# PROPER HOLOMORPHIC MAPPINGS IN TETRABLOCK

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Abstract. The theorem showing that there are no non-trivial proper holomorphic self-mappings in the tetrablock is proved. We obtain also some general extension results for proper holomorphic mappings and we prove that the Shilov boundary is invariant under proper holomorphic mappings between some classes of domains containing among others $(m_1, \ldots, m_n)$-balanced domains. It is also shown that the tetrablock is not $\mathbb{C}$-convex.

## 1. Introduction and statement of results

Throughout the paper $\Omega$ denotes the classical Cartan domain of the second type, i.e. $\Omega = \{z \in \mathcal{M}_{2\times 2}(\mathbb{C}) : \ z = z^t, \ ||z|| < 1\}$, where $|| \cdot ||$ is the operator norm. Put

$$\Pi : \mathcal{M}_{2\times 2}(\mathbb{C}) \ni z = (z_{i,j}) \to (z_{1,1}, z_{2,2}, \det z) \in \mathbb{C}^3.$$

We define $\mathbb{E} := \Pi(\Omega)$. The domain $\mathbb{E}$ is called the *tetrablock*.

In the paper we will use the notion of *circled* domains. Let $m_1, \ldots, m_n$ be relatively prime natural numbers. Recall that a domain $D \subset \mathbb{C}^n$ is said to be $(m_1, \ldots, m_n)$-circled if

$$(1) \qquad (\lambda^{m_1} x_1, \ldots, \lambda^{m_n} x_n) \in D \quad \text{for any} \quad |\lambda| = 1, \ x = (x_1, \ldots, x_n) \in D.$$

If the relation (1) holds with $|\lambda| \leq 1$, then $D$ is said to be $(m_1, \ldots, m_n)$-balanced.

The tetrablock is a $(1, 1, 2)$-balanced domain in $\mathbb{C}^3$ appearing in control engineering and produces problems of a function-theoretic character. Its geometric properties have been investigated in several papers (see e.g. [Ab-Wh-Yo], [Edi-Zwo2], [You] and references contained there). In [You] the author using Kaup's theorem obtained a description of the group of automorphisms of this domain.

Our paper is devoted to proving an Alexander-type theorem for the tetrablock showing that every proper holomorphic self-map of the tetrablock is an automorphism. As a side effect we obtain a natural correspondence between automorphisms of the tetrablock and of the classical domain of the second type indicated in Lemma 13. This correspondence gives, in particular, another method of deriving the explicit formulas for automorphisms of the tetrablock.

The methods used in the paper rely upon the investigation of proper holomorphic mappings between $(m_1, \ldots, m_n)$-circled domains. We start with generalizing the Bell's extension result (see [Bel2]). Next we analyze the behavior of the Shilov boundary under proper holomorphic mappings. As a consequence of our considerations we get that any proper holomorphic mapping between $(m_1, \ldots, m_n)$-balanced and $(k_1, \ldots, k_n)$-balanced bounded domains preserves the Shilov boundary. As we indicate in the sequel this result immediately gives the description of the Shilov boundary of many domains (like the symmetrized polydisc, the tetrablock etc.).

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Recall here that the Alexander-type theorem has been proved in [Hen-Nov] for the Cartan domain $\Omega$ appearing in the definition of the tetrablock. This result has been extended to a bigger class of irreducible bounded symmetric domains by Z.H. Tu (see [Tu1, Tu2, Tu3]) and N. Mok (see [Mok]).

Let us formulate the main result.

**Theorem 1.** *Let $\varphi : \mathbb{E} \to \mathbb{E}$ be a proper holomorphic mapping. Then $\varphi$ is an automorphism.*

In this paper in Remark 14 we also show that the tetrablock is not $\mathbb{C}$-convex. Recall that a consequence of the Lempert theorem is the fact that the Carathéodory pseudodistance and the Lempert function of a $\mathbb{C}$-convex domain with $\mathcal{C}^2$ boundary coincide (see [Jac]). Since results obtained in [Ab-Wh-Yo] (see also [Edi-Zwo2]) suggest that the equality between the Carathéodory pseudodistance and the Lempert function holds in the tetrablock, the tetrablock is the candidate for the first bounded pseudoconvex domain non-biholomorphically equivalent to a $\mathbb{C}$-convex domain.

It also seems to be interesting whether the tetrablock may be exhausted by domains biholomorphic to $\mathbb{C}$-convex domains.

Here is some notation. Throughout the paper $\mathbb{D}$ denotes the unit disc in the complex plane. The unit Euclidean ball in $\mathbb{C}^n$ is denoted by $\mathbb{B}_n$. Moreover $\operatorname{Prop}(D, G)$ is the set of proper holomorphic mappings between domains $D$ and $G$ and the Shilov boundary is denoted by $\partial_s$. For a subset $K$ of $\mathbb{C}^n$ we put $\tilde{K} := \{\overline{x} :\ x \in K\}$.

