QUANTUM HIDDEN SUBGROUP ALGORITHMS: AN ALGORITHMIC TOOLKIT

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ABSTRACT. One of the most promising and versatile approaches to creating new quantum algorithms is based on the quantum hidden subgroup (QHS) paradigm, originally suggested by Alexei Kitaev. This class of quantum algorithms encompasses the Deutsch-Jozsa, Simon, Shor algorithms, and many more.

In this paper, our strategy for finding new quantum algorithms is to decompose Shor's quantum factoring algorithm into its basic primitives, then to generalize these primitives, and finally to show how to reassemble them into new QHS algorithms. Taking an "alphabetic building blocks approach," we use these primitives to form an "algorithmic toolkit" for the creation of new quantum algorithms, such as wandering Shor algorithms, continuous Shor algorithms, the quantum circle algorithm, the dual Shor algorithm, a QHS algorithm for Feynman integrals, free QHS algorithms, and more.

Toward the end of this paper, we show how Grover's algorithm is most surprisingly "almost" a QHS algorithm, and how this result suggests the possibility of an even more complete "algorithmic tookit" beyond the QHS algorithms.

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1. INTRODUCTION

One major obstacle to the fulfillment of the promise of quantum computing is the current scarcity of quantum algorithms. Quantum computing researchers simply have not yet found enough quantum algorithms to determine whether or not future quantum computers will be general purpose or special purpose computing devices. As a result, much more research is crucially needed to determine the algorithmic limits of quantum computing.

One of the most promising and versatile approaches to creating new quantum algorithms is based on the quantum hidden subgroup (QHS) paradigm, originally suggested by Alexei Kitaev [20]. This class of quantum algorithms encompasses the Deutsch-Jozsa, Simon, Shor algorithms, and many more.

In this paper, our strategy for finding new quantum algorithms is to decompose Shor's quantum factoring algorithm into its basic primitives, then to generalize these primitives, and finally to show how to reassemble them into new QHS algorithms. Taking an "alphabetic building blocks approach," we will use these primitives to form an "algorithmic toolkit" for the creation of new quantum algorithms, such as wandering Shor algorithms, continuous Shor algorithms, the quantum circle algorithm, the dual Shor algorithm, a QHS algorithm for Feynman integrals, free QHS algorithms, and more.

Toward the end of this paper, we show how Grover's algorithm is most surprisingly "almost" a QHS algorithm, and how this suggests the possibility of an even more complete "algorithmic tookit" beyond the QHS algorithms.

2. An example of Shor's quantum factoring algorithm

Before discussing how Shor's algorithm can be decomposed into its primitive components, let's take a quick look at an example of the execution of Shor's factoring algorithm. As we discuss this example, we suggest that the reader, as an exercise, try to find the basic QHS primitives that make up this algorithm. Can you see them?

Shor's quantum factoring algorithm reduces the task of factoring a positive integer N to first finding a random integer a relatively prime to N, and then next to determining the period P of the following function

$$\begin{array}{rcl} \mathbb{Z} & \stackrel{\varphi}{\longrightarrow} & \mathbb{Z} \operatorname{mod} N \\ x & \longmapsto & a^x \operatorname{mod} N \end{array},$$

where \mathbb{Z} denotes the additive group of integers, and where $\mathbb{Z} \mod N$ denotes the integers mod N under multiplication¹.

¹A random integer *a* with gcd (a, N) = 1 is found by selecting a random integer, and then applying the Euclidean algorithm to determine whether or not it is relatively prime to *N*. If not, then the gcd is a non-trivial factor of *N*, and there is no need to proceed further. However, this possibility is highly unlikely if *N* is large.

Since \mathbb{Z} is an infinite group, Shor chooses to work instead with the finite additive cyclic group \mathbb{Z}_Q of order $Q = 2^m$, where $N^2 \leq Q < 2N^2$, and with the "approximating" map

$$\begin{array}{rcl} \mathbb{Z}_Q & \stackrel{\varphi}{\longrightarrow} & \mathbb{Z} \operatorname{mod} N \\ x & \longmapsto & a^x \operatorname{mod} N \ , & 0 \le x < Q \end{array}$$

Shor begins by constructing a quantum system with two quantum registers

|Left_Register> |Right_Register> ,

the left intended for holding the arguments x of $\tilde{\varphi}$, the right for holding the corresponding values of $\tilde{\varphi}$. This quantum system has been constructed with a unitary transformation

$$U_{\widetilde{\varphi}}:|x\rangle |1\rangle \longmapsto |x\rangle |\widetilde{\varphi}(x)\rangle$$

implementing the "approximating" map $\tilde{\varphi}$.

As an example, let us use Shor's algorithm to factor the integer N = 21, assuming that a = 2 has been randomly chosen. Thus, $Q = 2^9 = 512$.

Unknown to us, the period is P = 6, and hence, $Q = 6 \cdot 85 + 2$.

We proceed by executing the following steps:

 $\mathbb{STEP} \ \mathbf{0} \ \ \mathrm{Initialize}$

$$|\psi_0\rangle = |0\rangle |1\rangle$$

STEP 1 Apply the inverse Fourier transform²

$$\mathcal{F}^{-1}: |u\rangle \longmapsto \frac{1}{\sqrt{512}} \sum_{x=0}^{511} \omega^{-ux} |x\rangle$$

to the left register, where $\omega = \exp(2\pi i/512)$ is a primitive 512-th root of unity, to obtain

$$\left|\psi_{1}\right\rangle = \frac{1}{\sqrt{512}} \sum_{x=0}^{511} \left|x\right\rangle \left|1\right\rangle$$

STEP 2 Apply the unitary transformation

$$U_{\widetilde{\varphi}}: |x\rangle |1\rangle \longmapsto |x\rangle |2^x \operatorname{mod} 21\rangle$$

to obtain

$$|\psi_2\rangle = \frac{1}{\sqrt{512}} \sum_{x=0}^{511} |x\rangle |2^x \mod 21\rangle$$

 $^{^{2}}$ Actually, for this step, the original Shor algorithm uses instead the Hadamard transform, which for step 1, has the same effect as the 512-point Fourier transform.

STEP 3 Apply the Fourier transform

$$\mathcal{F}: |x\rangle \longmapsto \frac{1}{\sqrt{512}} \sum_{y=0}^{511} \omega^{xy} |y\rangle$$

to the left register to obtain

$$|\psi_{3}\rangle = \frac{1}{512} \sum_{x=0}^{511} \sum_{y=0}^{511} \omega^{xy} |y\rangle |2^{x} \mod 21\rangle = \frac{1}{512} \sum_{y=0}^{511} |y\rangle \left(\sum_{x=0}^{511} \omega^{xy} |2^{x} \mod 21\rangle\right)$$

$$=\frac{1}{512}\sum_{y=0}^{511}\left|y\right\rangle\left|\Upsilon\left(y\right)\right\rangle$$

where

$$|\Upsilon(y)\rangle = \sum_{x=0}^{511} \omega^{xy} |2^x \operatorname{mod} 21\rangle$$

STEP 4 Measure the left register. Then with Probability

$$Prob_{\widetilde{\varphi}}(y) = \frac{\langle \Upsilon(y) | \Upsilon(y) \rangle}{(512)^2}$$

the state will "collapse" to $|y\rangle$ with the value measured being the integer y, where $0\leq y < Q.$

A plot of $Prob_{\tilde{\varphi}}(y)$ is shown in Figure 1. (See [21] and [25] for details.)



Figure 1. A plot of $\operatorname{Prob}_{\widetilde{\varphi}}(\mathbf{y})$.

The peaks in the above plot of $Prob_{\tilde{\varphi}}(y)$ occur at the integers

y = 0, 85, 171, 256, 341, 427.

The probability that at least one of these six integers will occur is quite high. It is actually 0.78^+ . Indeed, the probability distribution has been intentionally engineered to make the probability of these particular integers as high as possible. And there is a good reason for doing so.

The above six integers are those for which the corresponding rational y/Q is "closest" to a rational of the form d/P. By "closest" we mean that

$$\left|\frac{y}{Q} - \frac{d}{P}\right| < \frac{1}{2Q} < \frac{1}{2P^2}$$

In particular,

$$\frac{0}{512}, \frac{85}{512}, \frac{171}{512}, \frac{256}{512}, \frac{341}{512}, \frac{427}{512}$$
are rationals respectively "closest" to the rationals
$$\frac{0}{6}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}.$$

The six rational numbers 0/6, 1/6, ..., 5/6 are "closest" in the sense that they are convergents of the continued fraction expansions of 0/512, 85/512, ..., 427/512, respectively. Hence, each of the six rationals 0/6, 1/6, ..., 5/6 can be found using the standard continued fraction recursion formulas.

But ... , we are not searching for rationals of the form d/P. Instead, we seek only the denominator P = 6.

Unfortunately, the denominator P = 6 can only be obtained from the continued fraction recursion when the numerator and denominator of d/P are relatively prime. Given that the algorithm has selected one of the random integers 0, 85, ..., 427, the probability that the corresponding rational d/P has relatively prime numerator and denominator is $\phi(6)/6 = 1/3$, where $\phi(-)$ denotes the Euler phi (totient) function. So the probability of finding P = 6 is actually not 0.78^+ , but is instead 0.23^- .

