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# THE PRE-HISTORY OF OPERADS

JIM STASHEFF

Peter May will tell you some of the influences that led to his origination of the concept of operad. I will try to supply some of the pre-history: before there were operads in name, there were examples.

For me, it all begins with Poincaré. As we teach our students, if we define based loops as maps

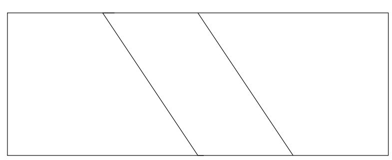
$$\lambda : [0,1] \to X$$

$$0, 1 \mapsto x_0,$$

then Poincaré's loop composition is not associative but does give associativity on homotopy classes. By the time of Serre's thesis [8] (1951), the set of based loops was considered as a topological space $\Omega X$ and the multiplication as a continuous map (that is, we had an H-space) with a homotopy for associativity:

$$I \times (\Omega X)^3 \to \Omega X,$$

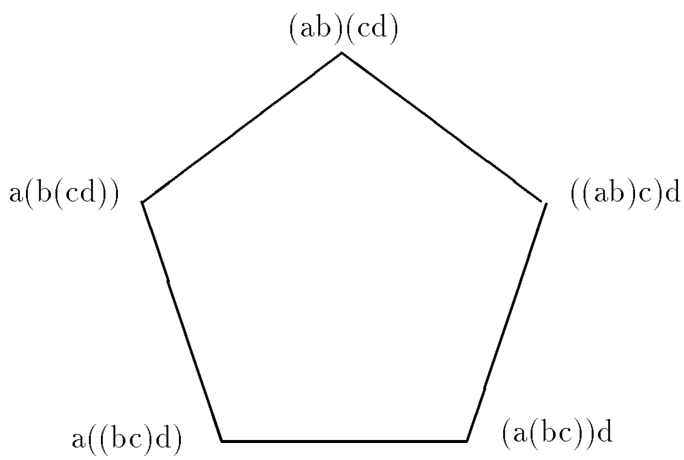
which was continuous because induced by the familiar parameter homotopy.



When I was a graduate student around 1958, John Moore suggested I look at the problem of when a cohomology class of a loop space $\Omega X$ was a 'suspension' or loop class, i.e. came from a cohomology class of $X$. It was known that it was necessary but not sufficient for the class to be primitive and that there was a relation to homotopy associativity [2].

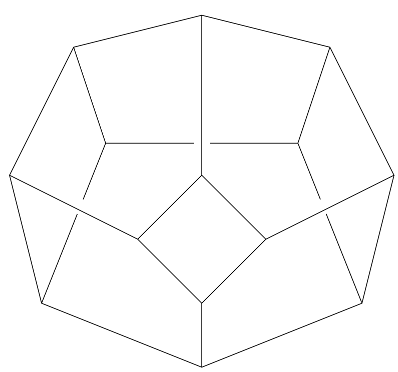
In pursueing this question, I was led to work of Sugawara [12] who had a *recognition principle* for characterizing loop spaces up to homotopy type. In the course of simplifying his criteria by eliminating the use of homotopy inverses, I was led to consider higher homotopies for associativity, e.g. the pentagon

Research

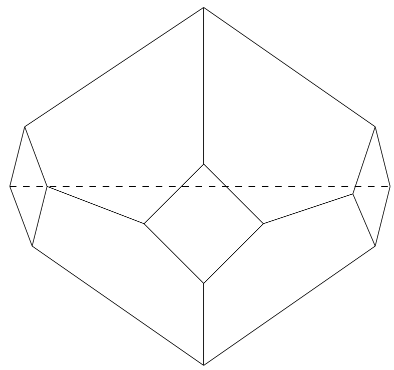



While I continued this work as a student in Oxford, Frank Adams visited and discussed his work with Mac Lane on PACTs and PROPs. With a key idea from Adams, I created the polytopes now known as *associahedra* [11]. The name is due to Gil Kalai, a geometric combinatorialist [6, 5].

The associahedron $K_n$ can be described now as a convex polytope with one vertex for each way of associating $n$ ordered variables, that is, ways of inserting parentheses in a meaningful way in a word of $n$ letters. For $n = 5$, here is a portrait I copy (literally) from Masahico Saito.



