Cut-by-curves criterion for the log extendability of overconvergent isocrystals

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Abstract

In this paper, we prove a 'cut-by-curves criterion' for an overconvergent isocrystal on a smooth variety over a field of characteristic $p > 0$ to extend logarithmically to its smooth compactification whose complement is a strict normal crossing divisor, under certain assumption. This is a p-adic analogue of a version of cut-by-curves criterion for regular singuarity of an integrable connection on a smooth variety over a field of characteristic 0. In the course of the proof, we also prove a kind of cut-by-curves criteria on solvability, highest ramification break and exponent of ∇-modules.

Contents

Introduction

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with ring of integers O_K and residue field k, and let $X \hookrightarrow \overline{X}$ be an open immersion of smooth

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k-varieties such that $Z = \overline{X} - X = \bigcup_{i=1}^r Z_i$ is a simple normal crossing divisor. Denote the log structure on $\overline{X}$ associated to Z by $M_{\overline{X}}$. Let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of $\mathbb{Z}_p^r$ which satisfies the conditions (NID) (non-integer difference) and (NLD) (p-adically non-Liouville difference). (For precise definition, see [S3, 1.8] or Subsection 1.5 in this paper.) In the previous paper [S3], we introduced the notion of 'having Σ-unipotent monodromy' for an overconvergent isocrystal on $(X, \overline{X})/K$ and proved that an overconvergent isocrystal on $(X, \overline{X})/K$ has Σ-unipotent monodromy if and only if it can be extended to an isocrystal on log convergent site $((\overline{X}, M_{\overline{X}})/O_K)_{\mathrm{conv}}$ with exponents in Σ. In this paper, we prove a 'cut-by-curves criterion' for an overconvergent isocrytal on $(X, \overline{X})/K$ to have Σ-unipotent monodromy.

Let us give the precise statement of our main theorem. For an open immersion of smooth k-curves $C \hookrightarrow \overline{C}$ such that $P := \overline{C} - C = \coprod_{i=1}^s P_i$ is a simple normal crossing divisor (= disjoint union of closed points P_i whose residue fields are separable over k), denote the log structure on $\overline{C}$ associated to P by $M_{\overline{C}}$. When we are given an exact locally closed immersion $\iota : (\overline{C}, M_{\overline{C}}) \hookrightarrow (\overline{X}, M_{\overline{X}})$ with $(\overline{C}, M_{\overline{C}})$ as above, ι induces a well-defined morphism of sets $\{1, ..., s\} \longrightarrow \{1, ..., r\}$, which we denote also by ι, by the rule $\iota(P_i) \subseteq Z_{\iota(i)}$. Then we define $\iota^*\Sigma$ by $\iota^*\Sigma := \prod_{i=1}^s \Sigma_{\iota(i)} \subseteq \mathbb{Z}_p^s$. Also, ι induces a morphism of pairs $(C, \overline{C}) \longrightarrow (X, \overline{X})$ which is also denoted by ι again. Therefore, for an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$, we can define the pull-back $\iota^*\mathcal{E}$ of $\mathcal{E}$ by ι, which is an overconvergent isocrystal on $(C, \overline{C})/K$. With this notation, we can state our main theorem as follows:

Theorem 0.1. *Let $K, k, X, \overline{X}, M_{\overline{X}}, \Sigma$ be as above and assume that k is uncountable. Then, for an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$, the following two conditions are equivalent*:

(1) *$\mathcal{E}$ has Σ-unipotent monodromy.*

(2) *For any $(C, \overline{C}), M_{\overline{C}}$ as above and for any exact locally closed immersion $\iota : (\overline{C}, M_{\overline{C}}) \hookrightarrow (\overline{X}, M_{\overline{X}})$, $\iota^*\mathcal{E}$ has $\iota^*\Sigma$-unipotent monodromy.*

Note that Theorem 0.1 is a p-adic analogue of a version of cut-by-curves criterion for regular singularity of an integrable connection on a smooth variety over a field of characteristic 0, which is shown in [De, II 4.4], [A-Ba, I 3.4.7].

Here we briefly sketch the proof. Since the implication (1) $\Longrightarrow$ (2) is a rather easy consequence of the results in [S3], the essential part is to prove (1) assuming (2). The condition (1) is equivalent to the condition that, for any i, the ∇-module associated to $\mathcal{E}$ on p-adic annulus around the generic point of Z_i has highest ramification break 0 and that any entry of its exponent (in the sense of Christol-Mebkhout) is contained in Σ_i. On the other hand, the condition (2) implies that, for any i and for any separable closed point $z \in Z$, the ∇-module associated to $\mathcal{E}$ on p-adic annulus around z transverse to Z_i has highest ramification 0 and that any component of its exponent is contained in Σ_i. We will prove the latter condition implies the former. In fact, we will prove stronger assertions: We prove a kind of 'cut-by-curves criteria' on solvability, highest ramification break and exponent of ∇-modules.

The content of each section is as follows: In Section 1, we give a review of the notions related to (log-)∇-modules on rigid analytic spaces due to Christol, Mebkhout, Dwork and Kedlaya which we need for the proof and recall the results in our previous paper [S3]. In Section 2, we prove cut-by-curves criteria for solvability, highest ramification break and exponent of ∇-modules and using these, we give a proof of the main theorem. We also prove a variant of the main theorem (Corollary 2.12) which treats the case where k is not necessarily uncountable.

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Convention

Throughout this paper, K is a fixed complete discrete valuation field with residue field k of characteristic $p > 0$ and let $|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$ be the fixed absolute value on K. Let Γ^* be $(\mathbb{Q} \otimes_{\mathbb{Z}} |K^\times|) \cup \{0\}$ and let O_K be the the ring of integers of K. For a p-adic formal scheme $\mathcal{P}$ topologically of finite type over O_K, we denote the associated rigid space by $\mathcal{P}_K$. A k-variety means a reduced separated scheme of finite type over k. A closed point in a k-variety X is called a separable closed point if its residue field is separable over k.

We use freely the notion concerning isocrystals on log convergent site and overconvergent isocrystals. For the former, see [S1], [S2] and [K1, §6]. For the latter, see [Be] and [K1, §2].

1 Preliminaries

In this section, we give a review of several notions and facts related to (log-)∇-modules on rigid analytic spaces due to Christol, Mebkhout, Dwork and Kedlaya which we need for the proof of our main result. Also, we recall the results in our previous paper [S3].

1.1 Differential modules

First we give a definition of differential modules and cyclic vectors, following [K2, 1.1.4]:

Definition 1.1. *Let L be a field of characteristic 0 endowed with non-Archimedean norm $|\cdot|$ and a non-zero differential ∂.*

(1) *A differential module on L is a finite-dimensional L-vector space V endowed with an action of ∂ satisfying the Leibniz rule.*

(2) *Let V be a differential module on L. $\mathbf{v} \in V$ is called a cyclic vector if and only if $\mathbf{v}, \partial(\mathbf{v}), ..., \partial^{\dim V-1}(\mathbf{v})$ forms a basis of V as L-vector space.*

It is known that there exists a cyclic vector for any non-zero differential module V on L.

When V is a differential module on L, we can define the operator norm $|\partial^n|_V$ of ∂^n on V ($n \in \mathbb{N}$) by fixing a basis of V. Then we define the spectral norm $|\partial|_{V,\mathrm{sp}}$ of ∂ on V by $|\partial|_{V,\mathrm{sp}} := \lim_{n\to\infty} |\partial^n|^{1/n}$. (In particular, we can define $|\partial|_L, |\partial|_{L,\mathrm{sp}}$.)

By using cyclic vector, we can calculate $|\partial|_{V,\mathrm{sp}}$ in certain case by the following proposition:

Proposition 1.2 ([Chr-Dw, Theorem 1.5])**.** *Let V be a differential module on L, $\mathbf{v}$ a cyclic vector of V and assume that we have the equation $\partial^{\dim V}(\mathbf{v}) = \sum_{i=0}^{\dim V-1} a_i\partial^i(\mathbf{v})$ ($a_i \in L$). Let r be the least slope of the lower convex hull of the set $\{(-i, -\log|a_i|) \mid 0 \le i \le \dim V - 1\}$ in $\mathbb{R}^2$. Then we have* $\max(|\partial|_L, |\partial|_{V,\mathrm{sp}}) = \max(|\partial|_L, e^{-r})$.

1.2 ∇-modules and generic radii of convergence

Let K be as in Convention and let L be a field of characteristic 0 containing K complete with respect to a norm (denoted also by $|\cdot|$) which extends the given absolute value of K. We define the notion of ∇-modules on rigid spaces over L as follows, following [K1, 2.3.4]:

Definition 1.3. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a morphism of rigid spaces over L. A ∇-module on $\mathfrak{X}$ relative to $\mathfrak{Y}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module E endowed with an integrable $f^{-1}\mathcal{O}_{\mathfrak{Y}}$-linear connection $\nabla : E \longrightarrow E \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathfrak{Y}}$. In the case $\mathfrak{Y} = \mathrm{Spm}\, L$, we omit the term 'relative to $\mathfrak{Y}$'.*

We call a subinterval $I \subseteq [0, \infty)$ aligned if any endpoint of I at which it is closed is contained in Γ^*. For an aligned interval I, we define the rigid space $A^n_L(I)$ by $A^n_L(I) := \{(t_1, ..., t_n) \in \mathbb{A}^{n,\mathrm{an}}_L \mid \forall i, |t_i| \in I\}$.