As it turns out, if he repeats the algorithm $O(\lg \lg N)$ times, we will obtain the desired period P with probability bounded below by approximately $4/\pi^2$. However, this is not the end of the story. Once we have in our possession a candidate P' for the actual period P = 6, the only way we can be sure we have the correct period P is to test P' by computing $2^{P'} \mod 21$. If the result is 1, we are certain we have found the correct period P. This last part of the computation is done by the repeated squaring algorithm³.

$$a^{P'} = \prod_{j} \left(a^{2^j}\right)^{P'_j},$$

where $P' = \sum_{j} P'_{j} 2^{j}$ is the radix 2 expansion of P'.

³By the repeated squaring algorithm, we mean the algorithm which computes $a^{P'} \mod N$ via the expression

3. Definition of quantum hidden subgroup (QHS) algorithms

Now that we have taken a quick look at Shor's algorithm, let's see how it can be decomposed into its primitive algorithmic components. We will first need to answer the following question:

What is a quantum hidden subgroup algorithm?

But before we can answer the this question, we need to provide an answer to an even more fundamental question:

What is a hidden subgroup problem?

Definition 1. A map $\varphi : G \longrightarrow S$ from a group G into a set S is said to have hidden subgroup structure if there exists a subgroup K_{φ} of G, called a hidden subgroup, and an injection $\iota_{\varphi} : G/K_{\varphi} \longrightarrow S$, called a hidden injection, such that the diagram

is commutative⁴, where G/K_{φ} denotes the collection of right cosets of K_{φ} in G, and where $\nu : G \longrightarrow G/K_{\varphi}$ is the natural surjection of G onto G/K_{φ} . We refer to the group G as the **ambient group** and to the set S as the **target set**. If K_{φ} is a normal subgroup of G, then $H_{\varphi} = G/K_{\varphi}$ is a group, called the **hidden quotient group**, and $\nu : G \longrightarrow G/K_{\varphi}$ is an epimorphism, called the **hidden epimorphism**. We will call the above diagram the **hidden subgroup structure** of the map $\varphi : G \longrightarrow S$. (See [25],[20].)

Remark 1. The underlying intuition motivating this formal definition is as follows: Given a natural surjection (or epimorphism) $\nu: G \longrightarrow G/K_{\varphi}$, an "archvillain with malice of forethought" hides the algebraic structure of ν by intentionally renaming all the elements of G/K_{φ} , and "maliciously tossing in for good measure" some extra elements to form a set S and a map $\varphi: G \longrightarrow S$.

The hidden subgroup problem can be stated as follows:

Hidden Subgroup Problem (HSP). Let $\varphi : G \longrightarrow S$ be a map with hidden subgroup structure. The problem of determining a hidden subgroup K_{φ} of G is called a **hidden subgroup problem (HSP)**. An algorithm solving this problem is called a **hidden subgroup algorithm**.

The corresponding quantum form of this HSP is stated as follows:

Hidden Subgroup Problem (Quantum Version). Let $\varphi : G \longrightarrow S$ be a map with hidden subgroup structure. Construct a quantum implementation of the map φ as follows:

⁴By saying that this diagram is commutative, we mean $\varphi = \iota_{\varphi} \circ \nu$. The notion generalizes in an obvious way to more complicated diagrams.

Let \mathcal{H}_G and \mathcal{H}_S be Hilbert spaces defined respectively by the orthonormal bases $\{|g\rangle : g \in G\}$ and $\{|s\rangle : s \in S\}$ and let $s_0 = \varphi(1)$, where 1 denotes the identity⁵ of the ambient group A. Finally, let U_{φ} be a unitary transformation such that

$$egin{array}{cccc} \mathcal{H}_G\otimes\mathcal{H}_S&\longrightarrow&\mathcal{H}_G\otimes\mathcal{H}_S\ |g
angle|s_0
angle&\mapsto&|g
angle|arphi(g)
angle \end{array}$$

Determine the hidden subgroup K_{φ} with bounded probability of error by making as few queries as possible to the blackbox U_{φ} . A quantum algorithm solving this problem is called a **quantum hidden subgroup** (QHS) algorithm.

4. The generic QHS algorithm

We are now in a position to construct one of the fundamental algorithmic primitives found in Shor's algorithm.

Let $\varphi : G \longrightarrow S$ be a map from a group G to a set S with hidden subgroup structure. We assume that all representations of G are equivalent to unitary representations⁶. Let \widehat{G} denote a **complete set of distinct irreducible unitary representations** of G. Using multiplicative notation for G, we let 1 denote the **identity** of G, and let s_0 denote its image in S. Finally, let $\widehat{1}$ denote the **trivial representation** of G.

Remark 2. If G is abelian, then $\widehat{\mathbf{G}}$ becomes the **dual group** of characters.

The generic QHS algorithm is given below:

Quantum Subroutine $QRAND(\varphi)$

Step 0. Initialization

$$\left|\psi_{0}\right\rangle = \left|\widehat{1}\right\rangle \left|s_{\mathbf{0}}\right\rangle \in \mathcal{H}_{\widehat{G}}\otimes \mathcal{H}_{S}$$

Step 1. Application of the inverse Fourier transform \mathcal{F}_G^{-1} of G to the left register

$$|\psi_1\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |s_0\rangle \in \mathcal{H}_G \otimes \mathcal{H}_S \ ,$$

where |G| denotes the cardinality of the group G.

Step 2. Application of the unitary transformation U_{φ}

$$|\psi_2\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |\varphi(g)\rangle \in \mathcal{H}_G \otimes \mathcal{H}_S$$

⁵We are using multiplicative notation for the group G.

 $^{^{6}}$ This is true for all finite groups as well as for a large class of infinite groups.

Step 3. Application of the Fourier transform \mathcal{F}_G of G to the left register

$$|\psi_{3}\rangle = \frac{1}{|G|} \sum_{\gamma \in \widehat{G}} |\gamma| \operatorname{Trace}\left(\sum_{\mathbf{g} \in \mathbf{G}} \gamma^{\dagger}(g) |\gamma\rangle\right) |\varphi(g)\rangle = \frac{1}{|G|} \sum_{\gamma \in \widehat{G}} |\gamma| \operatorname{Trace}\left(|\gamma\rangle \left|\Phi(\gamma^{\dagger})\right\rangle\right) \in \mathcal{H}_{\widehat{G}} \otimes \mathcal{H}_{S} \quad ,$$

where $|\gamma|$ denotes the degree of the representation γ , where γ^{\dagger} denotes the contragradient representation (i.e., $\gamma^{\dagger}(g) = \gamma \left(g^{-1}\right)^{T} = \overline{\gamma(g)}^{T}$), where $Trace\left(\gamma^{\dagger}|\gamma\rangle\right) = \sum_{i=1}^{|\gamma|} \sum_{j=1}^{|\gamma|} \overline{\gamma_{ji}(g)} |\gamma_{ij}\rangle$, and where $\left|\Phi\left(\gamma_{ij}^{\dagger}\right)\right\rangle = \sum_{g \in G} \overline{\gamma_{ji}(g)} |\varphi(g)\rangle$.

Step 4. Measurement of the left quantum register with respect to the orthonormal basis

$$\left\{ |\gamma_{ij}\rangle : \gamma \in \widehat{G}, \ 1 \le i, j \le |\gamma| \right\} \quad .$$

Thus, with probability

$$Prob_{\varphi}\left(\gamma_{ij}\right) = \frac{\left|\gamma\right|^{2} \left\langle \Phi\left(\gamma_{ij}^{\dagger}\right) | \Phi\left(\gamma_{ij}^{\dagger}\right) \right\rangle}{\left|G\right|^{2}}$$

the resulting measured value is the entry γ_{ij} , and the quantum system "collapses" to the state

$$\left|\psi_{4}\right\rangle = \frac{\left|\gamma_{ij}\right\rangle \left|\Phi\left(\gamma_{ij}^{\dagger}\right)\right\rangle}{\sqrt{\left\langle\Phi\left(\gamma_{ij}^{\dagger}\right)\left|\Phi\left(\gamma_{ij}^{\dagger}\right)\right.\right\rangle}} \in \mathcal{H}_{\widehat{G}} \otimes \mathcal{H}_{S}$$

Step 5. Step 5. Output γ_{ij} , and stop.

5. Pushing and Lifting Hidden subgroup problems (HSPs)

But Shor's algorithm consists of more than the primitive QRAND.

For many (but not all) hidden subgroup problems (HSPs) $\varphi: G \longrightarrow S$, the corresponding generic QHS algorithm QRAND either is not physically implementable or is too expensive to implement physically. For example, the HSP φ is usually not physically implementable if the ambient group is infinite (e.g., G is the infinite cyclic group \mathbb{Z}), and is too expensive to implement if the ambient group is too large (e.g., G is the symmetric group $\mathbb{S}_{10^{100}}$). In this case, there is a standard generic way of "tweaking" the HSP to get around this problem, which we will call **pushing**.

