^1 as a solution of

$$(z^1)^4 q^{\bar{3}} + (z^1)^3 q^{\bar{4}} + (z^1)^2 q^{\bar{2}} + z^1(q^4 - q^{\bar{1}}) - q^3 = 0.$$

Then z^1 is full and is holomorphic with respect to Hermitian structures parametrised by the function$\mu_1 = \mu_1(z^1, z^2)$. This example is additionally interesting because z^1 is invariant under radial translation $q \mapsto aq, 0 < a < \infty$, and so reduces to a full harmonic morphism on a domain in S^7 (see [3]). Generalisation of these constructions to arbitrary $m > 2$ is straightforward.

Global examples in even dimensions

The above examples show the richness and variety that abounds for $m > 2$. We may ask whether such diversity can occur if we insist our examples be globally defined on $\mathbf{R}^{2m}$. We now turn our attention to this case.

Let $m > 2$ and suppose $\mu_1 = \mu_1(z^3, \ldots, z^m), \mu_2 = \mu_2(z^4, \ldots, z^m), \ldots, \mu_{m-2} = \mu_{m-2}(z^m)$ with $\mu_{m-1} = a_{m-1}, \ldots, \mu_{m(m-1)/2} = a_{m(m-1)/2}$ arbitrary constants. Define h^k to be of the form $h^k = z^k + g_k(z^{k+1}, \ldots, z^m)$, for each $k = 1, \ldots, m-1$ and $h^m = z^m$, where the g_k are arbitrary globally defined holomorphic functions. Then M has the form

$$M = \begin{pmatrix} 0 & \mu_1 & \mu_2 & \cdots & \mu_{m-2} & a_{m-1} \\ -\mu_1 & 0 & a_m & \cdots & \vdots & \vdots \\ -\mu_2 & -a_m & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\mu_{m-2} & \vdots & \vdots & \cdots & \vdots & \vdots \\ -a_{m-1} & \cdots & \cdots & \cdots & \cdots & \end{pmatrix} \tag{47}$$

giving a family of cosymplectic structures defined globally on $\mathbf{R}^{2m}$. In fact

$$K = \begin{pmatrix} -1 & * & * & \cdots & * \\ 0 & -1 & * & \cdots & * \\ 0 & 0 & -1 & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & * \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \tag{48}$$

has determinant $(-1)^m$ everywhere. The system (28) determines globally defined harmonic holomorphic functions $z^1, \ldots, z^m$ which are found explicitly by first solving the mth equation for z^m, then substituting into the $(m-1)$st to obtain z^{m-1} etc. We fix our attention on a special case.

Example 4.13 Let $\mu_1 = z^m, \mu_2 = \cdots = \mu_{m(m-1)/2} = 0, h^1 = z^1, h^2 = z^2, \ldots, h^m = z^m$. Then the system of equations (28) takes the form

$$\left.\begin{array}{rcl} q^1 - z^m q^{\bar{2}} - z^1 & = & 0 \\ q^1 + z^m q^{\bar{1}} - z^2 & = & 0 \\ q^3 - z^3 & = & 0 \\ & \vdots & \\ q^m - z^m & = & 0 \end{array}\right\}$$

which has the solution

$$z^m = q^m, \ldots, z^3 = q^3, z^2 = q^2 + q^m q^{\bar{1}}, z^1 = q^1 - q^m q^{\bar{2}}.$$

All the maps z^k are globally defined and are holomorphic with respect to the Hermitian structure $J(\mu)$. In fact each is also holomorphic with respect to a Kähler structure (and so has superminimal fibres) and none are full if $m > 3$. However, define a holomorphic composition of the maps $z^1, \cdots, z^m$ by

$$\begin{array}{rcl} \phi & = & z^1 z^2 \cdots z^{m-1} \\ & = & (q^1 - q^m q^{\bar{2}})(q^2 + q^m q^{\bar{1}}) q^3 \cdots q^{m-1} \ . \end{array}$$

By Proposition 3.3 or 3.4, ϕ is holomorphic with respect to $J(\mu)$ and harmonic. We show that ϕ does not have superminimal fibres with respect to any almost complex structure and so is not holomorphic with respect to any Kähler (or parallel complex) structure. Furthermore ϕ is full. For ease of exposition we consider the case $m = 3$. By Proposition 4.6, $\nabla\phi = z^2 \nabla z^1 + z^1 \nabla z^2$ lies in a linear subspace

$$\langle b, \nabla\phi \rangle^{\mathbf{c}} = 0 \tag{49}$$

if and only if

$$z^2 \begin{vmatrix} b^1 - z^3 b^{\bar{2}} & 0 & -q^{\bar{2}} \\ b^2 + z^3 b^{\bar{1}} & -1 & q^{\bar{1}} \\ b^3 & 0 & -1 \end{vmatrix} + z^1 \begin{vmatrix} -1 & b^1 - z^3 b^{\bar{2}} & -q^{\bar{2}} \\ 0 & b^2 + z^3 b^{\bar{1}} & q^{\bar{1}} \\ 0 & b^3 & -1 \end{vmatrix} = 0 \tag{50}$$