For $\rho \in [0, +\infty)$, we denote the completion of $L(t)$ with respect to ρ-Gauss norm $|\cdot|_\rho$ by $L(t)_\rho$. This is a differential field with continuous differential operator ∂ satisfying $\partial(t) = 1$. It is easy to show the equalities $|\partial|_\rho = \rho^{-1}$, $|\partial|_{\rho,\mathrm{sp}} = p^{-1/(p-1)}\rho^{-1}$. For any closed aligned interval I containing ρ, we have the natural inclusion $\Gamma(A^1_L(I), \mathcal{O}) \hookrightarrow L(t)_\rho$. For an aligned interval I, a ∇-module E on $A^1_L(I)$ and $\rho \in I$, we define the differential module E_ρ on $L(t)_\rho$ by $E_\rho := \Gamma(A^1_L(I'), E) \otimes_{\Gamma(A^1_L(I'),\mathcal{O})} L(t)_\rho$ endowed with the action

$$\partial : E_\rho \xrightarrow{\nabla_\rho} E_\rho dt \longrightarrow E_\rho, \tag{1.1}$$

where I' is any closed aligned subinterval of I containing ρ, ∇_ρ is the connection on E_ρ induced by ∇ and the last map in (1.1) is the map $xdt \mapsto x$. Then we define the notion of generic radius of convergence as follows ([Chr-M3, 4.1]):

Definition 1.4. *Let $I \subseteq [0, +\infty)$ be an aligned interval and let E be a ∇-module on $A^1_L(I)$. Then we define the generic radius of convergence $R(E, \rho)$ of E at the radius ρ by*

$$R(E, \rho) := \rho|\partial|_{L(t)_\rho,\mathrm{sp}}/|\partial|_{E_\rho,\mathrm{sp}} = p^{-1/(p-1)}/|\partial|_{E_\rho,\mathrm{sp}}.$$

If we fix a basis $\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_\mu)$ of E_ρ and define G_n to be the matrix expression of ∂^n with respect to the basis $\mathbf{e}$, then we have the following equivalent definition:

$$R(E, \rho) = \min(\rho, \varliminf_{n\to\infty} |G_n/n!|_\rho^{-1/n}).$$

E is said to satisfy the Robba condition if $R(E, \rho) = \rho$ for any $\rho \in I$.

As for the behavior of $R(E, \rho)$, the following proposition is known.

Proposition 1.5 ([Chr-M3, 8.6])**.** *Let $I \subseteq [0, +\infty)$ be an aligned interval and let E be a ∇-module on $A^1_L(I)$ of rank μ. Let $f : -\log I = \{-\log x \,|\, x \in I\} \longrightarrow \mathbb{R}$ be the function defined by $f(r) := -\log R(E, e^{-r})$. Then f is continuous piecewise affine linear convex function whose slopes are in $\mathbb{Z}\cdot(\mu!)^{-1}$.*

For $\lambda \in [0, 1) \cap \Gamma^*$, we call a ∇-module E on $A^1_L[\lambda, 1)$ solvable if $\lim_{\rho\to 1^-} R(E, \rho) = 1$. As a corollary to Proposition 1.5, we have the following:

Corollary 1.6. *Let $\lambda \in [0, 1) \cap \Gamma^*$ and let E be a ∇-module on $A^1_L[\lambda, 1)$. Then:*

(1) *The following conditions are equivalent.*

(a) *E is solvable.*

(b) *There exist $b \geq 0$ and a strictly increasing sequence $\{\rho_m\}_{m\in\mathbb{N}} \subseteq [\lambda, 1)$ with $\lim_{m\to\infty} \rho_m = 1$ such that $R(E, \rho_m) = \rho_m^{b+1}$ for all $m \in \mathbb{N}$.*

(c) *There exists $b \geq 0$ such that $R(E, \rho) = \rho^{b+1}$ for any ρ sufficiently close to 1.*

If these conditions are satisfied, we call b the highest ramification break of E.

(2) *If E is solvable, the following are equivalent.*

(a) *E has highest ramification break b.*

(b) *There exist $\rho_1, \rho_2 \in [\lambda, 1)$ with $\rho_1 < \rho_2$ such that $R(E, \rho_i) = \rho_i^{b+1}\,(i = 1, 2)$.*

(3) *The following conditions are equivalent.*

(a) *E is not solvable.*

(b) *There exists a strictly increasing sequence $\{\rho_m\}_{m\in\mathbb{N}} \subseteq [\lambda, 1)$ with $\lim_{m\to\infty} \rho_m = 1$ such that $\varlimsup_{m\to\infty} R(E, \rho_m) < 1$.*

(c) *For any strictly increasing sequence* $\{\rho_m\}_{m\in\mathbb{N}} \subseteq [\lambda, 1)$ *with* $\lim_{m\to\infty} \rho_m = 1$, $\overline{\lim_{m\to\infty}}\, R(E, \rho_m) < 1$.

Proof. Although the proof is well-known, we give a proof for the reader's convenience. Let us put $f(r) := -\log R(E, e^{-r})$.

First we prove (1). (c) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (b) are obvious. Let us prove (b) $\Longrightarrow$ (c). Put $r_m := -\log \rho_m$. Then we have $f(r) = (b+1)r$ for $r \in (0, r_1]$: Indeed, if we have $f(r) > (b+1)r$ for some $r \in (0, r_1]$ with $r_m < r$, this contradicts the convexity of f on $[r_m, r_1]$. On the other hand, if we have $f(r) < (b+1)r$ for some $r \in (0, r_1]$, this contradicts the convexity of f on $[r, r_0]$. So we have proved (c).

We prove (a) $\Longrightarrow$ (c). If the slope of f is less than 1 at some $r_0 \in (0, -\log \lambda]$, then $f(r) = pr + q$ for some $p < 1, q \in \mathbb{R}$ around r_0. By definition of f, we have $r_0 \le f(r_0) = pr_0 + q$. So $q \ge (1-p)r_0 > 0$. Then, by convexity of f, we obtain the inequality $\overline{\lim_{r\to 0^+}}\, f(r) \ge \overline{\lim_{r\to 0^+}}\,(pr + q) = q > 0$ and this contradicts the solvability of E. So the slope of f is equal to or greater than 1 on $(0, -\log \lambda]$. Since the slope of f is always in $\mathbb{Z}\cdot(\mu!)^{-1}$ and decreasing as $r \to 0^+$, it is stationary around 0, that is, we have $f(r) = pr + q$ for some $p \ge 1, q \in \mathbb{R}$ on some $(0, r_1]$. Then, by the solvability of E, we have $p \ge 1$ and $q = 0$, which imply (c). So we have proved (1).

Next we prove (2). (a) $\Longrightarrow$ (b) is obvious. Let us prove (b) $\Longrightarrow$ (a). Let us assume (b) and assume that E has highest ramification break b'. Put $r_i := -\log \rho_i$ so that $f(r_i) = (b+1)r_i\,(i = 1, 2)$. Then, if we have $b' < b$, there exists $r_3 < r_2$ with $f(r_3) < (b+1)r_3$ and this contradicts the convexity of f on $[r_3, r_1]$. On the other hand, if we have $b' > b$, there exists $r_4 < r_3 < r_2$ with $f(r_i) = (b'+1)r_i\,(i = 3, 4)$ and this contradicts the convexity of f on $[r_4, r_2]$. Hence we have $b' = b$, that is, the assertion (a). So we have proved (2).

Finally we prove (3). Since (c) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (a) are obvious, it suffices to prove (a) $\Longrightarrow$ (c). To do this, first we prove the claim that $f(r)$ has the form $f(r) = pr + q$ for some $p \in \mathbb{R}, q > 0$ on some $[s, t] \subseteq [0, -\log \lambda)\,(s < t)$. If the slope of f is less than 1 at some $r_0 \in (0, -\log \lambda]$, we have $f(r) = pr + q$ for some $p < 1, q > 0$ around r_0, as we saw in the proof of (1). Hence the claim is true in this case. If the slope of f is equal to or greater than 1 on $(0, -\log \lambda]$, we have $f(r) = pr + q$ for some $p \ge 1, q \in \mathbb{R}$ on some $(0, r_1]$, which we also saw in the proof of (1). Then, if we have $q < 0$, we have $(1-p)r > q$ for some r and this implies the inequality $r > f(r)$, which contradicts the definition of f. Also, if $q = 0$, then E would be solvable and this is also contradiction. Hence we have $q > 0$ and the claim is true also in this case.

By the above claim and the convexity of f, we have $f(r) \ge pr + q$ for $r \in (0, s]$. Hence, if we put $r_m := -\log \rho_m$, we have $\underline{\lim_{m\to\infty}}\, f(r_m) \ge \underline{\lim_{m\to\infty}}\,(pr_m + q) = q > 0$. This implies the assertion (a) and so we are done. ∎

1.3 Frobenius antecedent

Let L be a field of characteristic 0 containing K and a primitive p-th root of unity ζ complete with respect to a norm (denoted also by $|\cdot|$) which extends the given absolute value of K. For an aligned interval $I \subseteq [0,1)$, let us put $I^p := \{a^p \,|\, a \in I\}$ and let $\varphi_L : A^1_L(I) \longrightarrow A^1_L(I^p)$ be the morphism over L induced by $t \mapsto t^p$, where t denotes the coordinate of the annuli.

Proposition-Definition 1.7 ([K4, 10.4.1, 10.4.4])**.** *Let $I \subseteq (0,+\infty)$ be a closed aligned interval and let E_L be a ∇-module on $A^1_L(I)$ satisfying $R(E_L,\rho) > p^{-1/(p-1)}\rho$ for any $\rho \in I$. Then there exists a unique ∇-module F_L on $A^1_L(I^p)$ such that $R(F_L,\rho) > p^{-p/(p-1)}\rho$ for any $\rho \in I^p$ and that $\varphi_L^* F_L = E_L$. Moreover, we have $R(F_L,\rho^p) = R(E_L,\rho)^p$ for any $\rho \in I$. We call this F_L the Frobenius antecedent of E_L.*

Corollary 1.8. *Let $I \subseteq (0,+\infty)$ be a closed aligned interval and let F_L be a ∇-module on $A^1_L(I^p)$ with $R(F_L,\rho^p) > p^{-p/(p-1)}\rho^p$ for $\rho \in I$. Then F_L is the Frobenius antecedent of $\varphi_L^* F_L$.*

Remark 1.9. In [K4], the above proposition is proved also in the case where L does not necessarily contain a primitive p-th root of unity, and also in the case where I contains 0.