Definition 2. Let $\varphi : G \longrightarrow S$ be a map from a group G to a set S. A map $\widetilde{\varphi} : \widetilde{G} \longrightarrow S$ from a group \widetilde{G} to the set S is said to be a **push** of φ , written

$$\widetilde{\varphi} = Push\left(\varphi\right) \quad ,$$

provided there exists an epimorphism $\nu : G \longrightarrow \widetilde{G}$ from G onto \widetilde{G} , and a transversal⁷ $\tau : \widetilde{G} \longrightarrow G$ of ν such that $\widetilde{\varphi} = \varphi \circ \tau$, i.e., such that the following diagram is commutative

$$egin{array}{ccc} G & \stackrel{\varphi}{\longrightarrow} & S \ \uparrow & \swarrow & \widetilde{arphi} & \ \widetilde{G} & & & \end{array} \ \end{array}$$

If the epimorphism μ and the transversal τ are chosen in an appropriate way, then execution of the generic QHS subroutine with input $\tilde{\varphi} = Push(\varphi)$, i.e., execution of

$$\operatorname{QRand}(\widetilde{\varphi})$$
,

will with high probability produce an irreducible representation $\tilde{\gamma}$ of the group \tilde{G} which is sufficiently close to an irreducible representation γ of the group G. If this is the case, then there is a polynomial time classical algorithm which upon input $\tilde{\gamma}$ produces the representation γ .

Obviously, much more can be said about pushing. But unfortunately that would take us far afield from the objectives of this paper. For more information on pushing, we refer the reader to [27].

It would be amiss not to mention that the above algorithmic primitive of pushing suggests the definition of a second primitive which we will call **lifting**.

Definition 3. Let $\varphi : G \longrightarrow S$ be a map from a group G to a set S. A map $\varphi: \underline{G} \longrightarrow S$ from a group \underline{G} to the set S is said to be a **lift** of φ , written

$$\underline{\varphi} = Lift\left(\varphi\right) \quad ,$$

provided there exists an morphism $\eta : \underline{G} \longrightarrow G$ from \underline{G} to G such that $\underline{\varphi} = \varphi \circ \eta$, *i.e.*, such that the following diagram is commutative

$$\begin{array}{ccc} \underline{G} \\ \eta \downarrow & \searrow \underline{\varphi} \\ G & \xrightarrow{\varphi} & S \end{array}$$

⁷Let $\nu : A \longrightarrow B$ be an epimorphism from a group A to a group B. Then a transversal τ of ν is is a map $\tau : B \longrightarrow A$ such that $\nu \circ \tau : B \longrightarrow A$ is the identity map $b \longmapsto b$. (It immediately follows that τ is an injection.) In othere words, a transversal τ of an epimorphism ν is a map which maps eacl element b of B to an element of A contained in the coset b, i.e., to a coset representative of b.



Figure 2. Pushing and Lifting HSPs.

6. Shor's algorithm revisited

We are now in position to describe Shor's algorithm in terms of its primitive components. In particular, we are now in a position to see that Shor's factoring algorithm is a classic example of a QHS algorithm created from the push of an HSP.

Let N be the integer to be factored. Let \mathbb{Z} denote the additive group of integers, and \mathbb{Z}_N^{\times} denote the integers mod N under multiplication.

Shor's algorithm is a QHS algorithm that solves the following HSP

$$\begin{array}{cccc} \varphi: \mathbb{Z} & \longrightarrow & \mathbb{Z}_N^\times \\ m & \longmapsto & a^m \operatorname{mod} N \end{array}$$

with unknown hidden subgroup structure given by the following commutative dia-gram

$$\begin{array}{cccc} \mathbb{Z} & \stackrel{\varphi}{\longrightarrow} & \mathbb{Z}_N^{\times} \\ \nu \searrow & & \swarrow \iota \\ & & \mathbb{Z}/P\mathbb{Z} \end{array}$$

where a is an integer relatively prime to N, where P is the hidden integer period of the map $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_N^{\times}$, where $P\mathbb{Z}$ is the additive subgroup of all integer multiples of P (i.e., the hidden subgroup), where $\nu : \mathbb{Z} \longrightarrow \mathbb{Z}/P\mathbb{Z}$ is the natural epimorpism of the integers onto the quotient group $\mathbb{Z}/P\mathbb{Z}$ (i.e., the hidden epimorphism), and where $\iota : \mathbb{Z}/P\mathbb{Z} \longrightarrow \mathbb{Z}_N^{\times}$ is the hidden monomorphism.

An obstacle to creating a physically implementable algorithm for this HSP is that the domain \mathbb{Z} of φ is infinite. As observed by Shor, a way to work around this difficulty is to push the HSP.

In particular, as illustrated by the following commutative diagram

$$\begin{array}{cccc} \mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z}_{N}^{\times} \\ \mu \searrow & \swarrow & \swarrow & \varphi = Push\left(\varphi\right) = \varphi \circ \tau \\ & \mathbb{Z}_{O} \end{array}$$

a push $\tilde{\varphi} = Push(\varphi)$ is constructed by selecting the epimorphism $\mu : \mathbb{Z} \longrightarrow \mathbb{Z}_Q$ of \mathbb{Z} onto the finite cyclic group \mathbb{Z}_Q of order Q, where the integer Q is the unique power of 2 such that $N^2 \leq Q < 2N^2$, and then choosing the transversal⁸

$$\begin{array}{cccc} \tau: \mathbb{Z}_Q & \longrightarrow & \mathbb{Z} \\ m \operatorname{mod} Q & \longmapsto & m \end{array}$$

where $0 \le m < Q$. This push $\tilde{\varphi} = Push(\varphi)$ is called **Shor's oracle**.

Shor's algorithm consists in first executing the quantum subroutine QRAND($\tilde{\varphi}$), thereby producing a random character

$$\gamma_{y/Q}: m \operatorname{mod} Q \longmapsto \frac{my}{Q} \operatorname{mod} 1$$

of the finite cyclic group \mathbb{Z}_Q . The transversal τ used in pushing has been engineered in such a way as to assure that the character $\gamma_{y/Q}$ is sufficiently close to a character

$$\gamma_{d/P}: k \mod P \longmapsto \frac{kd}{P} \mod 1$$

of the hidden quotient group $\mathbb{Z}/P\mathbb{Z} = \mathbb{Z}_P$. In this case "sufficiently close" means that

$$\left|\frac{y}{Q} - \frac{d}{P}\right| \le \frac{1}{2P^2}$$

which means that d/P is a continued fraction convergent of y/Q, and thus can be found found by the classical polynomial time continued fraction algorithm⁹.

7. WANDERING SHOR ALGORITHMS, A.K.A., VINTAGE SHOR ALGORITHMS.

Now let's use the primitives described in sections 3, 4, and 5 to create other new QHS algorithms, called wandering Shor algorithms.

Wandering Shor algorithms are essentially QHS algorithms on free abelian finite rank n groups A which, with each iteration, first select a random cyclic direct summand \mathbb{Z} of the group A, and then apply one iteration of the standard Shor algorithm to produce a random character of the "approximating" finite group $\widetilde{A} = \mathbb{Z}_Q$, called a **group probe**¹⁰. Three different wandering Shor algorithms are created in [25]. The first two wandering Shor algorithms given in [25] are quantum algorithms which find the order P of a maximal cyclic subgroup of the hidden quotient group H_{φ} .

,

⁸A **transversal** for an epimorphism $\alpha_{\varphi} : \mathbb{Z} \longrightarrow \mathbb{Z}_Q$ is an injection $\tau_{\varphi} : \mathbb{Z}_{\mathbb{Q}} \longrightarrow \mathbb{Z}$ such that $\alpha_{\varphi} \circ \tau_{\varphi}$ is the identity map on \mathbb{Z}_Q , i.e., a map that takes each element of \mathbb{Z}_Q onto a coset representative of the element in \mathbb{Z} .

⁹The characters $\gamma_{y/Q}$ and $\gamma_{d/P}$ can in the obvious way be identified with points of in the unit circle in the complex plane. With this identification, we can see that this inequality is equivalent to saying the the chordal distance between these two rational points on the unit circle is less than or equan to $1/2P^2$. Hence, Shor's algorithm is using the topology of the unit circle.

¹⁰By a group probe \widetilde{A} , we mean an epimorphic image of the ambient group A.

The first step in creating a wandering Shor algorithm is to find the right generalization one of the primitives found in Shor's algorithm, namely, the transversal $\iota: \mathbb{Z}_Q \longrightarrow \mathbb{Z}$ of Shor's factoring algorithm. In other words, we need to construct the "correct" generalization of the transversal from \mathbb{Z}_Q to a free abelian group Aof rank n. For this reason, we have created the following definition:

Definition 4. Let A be the free abelian group of rank n, let $\nu : A \longrightarrow \mathbb{Z}_Q$ onto the cyclic group \mathbb{Z}_Q of order Q with selected generator \tilde{a} . A transversal¹¹ $\iota : \mathbb{Z}_Q \longrightarrow A$ of ν is said to be a **Shor transversal** provided that:

- 1) $\iota(n\widetilde{a}) = n\iota(\widetilde{a})$ for all $0 \le n < Q$
- 2) For each (free abelian) basis a'_1, a'_2, \ldots, a'_n of A, the coefficients $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$ of $\iota(\widetilde{a}) = \sum_j \lambda'_j a'_j$ satisfy $gcd(\lambda'_1, \lambda'_2, \ldots, \lambda'_n) = 1$.