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## 2. Proofs

We start this section with recalling basic properties of balanced domains and the Bergman projection which will be useful in the sequel. By $k_D$ we shall denote the Bergman kernel associated to a domain $D$. Let moreover $P_D$ denote the Bergman projection for $D$. We use the notation

$$k_D^{\alpha}(z,w) = \partial^{\alpha} k_D(z,w) \quad \text{and} \quad k_D^{\overline{\alpha}}(z,w) = \partial^{\overline{\alpha}} k_D(z,w),$$

where $\partial^{\alpha}$ stands for $\frac{\partial^{\alpha}}{z^{\alpha}}$ and $\partial^{\overline{\alpha}}$ stands for $\frac{\partial^{\alpha}}{\overline{w}^{\alpha}}$.

For a given $(m_1, \ldots, m_n)$-balanced domain $D$ in $\mathbb{C}^n$, where $m_1, \ldots, m_n$ are relatively prime natural numbers, we define the *Minkowski functional*

$$(2)\quad \mu_D(x) := \inf\{\lambda > 0 :\ (\lambda^{-m_1} x_1, \ldots, \lambda^{-m_n} x_n) \in D\}, \quad x = (x_1, \ldots, x_m) \in \mathbb{C}^n.$$

This function has similar properties as the standard Minkowski functional for balanced domains. Some of them may be found in [Nik]. In particular,

$$(3)\qquad \mu_D(\alpha^{m_1} x_1, \ldots, \alpha^{m_n} x_n) = |\alpha| \mu_D(x), \quad x \in \mathbb{C}^n,\ \alpha \in \mathbb{C},$$

$$(4)\qquad D = \{x \in \mathbb{C}^n :\ \mu_D(x) < 1\}.$$

*Remark* 2. Let $D$ be a $(m_1, \ldots, m_n)$-balanced bounded domain whose Minkowski functional is continuous, $m_1, \ldots, m_n \in \mathbb{N}$. Put $D_r := \{x \in \mathbb{C}^n :\ \mu_D(x) < r\}$, $r > 0$. Since

$$k_D((r^{m_1} z_1, \ldots, r^{m_n} z_n), w) = k_D(z, (r^{m_1} w_1, \ldots, r^{m_n} w_n))$$

for $z, w \in D$, $r \in [0, 1]$, we easily find that the function $(z, w) \to k_D(z, \overline{w})$ may be extended holomorphically to $D_{1/r} \times \tilde{D}_r$ for any $0 < r \leq 1$.

Note also that the continuity of the Minkowski functional is equivalent to the fact that for every $0 < r < 1$ the domain $D$ is relatively compact in $D_{1/r}$. Therefore any $(m_1, \ldots, m_n)$-balanced domain fulfils the assumptions of Lemma 3.

It follows from [Bel1] that if $f : D \to G$ is a proper holomorphic mapping between bounded domains $D$, $G$ in $\mathbb{C}^n$, then for any $\Phi \in L^2(G)$ we have

$$P_D(\det[f'] \cdot (\Phi \circ f)) = \det[f'] \cdot ((P_G \Phi) \circ f). \tag{5}$$

Assume additionally that $G$ is an $(m_1, \dots, m_n)$-circled bounded domain containing the origin. Choose $\delta > 0$ such that $\delta \overline{\mathbb{B}_n} \subset G$. Let $\tilde{\theta}$ be a radial function in $\mathcal{C}^0(\delta \mathbb{B}_n)$ such that $\tilde{\theta} \geq 0$ and $\tilde{\theta} = 1$ in the neighborhood of 0. Since holomorphic functions assume their average values we find that there is an $A > 0$ such that for every $h \in \mathcal{O}(G) \cap L^2(G)$ :

$$\partial^\alpha h(0) = \int_G (\partial^\alpha h) \theta d\lambda^{2n} = \int_G h (-1)^{|\alpha|} \partial^\alpha \theta d\lambda^{2n}, \tag{6}$$

where $\theta = A\tilde{\theta}$. On the other hand $h(z) = \int_G k_G(z, w) h(w) d\lambda^{2n}(w),\ z \in G$. Since $k_G(z, \cdot)$ extends holomorphically to a neighborhood of $\overline{G}$ provided that $z$ is sufficiently close to 0 (compare with Remark 2), one may differentiate this formula at $z = 0$ to get that

$$\partial^\alpha h(0) = \int_G \partial^\alpha k_G(0, w) h(w) d\lambda^{2n}(w), \quad h \in \mathcal{O}(G) \cap L^2(G). \tag{7}$$

This relation together with (6) give

$$P_G((-1)^{|\alpha|} \overline{\partial}^\alpha \theta) = k_G^{\overline{\alpha}}(\cdot, 0). \tag{8}$$

The next lemma has been proved by S. Bell in the case when $D$ and $G$ are bounded circular domains and $0 \in G$ (see [Bel2]). It is interesting that after minor modifications the methods used by Bell yield a stronger result. We present the whole proof for the sake of completeness.