We explain how to construct F_L, following [K4]. Let ∂ be the composite $E_L \xrightarrow{\nabla} E_L dt \xrightarrow{=} E_L$ (where the second map sends xdt to x for $x \in E_L$). Let us define the action of the group $\mu_p = \{\underline{\zeta}^m \,|\, 0 \le m \le p-1\}$ of the p-th roots of unity in L on E_L by $\underline{\zeta}^m(x) := \displaystyle\sum_{i=0}^{\infty} \frac{((\zeta^m-1)t)^i}{i!}\partial^i(x)$ and let $P_j : E_L \longrightarrow E_L \,(0 \le j \le p-1)$ be the map defined by $P_j(x) := p^{-1}\displaystyle\sum_{i=0}^{p-1} \zeta^{-ij}(\underline{\zeta}^i(x))$. Then, it is straightforward to see that P_j's satisfy $P_j^2 = P_j, P_jP_{j'} = 0\,(j \ne j), \sum_{j=0}^{p-1} P_j = \mathrm{id}$. Hence we have the isomorphism $E \xrightarrow{\cong} \bigoplus_{j=0}^{p-1} \mathrm{Im}P_j$. Also, it is easy to see that each $\mathrm{Im}P_j$ has a natural structure of $\mathcal{O}_{A^1_L(I^p)}$-module and that the map $x \mapsto t^j x$ induces the isomorphism $\mathrm{Im}P_0 \xrightarrow{\cong} \mathrm{Im}P_j$ of $\mathcal{O}_{A^1_L(I^p)}$-modules. Moreover, $\mathcal{O}_{A^1_L(I)}$ is a free $\mathcal{O}_{A^1_L(I^p)}$-module with basis $1, t, ..., t^{p-1}$. These facts imply that we have the canonical isomorphism $\oplus_{j=0}^{p-1}\mathrm{Im}P_j \xleftarrow{=} \mathcal{O}_{A^1_L(I)} \otimes_{\mathcal{O}_{A^1_L(I^p)}} \mathrm{Im}P_0$. So, if we put $F_L := \mathrm{Im}P_0$, we have $\varphi_L^* F_L = E_L$. This F_L, endowed with the ∇-module structure $x \mapsto \dfrac{1}{pt^{p-1}}\partial(x)d(t^p)$ (where we denote the coodinate of $A^1_K(I^p)$ by t^p), gives the desired ∇-module on $A^1_L(I^p)$. (For the detail, see [K4, 10.4.1]).

We will slightly generalize the above construction. Let $\mathfrak{X} := \mathrm{Spm}\, A$ be a smooth affinoid rigid space over K which admits an injection $A \hookrightarrow L$ such that the norm on

L restricts to the supremum norm on $\mathfrak{X}$. For an aligned interval $I \subseteq [0,1)$, we can define the morphism $\varphi : \mathfrak{X} \times A^1_K(I) \longrightarrow \mathfrak{X} \times A^1_K(I^p)$ over $\mathfrak{X}$ by $t \mapsto t^p$.

Let I, E_L be as in Proposition-Definition 1.7 and assume that E_L is obtained from a ∇-module E on $\mathfrak{X} \times A^1_K(I)$ relative to $\mathfrak{X}$ via the 'pull-back' by $A^1_L(I) \longrightarrow \mathfrak{X} \times A^1_K(I)$. Then we have the following:

Proposition 1.10. *With the situation above, assume moreover that K contains ζ. Then there exists a ∇-module F on $\mathfrak{X} \times A^1_K(I^p)$ relative to $\mathfrak{X}$ with $\varphi^* F \cong E$ which induces F_L in Proposition-Definition 1.7 via the 'pull-back' by $A^1_L(I) \longrightarrow \mathfrak{X} \times A^1_K(I)$.*

Proof. We can define the maps $P_j : E \longrightarrow E \, (0 \leq j \leq p-1)$ in the same way as in the case of E_L and we can prove the isomorphisms

$$E \xrightarrow{\cong} \bigoplus_{j=0}^{p-1} \mathrm{Im} P_j \xleftarrow{\cong} \mathcal{O}_{\mathfrak{X} \times A^1_K(I)} \otimes_{\mathcal{O}_{\mathfrak{X} \times A^1_K(I^p)}} \mathrm{Im} P_0$$

in the same way. Then it suffices to put $F := \mathrm{Im} P_0$. □

1.4 Exponent in the sense of Christol-Mebkhout

In this subsection, we give a review of the exponent (in the sense of Christol-Mebkhout) for ∇-modules on annuli satisfying the Robba condition, following [Chr-M3, 10–12].

For $\alpha \in \mathbb{Z}_p$ and $h \in \mathbb{N}$, let $\alpha^{(h)}$ be the representative of α modulo p^h in $[(1-p^h)/2, (1+p^h)/2)$. For $\mu \in \mathbb{N}$ and $\Delta = (\Delta_i) \in \mathbb{Z}_p^\mu$, let us put $\Delta^{(h)} := (\Delta_i^{(h)}) \in \mathbb{Z}^\mu$, $\sigma(\Delta) := (\Delta_{\sigma(i)}) \in \mathbb{Z}_p^\mu \, (\sigma \in \mathfrak{S}_\mu)$. Let $|\cdot|_\infty$ be the usual absolute value on $\mathbb{Z}$ and for $\Delta = (\Delta_i) \in \mathbb{Z}^\mu$, put $|\Delta|_\infty := \max |\Delta_i|_\infty$.

With the above notation, we define the equivalence relation $\overset{e}{\sim}$ on $\mathbb{Z}_p^\mu$ as follows: For $\Delta, \Delta' \in \mathbb{Z}_p^\mu$, $\Delta \overset{e}{\sim} \Delta'$ if and only if there exists a sequence $(\sigma_h)_{h \in \mathbb{N}}$ of elements in $\mathfrak{S}_\mu$ such that $|\Delta'^{(h)} - \Delta^{(h)}|_\infty / h \, (h \in \mathbb{N})$ is bounded.

Let K be as in Convention and let L be a field of characteristic 0 containing K complete with respect to a norm (denoted also by $|\cdot|$) which extends the given absolute value of K. Let L_∞ be the completion of $L(\mu_{p^\infty})$, where μ_{p^∞} is the set of p-power roots of unity. Let $I \subseteq (0, +\infty)$ be a closed aligned interval and let E be a ∇-module on $A^1_L(I)$ of rank μ satisfying the Robba condition. (E is automatically a free module on $A^1_L(I)$ of rank μ.) Let ∂ be the composite $E \xrightarrow{\nabla} E dt \xrightarrow{=} E$ (where the second map sends $x dt$ to x for $x \in E$). Fixing a basis $\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_\mu)$ of E, we define the resolvent $Y_{\mathbf{e}}(x,y)$ associated to $\mathbf{e}$ by $Y_{\mathbf{e}}(x,y) := \sum_{n=0}^\infty G_n(y) \dfrac{(x-y)^n}{n!}$, where $G_n \in \mathrm{Mat}_\mu(\mathcal{O}_{A^1_L(I)})$ denotes the matrix expression of ∂^n with respect to the basis $\mathbf{e}$. ($Y_{\mathbf{e}}(x,y)$ is defined on $\{(x,y) \in \mathbb{A}^{2,\mathrm{an}}_L \,|\, |y| \in I, |x-y| < |y|\}$.) With these notations, we define the exponent of E as follows:

Theorem-Definition 1.11 ([Chr-M3, 11.3])**.** *Let I, E, μ be as above, take a basis $\mathbf{e}$ of E and let $Y_{\mathbf{e}}(x, y)$ be the resolvent of E associated to $\mathbf{e}$. Then the set of elements Δ in $\mathbb{Z}_p^\mu$ for which there exist a sequence $(S_h)_{h\in\mathbb{N}}$ in $\mathrm{Mat}_\mu(\mathcal{O}_{A^1_{L_\infty}(I)})$ and constants $c_1, c_2 > 0$ satisfying the conditions*

(1) *$\zeta^\Delta S_h(x) = S_h(\zeta x) Y_{\mathbf{e}}(\zeta x, x)$ for any $h \in \mathbb{N}$ and any ζ with $\zeta^{p^h} = 1$, where ζ^Δ denotes the diagonal $\mu \times \mu$ matrix with diagonal entries ζ^{Δ_i} $(1 \le i \le \mu)$.*

(2) *$|S_h|_\rho \le c_1^h$ for any $h \in \mathbb{N}$ and $\rho \in I$, where $|\cdot|_\rho$ denotes the ρ-Gauss norm.*

(3) *There exists some $\rho_0 \in I$ such that $|\det(S_h)|_{\rho_0} \ge c_2$ for any $h \in \mathbb{N}$.*

is non-empty, independent of the choice of $\mathbf{e}$ and contained in one equivalence class for the relation $\stackrel{e}{\sim}$. We call this class the exponent of E and denote it by $\mathrm{Exp}(E)$.

We recall the construction of Δ and $(S_h)_{h\in\mathbb{N}}$, following [Chr-M3, 11.2.1]. (See also [Dw].) For $h \in \mathbb{N}$ and $\Delta_h \in (\mathbb{Z}/p^h\mathbb{Z})^\mu$, let us define $S_{h,\Delta_h}(x)$ by $S_{h,\Delta_h} := p^{-h} \sum_{\zeta^{p^h}=1} \zeta^{-\Delta_h} Y_{\mathbf{e}}(\zeta x, x)$. Then we have

$$\det(S_{h,\Delta_h}) = \sum_{\Delta' \in (\mathbb{Z}/p^{h+1}\mathbb{Z})^\mu, \equiv \Delta_h \bmod p^h} \det(S_{h+1,\Delta'}).$$

Fix any $\rho_0 \in I$. Then, from the above equation, we can choose $\Delta \in \mathbb{Z}_p^\mu$ such that, if we put $\Delta_h := \Delta \bmod p^h$, we have the inequalities $|\det(S_{h,\Delta_h})|_{\rho_0} \le |\det(S_{h+1,\Delta_{h+1}})|_{\rho_0}$ for any $h \in \mathbb{N}$. Then, if we put $S_h := S_{h,\Delta_h}$, this Δ and $(S_h)_{h\in\mathbb{N}}$ satisfy the conditions (1), (2), (3) in Theorem-Definition 1.11. (For detail, see [Chr-M3] and the references which are quoted there.)