Remark 3. Later, when we construct a generalization of Shor transversals to free groups of finite rank n, we will see that the first condition simply states that a Shor transversal is nothing more than a 2-sided Schreier transversal. The second condition of the above definition simply says that ι maps the generator \tilde{a} of \mathbb{Z}_Q onto a generator of a free direct summand \mathbb{Z} of A. (For more details, please refer to section 12 of this paper.)

Remark 4. In [25], we show how to use the extended Euclidean algorithm to construct the epimorphism $\nu : A \longrightarrow \mathbb{Z}_Q$ and the transversal $\iota : \mathbb{Z}_Q \longrightarrow A$.

¹¹Let $\nu : A \longrightarrow B$ be an epimorphism from a group A to a group B. Then a transversal τ of ν is is a map $\tau : B \longrightarrow A$ such that $\nu \circ \tau : B \longrightarrow A$ is the identity map $b \longmapsto b$. (It immediately follows that τ is an injection.) In othere words, a transversal τ of an epimorphism ν is a map which maps eacl element b of B to an element of A contained in the coset b, i.e., to a coset representative of b.



Figure 3. Flowchart for the first wandering Shor algorithm (a.k.a., a vintage Shor algorithm). This algorithm finds the order P of a maximal cyclic subgroup of the hidden quotient group H_{φ} .

Flow charts for the three wandering Shor algorithms created in [25] are given in figures 3 through 5. In [25], these were also called **vintage Shor algorithms**.



Figure 4. Flowchart for the second wandering Shor algorithm (a.k.a., a vintage Shor algorithm). This algorithm finds the order P of a maximal cyclic subgroup of the hidden quotient group H_{φ} .



Figure 5. Flowchart for the third wandering Shor algorithm, a.k.a., a vintage Shor algorithm. This algorithm finds the entire hidden quotient group H_{φ} .

The algorithmic complexities of the above wandering Shor algorithms is given in [25]. For example, the first wandering Shor algorithm is of time complexity

$$O\left(n^2 \left(\lg N\right)^3 \left(\lg \lg N\right)^{n+1}\right)$$

where n is the rank of the free abelian group A. This can be readily deduced from the abbreviated flowchart given in figure 6.



Figure 6. Abbreviated flowchart for the first wandering Shor algorithm.

8. CONTINUOUS (VARIABLE) SHOR ALGORITHMS

In in [27] and in [29], the algorithmic primitives found in above sections of this paper were used to create a class of algorithms called continuous Shor algorithms. By a **continuous variable Shor algorithm**, we mean a quantum hidden subgroup algorithm that finds the hidden period P of an admissible function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ from the reals \mathbb{R} to itself.

Remark 5. By an admissible function, we mean a function belonging to any sufficiently well behaved class of functions. For example, the class of functions which are Lebesgue integrable on every closed interval of \mathbb{R} . There, are many other classes of functions that work equally as well.

Actually, the papers [27], [29] give in succession three such continuous Shor algorithms, each successively more general than the previous.

For the first algorithm, we assume that the unknown hidden period P is an integer. The algorithm is then constructed by using rigged Hilbert spaces[4], [10], linear combinations of Dirac delta functions, and a subtle extension of the Fourier transform found in the generic QHS subroutine QRAND(φ), which has been described previously in section 4 of this paper. In Step 5 of QRAND(φ), the observable

$$A=\int\limits_{-\infty}^{\infty}dy\frac{\lfloor Qy\rfloor}{Q}\left|y\right\rangle \left\langle y\right|$$

is measured, where Q is an integer chosen so that $Q \ge 2P^2$. It then follows that the output of this algorithm is a rational m/Q which is a convergent of the continued fraction expansion of a rational of the form n/P.

The above quantum algorithm is then extended to a second quantum algorithm that finds the hidden period P of functions $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, where the unknown period P is a rational.

Finally, the second algorithm is extended to a third algorithm which finds the hidden period P of functions $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, when P is an *arbitrary real number*. We point out that for the third and last algorithm to work, we must impose a very restrictive condition on the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, i.e., the condition that the map φ is continuous.

9. The quantum circle and the dual Shor algorithms.

We have shown in previous sections how the mathematical primitives of pushing and lifting can be used to create new quantum algorithms. In particular, we have described how pushing and lifting can be used to derive new HSPs from an HSP $\varphi : G \longrightarrow S$ on an arbitrary group G. We now see how group duality can be exploited by these two primitives to create even more quantum algorithms.



Figure 7. Using duality to create new QHS algorithms.

To this end, we assume that G is an *abelian* group. Hence, its dual group of characters \widehat{G} exists¹². It now follows that pushing and lifting can also be used to derive new HSPs from an arbitrary HSP $\Phi : \widehat{G} \longrightarrow S'$ on the dual group \widehat{G} . In [27], this method is used to create a number of new quantum algorithms derived from Shor-like HSPs $\varphi : \mathbb{Z} \longrightarrow S$.

A roadmap is shown in figure 8 of the developmental steps taken to find and to create a new QHS algorithm on \mathbb{Z}_Q , which is (in the sense described below) dual to Shor's original algorithm. We call the algorithm developed in the final step of figure 8 the **dual Shor algorithm**.



Figure 8. Roadmap for creating the dual Shor algorithm.

¹²If G is non-abelian, then its dual is not a group, but instead the representation algebra \mathcal{A} over the group ring $\mathbb{C}G$. The methods described in this section can also be used to create new quantum algorithms for HSPs $\Phi : \mathcal{A} \longrightarrow S$ on on the representation algebra \mathcal{A} .

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As indicated in figure 5, our first step is to create an intermediate QHS algorithm based on a Shor-like HSP $\varphi : \mathbb{Z} \longrightarrow S$ from the additive group of integers \mathbb{Z} to a target set S. The resulting algorithm "lives" in the infinite dimensional space $\mathcal{H}_{\mathbb{Z}}$ defined by the orthonormal basis $\{\langle n | : n \in \mathbb{Z}\}$. This is a physically unemplementable quantum algorithm created as a first steping stone in our algorithmic development sequence. Intuitively, this algorithm can be viewed as a "distillation" or a "purification" of Shor's original algorithm.

As a next step, **duality** is used to create the **quantum circle algorithm**. This is accomplished by devising a QHS algorithm for an HSP $\Phi : \mathbb{R}/\mathbb{Z} \longrightarrow S$ on the dual group \mathbb{R}/\mathbb{Z} of the additive group of integers \mathbb{Z} . (By \mathbb{R}/\mathbb{Z} , we mean the **additive group of reals mod1**, which is isomorphic to the multiplicative group $\{e^{2\pi i\theta} : 0 \le \theta < 1\}$, i.e., the **unit circle** in the complex plane.) Once again, this is probably a physically unemplementable quantum algorithm¹³. But its utility lies in the fact that it leads to the physically implementable quantum algorithm created in the last and final developmental step, as indicated in figure 8. For in the final step, a physically implementable QHS algorithm is created by **lifting** the HSP $\Phi : \mathbb{R}/\mathbb{Z} \longrightarrow S$ to an HSP $\widetilde{\Phi} : \mathbb{Z}_Q \longrightarrow S$. For the obvious reason, we call the resulting algorithm a **dual Shor algorithm**.

For detailed descriptions of each of these quantum algorithms, i.e., the "distilled" Shor, the quantum circle, and the dual Shor algorithms, the reader is referred to [27] and [29].

We give below brief descriptions of the quantum circle and the dual Shor algorithms.

For the **quantum circle algorithm**, we make use of the following spaces (each of which is used in quantum optics):

- The rigged Hilbert space $H_{\mathbb{R}/\mathbb{Z}}$ with orthonormal basis $\{|x\rangle : x \in \mathbb{R}/\mathbb{Z}\}$. By "orthonormal" we mean that $\langle x|y\rangle = \delta(x-y)$, where " δ " denotes the Dirac delta function. The elements of $H_{\mathbb{R}/\mathbb{Z}}$ are **formal integrals** of the form $\oint dx f(x) |x\rangle$. (The physicist Dirac in his classic book[6] on quantum mechanics refers to these integrals as infinite sums. See also [4] and [10].)
- The complex vector space $H_{\mathbb{Z}}$ of formal sums

$$\left\{\sum_{n=-\infty}^{\infty} a_n \left| \mathbf{n} \right\rangle : a_n \in C \ \forall n \in \mathbb{Z}\right\}$$

with orthonormal basis $\{|n\rangle : n \in Z\}$. By "orthonormal" we mean that $\langle n|m\rangle = \delta_{nm}$, where δ_{nm} denotes the Kronecker delta.