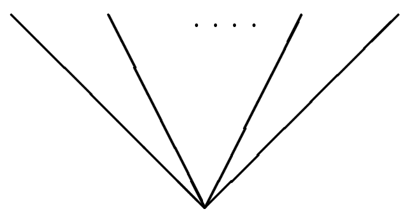
From a slighly different perspective, the 6-fold cyclic symmetry is more manifest, as in this portrait I learned from John Harer a very few years ago. (It also appears in Kapranov's paper on the permutoassociahedron [4]).



To describe all the cells of $K_n$, the language of planar rooted trees is helpful, as Adams indicated to me. The cells of $K_n$ are all of the form

$$\Pi\ K_{n_i} \quad with \quad \Sigma\ n_i = n - k + 1.$$

In particular all the facets (cells of codimension 1) are of the form $K_r \times K_s$. If we let $K_n$ be indexed by the *corolla*



with $n$ branches, then the facets of e.g. $K_4$ are labelled as follows:

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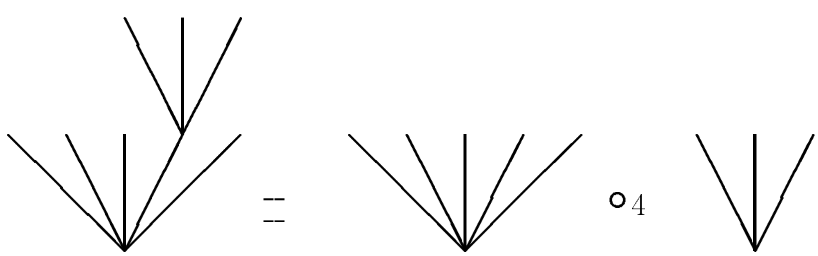
In general, the facets are labelled by grafting the $s$-corolla to a leaf $i$ of the $r$-corolla. Denote the grafting operation by



This makes the set of rooted planar trees into the non-symmetric **tree operad**. The corresponding inclusion of facets

$$\circ_i : K_r \times K_s \hookrightarrow K_{r+s-1}$$

makes the set of associahedra into an operad. Cells of lower dimension are indexed by trees obtained by iterated grafting and have the product from indicated above.

(Perhaps because of my emphasis originally on the $\circ_i$ inclusions of facets and/or the lack of symmetric group actions, the structure of an operad here went unrecognized for some time after May's introduction of the concept. The extension to a full operad can readily be achieved by free generation: replace $K_n$ by $\Sigma_n \times K_n$ with the obvious action on the $\circ_i$.)

Jumping ahead in time, there was an 'open' problem in combinatorial geometry from at least 1978 until solved in 1984 by Haiman as to whether there was a *convex polytope* realizing the associahedron. As I originally constructed them, the $K_n$ were convex but curvilinear, although linearized by Milnor (unpublished) early on. Thanks to Kapranov who bridged the two communities, the combinatorialists realized that the solution predated the question. (A particular realization as a truncation of the standard simplex and related to symplectic moment maps and toric varieties was presented (on the cover of their book [9]) by Shnider and Sternberg; it is described in more detail in an appendix to my talk in these proceedings where it is related to a corresponding realization of the cyclohedra [10].)

The main result of my dissertation with regard to the associahedra was a simplification of Sugawara's recognition principle.

Theorem: A connected space $Y$ (of the homotopy type of a CW-complex) has the homotopy type of a loop space $\Omega X$ for some $X$ if and only if there exist suitably compatible maps

$$K_n \times Y^n \to Y.$$

In fact, such a space $X$ was constructed as a quotient of $\coprod K_n \times Y^n$.

Because I was dealing here with associativity only, planar trees sufficed - there was no room nor need for symmetry. I next looked at when $\Omega X$was the loop space on an H-space $X$. Obviously this required homotopy commutativity. Again generalizing work of Sugawara [13], I introduced the hexagon which in turn was generalized to higher dimensions in Milgram's permutahedra.

From this point, the subject began to diverge. Another hexagon appears which together with the pentagon is crucial in Mac Lane's first coherence theorem [7] in category theory. My attempts to describe even two-fold loop spaces by specific cell complexes became too complicated, although at this conference we have with hindsight seen that things could have been done that way [3]. For infinite loop spaces, Boardman and Vogt [1] introduced what is now known as the linear isometries operad on $R^\infty$.

Thus we come finally to the dawning of the age of operads, for which history I'll turn things over to Peter May, but recent reincarnations of the associahedra will appear in my second talk along with the remarkable appearance of operads in mathematical physics.

## References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Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

*E-mail address*: jds@math.unc.edu