Let $\lambda \in (0, 1) \cap \Gamma^*$ and let E be a ∇-module on $A^1_L[\lambda, 1)$ with highest ramification break 0. Then there exists some $\lambda' \in [\lambda, 1) \cap \Gamma^*$ such that, for any closed aligned subinterval I in $[\lambda', 1)$, E satisfies the Robba condition on $A^1_L(I)$ (see Corollary 1.6). Hence we can define the exponent $\mathrm{Exp}(E)$ of E and it is known that this defininition of $\mathrm{Exp}(E)$ is independent of the choice of λ' and I.

As for the relation of the equivalence $\stackrel{e}{\sim}$ with p-adic non-Liouvilleness, we have the following:

Proposition 1.12 ([Chr-M3, 10.5])**.** *Let $\Delta = (\Delta_i), \Delta' = (\Delta'_i) \in \mathbb{Z}_p^\mu$ and assume that $\Delta_i - \Delta_j$ are p-adically non-Liouville for any i, j. Then we have $\Delta \stackrel{e}{\sim} \Delta'$ if and only if there exists an element $\sigma \in \mathfrak{S}_\mu$ with $\Delta' - \sigma(\Delta) \in \mathbb{Z}^\mu$.*

1.5 Overconvergent isocrystals and log-extendability

In this subsection, we recall the notion of overconvergent isocrystals having Σ-unipotent monodromy and isocrystals on log convergent site with exponents in Σ for $\Sigma \subseteq \mathbb{Z}_p^r$. Also, we recall several results proved in [S3]. In this subsection, we fix an open immersion $X \hookrightarrow \overline{X}$ of smooth k-varieties with $Z := \overline{X} - X = \bigcup_{i=1}^r Z_i$ a simple normal crossing divisor and let us denote the log structure on $\overline{X}$ associated to Z by $M_{\overline{X}}$. First we recall the notion concerning frames.

Definition 1.13 ([K1, 2.2.4, 4.2.1])**.** (1) *A frame (or affine frame) is a tuple $(U, \overline{U}, \mathcal{P}, i, j)$, where $U, \overline{U}$ are k-varieties, $\mathcal{P}$ is a p-adic affine formal scheme topologically of finite type over O_K, $i : \overline{U} \hookrightarrow \mathcal{P}$ is a closed immersion over O_K, $j : U \hookrightarrow \overline{U}$ is an open immersion over k such that $\mathcal{P}$ is formally smooth over O_K on a neighborhood of X. We say that the frame encloses the pair $(U, \overline{U})$.*

(2) *A small frame is a frame $(U, \overline{U}, \mathcal{P}, i, j)$ such that $\overline{U}$ is isomorphic to $\mathcal{P} \times_{\mathrm{Spf}\, O_K} \mathrm{Spec}\, k$ via i and that there exists an element $f \in \Gamma(\overline{U}, \mathcal{O}_{\overline{U}})$ with $U = \{f \neq 0\}$.*

Definition 1.14 ([S3, 3.3])**.** *Let $U \hookrightarrow \overline{U}$ be an open immersion of smooth k-varieties such that $Y := \overline{U} - U$ is a simple normal crossing divisor and let $Y = \bigcup_{i=1}^r Y_i$ be a decomposition of Y such that $Y = \bigcup_{\substack{1 \le i \le r \\ Y_i \neq \emptyset}} Y_i$ gives the decomposition of Y into irreducible components. A standard small frame enclosing $(U, \overline{U})$ is a small frame $\mathbf{P} := (U, \overline{U}, \mathcal{P}, i, j)$ enclosing $(U, \overline{U})$ which satisfies the following condition: There exist $t_1, ..., t_r \in \Gamma(\mathcal{P}, \mathcal{O}_{\mathcal{P}})$ such that, if we denote the zero locus of t_i in $\mathcal{P}$ by $\mathcal{Q}_i$, each $\mathcal{Q}_i$ is irreducible (possibly empty) and that $\mathcal{Q} = \bigcup_{i=1}^r \mathcal{Q}_i$ is a relative simple normal crossing divisor of $\mathcal{P}$ satisfying $Y_i = \mathcal{Q}_i \times_{\mathcal{P}} \overline{U}$. We call a pair $(\mathbf{P}, (t_1, ..., t_r))$ a charted standard small frame. When $r = 1$, we call $\mathbf{P}$ a smooth standard small frame and the pair $(\mathbf{P}, t_1)$ a charted smooth standard small frame.*

Let $\overline{U}$ be an open subscheme of $\overline{X}$, let $U := X \cap \overline{U}$ and let $((U, \overline{U}, \mathcal{P}, i, j), t)$ be a charted smooth standard small frame. Then an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$ induces a ∇-module on $\mathfrak{U}_\lambda := \{x \in \mathcal{P}_K \,|\, |t(x)| \geq \lambda\}$ for some $\lambda \in (0, 1) \cap \Gamma^*$, which we denote by $E_{\mathcal{E}} = (E_{\mathcal{E}}, \nabla)$. Since $\mathfrak{U}_\lambda$ contains a relative annulus $\mathcal{Q}_K \times A_K^1[\lambda, 1)$, we can restrict $E_{\mathcal{E}}$ to this space. Furthurmore, if $\mathcal{Q}_K$ is non-empty and if we are given an injection $\Gamma(\mathcal{Q}_K, \mathcal{O}_{\mathcal{Q}_K}) \hookrightarrow L$ into a field L endowed with a complete norm which restricts to the supremum norm on $\mathcal{Q}_K$ (e.g., when L is the completion of the fraction field of $\Gamma(\mathcal{Q}_K, \mathcal{O}_{\mathcal{Q}_K})$ with respect to the supremum norm), we can further restrict $E_{\mathcal{E}}$ to $A_L^1[\lambda, 1)$, which we denote by $E_{\mathcal{E},L}$. Then we have the following proposition:

Proposition 1.15. *With the above notation, the ∇-module $E_{\mathcal{E},L}$ on $A_L^1[\lambda, 1)$ is solvable.*

Proof. Since we may shrink $\mathcal{P}$ (since, for dense open immersion $\mathcal{Q}' \hookrightarrow \mathcal{Q}$, induced map $\Gamma(\mathcal{Q}_K, \mathcal{O}_{\mathcal{Q}_K}) \hookrightarrow \Gamma(\mathcal{Q}'_K, \mathcal{O}_{\mathcal{Q}'_K})$ respects the supremum norm), we may assume that $\Omega^1_{\mathcal{P}_K}$ is freely generated by $dx_1, ..., dx_n, dt$ for some $x_1, ..., x_n \in \Gamma(\mathcal{P}_K, \mathcal{O}_{\mathcal{P}_K})$. Let $\partial : E_{\mathcal{E}} \longrightarrow E_{\mathcal{E}}$ be the composite

$$E_{\mathcal{E}} \xrightarrow{\nabla} E_{\mathcal{E}} \otimes \Omega^1_{\mathfrak{U}_\lambda} \xrightarrow{\pi} E_{\mathcal{E}} dt \xrightarrow{=} E,$$

where π sends dx_i (resp. dt) to 0 (resp. dt) and the last map sends $xdt\,(x \in E_{\mathcal{E}})$ to x. Then, by [Be, 2.2.13] and [K1, 2.5.6], we have the following: For any $\eta < 1$, there exists $\lambda_0 < 1$ such that for any $\lambda \in (\lambda_0, 1) \cap \Gamma^*$ and for any $e \in \Gamma(\mathfrak{U}_\lambda, E_{\mathcal{E}})$, we have $\lim_{i\to\infty} \|\partial^i(e)/i!\|\eta^i = 0$, where $\|\cdot\|$ denotes any Banach norm on $\Gamma(\mathfrak{U}_\lambda, E_{\mathcal{E}})$. Hence, for any $\eta < 1$, there exists $\rho_0 < 1$ such that for any $\rho \in (\rho_0, 1) \cap \Gamma^*$ and for any $e \in E_{\mathcal{E},L,\rho}$, we have $\lim_{i\to\infty} |\partial^i(e)/i!|_\rho \eta^i = 0$, where $|\cdot|_\rho$ denotes any norm on $E_{\mathcal{E},L,\rho}$ induced by the norm on $L(t)_\rho$(= the completion of $L(t)$ with repsect to the ρ-Gauss norm). This implies the inequality $p^{1/(p-1)}|\partial|_{\rho,\mathrm{sp}}\eta \leq 1$. Hence, for any $\eta < 1$, we have $\eta \leq R(E_{\mathcal{E},L}, \rho)$ for ρ sufficiently close to 1, that is, $E_{\mathcal{E},L}$ is solvable. $\square$

Next we recall the notion of Σ-unipotent ∇-modules. For aligned interval $I \subseteq (0, +\infty)$ and $\xi \in \mathbb{Z}_p$, we define the ∇-module $M_\xi := (M_\xi, \nabla_{M_\xi})$ on $A^1_K(I)$ (whose coordinate is t) as the ∇-module $(\mathcal{O}_{A^1_K(I)}, d + \xi \mathrm{dlog}\, t)$.

Definition 1.16 ([S3, 1.3])**.** *Let $\mathfrak{X}$ be a smooth rigid space. Let $I \subseteq (0, \infty)$ be an aligned interval and fix $\Sigma \subseteq \mathbb{Z}_p$. A ∇-module E on $\mathfrak{X} \times A^1_K(I)$ is called Σ-unipotent if it admits a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

by sub log-∇-modules whose successive quotients have the form $\pi_1^* F \otimes \pi_2^* M_\xi$ for some ∇-module F on $\mathfrak{X}$ and $\xi \in \Sigma$, where $\pi_1 : \mathfrak{X} \times A^1_K(I) \longrightarrow \mathfrak{X}$, $\pi_2 : \mathfrak{X} \times A^1_K(I) \longrightarrow A^1_K(I)$ denote the projections.