We can now design an algorithm which solves the following hidden subgroup problem:

Hidden Subgroup Problem for the Circle. Let $\Phi : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}$ be an admissible function from the circle group \mathbb{R}/\mathbb{Z} to the complex numbers \mathbb{C} with hidden

¹³There is a possibility that the quantum circle algorithm may have a physical implementation in terms of quantum optics.

rational period $\alpha \in \mathbb{Q}/\mathbb{Z}$, where $\alpha \in \mathbb{Q}/\mathbb{Z}$ denotes the rational circle, i.e., the rationals mod 1.

Remark 6. By an admissible function, we mean a function belonging to any sufficiently well behaved class of functions. For example, the class of functions which are Lebesgue integrable on \mathbb{R}/\mathbb{Z} . There, are many other classes of functions that work equally as well.

Proposition 1. If $\alpha = a_1/a_2$ (with gcd $(a_1, a_2) = 1$) is a rational period of a function $\Phi : \mathbb{R}/\mathbb{Z} \longrightarrow C$, then $1/a_2$ is also a period of Φ . Hence, the minimal rational period of Φ is always a reciprocal integer mod 1.

The following quantum algorithm finds the reciprocal integer period of the function Φ .

CIRCLE-ALGORITHM(Φ)

Step 0. Initialization

$$|\psi_0\rangle = |0\rangle |0\rangle \in H_{\mathbb{Z}} \otimes H_{\mathbb{C}}$$

Step 1. Application of the inverse Fourier transform $\mathcal{F}^{-1} \otimes 1$

$$\left|\psi_{1}\right\rangle = \int dx \ e^{2\pi i \cdot 0} \left|x\right\rangle \left|0\right\rangle = \int dx \ \left|x\right\rangle \left|0\right\rangle \in H_{\mathbb{R}/\mathbb{Z}} \otimes H_{\mathbb{C}}$$

Step 2. Step 2. Application of the unitary transformation $U_{\varphi} : |x\rangle |u\rangle \mapsto |x\rangle |u + \Phi(x)\rangle$

$$|\psi_2\rangle = \int dx \ |x\rangle |\Phi(x)\rangle \in H_{\mathbb{R}/\mathbb{Z}} \oplus H_{\mathbb{C}}$$

Step 3. Application of the Fourier transform $\mathcal{F} \otimes 1$

Remark 7. Remark. Letting $x_m = x - \frac{m}{a}$, we have $\int dx \ e^{2\pi i n x} |\Phi(x)\rangle = \sum_{m=0}^{a-1} \int_{m/a}^{(m+1)/a} dx \ e^{-2\pi i n x} |\Phi(x)\rangle$ $= \sum_{m=0}^{a-1} \int_{0}^{1/a} dx_m \ e^{-2\pi i n \left(x_m + \frac{m}{a}\right)} \left|\Phi\left(x_m + \frac{m}{a}\right)\right\rangle$ $= \left(\sum_{m=0}^{a-1} e^{-2\pi i n m/a}\right) \int_{0}^{1/a} dx \ e^{-2\pi i n x} |\Phi(x)\rangle$ where 1/a is the unknown reciprocal period. But

$$\sum_{m=0}^{a-1} e^{-2\pi i n m/a} = a\delta_{n=0 \mod a} = \begin{cases} a & \text{if } n = 0 \mod a \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} |\psi_{\mathbf{3}}\rangle &= \sum_{n \in \mathbb{Z}} |n\rangle \int dx \ e^{-2\pi i n x} \left| \Phi\left(x\right) \right\rangle = \left(\sum_{n \in Z} |n\rangle \,\delta_{n=0 \, \text{mod} \, a} \right) \int_{0}^{1/a} dx \ e^{-2\pi i n x} \left| \Phi\left(x\right) \right\rangle \\ &= \left(\sum_{\ell \in Z} |\ell a\rangle \right) \left(\int_{0}^{1/a} dx \ e^{-2\pi i n x} \left| \Phi\left(x\right) \right\rangle \right) = \sum_{\ell \in \mathbb{Z}} |\ell a\rangle \left| \Omega\left(\ell a\right) \right\rangle \end{aligned}$$

Step 4. Measurement of

$$\left|\psi_{\mathbf{3}}\right\rangle = \sum_{\ell \in \mathbb{Z}} \left|\ell a\right\rangle \left|\mathbf{\Omega}\left(\ell \mathbf{a}\right)\right\rangle \in \mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{C}}$$

with respect to the observable

$$\sum_{n\in\mathbb{Z}}n\left|n\right\rangle\left\langle n\right.$$

to produce a random eigenvalue ℓa .

Remark 8. The above quantum circle algorithm can be extended to a quantum algorithm which finds the hidden period α of a function $\Phi : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}$, when α is an arbitrary real number mod 1. But in creating this extended quantum algorithm, a very restrictive condition must be imposed on the map $\Phi : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}$, namely, the condition that Φ be continuous.

We now give a brief description of the **dual Shor algorithm**.

The dual Shor algorithm is a QHS algorithm created by making a discrete approximation of the quantum circle algorithm. More specifically, it is created by lifting the QHS circle algorithm for $\varphi : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}$ to the finite cyclic group \mathbb{Z}_Q , as illustrated in the commutative diagram given below:

$$\begin{array}{ll} \mathbb{Z}_Q \\ \mu \downarrow & \searrow \widetilde{\varphi} = Push\left(\varphi\right) = \varphi \circ \mu \\ \mathbb{R}/\mathbb{Z} & \longrightarrow S \end{array}$$

Intuitively, just as in Shor's algorithm, the circle group \mathbb{R}/\mathbb{Z} is "approximated" with the finite cyclic group \mathbb{Z}_Q , where the group \mathbb{Z}_Q is identified with the additive group

$$\left\{\frac{0}{Q}, \frac{1}{Q}, \dots, \frac{Q-1}{Q}\right\} \mod 1 \quad ,$$

and where the hidden subgroup \mathbb{Z}_P is identified with the additive group

$$\left\{\frac{0}{P}, \frac{1}{P}, \dots, \frac{P-1}{P}\right\} \mod 1 \quad ,$$

with $P = a_2$.

This is a physically implementable quantum algorithm. In a certain sense, it is actually faster than Shor's algorithm. For the last step of Shor's algorithm uses the standard continued fraction algorithm to determine the unknown period. On the other hand, the last step of the dual Shor algorithm uses the much faster Euclidean algorithm to compute the greatest common divisor of the integers $\ell_1 a, \ell_2 a, \ell_3 a, \ldots$, thereby determining the desired reciprocal integer period 1/a. For more details, please refer to [27] and [29].

10. A QHS ALGORITHM FOR FEYNMAN INTEGRALS.

We now discuss a QHS algorithm based on Feynman path integrals. This quantum algorithm was developed at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California when one of the authors of this paper was challenged with an invitation to give a talk on the relation between Feynmann path integrals and quantum computing at an MSRI conference on Feynman path integrals.

Until recently, both authors of this paper thought that the quantum algorithm to be described below was a highly speculative quantum algorithm. For the existence of Feynman path integrals is very difficult (if not impossible) to determine in a mathematically rigorous fashion. But surprisingly, Jeremy Becnel in his doctoral dissertation [1] actually succeeded in creating a firm mathematical foundation for this algorithm.

We should mention, however, that the physical implementability of this algorithm is still yet to be determined.

Definition 5. Definition. Let PATHS be the real vector space of all continuous paths $x : [0,1] \longrightarrow \mathbb{R}^n$ which are L^2 with respect to the inner product

$$x\cdot y = \int_0^1 ds \ x(s)y(s)$$

with scalar multiplication and vector sum defined as

- $(\lambda x)(s) = \lambda x(s)$
- (x+y)(s) = x(s) + y(s)

We wish to create a QHS algorithm for the following hidden subgroup problem:

Hidden Subgroup Problem for PATHS. Let φ : PATHS $\longrightarrow C$ be a functional with a hidden subspace V of PATHS such that

$$\varphi\left(x+v\right) = \varphi\left(x\right) \quad \forall v \in V$$

Our objective is to create a QHS algorithm which solves the above problem, i.e., which finds the hidden subspace V.

Definition 6. Let H_{PATHS} be the rigged Hilbert space with orthonormal basis $\{|x\rangle : x \in \text{PATHS}\}$, and with bracket product $\langle x|y \rangle = \delta(x-y)$.

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We will use the following observation to create the QHS algorithm:

Observation. PATHS = $\bigcup_{v \in V} (v + V^{\perp})$, where V^{\perp} denotes the orthogonal complement of the hidden vector subspace V.