**Lemma 3.** *Let $D, G$ be bounded domains in $\mathbb{C}^n$. Suppose that $G$ is $(m_1, \dots, m_n)$-circled and contains the origin. Assume moreover that the domain $D$ satisfies the following property: for any open $K \Subset D$ there is an open set $U$ containing $\overline{D}$ such that $(z, w) \to k_D(z, \overline{w})$ extends to be holomorphic on $U \times \tilde{K}$.*

*Then any proper holomorphic mapping $f : D \to G$ extends holomorphically to a neighborhood of $\overline{D}$.*

*Proof.* Let $m = (m_1, \dots, m_n)$. Properties of the Bergman kernel and a standard argument imply that the equation

$$k_G((\lambda^{m_1} z_1, \dots, \lambda^{m_n} z_n), w) = k_G(z, (\overline{\lambda^{m_1}} w_1, \dots, \overline{\lambda^{m_n}} w_n)) \tag{9}$$

holds for any $z, w \in G$ and $|\lambda|$ sufficiently close to 1. Differentiating this formula several times with respect to $\overline{w}_i$ and putting $w = 0$ we find that

$$\frac{\partial^\alpha k_G}{\partial \overline{w}^\alpha}((\lambda^{m_1} z_1, \dots, \lambda^{m_n} z_n), 0) = \lambda^{\langle \alpha, m \rangle} \frac{\partial^\alpha k_G}{\partial \overline{w}^\alpha}(z, 0) \tag{10}$$

for $\alpha \in \mathbb{N}^n$, $z \in G$ and $|\lambda|$ sufficiently close to 1.

Whence a standard argument shows that there are $c_\beta \in \mathbb{C}$ such that

$$k_G^{\overline{\alpha}}(z, 0) = \sum c_\beta z^\beta,\ z \in G, \tag{11}$$

where the sum is taken over $\beta \in \mathbb{N}^n$ satisfying the relation $\langle \beta, m \rangle = \langle \alpha, m \rangle$. Therefore, the linear independence of $k_G^{\overline{\alpha}}(z, 0)$ (see (7)) implies that for every $\beta \in \mathbb{N}^n$ there are $\tilde{c}_\alpha$ such that

$$z^\beta = \sum \tilde{c}_\alpha k_G^{\overline{\alpha}}(z, 0), \tag{12}$$

where the sum is taken over $\alpha \in \mathbb{N}^n$ satisfying the relation $\langle \alpha, m \rangle = \langle \beta, m \rangle$.

Now (12) together with (8) provide us with the function $\phi_{i,k} \in \mathcal{C}_0(\delta\mathbb{B}_n)$ such that

$$z_i^k = P_G(\Phi_{i,k}), \quad i = 1, \dots, n, \; k \in \mathbb{N}. \tag{13}$$

Making use of the above relations we infer that

$$\det[f'(z)] f_i^k(z) = \det[f'(z)](z_i^k \circ f(z)) = \int_D k_D(z,w) \det[f'(w)] \Phi_{i,k}(f(w)) d\lambda^{2n}(w), \tag{14}$$

for $i = 1, \dots, n$, and $k \in \mathbb{N}$. From these relations and the assumption on $k_D$ we easily conclude that all the functions appearing in the left side of (14) extend holomorphically to some open connected neighborhood $U$ of $\overline{D}$.

We shall show that $f_i$ extends holomorphically to the domain $U$. Putting $u = \det[f']$ we have the following situation

$$u \in \mathcal{O}(U), \quad u \not\equiv 0 \quad \text{and} \quad u f_i^k \in \mathcal{O}(U), \; k \in \mathbb{N}.$$

Fix any point $x \in U$ such that $u(x) = 0$. Changing, if necessary, the coordinates system we may assume that both $u$ and $uf_i$ satisfy the assumptions of Weierstrass Preparation Theorem near $x$. Observe that the Weierstrass polynomial associated to $u$ divides the Weierstrass polynomial associated to $uf_i$. This in particular means that $f_i$ is locally bounded near the analytic set $\{u = 0\}$, so the assertion follows from the Riemann's removable singularity theorem. □

**Corollary 4.** *Any proper holomorphic mapping $f : \Omega \to \mathbb{E}$ may be extended holomorphically to a neighborhood of $\overline{\Omega}$.*

*Proof.* It is sufficient to use the properties of the Bergman kernel for symmetric domains (see e.g. [Hua] or use Remark 2) and then to apply Lemma 3. □

**Lemma 5.** *Let $D$ and $G$ be bounded domains in $\mathbb{C}^n$ and let $f : D \to G$ be a proper holomorphic mapping extending continuously to $\overline{D}$. Assume that there is an increasing family of domains $\{G_m\}$, $G_m \Subset G_{m+1}$, such that $\bigcup G_m = G$ and $\overline{(\bigcup_m \partial_s G_m)} \cap \partial G = \partial_s G$.*

*Then $f(\partial_s D) = \partial_s G$.*

*Remark* 6. Observe that $x \in \overline{(\bigcup_m \partial_s G_m)} \cap \partial G$ if and only if there is a subsequence $(n_k)$ and there are $x_{n_k} \in \partial_s G_{n_k}$ such that $x_{n_k} \to x$.