We call a subset $\Sigma := \prod_{i=1}^r \Sigma_i$ in $\mathbb{Z}_p^r$ (NID) (resp. (NLD)) if for any $1 \leq i \leq r$ and any $\alpha, \beta \in \Sigma_i$, $\alpha - \beta$ is not a non-zero integer (resp. p-adically non-Liouville). Under the assumption of (NID) and (NLD), we have the following generization property for Σ-unipotent ∇-modules:

Proposition 1.17 ([S3, 2.4])**.** *Let $\mathfrak{X}$ be a smooth affinoid rigid space and let $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \hookrightarrow L$ be an injection into a field complete with respect to a norm which restricts to the supremum norm on $\mathfrak{X}$. Let $I \subseteq (0, 1)$ be an open interval, let Σ be a subset of $\mathbb{Z}_p$ which is* (NID), (NLD) *and let E be a ∇-module on $X \times A^1_K(I)$ whose restriction to $A^1_L(I)$ is Σ-unipotent. Then E is also Σ-unipotent.*

For a subset Λ of $\mathbb{Z}_p$ and $\mu \in \mathbb{N}$, let $\overline{\Lambda} \subseteq \mathbb{Z}_p^\mu / \overset{e}{\sim}$ be the image of Λ^μ by the projection $\mathbb{Z}_p^\mu \longrightarrow \mathbb{Z}_p^\mu / \overset{e}{\sim}$. Then, by [Chr-M3, 12.1], we have the following characterization of Σ-unipotence:

Proposition 1.18. *Let L be a field containing K complete with respect to a norm which extends the given absolute value of K. Let $I \subseteq (0, 1)$ be an open interval and let E be a ∇-module on $A^1_L(I)$. Then, for a subset $\Sigma \subseteq \mathbb{Z}_p$ which is* (NID) *and* (NLD)*, the following are equivalent*:

(1) *E is Σ-unipotent.*

(2) *E satisfies the Robba condition and* $\mathrm{Exp}(E) \in \overline{\Sigma}$.

Proof. (2) $\Longrightarrow$ (1) follows from Proposition 1.12 and [Chr-M3, 12.1]. In rank one case, (1) $\Longrightarrow$ (2) follows from [K4, 9.5.2, 13.5.3] and the general case follows from the rank one case and [Chr-M3, 4.5, 11.7]. $\square$

Now we recall the definition of overconvergent isocrystals having Σ-unipotent monodromy.

Definition 1.19 ([S3, 3.9]). *Let $(X, \overline{X}), Z = \bigcup_{i=1}^r Z_i$ be as above and let Z_{sing} be the set of singular points of Z. Let $\Sigma = \prod_{i=1}^r \Sigma_i$ be a subset of $\mathbb{Z}_p^r$. Then we say that an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$ has Σ-unipotent monodromy if there exist an affine open covering $\overline{X} - Z_{\mathrm{sing}} = \bigcup_{\alpha \in \Delta} \overline{U}_\alpha$ and charted smooth standard small frames $((U_\alpha, \overline{U}_\alpha, \mathcal{P}_\alpha, i_\alpha, j_\alpha), t_\alpha)$ enclosing $(U_\alpha, \overline{U}_\alpha)$ ($\alpha \in \Delta$, where we put $U_\alpha := X \cap \overline{U}_\alpha$) such that, for any $\alpha \in \Delta$, there exists some $\lambda \in (0,1) \cap \Gamma^*$ such that the ∇-module $E_{\mathcal{E},\alpha}$ associated to $\mathcal{E}$ is defined on $\{x \in \mathcal{P}_{\alpha,K} \,|\, |t_\alpha(x)| \geq \lambda\}$ and that the restriction of $E_{\mathcal{E},\alpha}$ to $\mathcal{Q}_{\alpha,K} \times A_K^1[\lambda, 1)$ is Σ_i-unipotent (where $\mathcal{Q}_\alpha$ is the zero locus of t_α in $\mathcal{P}_\alpha$ and i is any index with $\overline{U}_\alpha \cap Z \subseteq Z_i$, which is unique if $\overline{U}_\alpha \cap Z$ is non-empty. When $\underline{U}_\alpha \cap Z$ is empty, we regard this last condition as vacuous one.)*

When Σ is (NID) and (NLD), we have the following characterization of overconvergent isocrystals having Σ-unipotent monodromy:

Proposition 1.20 ([S3, 3.10]). *Let $(X, \overline{X}), Z = \bigcup_{i=1}^r Z_i, \Sigma$ be as above and assume that Σ is* (NID) *and* (NLD). *Then an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$ has Σ-unipotent monodromy if and only if the following condition is satisfied: For any affine open subscheme $\overline{U} \hookrightarrow \overline{X} - Z_{\mathrm{sing}}$ and any charted smooth standard small frame $((U, \overline{U}, \mathcal{P}, i, j), t)$ enclosing $(U, \overline{U})$ (where we put $U := X \cap \overline{U}$), there exists some $\lambda \in (0,1) \cap \Gamma^*$ such that the ∇-module $E_{\mathcal{E}}$ associated to $\mathcal{E}$ is defined on $\{x \in \mathcal{P}_K \,|\, |t(x)| \geq \lambda\}$ and that the restriction of $E_{\mathcal{E}}$ to $\mathcal{Q}_K \times A_K^1[\lambda, 1)$ is Σ_i-unipotent (where $\mathcal{Q}$ is the zero locus of t in P and i is any index with $\overline{U} \cap Z \subseteq Z_i$, which is unique if $\overline{U} \cap Z$ is non-empty. When $\underline{U} \cap Z$ is empty, we regard this last condition as vacuous one.)*

Next, we proceed to recall the notion of isocrystals on log convergent site with exponents in Σ. To do so, first we should recall the definition of log-∇-modules and exponents of them.

Definition 1.21 ([K1, 2.3.7]). *Let $\mathfrak{X}$ be a rigid space over K and let $x_1, ..., x_r$ be elements in $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Then a log-$\nabla$-module on $\mathfrak{X}$ with respect to $x_1, ..., x_r$ is a locally free $\mathcal{O}_{\mathfrak{X}}$-module E endowed with an integrable K-linear log connection $\nabla : E \longrightarrow E \otimes_{\mathcal{O}_{\mathfrak{X}}} \omega_{\mathfrak{X}}^1$, where $\omega_{\mathfrak{X}}^1$ denotes the sheaf of continuous log differentials with respect to* $\mathrm{dlog}\, x_i$ $(1 \leq i \leq r)$.

When we are given $\mathfrak{X}, x_1, ..., x_r, (E, \nabla)$ as above, we can define the residue res_i of E along $\mathfrak{D}_i = \{x_i = 0\}$ $(1 \leq i \leq r)$, which is an element in $\mathrm{End}_{\mathcal{O}_{\mathfrak{D}_i}}(E|_{\mathfrak{D}_i})$, in the same way as in the case of integrable log connection on algebraic varieties. When $\mathfrak{X}$ is smooth, zero loci of x_i are affinoid smooth and meet transversally, there exists a minimal monic polynomial $P_i(x) \in K[x]$ with $P_i(\mathrm{res}_i) = 0\,(1 \leq i \leq r)$ by the argument of [Ba-Chi, 1.5.3]. We call the roots of $P_i(x)$ the exponents of (E, ∇) along $\mathfrak{D}_i$.

If $\overline{U}$ is an open subscheme of $\overline{X}$ and $((U, \overline{U}, \mathcal{P}, i, j), (t_1, ..., t_r))$ is a charted standard small frame enclosing $(U, \overline{U})$ (where $U := X \cap \overline{U}$), an isocrystal $\mathcal{E}$ on log convergent site $((\overline{X}, M_{\overline{X}})/O_K)_{\mathrm{conv}}$ induces a log-∇-module on $\mathcal{P}_K$ with respect to $t_1, ..., t_r$, which we denote by $E_{\mathcal{E}} = (E_{\mathcal{E}}, \nabla)$. Using this construction, we can define the notion of isocrystals on log convergent site with exponents in Σ in the following way.

Definition 1.22. *Let $(X, \overline{X}), M_{\overline{X}}$ be as above and let $\Sigma = \prod_{i=1}^{r} \Sigma_i$ be a subset of $\mathbb{Z}_p^r$. Then we say that a locally free isocrystal $\mathcal{E}$ on $((\overline{X}, M)/O_K)_{\mathrm{conv}}$ has exponents in Σ if there exist an affine open covering $\overline{X} = \bigcup_{\alpha\in\Delta} \overline{U}_\alpha$ and charted standard small frames $((U_\alpha, \overline{U}_\alpha, \mathcal{P}_\alpha, i_\alpha, j_\alpha), (t_{\alpha,1}, ..., t_{\alpha,r}))$ enclosing $(U_\alpha, \overline{U}_\alpha)$ ($\alpha \in \Delta$, where we put $U_\alpha := X \cap \overline{U}_\alpha$) such that, for any $\alpha \in \Delta$ and any i ($1 \leq i \leq r$), all the exponents of the log-∇-module $E_{\mathcal{E},\alpha}$ on $\mathcal{P}_{\alpha,K}$ induced by $\mathcal{E}$ along the locus $\{t_{\alpha,i} = 0\}$ are contained in Σ_i.*

Then the main result of the paper [S3] is as follows:

Theorem 1.23 ([S3, 3.16]). *Let $(X, \overline{X}), Z = \bigcup_{i=1}^{r} Z_i, M_{\overline{X}}$ be as above and let $\Sigma := \prod_{i=1}^{r} \Sigma_i$ be a subset of $\mathbb{Z}_p^r$ which is* (NID) *and* (NLD)*. Then we have the canonical equivalence of categories*

$$j^\dagger : \begin{pmatrix} \text{isocrystals on} \\ ((\overline{X}, M)/O_K)_{\mathrm{conv}} \\ \text{with exponents in } \Sigma \end{pmatrix} \xrightarrow{=} \begin{pmatrix} \text{overcongent isocrystals} \\ \text{on } (X, \overline{X})/K \text{ having} \\ \Sigma\text{-unipotent monodromy} \end{pmatrix}, \tag{1.2}$$

which is defined by the restriction.

2 Proofs

In this section, we give a proof of Theorem 0.1. In the course of the proof, we prove a kind of cut-by-curves criteria on solvability, highest ramification break and exponent of ∇-modules.

2.1 Geometric set-up

In this subsection, we give a geometric set-up which we use in the following two subsections. To do this, first let us recall the notion of tubular neighborhood (for schemes), which is introduced in [A-Ba].