The QHS algorithm for Feynman path integral is given below:

 $\operatorname{Feynman}(\varphi)$

Step 0. Initialize

$$|\psi_0\rangle = |0\rangle |0\rangle \in H_{Paths} \otimes H_{\mathbb{C}}$$

Step 1. Apply $\mathcal{F}^{-1} \otimes 1$

$$|\psi_1\rangle = \int_{\text{PATHS}} \mathcal{D}x \ e^{2\pi i x \cdot 0} |x\rangle |0\rangle = \int_{\text{PATHS}} \mathcal{D}x \ |x\rangle |0\rangle$$

Step 2. Apply $U_{\varphi}: |x\rangle |u\rangle \mapsto |x\rangle |u + \varphi(x)\rangle$

$$|\psi_{\mathbf{2}}\rangle = \int_{\text{PATHS}} \mathcal{D}x |x\rangle |\varphi(x)\rangle$$

Step 3. Apply
$$\mathcal{F} \otimes 1$$

$$\begin{aligned} |\psi_{3}\rangle &= \int\limits_{\text{PATHS}} \mathcal{D}y \int\limits_{\text{PATHS}} \mathcal{D}x \ e^{-2\pi i x \cdot y} \left|y\right\rangle \left|\varphi\left(x\right)\right\rangle \\ &= \int\limits_{\text{PATHS}} \mathcal{D}y \ \left|y\right\rangle \int\limits_{\text{PATHS}} \mathcal{D}x \ e^{-2\pi i x \cdot y} \left|\varphi\left(x\right)\right\rangle \end{aligned}$$

But

$$\int_{\text{PATHS}} \mathcal{D}x \ e^{-2\pi i x \cdot y} \left| \varphi \left(x \right) \right\rangle = \int_{V} \mathcal{D}v \int_{v+V^{\perp}} \mathcal{D}x \ e^{-2\pi i x \cdot y} \left| \varphi \left(x \right) \right\rangle$$
$$= \int_{V} \mathcal{D}v \int_{V^{\perp}} \mathcal{D}x \ e^{-2\pi i (\mathbf{v} + \mathbf{x}) \cdot y} \left| \varphi \left(v + x \right) \right\rangle$$
$$= \int_{V} \mathcal{D}v \ e^{-2\pi i v \cdot y} \int_{V^{\perp}} \mathcal{D}x \ e^{-2\pi i x \cdot y} \left| \varphi \left(x \right) \right\rangle$$

However,

$$\int_{V} \mathcal{D}v \ e^{-2\pi i v \cdot y} = \int_{V^{\perp}} \mathcal{D}u \ \delta(y-u)$$

So,

$$\begin{split} |\psi_{3}\rangle &= \int_{\text{PATHS}_{n}} \mathcal{D}y \ |y\rangle \int_{V} \mathcal{D}v \ e^{-2\pi i v \cdot y} \int_{V^{\perp}} \mathcal{D}x \ e^{-2\pi i x \cdot y} |\varphi(x)\rangle \\ &= \int_{\text{PATHS}_{n}} \mathcal{D}y \ |y\rangle \int_{V^{\perp}} \mathcal{D}u \ \delta(y-u) \int_{V^{\perp}} \mathcal{D}x \ e^{-2\pi i x \cdot y} |\varphi(x)\rangle \\ &= \int_{V^{\perp}} \mathcal{D}u \ |u\rangle \int_{V^{\perp}} \mathcal{D}x \ e^{-2\pi i x \cdot u} |\varphi(x)\rangle \\ &= \int_{V^{\perp}} \mathcal{D}u \ |u\rangle \int_{V^{\perp}} \mathcal{D}x \ e^{-2\pi i x \cdot u} |\varphi(x)\rangle \\ &= \int_{V^{\perp}} \mathcal{D}u \ |u\rangle |\Omega(u)\rangle \end{split}$$

Step 4. Measure

$$\left|\psi_{3}\right\rangle = \int\limits_{V^{\perp}} \mathcal{D}u \left|u\right\rangle \left|\Omega\left(u\right)\right\rangle$$

with respect to the observable

$$A = \int_{\text{PATHS}} \mathcal{D}w |w\rangle \langle w|$$

to produce a random element of V^{\perp}

The above algorithm suggests an intriguing question. Can the above QHS Feynman integral algorithm be modified in such a way as to create a quantum algorithm for the Jones polynomial? In other words, can it be modified by replacing PATHS with the space of gauge connections, and making suitable modifications?

This question is motivated by the fact that the integral over gauge transformations

$$\widehat{\psi}(K) = \int \mathcal{D}A \ \psi(A) \mathcal{W}_K(A)$$

looks very much like a Fourier transform, where

$$\mathcal{W}_{K}(A) = tr\left(P\exp\left(\oint_{K}A\right)\right)$$

denotes the **Wilson loop** over the knot K.

11. QHS Algorithms on free groups

In this and the following section of this paper, our objective is to show that a free group is the the most natural domain for QHS algorithms. In retrospect, this is not so surprising if one takes a discerning look at Shor's factoring algorithm. For in section 6, we have seen that Shor's algorithm is essentially a QHS algorithm on the free group \mathbb{Z} which has been pushed onto the finite group \mathbb{Z}_{Q} .

In particular, let $\varphi: G \longrightarrow S$ be a map with hidden subgroup structure from a finitely generated (f.g.) group G to a set S. We assume that the hidden subgroup

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K is a normal subgroup of G of finite index. Then the objectives of this section are to demonstrate the following:

- Every hidden subgroup problem (HSP) $\varphi : G \longrightarrow S$ on an arbitray f.g. group G can be lifted to an HSP $\tilde{\varphi} : F \longrightarrow S$ on a free group F of finite rank.
- Moreover, a solution for the lifted HSP $\tilde{\varphi} : F \longrightarrow S$ is for all practical purposes the same as the solution for the original HSP $\varphi : G \longrightarrow S$.

Thus, one need only investigate QHS algorithms for free groups of finite rank!

Before we can describe the above results, we need to review a number of definitions. We begin with the definition of a free group:

Definition 7 (Universal Definition). A group F is said to be **free** of finite rank n if there exists a finite set of n generators $X = \{x_1, x_2, \ldots, x_n\}$ such that, for every group G and for every map $f : X \longrightarrow G$ of the set X into the group G, the map f extends to a morphism $\tilde{f} : F \longrightarrow G$. We call the set X a **free basis** of the group F, and frequently denote the group F by $F(x_1, x_2, \ldots, x_n)$, . It follows from this definition that the morphism \tilde{f} is unique.

The intuitive idea encapsulated by this definition is that a free group is an unconstrained group (very much analogous to a physical system without boundary conditions.) In other words, a group is free provided it has a set of generators such that the only relations among those generators are those required for F to be a group. For example,

- $x_i x_i^{-1} = 1$ is an allowed relation
- $x_i x_j = x_j x_i$ is not an allowed relation for $i \neq j$
- $x_i^3 = 1$ is not an allowed relation

As an immediate consequence of the above definition, we have the following proposition:

Proposition 2. Let G be an arbitrary f.g. group with finite set of n generators $\{g_1, g_2, \ldots, g_n\}$, and let $F = F(x_1, x_2, \ldots, x_n)$ be the free group of rank n with free basis $\{x_1, x_2, \ldots, x_n\}$.

Then by the above definition, the map $x_j \mapsto g_j$ (j = 1, 2, ..., n) induces a uniques epimorphism $\nu : F \longrightarrow G$ from F onto G. With this epimorphism, every $HSP \varphi : G \longrightarrow S$ on the group G uniquely lifts to the HSP $\tilde{\varphi} = \varphi \circ \nu : F \longrightarrow S$ on the free group F.

Moreover, if K and \widetilde{K} are the hidden subgroups of the HSPs φ and $\widetilde{\varphi}$, respectively, the corresponding hidden quotient groups G/K and F/\widetilde{K} of these two HSPs are isomorphic. Hence, every solution of the HSP $\widetilde{\varphi} : F \longrightarrow S$ immediately produces a solution of the original HSP $\varphi : G \longrightarrow S$.

We close this section with the definition of a group resentation, a concept that will be needed in the next section for generalizing Shor's algorithm to free groups.

Definition 8. Let G be a group. A group presentation

 $(x_1, x_2, \ldots, x_n : r_1, r_2, \ldots, r_m)$

for G is a set of free generators x_1, x_2, \ldots, x_n of a free group F and a set of words r_1, r_2, \ldots, r_n in $F(x_1, x_2, \ldots, x_n)$, called **relators**, such that the group G is isomorphic to the quotient group $F(x_1, x_2, \ldots, x_n) / Cons(r_1, r_2, \ldots, r_n)$, where $Cons(r_1, r_2, \ldots, r_n)$, called the **consequence** of r_1, r_2, \ldots, r_n , is the smallest normal subgroup of $F(x_1, x_2, \ldots, x_n)$ containing the relators r_1, r_2, \ldots, r_n .

The intuition captured by the above definition is that x_1, x_2, \ldots, x_n are the generators of G, and $r_1 = 1, r_2 = 1, \ldots, r_n = 1$ is a complete set of relations among these generators, i.e., every relation among the generators of G is a **consequence** of (derivable from) the relations $r_1 = 1, r_2 = 1, \ldots, r_n = 1$. For example,

- $(x_1, x_2, \dots, x_n :)$ and $(x_1, x_2, \dots, x_n : x_1 x_1^{-1}, x_2^5 x_2^{-5}, x_3 x_4 x_4^{-1} x_3^{-1})$ are both presentations of the free group $F(x_1, x_2, \dots, x_n)$
- $(x: x^Q)$ and $(x: x^a, x^b)$ are both presentations of the cyclic group \mathbb{Z}_Q of order Q, where a and b are integers such that gcd(a, b) = Q.
- $(x_1, x_2 : x_1^3, x_2^2, (x_1x_2)^2)$ is a presentation of the symmetric group S_3 on three symbols.