One may very easily show that for any bounded domain $G$ and any increasing family of domains $\{G_m\}$ such that $\bigcup G_m = G$, the Shilov boundary of $G$ is contained in $\overline{(\bigcup_m \partial_s G_m)} \cap \partial G$.

Note moreover that Lemma 5 does not remain valid if we remove the assumption $\overline{(\bigcup_m \partial_s G_m)} \cap \partial G = \partial_s G$. As an example one may take $D = \mathbb{D} \cap \{z \in \mathbb{D} : \operatorname{Im} z > 0\}$, $G = \mathbb{D} \setminus [0, 1)$ and $f(z) = z^2$.

*Proof of Lemma 5.* The inclusion $\partial_s G \subset f(\partial_s D)$ follows immediately from the definition of the Shilov boundary. We shall prove that $\partial_s D \subset f^{-1}(\partial_s G)$. Assume the contrary i.e. there is a $\psi \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$ such that

$$|\psi(x_0)| > \max\{|\psi(x)| : \; x \in f^{-1}(\partial_s G)\}, \tag{15}$$

for some $x_0 \in \partial D$. Note that

$$\limsup_{m \to \infty} \max\{|\psi(x)| : \; x \in D \cap f^{-1}(\partial_s G_m)\} \leq \max\{|\psi(x)| : \; x \in \partial D \cap f^{-1}(\partial_s G)\}.$$

Actually, otherwise there would exist a subsequence $(m_k) \subset \mathbb{N}$, $\epsilon > 0$ and $x_{m_k} \in D \cap f^{-1}(\partial_s G_{m_k})$ such that

$$|\psi(x_{m_k})| > \max\{|\psi(x)| : \; x \in \partial D \cap f^{-1}(\partial_s G)\} + \epsilon.$$

Passing, if necessary, to a subsequence we can assume that $x_{m_k}$ converges to some $x_0$. Using the assumptions on the domain $G$ and the mapping $f$ we infer that $f(x_0) \in \partial_s G$. Thus

$$|\psi(x_0)| \geq \max\{|\psi(x)|:\ x \in \partial D \cap f^{-1}(\partial_s G)\} + \epsilon \quad \text{and} \quad x_0 \in \partial D \cap f^{-1}(\partial_s G),$$

which gives an obvious contradiction.

Therefore, taking $m$ big enough and replacing $x_0$ by a point $x_0' \in f^{-1}(\overline{G}_m)$ sufficiently close to $x_0$ at which the mapping $f$ is non-degenerate we get from (15)

$$(16) \qquad |\psi(x_0')| > A := \max\{|\psi(x)|:\ x \in D \cap f^{-1}(\partial_s G_m)\}, \quad \# f^{-1}(f(x_0')) = k,$$

where $k$ denotes the multiplicity of $f$.

Let $h_i,\ i = 1,\dots,k$, be holomorphic mappings in the neighborhood of $f(x_0')$ given by $f^{-1} = \{h_i:\ i = 1,\dots,k\}$. Making use of (16) one may show the existence of a natural number $d$ such that

$$(17) \qquad |\psi(h_1(f(x_0')))^d + \dots + \psi(h_k(f(x_0')))^d| > kA^d.$$

To prove it put $a_j = \psi(h_j(f(x_0'))),\ j = 1,\dots,k$. Change, if necessary, the order of $a_j$ so that $|a_1| = \dots = |a_k|$ and $|a_j| < |a_1|$ for $j = k+1,\dots,n$. Dividing all $a_j$ by $a_1$ we reduce ourselves to the following situation:

$$a_j = e^{i\theta_j},\ j = 1\dots,k, \quad |a_j| < 1,\ j = k+1,\dots,n \quad \text{and} \quad A < 1,$$

where $\theta_j \in \mathbb{R}$, $j = 1,\dots,k$.

Changing the order once again we may assume that $\theta_1,\dots,\theta_{k_1}$ are $\mathbb{Q}$-linearly independent, $k_1 \leq k$, and $\theta_j = \sum_{\iota=1}^{k_1} \frac{q_{j,\iota}}{N}\theta_\iota$, $j = l+1,\dots,k$, where $q_{j,\iota} \in \mathbb{Z}$ and $N \in \mathbb{N} \setminus \{0\}$. Put $M = \max\{|q_{i,\iota}|, N\}$.