Now

$$q^{\bar{1}} = \frac{z^{\bar{1}} + z^2 z^{\bar{3}}}{1 + |z^3|^2}, q^{\bar{2}} = \frac{z^{\bar{2}} - z^1 z^{\bar{3}}}{1 + |z^3|^2}.$$

Then setting $z^1 z^2 = a$ where a is a constant and eliminating z^1 gives that (49) is satisfied if and only if

$$\begin{array}{rcl} |z^2|^2 \left[(b^1 - z^3 b^{\bar{2}})(1 + |z^3|^2) z^2 - b^3(|z^2|^2 - a z^{\bar{3}}) \right] & + & \\ |z^2|^2 \left[(b^2 + z^3 b^{\bar{1}})(1 + |z^3|^2) z^{\bar{2}} + b^3(\bar{a} + |z^2|^2 + |z^2|^2 z^{\bar{3}}) \right] & = & 0. \end{array}$$

Comparing coefficients, provided a is non-zero, yields $b^1 = b^2 = b^3 = b^{\bar{1}} = b^{\bar{2}} = 0$, leaving $b^{\bar{3}}$ arbitrary. Thus along each fibre over $a \neq 0$, the smallest subspace of $\mathbf{C}^6$ containing $\nabla\phi$ is a $\mathbf{C}^5$, showing that such a fibre is not superminimal with respect to any almost complex structure.

To show ϕ is full, by Proposition 4.2 , we follow the above calculation, but now suppose $b^{\bar{k}} = \bar{b^k}$ and then $b \equiv 0$. For general m, the smallest subspace of $\mathbf{C}^{2m}$ containing $\nabla\phi$ is a $\mathbf{C}^{m+2}$. Thus ϕ provides an example of a harmonic morphism defined globally on $\mathbf{R}^{2m}$ which is both full and does not have superminimal fibres with respect to any almost complex structure.

By modifying ϕ to be

$$\tilde{\phi} = z^1 \cdots z^{m-1} + z^m,$$

we obtain a *submersive* example with the same properties as ϕ (full, with non-superminimal fibres). In particular the fibres of $\tilde{\phi}$ form a global conformal foliation of $\mathbf{R}^{2m}$ by minimal submanifolds of codimension 2, which does not arise from any Kähler structure. That is the leaves are not the fibres of a map, holomorphic with respect to a Kähler structure.

Global examples in odd dimensions

To construct examples on $\mathbf{R}^{2m-1}$ we proceed as follows.

Example 4.14 Let μ_i, h^i be defined as in Example 4.13. Define

$$\begin{aligned} \phi &= (z^1 z^m - z^2) z^3 \cdots z^{m-1} \\ &= \big((q^1 - q^{\bar{1}}) q^m - (q^m)^2 q^{\bar{2}} - q^2\big) q^3 \cdots q^{m-1}. \end{aligned}$$

Then ϕ is invariant in the x^1-direction ($x^1 = \operatorname{Re} q^1$). Calculations similar to the last example above show that along almost every fibre, the smallest subspace of $\mathbf{C}^{2m}$ containing $\nabla\phi$ is a $\mathbf{C}^{m+1}$ and so the fibres are not superminimal with respect to any almost complex structure. Furthermore ϕ is not invariant under any other direction apart from x^1 and so reduces to a globally defined, full harmonic morphism on $\mathbf{R}^{2m-1}$.

The modified example

$$\tilde{\phi} = (z^1 z^m - z^2) + (z^3)^2 + \cdots + (z^{m-2})^2 + z^{m-1}$$

gives a *submersive* harmonic morphism on $\mathbf{R}^{2m}$, with fibres not superminimal with respect to any almost complex structure, which reduces to $\mathbf{R}^{2m-1}$ and is full on that space. Thus the fibres of $\tilde{\phi}$ form a global conformal foliation by submanifolds of codimension 2 of $\mathbf{R}^{2m-1}$ which does not arise from any Kähler structure.

4.3 Results

From the above examples we deduce the existence of various sorts of harmonic morphism. First the local case:

Theorem 4.15 *For any $m > 2$ there are full harmonic morphisms $\phi : U \to \mathbf{C}$ defined on open subsets U of $\mathbf{R}^{2m}$ which are holomorphic with respect to a Hermitian structure J on U but not holomorphic with respect to any Kähler structure on U. There are such examples with*

- *the map submersive and fibres superminimal with respect to J or not superminimal with respect to any Hermitian structure,*
- *J cosymplectic or not,*
- *holomorphic with respect to a family of Hermitian structures, with fibres superminimal with respect to some of the family but not all.*

Remarks 4.16 This is not true for $m = 2$ where a submersive harmonic morphism is holomorphic with respect to precisely one Hermitian structure, or precisely two of different orientations if the fibres are totally geodesic, and the fibres are always superminal with respect to those Hermitian structures.

Next the global results. First for even dimensions.