12. Generalizing Shor's algorithm to free groups

The objective of this section is to generalize Shor's algorithm to free groups of finite rank¹⁴. The chief obstacle to accomplishing this goal is finding a correct generalization of the Shor transversal

$$\begin{array}{cccc} \mathbb{Z}_Q & \xrightarrow{\tau} & \mathbb{Z} \\ n \operatorname{mod} Q & \longmapsto & n \end{array} (0 \le n < Q)$$

Unfortunately, there appear to be few mathematical clues indicating how to go about making such a generalization. However, as we shall see, the generalization of the Shor transversal to the transversal found in the wandering Shor algorithm does provide a crucial clue, suggesting that a generalized Shor transversal must be a 2-sided Schreier transversal. (See section 7.)

We begin by formulating a constructive approach to free groups:

Definition 9. Let $F(x_1, x_2, \ldots, x_n)$ be a free group with free basis x_1, x_2, \ldots, x_n . Then a word is a finite string of the symbols $x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}$. A reduced word is a word in which there is no substring of the form $x_j x_j^{-1}$ or $x_j^{-1} x_j$. Two words are said to be equivalent if one can be transformed into the other by applying a finite number of substring insertions or deletions of the form $x_j x_j^{-1}$ or $x_j^{-1} x_j$. We denote an arbitrary word w by $w = a_1 a_2 \cdots a_\ell$, where each $a_j = x_{k_j}^{\pm 1}$.

¹⁴We remind the reader that, in section 6, we showed that Shor's algorithm is essentially a QHS algorithm on the free group \mathbb{Z} of rank 1 constructed by a push onto the cyclic group \mathbb{Z}_Q . In light of this and of the results outlined in the previous section, it is a natural objective to generalize Shor's algorithm to free groups of finite rank.

The **length** |w| of a word $w = a_1 a_2 \cdots a_\ell$ is number of symbols $x_{k_j}^{\pm 1}$ that appear in w, i.e., $|w| = \ell$.

For example, $x_2x_1^{-1}x_1x_1^{-1}x_5^{-1}x_5^{-1}x_5^{-1}x_5$ is a word of length 8 which is equivalent to the reduced word $x_2x_1^{-1}x_5^{-1}x_5^{-1}$ of length 4.

It easily follows that:

Proposition 3. A free group $F(x_1, x_2, ..., x_n)$ is simply the set of reduced words together with the obvious definition of product, i.e., concatenation followed by full reduction.

We can now use this constructive approach to create a special kind of transversal $\tau : G \longrightarrow F$ of an epimorphism $\nu : F \longrightarrow G$, called a 2-sided Scheier transversal[14]:

Definition 10. A set W of reduced words in a free group $F = F(x_1, x_2, ..., x_n)$ is said to be a 2-sided Schreier system provided

- The empty word 1 lies in W.
- $w = a_1 a_2 \cdots a_{\ell-1} a_\ell \in \mathcal{W} \Rightarrow w_{Left} = a_1 a_2 \cdots a_{\ell-1} \in \mathcal{W}$, and
- $w = a_1 a_2 \cdots a_{\ell-1} a_\ell \in \mathcal{W} \Rightarrow w_{Right} = a_2 \cdots a_{\ell-1} a_\ell \in \mathcal{W}$

Given an epimorphism $\nu : F \longrightarrow G$ of the free group F onto a group G, a 2-sided Schreier transversal $\tau : G \longrightarrow F$ for ν is a transversal of ν for which there exists a 2-sided Schreier system such that $\tau(G) = W$. A 2-sided Schreier transversal is said to be **minimal** provided the length of each word w is less than or equal to the length of each reduced word in the coset $wKer(\nu) = Ker(\nu)w$, where $Ker(\nu)$ denotes the kernel of the epimorpism ν .

The wandering Shor algorithm found in section 7 suggests that a correct generalization of the Shor transversal $n \mod N \longmapsto n$ $(0 \le n < Q)$ must at least have the property that it is a minimal 2-sided Schreier transversal. Whatetever other additional properties this generalization must have is simply not clear.

In [31], we construct and investigate a number of different QHS algorithms on free groups that arise from the application of various additional conditions imposed upon the minimal 2-sided Schreier transversal requirement. In this section, we only give a descriptive sketch of the simplest of these algorithms, i.e., a QHS algorithm on free groups with only the minimal 2-sided Schreier transversal requirement imposed.

Let $F = F(x_1, x_2, ..., x_n)$ be the free group of finite rank n with free basis $X = \{x_1, x_2, ..., x_n\}$, and let $\varphi : F \longrightarrow S$ be an HSP on the free group F. We assume that the hidden subgroup K is normal and of finite index in F. (Please note that $K = Ker(\varphi) = \varphi^{-1}\varphi(1)$.)

- Choose a finite group probe G with presentation $(x_1, x_2, \ldots, x_n : r_1, r_2, \ldots, r_m)_{\nu}$, where the subscript ν denotes the epimorphism $\nu : F \longrightarrow G$ induced by the map $x_j \longmapsto x_j Cons(r_2, \ldots, r_m)$.
- Choose a minimal 2-sided Schreier transversal $\tau : G \longrightarrow F$ of the epimorphism $\nu : F \longrightarrow G$.

• Finally, construct the push

$$\widetilde{\varphi} = Push\left(\varphi\right) = \varphi \circ \tau : G \longrightarrow S$$

Our generalized Shor algorithm for the free group F consists of the following steps:

- **Step 1.** Call QRAND($\tilde{\varphi}$) to produce a word s'_j in F close to a word s_j lying in $\varphi^{-1}\varphi(1)$.
- **Step 2.** With input s'_j , use a polytime classical algorithm to determine s_j . (See [31].)
- **Step 3.** Repeat Steps 1 and 2 until enough relators s_j 's are found to produce a presentation

$$(x_1, x_2, \ldots, x_n : s_1, s_2, \ldots, s_\ell)$$

of the hidden subgroup F/K, then output the presentation $(x_1, x_2, \ldots, x_n : s_1, s_2, \ldots, s_\ell)$, and STOP.

Obviously, much more needs to be said. For, example, we have not explained how one chooses the relators r_j so that $G = (x_1, x_2, \ldots, x_n : r_1, r_2, \ldots, r_m)$ is a good group probe. Moreover, we have not explained what classical algorithm is used to transform the words s'_j into the relators s_j . For more details, we refer the reader to [31].

13. IS GROVER'S ALGORITHM A QHS ALGORITHM?

In this section, our objective is to factor Grover's algorithm into the QHS primitives developed in the previous sections of this paper. As a result, we will show that Grover's algorithm is more closely related to Shor's algorithm than one might at first expect. In particular, we will show that Grover's algorithm is a QHS algorithm in the sense that it solves an HSP $\varphi : S_N \longrightarrow S$, which we will refer to as the **Grover HSP.** However, we will then show that the standard QHS algorithm for this HSP cannot possibly find a solution.

We begin with a question:

Does Grover's algorithm have symmetries that we can exploit?

The problem solved by Grover's algorithm [24], [11], [12], [13] is that of finding an unknown integer label j_0 in an unstructured database with items labeled by the integers:

$$0, 1, 2, \ldots, j_0, \ldots, N-1 = 2^n - 1$$
,

given the oracle

$$f(j) = \begin{cases} 1 & \text{if } j = j_0 \\ 0 & \text{otherwise} \end{cases}$$

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Let \mathcal{H} be the Hilbert space with orthonormal basis $|0\rangle, |1\rangle, |2\rangle, \dots, |N-1\rangle$. Grover's oracle is essentially given by the unitary transformation

$$\begin{array}{cccc} I_{|j_0\rangle} : \mathcal{H} & \longrightarrow & \mathcal{H} \\ |j\rangle & \longmapsto & (-1)^{f(j)} |j\rangle \end{array}$$

where $I_{|j_0\rangle} = I - 2 |j_0\rangle \langle j_0|$ is inversion in the hyperplane orthogonal to $|j\rangle$. Let W denote the Hadamard transformation on the Hilbert space H. Then Grover's algorithm is as follows:

Step 0. (Initialization)

$$\begin{aligned} |\psi\rangle & \longleftarrow \quad W |0\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle \\ k & \longleftarrow \quad 0 \end{aligned}$$

Step 1. Loop until $k \approx \pi \sqrt{N}/4$

$$\begin{aligned} |\psi\rangle & \longleftarrow \quad Q |\psi\rangle = -WI_{|0\rangle}WI_{|j_0\rangle} |\psi\rangle \\ k & \longleftarrow \quad k+1 \end{aligned}$$

Step 2. Measure $|\psi\rangle$ with respect to the standard basis

$$|0\rangle, |1\rangle, |2\rangle, \ldots, |N-1\rangle$$

to obtain the unknown state $|j_0\rangle$ with

$$Prob \ge 1 - \frac{1}{N}$$

But where is the hidden symmetry in Grover's algorithm?