Definition 2.1 ([A-Ba, I 1.3.1]). *Let $\overline{X}$ be a smooth k-variety and let $i : Z \hookrightarrow \overline{X}$ be a smooth divisor. Then $(f : \overline{X} \longrightarrow Y, t : \overline{X} \longrightarrow \mathbb{A}^1_k)$ is a coodinatized tubular neighborhood of Z in X if it admits the following commutative diagram*

(2.1)
$$\begin{tikzcd}
Z \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{X} \arrow[d, "\phi"] & \\
Y \arrow[r, hook] \arrow[dr, equal] & \mathbb{A}^1_Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \arrow[dr, phantom, "\square"] & \mathbb{A}^1_k \arrow[d] \\
& Y \arrow[r] & \operatorname{Spec} k,
\end{tikzcd}$$

where Y is an affine connected smooth k-variety, ϕ is etale, two squares are Cartesian and $f = \mathrm{pr}_2 \circ \phi, t = \mathrm{pr}_1 \circ \phi$.

Remark 2.2. The smoothness of $\overline{X}, Z, Y$ and the affineness, connectedness of Y are not assumed in [A-Ba]. However, this change does not cause any problem for our purpose. Also, the above diagram looks different from that given in [A-Ba, I 1.3.1], but they are in fact equivalent.

We have the following result on the existence of coodinatized tubular neighborhoods.

Lemma 2.3 ([A-Ba, I 1.3.2]). *Let $\overline{X}$ be a smooth k-variety and let Z be a smooth divisor on it. Then there exists an open covering $\overline{X} = \bigcup_\alpha \overline{X}_\alpha$ such that each $Z \cap \overline{X}_\alpha \hookrightarrow \overline{X}_\alpha$ admits a structure of a coodinatized tubular neighborhood.*

In the next subsection, we will consider the following geometric situation:

Hypothesis 2.4. Let $\overline{X}$ be a connected affine smooth k-variety, let $i : Z \hookrightarrow \overline{X}$ be a non-empty connected smooth divisor admitting a structure of a coodinatized tubular neighborhood and fix a diagram (2.1). Then there exists a lifting of the diagram (2.1) to a diagram of smooth formal schemes over $\operatorname{Spf} O_K$, that is, there exists a diagram

(2.2)
$$\begin{tikzcd}
\mathcal{Z} \arrow[r, hook, "\widehat{i}"] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{X}} \arrow[d, "\widehat{\phi}"] & \\
\mathcal{Y} \arrow[r, hook] \arrow[dr, equal] & \widehat{\mathbb{A}}^1_{\mathcal{Y}} \arrow[r, "\widehat{\mathrm{pr}}_1"] \arrow[d, "\widehat{\mathrm{pr}}_2"] \arrow[dr, phantom, "\square"] & \widehat{\mathbb{A}}^1_{O_K} \arrow[d] \\
& \mathcal{Y} \arrow[r] & \operatorname{Spf} O_K
\end{tikzcd}$$

consisting of smooth formal schemes over $\operatorname{Spf} O_K$ whose special fiber gives the diagram (2.1) such that $\widehat{\phi}$ is etale and two squares are Cartesian. We fix this diagram.

Also, we denote the composite $\widehat{\mathrm{pr}}_1 \circ \widehat{\phi}$ also by t, by abuse of notation. Let us put the open immersion $X := \overline{X} - Z \hookrightarrow \overline{X}$ by j and the closed immersion $\overline{X} \hookrightarrow \overline{\mathcal{X}}$ by ι. Also, we fix an inclusion $\Gamma(\mathcal{Z}_K, \mathcal{O}_{\mathcal{Z}_K}) \hookrightarrow L$, where L is a field complete with respect to a norm which restricts to the supremum norm on $\mathcal{Z}_K$.

For a separable closed point y in Y, a lift of y in $\mathcal{Y}$ is a closed sub formal scheme $\widetilde{y} \hookrightarrow \mathcal{Y}$ etale over $\mathrm{Spf}\, O_K$ such that $\widetilde{y} \times_{\mathcal{Y}} Y = y$. For a separable closed point y in Y and its lift $\widetilde{y}$ in $\mathcal{Y}$, denote the pull-back by $\mathbb{A}^1_y \hookrightarrow \mathbb{A}^1_Y, \mathbb{A}^1_{\widetilde{y}} \hookrightarrow \mathbb{A}^1_{\mathcal{Y}}$ of the upper square of (2.1), (2.2) by

(2.3)
$$\begin{tikzcd} Z_y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{X}_y \arrow[d] \\ y \arrow[r, hook] & \mathbb{A}^1_y, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Z}_{\widetilde{y}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{X}}_{\widetilde{y}} \arrow[d] \\ \widetilde{y} \arrow[r, hook] & \hat{\mathbb{A}}^1_{\widetilde{y}}, \end{tikzcd}$$

respectively, and let us put $X_y := \overline{X}_y - Z_y$.

Under Hypothesis 2.4, $((X, \overline{X}, \overline{\mathcal{X}}, \iota, j), t)$ forms a charted smooth standard small frame. So, an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$ induces a ∇-module on $\mathfrak{U}_\lambda = \{x \in \overline{\mathcal{X}}_K \,|\, |t(x)| \geq \lambda\}$ for some $\lambda \in (0,1) \cap \Gamma^*$, hence a ∇-module $E_{\mathcal{E}}$ on $\mathcal{Z}_K \times A^1_K[\lambda, 1)$. We denote the induced ∇-module on $A^1_L[\lambda, 1)$ by $E_{\mathcal{E},L}$. On the other hand, for a separable closed point $y \in Y$ and its lift $\widetilde{y}$ in $\mathcal{Y}$, we can restrict $\mathcal{E}$ to an overconvergent isocrystal $\mathcal{E}_y$ on $(X_y, \overline{X}_y)/K$, and by using the diagrams (2.3), we can define the ∇-module $E_{\mathcal{E},\widetilde{y}}$ on $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$ for some λ. $E_{\mathcal{E},\widetilde{y}}$ is nothing but the restriction of $E_{\mathcal{E}}$ to $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$.

Note that, in the situation of Hypothesis 2.4, we have a closed immersion $(\overline{X}_y, M_{\overline{X}_y}) \hookrightarrow (\overline{X}, M_{\overline{X}})$, where $M_{\overline{X}_y}, M_{\overline{X}}$ denotes the log structure on $\overline{X}_y, \overline{X}$ associated to Z_y, Z, respectively. Note also that $\overline{X}_y$ is a curve, Z_y is a smooth divisor on it and that $\mathcal{Z}_{\widetilde{y}}$ is a formal scheme etale over $\mathrm{Spf}\, O_K$. Hence $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$ has the form $\coprod_{j=1}^a A_{K_j}[\lambda, 1)$ for some finite extensions K_j $(1 \leq j \leq a)$ of K.

2.2 Criteria on solvability, highest ramification break and exponent

Let the notations be as in Hypothesis 2.4 and fix $\lambda \in (0,1) \cap \Gamma^*$. Let E be a ∇-module on $\mathcal{Z}_K \times A^1_K[\lambda, 1)$ relative to $\mathcal{Z}_K$ and let E_L be the ∇-module on $A^1_L[\lambda, 1)$ induced by E. On the other hand, for a separable closed point y in Y and its lift $\widetilde{y}$ in $\mathcal{Y}$, let $E_{\widetilde{y}}$ be the restriction of E to $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$. First we prove a kind of cut-by-curves criterion on highest ramification break, assuming certain solvability:

Theorem 2.5. *Let the notations be as above and assume the condition* $(*)$ *below:*

$(*)$ *E_L is solvable and for any separable closed point $y \in Y$ and its lift $\widetilde{y}$ in $\mathcal{Y}$, $E_{\widetilde{y}}$ is also solvable on any connected component of $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$.*

Then the following are equivalent:

(1) *E_L has highest ramification break b.*

(2) *There exists a dense open subset $U \subseteq Y$ such that, for any separable closed point y in U and its lift $\widetilde{y}$ in $\mathcal{Y}$, $E_{\widetilde{y}}$ has highest ramification break b on any connected component of $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$.*

Remark 2.6. The condition $(*)$ is satisfied when E is the ∇-module induced from an overconvergent isocrystal on $(X, \overline{X})/K$, by Proposition 1.15.

Remark 2.7. The solvability and the highest ramification break for E_L is independent of the choice of L in Hypothesis 2.4: If we denote the completion of $\mathrm{Frac}\,\Gamma(\mathcal{Z}_K, \mathcal{O}_{\mathcal{Z}_K})$ with respect to the supremum norm by L_0, the inclusion $\Gamma(\mathcal{Z}_K, \mathcal{O}_{\mathcal{Z}_K}) \hookrightarrow L$ factors as $\Gamma(\mathcal{Z}_K, \mathcal{O}_{\mathcal{Z}_K}) \hookrightarrow L_0 \hookrightarrow L$ and the norms of L_0 and L are compatible. Hence, by definition, the solvability of E_L is equivalent to that of E_{L_0} and the highest ramification break of E_L is equal to that of E_{L_0}.

Before giving the proof, we first prove a technical lemma.

Lemma 2.8. *Let K' be a field containing K which is complete with respect to a norm which extends the given norm on K and let $O_{K'}$ be the ring of integers of K'. Let $I \subseteq (0, 1)$ be a closed aligned interval of positive length and let $a = \sum_{n\in\mathbb{Z}} a_n t^n$ be a non-zero element of $\Gamma(\mathcal{Z}_{K'} \times A^1_{K'}(I), \mathcal{O})$. (Here $\mathcal{Z}_{K'}$ denotes the rigid analytic space over K' associated to $\mathcal{Z}\widehat{\otimes}_{O_K} O_{K'}$.) Then there exist an open dense sub affine formal scheme $\mathcal{U} \subseteq \mathcal{Z}$ and a closed aligned subinterval $I' \subseteq I$ of positive length satisfying the following conditions*:

(1) *$a \in \Gamma(\mathcal{U}_{K'} \times A^1_{K'}(I'), \mathcal{O}^\times)$.*

(2) *For any $u \in \mathcal{U}_{K'}$ and $\rho \in I'$, we have $|a(u)|_\rho = |a|_\rho$, where $a(u) := \sum_{n\in\mathbb{Z}} a_n(u)t^n \in \Gamma(u \times A^1_{K'}(I'), \mathcal{O})$ and $|\cdot|_\rho$ denotes the ρ-Gauss norm.*

Proof. In this proof, $|\cdot|$ denotes the supremum norm. Let us write $I = [\alpha, \beta]$. By [K1, 3.1.7, 3.1.8], we have

$$|a| = \max(\sup_n(|a_n|\alpha^n), \sup_n(|a_n|\beta^n)) = \max(\sup_{n\le 0}(|a_n|\alpha^n), \sup_{n\ge 0}(|a_n|\beta^n)).$$

Let us define finite subsets $A \subseteq \mathbb{Z}_{\le 0}$, $B \subseteq \mathbb{Z}_{\ge 0}$ by $A := \{n \le 0 \,|\, |a_n|\alpha^n = |a|\}$, $B := \{n \ge 0 \,|\, |a_n|\beta^n = |a|\}$. Then we have $A \cup B \neq \emptyset$.