According to the Kronecker Theorem (see e.g. [Har-Wri]) there is a sequence of natural numbers $(\tilde{d}_\mu)$ such that $-\frac{1}{(\mu+1)kM} < \arg(e^{i\tilde{d}_\mu\theta_j}) < \frac{1}{(\mu+1)kM}$, $j = 1,\dots,k_1$, $\mu \in \mathbb{N}$. In particular, $-\frac{1}{\mu+1} < \arg(e^{id_\mu\theta_j}) < \frac{1}{\mu+1}$ for $j = 1,\dots,k$, $\mu \in \mathbb{N}$, where $d_\mu = N\tilde{d}_\mu$.

Properties of $(d_\mu)$ guarantee that $|a_1^{d_\mu} + \dots + a_k^{d_\mu}| \to k$ as $\mu \to \infty$. Since $d_\mu \to \infty$, we find that $kA^{d_\mu} \to 0$ and $|a_{k+1}^{d_\mu} + \dots + a_n^{d_\mu}| \to 0$, $\mu \to \infty$. Therefore $|a_1^{d_\mu} + \dots + a_n^{d_\mu}| - kA^{d_\mu} \to k > 0$, which obviously proves the existence of a natural number $d$ fulfilling (17).

Put

$$(18) \qquad \zeta(x) = x_1^d + \dots + x_k^d, \quad \text{for} \quad x = (x_1,\dots,x_k) \in \mathbb{C}^k.$$

A well known argument shows that the formula $\varphi = \zeta \circ (\psi \times \dots \times \psi) \circ f^{-1}$ defines a holomorphic function on $G$. It follows from (16) and (17) that

$$|\varphi(f(x_0'))| > \max\{|\varphi(x)|:\ x \in \partial_s G_m\};$$

a contradiction. □

*Remark* 7. It is clear that the proof remains valid if the assumption $\overline{(\bigcup_m \partial_s G_m)} \cap \partial G = \partial_s G$ occurring in Lemma 5 is replaced by a weaker condition $\overline{(\bigcup_m \partial_b G_m)} \cap \partial G = \partial_s G$, where $\partial_b$ denotes the Bergman boundary.

Based on the former idea we obtain the following result.

**Lemma 8.** *Let $f : D \to G$ be a proper holomorphic mappings between domains in $\mathbb{C}^n$. Let $L$ be a domain relatively compact in $G$. Put $K = f^{-1}(L)$. Then*

$$(19) \qquad f(\partial_s K) = \partial_s L \quad \text{and} \quad f(\partial_b K) = \partial_b L.$$

*Proof.* It suffices to repeat a construction from the proof of Lemma 5. □

We have the following

**Corollary 9.** *a) Let $D \subset \mathbb{C}^n$ be a bounded domain and let $G$ be a bounded $(m_1, \ldots, m_n)$-balanced domain in $\mathbb{C}^n$. Assume that the Minkowski functional associated to $G$ is continuous and for any open $K \Subset D$ there is an open neighborhood $U$ of $\overline{D}$ such that $k_D(z, \overline{w})$ extends holomorphically on $U \times \tilde{K}$.*

*Then every proper holomorphic mapping $f : D \to G$ maps $\partial_s D$ onto $\partial_s G$.*

*b) Let $D$ and $G$ be bounded $(m_1, \ldots, m_n)$ and $(k_1, \ldots, k_m)$-balanced domains. If the Minkowski functionals of $D$ and $G$ are continuous, then any proper holomorphic mapping between $D$ and $G$ preserves the Shilov boundary.*

*Proof. a)* Define

$$G_m := \left\{ x \in \mathbb{C}^n : \ \mu_G(x) < 1 - \frac{1}{m} \right\}, \quad m = 2, 3 \ldots.$$

It is clear that the family $\{G_m\}_m$ satisfies the assumptions of Lemma 5. So applying Lemma 3 we reduce the situation to the one occurring in Lemma 5.

*b)* It is a direct consequence of *a)* and Remark 2 □

*Remark* 10. Note that Lemma 8 and Corollary 9 allow us to determine the Shilov boundary of some classes of domains containing the symmetrized polydisc (see [Edi-Zwo1]) and the tetrablock. For example $\partial_s \mathbb{E} = \Pi(\partial_s \Omega) = \Pi(\mathcal{U})$, where $\mathcal{U}$ consists of unitary symmetric matrices (see also [You], where the author using elementary methods computed the Shilov boundary of the tetrablock).

It is also interesting that Lemma 5 may be used for showing the non-existence of proper holomorphic mappings between some domains. For instance, using Corollary 9 we immediately see that $\mathrm{Prop}(\mathbb{D}^n, \mathbb{B}_n)$ and $\mathrm{Prop}(\mathbb{B}_n, \mathbb{D}^n)$ are empty for $n \geq 2$ (see also [Nar]). As an other example of the application of this result, observe that the theorem showing that there are no proper holomorphic mappings between $\mathbb{B}_n \times \mathbb{B}_m$ and $\mathbb{B}_{n+m}$ follows directly from Corollary 9.