Theorem 4.17 *For any $m > 2$ there are full harmonic morphisms $\mathbf{R}^{2m} \to \mathbf{C}$ defined on the whole of $\mathbf{R}^{2m}$ which are holomorphic with respect to a Hermitian structure J on $\mathbf{R}^{2m}$ but not holomorphic with respect to any Kähler structure on $\mathbf{R}^{2m}$. These can be chosen to be submersive.*

Remarks 4.18 Again this is not true for $m = 2$, see [17] where it is proved that any globally defined submersive harmonic morphism $\mathbf{R}^4 \to \mathbf{C}$ is holomorphic with respect to a Kähler structure; it is conjectured that this is true for any harmonic morphism whether submersive or not.

Since the fibres of a harmonic morphism to a surface give a conformal foliation by minimal submanifolds [2], we deduce

Corollary 4.19 *For any $m > 2$ there are full conformal foliations by minimal submanifolds of codimension 2 of the whole of $\mathbf{R}^{2m}$ which are holomorphic with respect to a Hermitian structure on $\mathbf{R}^{2m}$ but not with respect to any Kähler structure.*

Next for odd dimensions:

Theorem 4.20 *For each $m = 1, 2, \ldots$ there are harmonic maps $\mathbf{R}^{2m} \to \mathbf{C}$ defined on the whole of $\mathbf{R}^{2m}$ which are holomorphic with respect to a Hermitian structure J on $\mathbf{R}^{2m}$ but not holomorphic with respect to any Kähler structure on $\mathbf{R}^{2m}$ and which factor to full harmonic morphisms from $\mathbf{R}^{2m-1}$ to $\mathbf{C}$. These harmonic morphisms can be chosen to be submersions.*

We can combine these results for n even and odd into a single statement by using the notion of *arising from a Kähler structure* (Definition 4.8):

Corollary 4.21 *For any $n > 4$ there are full harmonic morphisms from $\mathbf{R}^n$ to $\mathbf{C}$ which do not arise from Kähler structures.*

The consequence of this for foliations is

Corollary 4.22 *For any $n > 4$ there are full conformal foliations of $\mathbf{R}^n$ by minimal submanifolds of codimension 2 which do not arise from Kähler structures.*

4.4 Harmonic morphisms from tori

A map $\Phi : \mathbf{R}^{2m} \to \mathbf{C}$ descends to a map $\phi : \mathbf{R}^{2m}/\Gamma \to \mathbf{C}/\Gamma'$ of tori (where Γ and Γ' are lattices) if and only if $\Phi(\Gamma) \subset \Gamma'$. Then ϕ is a harmonic morphism if and only if Φ is and all harmonic morphisms ϕ arise in this way. The following Proposition describes such maps.

Proposition 4.23 *Let $\phi : \mathbf{R}^{2m}/\Gamma \to \mathbf{C}/\Gamma'$ be a harmonic morphism of tori. Then*

(i) With respect to suitable Euclidean coordinates q^I on $\mathbf{R}^{2m}$, $\Phi = cq^1$ for some constant $c > 0$.

(ii) For $m > 2$, ϕ is holomorphic with respect to infinitely many Kähler and infinitely many non-Kähler Hermitian structures J on the torus. If $m > 3$, we can choose J such that the fibres of ϕ are not superminimal with respect to it.

Proof (i) Since ϕ is harmonic, Φ must be linear, say, $\Phi = \sum_I a_I q^I$; then, if ϕ is a harmonic morphism, horizontal weak conformality (2) implies that the vector with components a_I is isotropic. So we can choose an orthonormal change of coordinates to make it $(1, 0, \ldots, 0)$.

(ii) The Kähler assertion is obvious. For the other assertion define $M : \mathbf{R}^{2m}/\Gamma \to so(m, \mathbf{C})$ by $M^2_{\bar{3}} = -M^3_{\bar{2}} = \lambda\mathcal{P}(q^1)$ where $\mathcal{P}$ is a holomorphic function $\mathbf{C}/\Gamma' \to \mathbf{C} \cup \{\infty\} = \mathbf{C}P^1$ (e.g. the Weierstrass $\mathcal{P}$ function) and other entries of M zero. Then set $J = J(M)$ and note that J is well-defined at poles of $\mathcal{P}$ since it is of the form

$$\mathbf{R}^{2m}/\Gamma \stackrel{q^1}{\to} \mathbf{C}/\Gamma' \stackrel{\mathcal{P}}{\to} \mathbf{C} \cup \{\infty\} \stackrel{\times\lambda}{\to} \mathbf{C} \cup \{\infty\} \stackrel{stereo}{\to} \mathbf{C}P^1 \hookrightarrow \mathcal{J}^+(\mathbf{R}^{2m}) \ .$$

Then J is easily seen to be Hermitian, and non-Kähler if $\lambda \neq 0$. Note that we can see $\phi = q^1$ as the first component z^1 of the solution $z(q)$ of the equation (18) with $M^2_3 = -M^3_2 = \lambda \mathcal{P}(z^1)$ and other entries of M zero, and $f^1 = \mathcal{P}(w^1) - z^1$, $f^i = w^i - z^i$ for $i > 1$. For $m > 3$, we may replace q^1 by q^2 in the definition of M.

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