Let us first consider the case $A \neq \emptyset$. Let n_0 be the maximal element of A. Then, since $a_{n_0} \neq 0$, there exists an element $b \in K^\times$ such that $ba_{n_0} \in \Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ and that the image $\overline{ba_{n_0}}$ of ba_{n_0} in $\Gamma(Z, \mathcal{O}_Z)$ is non-zero. Let $\mathcal{U} \subseteq \mathcal{Z}$ be the open dense affine sub formal scheme such that $\overline{ba_{n_0}}$ is invertible on $\mathcal{U} \times_{\mathcal{Z}} Z$. Then we have $ba_{n_0} \in \Gamma(\mathcal{U}, \mathcal{O}^\times_{\mathcal{U}})$. So, for all $u \in \mathcal{U}_{K'}$, we have $|a_{n_0}(u)| = |b^{-1}|$ and hence

$|a_{n_0}(u)| = |a_{n_0}|$. (Here note that, for elements in $\Gamma(\mathcal{Z}_{K'}, \mathcal{O})$, its supremun norm on $\mathcal{Z}_{K'}$ is the same as that on $\mathcal{U}_{K'}$.) Next we prove the following claim:

claim. There exists a closed aligned subinterval $I' \subseteq I$ of positive length such that $|a_n|\rho^n < |a_{n_0}|\rho^{n_0}$ for any $n \in \mathbb{Z}, \neq n_0$ and $\rho \in I'$.

Let us put $C := \{n \in \mathbb{Z} \,|\, \max(|a_n|\alpha^n, |a_n|\beta^n) \geq |a_{n_0}|\beta^{n_0}\}$. Then C is a finite set containing A. If $n \in A, \neq n_0$, we have $|a_n|\alpha^n = |a_{n_0}|\alpha^{n_0}$ and $n < n_0 \leq 0$. Hence we have $|a_n|\rho^n < |a_{n_0}|\rho^{n_0}$ for any $\rho \in (\alpha, \beta]$. For $n \in C - A$, we have $|a_n|\alpha^n < |a_{n_0}|\alpha^{n_0}$. So there exists $\beta' \in (\alpha, \beta]$ such that, for any $n \in C - A$ and for any $\rho \in [\alpha, \beta']$, we have $|a_n|\rho^n < |a_{n_0}|\rho^{n_0}$. For $n \notin C$, we have, for any $\rho \in I$, the inequalities

$$|a_n|\rho^n \leq \max(|a_n|\alpha^n, |a_n|\beta^n) < |a_{n_0}|\beta^{n_0} \leq |a_{n_0}|\rho^{n_0}.$$

Summing up these, we see the claim.

Let us put $f := \sum_{n \neq n_0} (a_n/a_{n_0}) t^{n-n_0} \in \Gamma(\mathcal{U}_{K'} \times A^1_{K'}(I'), \mathcal{O})$ and take any $u \in \mathcal{U}_{K'}$, $\rho \in I'$. Then we have

$$|f(u)|_\rho \leq \frac{\sup_{n \neq n_0}(|a_n|\rho^n)}{|a_{n_0}(u)|\rho^{n_0}} = \frac{\sup_{n \neq n_0}(|a_n|\rho^n)}{|a_{n_0}|\rho^{n_0}} < 1.$$

So we have $|f| < 1$. So we have $a = a_{n_0} t^{n_0}(1 + f) \in \Gamma(\mathcal{U}_{K'} \times A^1_{K'}(I'), \mathcal{O}^\times)$. Moreover, for any $u \in \mathcal{U}_K$ and $\rho \in I'$, we have

$$|a(u)|_\rho = \sup_n(|a_n(u)|\rho^n) = |a_{n_0}(u)|\rho^{n_0} = |a_{n_0}|\rho^{n_0} = \sup_n(|a_n|\rho^n) = |a|_\rho.$$

We can prove the lemma in the case $B \neq \emptyset$ in the same way. (In this case, we define n_0 to be the minimal element of B.) So we are done. □

Now we give a proof of Theorem 2.5.

Proof of Theorem 2.5. First we prove (2) $\Longrightarrow$ (1), assuming (1) $\Longrightarrow$ (2). Assume (2) and assume that E_L has highest ramification break b'. Then, since we assumed the implication (1) $\Longrightarrow$ (2), there exists a dense open subset $U' \subseteq Y$ such that, for any separable closed point y in U' and its lift $\widetilde{y}$ in $\mathcal{Y}$, $E_{\widetilde{y}}$ has highest ramification break b'. Since $U \cap U'$ contains a separable closed point, this implies the equality $b = b'$ as desired. So it suffices to prove (1) $\Longrightarrow$ (2). We will prove this.

For a separable closed point x in Z, we define a lift of x in $\mathcal{Z}$ as a closed sub formal scheme $\widetilde{x} \hookrightarrow \mathcal{Z}$ etale over $\mathrm{Spf}\, O_K$ such that $x = \widetilde{x} \times_{\mathcal{Z}} Z$. Then, to prove the assertion, it suffices to prove that there exists an open dense subscheme $V \subseteq Z$ such that, for any separable closed point x and its lift $\widetilde{x}$ in $\mathcal{Z}$, the restriction $E_{\widetilde{x}}$ of E to $\widetilde{x}_K \times A^1_K[\lambda, 1)$ has highest ramification break b. Indeed, if this is true, we have the assertion if we put $U := Y - \overline{\phi(Z - V)}$. So we will prove this claim.

First we prove the above claim in the case $b = 0$. We may assume that $R(E_{\mathcal{E},L}, \rho) = \rho$ for any $\rho \in [\lambda, 1)$. Take any closed aligned subinterval $I \subseteq [\lambda, 1)$ of positive length and put $A := \Gamma(\mathcal{Z}_K \times A_K(I), \mathcal{O})$, $\mathbf{E} := \Gamma(\mathcal{Z}_K \times A_K(I), E)$. Then A is an integral domain and $\mathbf{E}$ is a finitely generated A-module. Let $\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_\mu)$ be a basis of $\mathrm{Frac}\, A \otimes_A \mathbf{E}$ as $\mathrm{Frac}\, A$-vector space and let $(\mathbf{f}_1, ..., \mathbf{f}_\nu)$ be a set of generator of $\mathbf{E}$ as A-module. Then there exist $b_{ij} := b'_{ij}/b''_{ij}, c_{ji} := c'_{ji}/c''_{ji} \in \mathrm{Frac}\, A\,(1 \leq i \leq \mu, 1 \leq j \leq \nu)$ such that $\mathbf{e}_i = \sum_{j=1}^{\nu} b_{ij}\mathbf{f}_j\,(\forall i), \mathbf{f}_j = \sum_{i=1}^{\mu} c_{ji}\mathbf{e}_i\,(\forall j)$. By Lemma 2.8, there exist an open dense sub affine formal scheme $\mathcal{V} \subseteq \mathcal{Z}$ and closed aligned subinterval $I' \subseteq I$ of positive length such that $b''_{ij}, c''_{ji} \in \Gamma(\mathcal{V}_K \times A^1_K(I'), \mathcal{O}^\times)$. Then we see that $\mathbf{e}$ forms a basis of $\Gamma(\mathcal{V}_K \times A^1_K(I'), E)$ as $\Gamma(\mathcal{V}_K \times A^1_K(I'), \mathcal{O})$-module. Then let us put $V := \mathcal{V} \times_{\mathcal{Z}} Z$ and let $\partial : E \longrightarrow E$ be a morphism on $\mathcal{V}_K \times A^1_K(I')$ defined as the composite

$$E \xrightarrow{\nabla} E \otimes \Omega^1_{\mathcal{V}_K \times A^1_K(I')/\mathcal{V}_K} = Edt \xrightarrow{=} E,$$

where the last map is defined by $xdt \mapsto x$. For $n \in \mathbb{N}$, let G_n be the matrix expression of ∂^n with respect to the basis $\mathbf{e}$ (so we have $G_n \in \mathrm{Mat}_\mu(\Gamma(\mathcal{V}_K \times A^1_K(I'), \mathcal{O})) \subseteq \mathrm{Mat}_\mu(\Gamma(A^1_L(I'), \mathcal{O})))$.

For any separable closed point $x \in V$ and its lift $\widetilde{x}$ in $\mathcal{Z}$, E restricts to a ∇-module $E_{\widetilde{x}}$ on $\widetilde{x}_K \times A^1_K(I')$, ∂ restricts to the morphism $\partial_{\widetilde{x}} : E_{\widetilde{x}} \longrightarrow E_{\widetilde{x}}$ induced from the ∇-module structure of $E_{\widetilde{x}}$ and the matrix $G(\widetilde{x}_K) \in \mathrm{Mat}_\mu(\Gamma(\widetilde{x}_K \times A^1_K(I'), \mathcal{O}))$ gives the matrix expression of $\partial_{\widetilde{x}}$ with respect to the basis $\mathbf{e}$. By definition of the supremum norm and [K1, 3.1.7, 3.1.8], we have the inequality $|G_n(\widetilde{x}_K)|_\rho \leq |G_n|_\rho$ for any $\rho \in I' \cap \Gamma^*$. Hence we have

$$\begin{aligned} \rho \geq R(E_{\widetilde{x}}, \rho) &= \min(\rho, \varliminf_{n\to\infty} |G_n(\widetilde{x}_K)/n!|_\rho^{-1/n}) \\ &\geq \min(\rho, \varliminf_{n\to\infty} |G_n/n!|_\rho^{-1/n}) \overset{(*)}{=} R(E_L, \rho) = \rho, \end{aligned}$$

that is, $R(E_{\widetilde{x}}, \rho) = \rho$. (As for $(*)$, note that the map $\Gamma(\mathcal{Z}_K \times A^1_K(I), \mathcal{O}) \longrightarrow L(t)_\rho$ factors through $\Gamma(\mathcal{V}_K \times A^1_K(I'), \mathcal{O})$.) Since this is true for any $\rho \in I' \cap \Gamma^*$, we see that the highest ramification break of $E_{\widetilde{x}}$ is equal to 0, by Corollary 1.6 (2). So we have proved the desired claim in the case $b = 0$.