The next result has been proved in [Rud2] for the Euclidean ball in $\mathbb{C}^n$. We would like to mention here that for our purposes a much weaker result of Tumanov and Henkin proved in [Tum-Hen] is sufficient. However, it seems to be interesting that after some modifications the Rudin's idea may be applied to the symmetric domains.

First recall a well known classical result.

**Lemma 11** (see [Rud1], Theorem 8.1.2)**.** *Suppose that $\Omega_1$ and $\Omega_2$ are balanced domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Suppose moreover that $\Omega_2$ is convex and bounded and $F : \Omega_1 \to \Omega_2$ is holomorphic. Then $F'(0)$ maps $\Omega_1$ into $\Omega_2$.*

*If additionally $F(0) = 0$, then $F(\lambda\Omega_1) \subset \lambda\Omega_2$, $0 \leq \lambda \leq 1$.*

**Lemma 12.** *Let $a_0$, $b_0$ be any unitary matrices. Let $U$, $V$ be open neighborhoods of $a_0$ and $b_0$ respectively. Let $\varphi : U \cap \Omega \to V \cap \Omega$ be a biholomorphic mapping. If $\varphi(a_k) \to b_0$ for some $a_k \to a_0$, then $\varphi$ extends to an automorphism of $\Omega$.*

*Proof.* A direct computation shows that for any symmetric unitary matrix $a$ there is a unitary matrix $u$ such that $uu^t = a$. Since any of the mappings $\Omega \ni x \to uxu^t \in \Omega$, where $u$ is unitary, is an automorphism of $\Omega$, we may assume that $a_0 = b_0 = 1$.

For every $a \in \Omega$ the mapping

$$\varphi_a(x) = -a + (1 - aa^*)^{1/2} x (1 - a^* x)^{-1} (1 - a^* a)^{1/2} \tag{20}$$

is an automorphism of $\Omega$, its inverse is given by $\varphi_a^{-1} = \varphi_{-a}$ and $\varphi_a(0) = -a$.

Put $b_k = \varphi(a_k)$ and define $G_k := \varphi_{b_k} \circ \varphi \circ \varphi_{-a_k} : \varphi_{a_k}(U \cap \Omega) \to \varphi_{b_k}(V \cap \Omega)$, $k \in \mathbb{N}$. Note that $G_k$ is a biholomorphic mapping, $G_k(0) = 0$. Clearly $\varphi_{-a}(x) \to 1$ locally uniformly whenever $a \to 1$, so a compactness argument gives the existence of $\delta_k > 0$

such that $\delta_k \to 1$, as $k \to \infty$, and both $\varphi_{a_k}(U \cap \Omega)$, $\varphi_{b_k}(V \cap \Omega)$ contain a domain $\delta_k \Omega$. Properly scaled Lemma 11 implies that $\delta_k^3 \leq |\det G_k'(0)| \leq \delta_k^{-3}$.

Since $G_k(0) = 0$, it follows that there exists a subsequence of $\{G_k\}$ (also denote by $\{G_k\}$) converging locally uniformly to $G : \Omega \to \Omega$. Clearly $|\det G'(0)| = 1$ and $G(0) = 0$, so by Lemma 11 the domain $\Omega$ is mapped by $G'(0)$ into $\Omega$. Since $|\det G'(0)| = 1$, the mapping $G'(0)$ preserves the volume. Hence $G'(0)$ maps $\Omega$ onto $\Omega$, in particular it is a unitary operator. Compose $G$ with $(G'(0))^{-1}$ and then apply the Cartan theorem in order to find that $G$ is also unitary.

Let $\mathcal{N} = \{z \in \mathcal{M}_{2\times 2}(\mathbb{C}) : \ z = z^t, \ ||z|| \neq \rho(z)\}$. Note that $\mathcal{N} \cap \Omega$ is open and dense in $\Omega$. Moreover $\lambda z \in \mathcal{N}$ for any $z \in \mathcal{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. For $z \in \Omega$ define $D_z = \{\lambda z : \ ||\lambda z|| < 1\} \subset \Omega$.

Let $K$ be any compact subset of $\mathcal{N}$. Observe that

$$\bigcup \{D_z : \ z \in K\} \subset \varphi_a(\Omega \cap U) \tag{21}$$

for $a \in \Omega$ sufficiently close to 1. Indeed, otherwise there would exist sequences $(\lambda_n) \subset \mathbb{C}$, $(z_n) \subset K$ and $(a_n) \subset \Omega$ such that $a_n \to 1$ and $\lambda_n \to \lambda_0 \in \mathbb{C}$, $z_n \to z_0 \in K$ (pass to subsequences, if necessary). If $\lambda_0 = 0$, then the contradiction is obvious. In the other case $\lambda_0 z_0 \in \mathcal{N} \cap \overline{\Omega}$. It follows that $\det(1 - \lambda_0 z_0) \neq 0$ (otherwise $\rho(\lambda_0 z_0) \geq 1 \geq ||\lambda_0 z_0||$). This in particular means that $\varphi_{a_n}^{-1}(\lambda_n z_n)$ converges to 1 (use the formula (20)). Whence $\lambda_n z_n \in \varphi_{a_n}(\Omega \cap U)$; a contradiction.