Let S_N be the symmetric group on the symbols $0, 1, 2, \ldots, N-1$. Then Grover's algorithm is invariant under the **hidden subgroup** $Stab_{j_0} = \{g \in S_N : g(j_0) = j_0\} \subset S_N$, called the **stabilizer subgroup** for j_0 , i.e., Grover's algorithm is invariant under the group action

$$\begin{array}{ccc} Stab_{j_0} \times H & \longrightarrow & H \\ \left(g, \sum_{j=0}^{N-1} a_j \left| j \right\rangle \right) & \longmapsto & \sum_{j=0}^{N-1} a_j \left| g\left(j \right) \right\rangle \end{array}$$

Moreover, if we know the hidden subgroup $Stab_{j_0}$, then we know j_0 , and vice versa. In other words, the problem of finding the unknown label j_0 is informationally the same as the problem of finding the hidden subgroup $Stab_{j_0}$.

Let $(ij) \in S_N$ denote the permutation that interchanges integers i and j, and leaves all other integers fixed. Thus, (ij) is a transposition if $i \neq j$, and the identity permutation 1 if i = j.

Proposition 4. The set $\{(0j_0), (1j_0), (2j_0), \ldots, ((N-1)j_0)\}$ is a complete set of distinct coset representatives for the hidden subgroup $Stab_{j_0}$ of S_N , i.e., the coset space $S_N/Stab_{j_0}$ is given by the following complete set of distinct cosets:

$$S_N/Stab_{j_0} = \{(0j_0) Stab_{j_0}, (1j_0) Stab_{j_0}, (2j_0) Stab_{j_0}, \dots, ((N-1) j_0) Stab_{j_0}\}$$

We can now see that Grover's algorithm is a hidden subgroup algorithm in the sense that it is a quantum algorithm which solves the following hidden subgroup problem:

Grover's Hidden Subgroup Problem. Let $\varphi : S_N \longrightarrow S$ be a map from the symmetric group S_N to a set $S = \{0, 1, 2, ..., N-1\}$ with hidden subgroup structure given by the commutative diagram

$$egin{array}{cccc} S_N & \longrightarrow & S \
u_{j_0}\searrow & & \nearrow \iota & , \ & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & &$$

where $\nu_{j_0}: S_N \longrightarrow S_N / Stab_{j_0}$ is the natural surjection of S_N on to the coset space $S_N / Stab_{j_0}$, and where

$$\begin{array}{cccc} \iota : \ S_N / Stab_{j_0} & \longrightarrow & S \\ (jj_0) \ Stab_{j_0} & \longmapsto & j \end{array}$$

is the unknown relabeling (bijection) of the coset space $S_N/Stab_{j_0}$ onto the set S. Find the hidden subgroup $Stab_{j_0}$ with bounded probability of error.

Now let us compare Shor's algorithm with Grover's.

From section 6, we know that Shor's algorithm [21], [25], [35], [36] solves the hidden subgroup problem $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_N$ with hidden subgroup structure

$$\begin{array}{cccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}_{I} \\ \nu \searrow & & \nearrow \iota \\ & \mathbb{Z}/P\mathbb{Z} \end{array}$$

Moreover, as stated in section 6, Shor has created his algorithm by pushing¹⁵ the above hidden subgroup problem $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_N$ to the hidden subgroup problem $\tilde{\varphi} : \mathbb{Z}_Q \longrightarrow \mathbb{Z}_N$ (called Shor's oracle), where the hidden subgroup structure of $\tilde{\varphi}$ is given by the commutative diagram

$$\begin{array}{cccc} Z & \longrightarrow & Z_N \\ \alpha \searrow & & \swarrow & \widetilde{\varphi} = \varphi \circ \tau \\ & & & Z_Q \end{array} ,$$

where α is the natural epimorphism of \mathbb{Z} onto \mathbb{Z}_Q , and where τ is Shor's chosen transversal for the epimorphism α .

Surprisingly, Grover's algorithm, viewed as an algorithm that solves the Grover hidden subgroup problem, is very similar to Shor's algorithm.

Like Shor's algorithm, Grover's algorithm solves a hidden subgroup problem, i.e., the Grover hidden subgroup problem $\varphi: S_N \longrightarrow S$ with hidden subgroup structure

$$\begin{array}{cccc} S_N & \longrightarrow & S \\ \nu \searrow & & \nearrow \iota \\ & & & & \\ S_N/Stab_{j_0} \end{array}$$

where $S = \{0, 1, 2, ..., N - 1\}$ denotes the set resulting from an unknown relabeling (bijection)

$$(jj_0) \operatorname{Stab}_{j_0} \longmapsto j$$

¹⁵See Section II.A.6 for a definition of pushing.

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of the coset space

$$S_N/Stab_{j_0} = \{(0j_0) Stab_{j_0}, (1j_0) Stab_{j_0}, (2j_0) Stab_{j_0}, \dots, ((N-1) j_0) Stab_{j_0}\}$$

Also, like Shor's algorithm, we can think of Grover's algorithm as one created by pushing the Grover hidden subgroup problem $\varphi : S_N \longrightarrow S$ to the hidden subgroup problem $\tilde{\varphi} : S_N/Stab_{j_0} \longrightarrow S$, where the pushing is defined by the following commutative diagram

$$\begin{array}{ccc} S_N & \longrightarrow & S = S_N / Stab_{j_0} \\ \alpha \searrow & & \swarrow & & & \swarrow & & & & & & & & \\ S_N / Stab_0 & & & & & & & & & & & & \\ \end{array}$$

where $\alpha : S_N \longrightarrow S_N/Stab_0$ denotes the natural surjection of S_N onto the coset space $S_N/Stab_0$, and where $\tau : S_N/Stab_0 \longrightarrow S_N$ denotes the transversal of α given by

$$\begin{array}{cccc} S_N/Stab_0 & \longrightarrow & S_N \\ (j0) Stab_0 & \longmapsto & (j0) \end{array}$$

Again also like Shor's algorithm, the map $\tilde{\varphi}$ given by

$$S_N/Stab_0 \longrightarrow S_N/Stab_{j_0} = S$$

$$(j0) Stab_0 \mapsto (jj_0) Stab_{j_0} = j$$

is (if $j_0 \neq 0$) actually a disguised Grover's oracle. For the map $\tilde{\varphi}$ can easily be shown to simply to

$$\widetilde{\varphi}\left((j0)Stab_{0}\right) = \begin{cases} (j0)Stab_{j_{0}} & \text{if } j = j_{0} \\ Stab_{j_{0}} & \text{otherwise }, \end{cases}$$

which is informationally the same as Grover's oracle

$$f(j) = \begin{cases} j & \text{if } j = j_0 \\ 1 & \text{otherwise} \end{cases}$$

Hence, we can conclude that Grover's algorithm is a quantum algorithm very much like Shor's algorithm, in that it is a quantum algorithm that solves the Grover hidden subgroup problem.

However, ..., this appears to be where the similarity between Grover's and Shor's algorithms ends. For the standard non-abelian QHS algorithm for S_N cannot find the hidden subgroup $Stab_{j_0}$ for each of following two reasons:

- Since the subgroups $Stab_j$ are not normal subgroups of S_N , it follows from the work of Hallgren et al [16], [17] that the standard non-abelian hidden subgroup algorithm will find the largest normal subgroup of S_N lying in $Stab_{j_0}$. But unfortunately, the largest normal subgroup of S_N lying in $Stab_j$ is the trivial subgroup of S_N .
- The subgroups $Stab_0, Stab_1, \ldots, Stab_{N-1}$ are mutually conjugate subgroups of S_N . Moreover, one can not hope to use this QHS approach to Grover's algorithm to find a faster quantum algorithm. For Zalka [40] has shown that Grover's algorithm is optimal.

,

The arguments given above suggest that Grover's and Shor's algorithms are more closely related that one might at first expect. Although the standard non-abelian QHS algorithm on S_N can not solve the Grover hidden subgroup problem, there does remain an intriguing question:

Question. Is there some modification of (or extension of) the standard QHS algorithm on the symmetric group S_N that actually solves Grover's hidden subgroup problem?

For a more in-depth discussion of the results found in this section, we refer the reader to [30].

14. Beyond QHS algorithms: The suggestions of a meta-scheme for creating new quantum algorithms

In this paper, we have decomposed Shor's quantum factoring algorithm into primitives, generalized these primitives, and then reassembled them into a wealth of new QHS algorithms. But as the results found in the previous section suggest, this list of quantum algorithmic primitives is far from complete. This is expressed by the following question:

Where can we find more algorithmic primitives to create a more well rounded toolkit for quantum algorithmic development?

The previous section suggests that indeed all quantum algorithms may well be hidden subgroup algorithms in the sense that they all find hidden symmetries, i.e., hidden subgroups. This is suggestive of the following meta-procedure for quantum algorithm development:

Meta-Step 1. Explicitly state the problem to be solved.Meta-Step 2. Rephrase the problem as a hidden symmetry problem.Meta-Step 3. Create a quantum algorithm to find the hidden symmetry.

Can this meta-procedure be made more explicit?

Perhaps some reader to this paper will be able to answer this question.

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