Since $G$ is unitary, $G^{-1}(\mathcal{N}) \cap \mathcal{N}$ is open and dense in $\Omega$. Let $B = \{z \in \mathcal{M}_{2\times 2}(\mathbb{C}) : ||z - p|| < 2c\}$, where $p$ and $c$ are chosen such that

$$B \Subset G^{-1}(\mathcal{N}) \cap \mathcal{N} \quad \text{and} \quad ||z|| < 1 - c \quad \text{for} \quad z \in B.$$

Property (21) yields the existence of an $n$ such that

$$D_z \subset \varphi_{a_n}(U \cap \Omega) \quad \text{and} \quad D_{G(z)} \subset \varphi_{b_n}(V \cap \Omega), \quad z \in B. \tag{22}$$

We may assume that for such chosen $n : ||G_n(z) - G(z)|| < c$ whenever $||z|| \leq 1 - c$.

Then for $||z - p|| < c$, $D_z \subset \varphi_{a_n}(U \cap \Omega)$. Since $||G^{-1}(G_n(z)) - p|| = ||G_n(z) - G(p)|| < 2c$ we see that $G^{-1}(G_n(z)) \in B$. Therefore, making use of (22) we get that $D_{G_n(z)} \subset \varphi_{b_n}(V \cap \Omega)$.

Thus we may use a standard argument to the mapping $G_n : D_z \to \Omega$, where $z$ is such that $||z - p|| < c$, in order to find that $||G_n(z)|| \leq ||z||$. The same argument applied to $G_n^{-1} : D_{G_n(z)} \to \Omega$ together with the previous inequality give $||G_n(z)|| = ||z||$ for $||z - p|| < c$. Obviously this equality remains validate on the whole $\varphi_{a_n}(\Omega \cap U)$.

Choose $r$ such that a ball $r\Omega$ is contained in $\varphi_{a_n}(\Omega \cap U) \cap \varphi_{b_n}(\Omega \cap V)$. The restriction of $G_k$ to $r\Omega$ is an automorphism of $r\Omega$ fixing 0. So we conclude from the description of the group of automorphism of classical Cartan domain of the second type that $G_k$ is unitary. From this piece of information we immediately get the assertion. $\square$

The next lemma describes proper holomorphic self-mappings in the tetrablock.

**Lemma 13.** *Let $\varphi : \mathbb{E} \to \mathbb{E}$ be a proper holomorphic mapping. Then, there is an automorphism $\psi \in \mathrm{Aut}(\Omega)$ of Cartan symmetric domain of the second type such that*

$$\varphi \circ \Pi = \Pi \circ \psi \tag{23}$$

*Proof.* First observe that $\Pi^{-1}(\mathbb{E}) = \Omega$, so it is very easy to see that $\Pi : \Omega \to \mathbb{E}$ is proper. Put $f := \varphi \circ \Pi$. By Corollary 4 the mapping $f$ extends to an open neighborhood $\Omega_1$ of $\overline{\Omega}$. Define

$$\mathcal{J} := \{x \in \Omega_1 : \ \det[f'(x)] \neq 0 \text{ and } f_1(x) f_2(x) \neq f_3(x)\}. \tag{24}$$

Since every proper holomorphic mapping is non-degenerate, properties of the Shilov boundary show that the intersection of domains $\mathcal{J}$ and $\partial_s\Omega$ is non-empty. Take any $x_0 \in \mathcal{J} \cap \partial_s\Omega \neq \emptyset$.

Fix any $y_0$ such that $\Pi(y_0) = f(x_0)$. The choice of $x_0$ and properties of covering maps allow us to choose open neighborhoods $U$, $V$ of $x_0$, $y_0$ respectively and a biholomorphic mapping $\psi : U \to V$, $\psi(x_0) = y_0$, such that

$$f = \Pi \circ \psi \quad \text{on } U. \tag{25}$$

We find from Corollary 9 that $f(x_0)$ lies in the Shilov boundary of the tetrablock. So $y_0$ is unitary.

Lemma 12 and the identity principle finish the proof. □

Now we are in position to bring the proof to the end.

*Proof of Theorem 1.* It suffices to apply Lemma 13 to get that the mapping $\varphi\circ\Pi$ has multiplicity 2. Since $\Pi$ also has multiplicity 2 we infer that $\varphi$ is an automorphism. □

*Remark* 14. Note that the tetrablock is not $\mathbb{C}$-convex. Actually, let

$$\gamma(x) = |x_1 - \overline{x_2}x_3| + |x_1x_2 - x_3| + |x_3|^2, \quad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbb{C}^3. \tag{26}$$

As shown in [Ab-Wh-Yo], $x \in \mathbb{E}$ if and only if $\gamma(x) < 1$.

For $\zeta \in \mathbb{C}$ put

$$\varphi(\zeta) := \left(\frac{1-i}{2}\zeta + \frac{1+i}{2}, \frac{1+i}{2}\zeta + \frac{i-1}{2}, i\zeta\right).$$

Obviously $\varphi(1),\ \varphi(-1) \in \overline{\mathbb{E}}$. Moreover $\varphi(i\zeta) = \left(\frac{1+i}{2}(\zeta+1), \frac{i-1}{2}(\zeta+1), -\zeta\right)$. An easy computation shows that for any $\zeta \in \mathbb{R}$ :

$$\gamma(\varphi(i\zeta)) = \left|\frac{1+i}{2}(\zeta+1) - \frac{i+1}{2}(\zeta+1)\zeta\right| + \left|-1/2(\zeta+1)^2 + \zeta\right| + \zeta^2 =$$

$$= \frac{\sqrt{2}}{2}|1-\zeta^2| + |\frac{\zeta^2+1}{2}| + \zeta^2 = \frac{\sqrt{2}}{2}|1-\zeta^2| + \frac{3}{2}\zeta^2 + 1/2.$$

In particular $\gamma(\varphi(z)) > 1$ for any $z \in \{x \in \mathbb{C} : \ \operatorname{Re} x = 0\}$, so $\mathbb{E} \cap \varphi(\mathbb{C})$ is not connected.

## References


[Ab-Wh-Yo] A. Abouhajar, M. White and N. Young, *A Schwarz lemma for a domain related to mu-synthesis*, Journal of Geometric Analysis **17** (2007), no. 4, 717–750.

[Bel1] S. Bell, *Analytic hypoellipticity of the $\bar{\partial}$-Neumann problem and extendability of holomorphic mappings*, Acta Math. **147** (1981), no. 1-2, 109–116.

[Bel2] S. Bell, *Proper holomorphic mappings between circular domains*, Comment. Math. Helv. **57** (1982), no. 4, 532–538.

[Edi-Zwo1] A. Edigarian and W. Zwonek, *Geometry of the symmetrized polydisc*, Arch. Math. **84** (2005) 364-374.

[Edi-Zwo2] A. Edigarian and W. Zwonek, *Schwarz lemma for the tetrablock*, Bulletin of the London Mathematical Society, **41** (2009), no. 3, 506-514.

[Har-Wri] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publ., 1978.

[Hen-Nov] G.M. Henkin and R. Novikov, *Proper mappings of classical domains*, Lecture Notes in Math. **1043**, (1984), 625–627.

[Hua] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, AMS. (1963), Providence.

[Jac] D. Jacquet *$\mathbb{C}$-convex domains with $C^2$ boundary*, Complex Var. Elliptic Equ. **51** (2006), no. 4, 303–312.

[Lem] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427–474.

[Mok] N. Mok, *Nonexistence of proper holomorphic maps between certain classical bounded symmetric domains*, Chinese Annals of Mathematics - *Series B* **29**, No.2 (2008), 135–146.

[Nar] R. Narasimhan, *Several Complex Variables,* Chicago Lectures in Mathematics, 1971.

[Nik] N. Nikolov, *The symmetrized polydisc cannot be exhausted by domains biholomorphic to convex domains*, Ann. Polon. Math., **88** (2006), 279–283.

[Rud1] W. Rudin, *Fuction theory in the unit ball of* $\mathbb{C}^n$, Grundlehren der mathematischen Wissenschaften 241 (1980).

[Rud2] W. Rudin, *Holomorphic maps that extend to automorphisms of a ball*, Proc. Amer. Math. Soc. **81** (1981), no. 3, 429–432.

[Tu1] Z.-H. Tu, *Rigidity of proper holomorphic mappings between equidimensional bounded symmetric domains*, Proc. Amer. Math. Soc. **130** (2002), 1035–1042.

[Tu2] Z.-H. Tu, *Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains*, Math. Z. **240** (2002), 13–35.

[Tu3] Z.-H. Tu, *Rigidity of proper holomorphic mappings between bounded symmetric domains*, Geometric function theory in several complex variables, pp.310-316, World Sci. Publishing, River Edge, NJ, 2004.

[Tum-Hen] A.E. Tumanov and G.M. Khenkin, *Local characterization of holomorphic automorphisms of Siegel domains*, (Russian) Funktsional. Anal. i Prilozhen. **17**0 (1983), no. 4, 49–61.

[You] N. Young, *The automorphism group of the tetrablock*, Journal of the London Mathematical Society , **77** (2008), no. 3, 757-770.

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