Next we treat the case $b > 0$. (The proof in this case partly follows the argument in [K3, 1.3.1].) Let ζ be a primitive p-th root of unity. Let us put $K' := K(\zeta)$, let $O_{K'}$ be the ring of integers of K' and let $\mathcal{Z}_{K'}$ be the rigid space over K' associated to $\mathcal{Z}\widehat{\otimes}_{O_K} O_{K'}$. Also, let us take an inclusion $\Gamma(\mathcal{Z}_{K'}, \mathcal{O}) \hookrightarrow L'$, where L' is a field complete with respect to a norm which restricts to the supremum norm on $\mathcal{Z}_{K'}$. Since the supremum norm on $\mathcal{Z}_{K'}$ is compatible with that on $\mathcal{Z}_K$, the restriction of the norm on L' to $\Gamma(\mathcal{Z}_K, \mathcal{O})$ gives the supremum norm on $\mathcal{Z}_K$. Hence, by Remark 2.7, the ∇-module $E_{L'}$ on $A_{L'}[\lambda, 1)$ induced by E has solvable and it has highest ramification break b. So we may assume that $R(E_{L'}, \rho) = \rho^{b+1}$ for any $\rho \in [\lambda, 1)$. Then there exists a closed aligned subinterval $I \subseteq [\lambda, 1)$ of positive length such that there exists a positive integer m with

$$p^{-1/p^{m-1}(p-1)}\rho < R(E_{L'}, \rho) < p^{-1/p^m(p-1)}\rho \quad (\forall \rho \in I).$$

Let F be a ∇-module on $\mathcal{Z}_{K'} \times A^1_{K'}(I^{p^m})$ with $\varphi^{m*}F = E_{K'}$ (where $E_{K'}$ denotes the restriction of E to $\mathcal{Z}_{K'} \times A^1_{K'}(I)$ and φ denotes the morphism $\mathcal{Z}_{K'} \times A^1_{K'}(I) \longrightarrow \mathcal{Z}_{K'} \times A^1_{K'}(I^p)$ over $\mathcal{Z}_{K'}$ defined by $t \mapsto t^p$) such that the induced ∇-module $F_{L'}$ on $A^1_{L'}(I^{p^m})$ is the m-fold Frobenius antecedent of $E_{L'}$. (The existence of such F is assured by Proposition 1.10.) Then we have $R(F_{L'}, \rho^{p^m}) = R(E_{L'}, \rho)^{p^m}$. Hence we have the inequality $p^{-p/(p-1)}\rho^{p^m} < R(F_{L'}, \rho^{p^m}) < p^{-1/(p-1)}\rho^{p^m}$ $(\rho \in I)$. This implies the inequality

$$|\partial|_{F_{L',\rho^{p^m}},\mathrm{sp}} > \rho^{-p^m} = |\partial|_{L'(t)_{\rho^{p^m}}}. \tag{2.4}$$

Let us put $A := \Gamma(\mathcal{Z}_{K'} \times A^1_{K'}(I^{p^m}), \mathcal{O})$, $\mathbf{F} := \Gamma(\mathcal{Z}_{K'} \times A_{K'}(I), F)$. Then we can define $\partial : F \longrightarrow F$ on $\mathcal{Z}_{K'} \times A^1_{K'}(I^{p^m})$ as before and it induces the map $\partial : \mathbf{F} \longrightarrow \mathbf{F}$. Let $\mathbf{v}$ be a cyclic vector of $\mathrm{Frac}\, A \otimes_A \mathbf{F}$, put $\partial^\mu(\mathbf{v}) = \sum_{i=0}^{\mu-1} a_i \partial^1(\mathbf{v})$ (where μ is the rank of F) with $a_i = a'_i/a''_i \in \mathrm{Frac}\, A$ and let $(\mathbf{f}_1, ..., \mathbf{f}_\nu)$ be a set of generators of $\mathbf{F}$ as A-module. Then there exist $b_{ij} := b'_{ij}/b''_{ij}, c_{ji} := c'_{ji}/c''_{ji} \in \mathrm{Frac}\, A$ $(0 \le i \le \mu-1, 1 \le j \le \nu)$ such that $\partial^i(\mathbf{v}) = \sum_{j=1}^{\nu} b_{ij}\mathbf{f}_j\ (\forall i), \mathbf{f}_j = \sum_{i=0}^{\mu-1} c_{ji}\partial^i(\mathbf{v})\ (\forall j)$. By Lemma 2.8, there exists an open dense sub affine formal scheme $\mathcal{V} \subseteq \mathcal{Z}$ and a closed aligned subinterval $I' \subseteq I$ of positive length such that $a''_i, b''_{ij}, c''_{ji} \in \Gamma(\mathcal{V}_{K'} \times A^1_{K'}(I'^{p^m}), \mathcal{O}^\times)$ (where $\mathcal{V}_{K'}$ is defined in the same way as $\mathcal{Z}_{K'}$, by using $\mathcal{V}$ instead of $\mathcal{Z}$). Then we see that $\mathbf{v}, \partial(\mathbf{v}), ..., \partial^{\mu-1}(\mathbf{v})$ form a basis of $\Gamma(\mathcal{V}_{K'} \times A^1_{K'}(I'^{p^m}), F)$ as $\Gamma(\mathcal{V}_{K'} \times A^1_{K'}(I'^{p^m}), \mathcal{O})$-module. By shrinking $\mathcal{V}$ and I' further and using Lemma 2.8 again, we may assume that $|a_i(u)|_{\rho^{p^m}} = |a_i|_{\rho^{p^m}}$ for any $u \in \mathcal{V}_{K'}$ and $\rho \in I'$.

Now let us put $V := \mathcal{V} \times_{\mathcal{Z}} Z$, take any separable closed point $x \in V$, its lift $\widetilde{x}$ in $\mathcal{V}$, a point $\widetilde{x}_{K'} \in \mathcal{V}_{K'}$ which lies above $\widetilde{x}_K \in \mathcal{V}_K$ and any $\rho \in I'$. Let r be the least slope of the lower convex hull of the set $\{(-i, \log|a_i|_{\rho^{p^m}}) \,|\, 0 \le i \le \mu - 1\} = \{(-i, \log|a_i(\widetilde{x}_{K'})|_{\rho^{p^m}}) \,|\, 0 \le i \le \mu - 1\} \subseteq \mathbb{R}^2$. Then we have

$$\max(|\partial|_{F_{L',\rho^{p^m}},\mathrm{sp}}, |\partial|_{L'(t)_{\rho^{p^m}}}) = \max(|\partial|_{F_{\widetilde{x},\rho^{p^m}},\mathrm{sp}}, |\partial|_{K'(t)_{\rho^{p^m}}}) = e^{-r}$$

(where $F_{\widetilde{x}}$ is the ∇-module on $\widetilde{x}_{K'} \times A^1_{K'}(I'^{p^m})$ induced by F) and by (2.4), this implies the equality $|\partial|_{F_{L',\rho^{p^m}},\mathrm{sp}} = e^{-r} = |\partial|_{F_{\widetilde{x},\rho^{p^m}},\mathrm{sp}}$. Hence we have $R(F_{L'}, \rho^{p^m}) = R(F_{\widetilde{x}}, \rho^{p^m})$. In particular, we have $R(F_{\widetilde{x}}, \rho^{p^m}) > p^{-p/(p-1)}\rho^{p^m}$. Hence $F_{\widetilde{x}}$ is the m-fold Frobenius antecedent of $\varphi^{m*}F_{\widetilde{x}} = E_{\widetilde{x},K'}$, where $E_{\widetilde{x},K'}$ is the restriction of $E_{\widetilde{x}}$ to $\widetilde{x}_{K'} \times A^1_{K'}(I')$. Noting the equality $R(E_{\widetilde{x}}, \rho) = R(E_{\widetilde{x},K'}, \rho)$, we obtain the equalities

$$R(E_{\widetilde{x}}, \rho) = R(E_{\widetilde{x},K'}, \rho) = R(F_{\widetilde{x}}, \rho^{p^m})^{p^{-m}} = R(F_{L'}, \rho^{p^m})^{p^{-m}} = R(E_{L'}, \rho) = \rho^{b+1}.$$

Since this is true for any $\rho \in I'$, we can conclude that the highest ramification break of $E_{\widetilde{x}}$ is b by Corollary 1.6 (2). So we have finished the proof of the theorem. $\square$

Even if we do not assume the solvability assumption $(*)$, we still have the following criterion for solvability and highest ramification break:

Theorem 2.9. *Let the notations be as in the beginning of this subsection and assume that k is uncountable. Then the following are equivalent*:

(1) *E_L is solvable with highest ramification break b (resp. not solvable).*

(2) *There exists a decreasing sequence of dense open subsets $\{U_i\}_{i\in\mathbb{N}}$ in Y such that, for any separable closed point y in $\bigcap_{i\in\mathbb{N}} U_i$ and its lift $\widetilde{y}$ in $\mathcal{Y}$, $E_{\widetilde{y}}$ has highest ramification break b (resp. not solvable) on any connected component of $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$.*

Until the end of this subsection, we will sometimes say by abuse of terminology that E_L or $E_{\widetilde{y}}$ has highest ramification break $+\infty$ if it is not solvable.

Proof. The proof is similar to the proof of Theorem 2.5. First we prove $(2) \Longrightarrow (1)$, assuming $(1) \Longrightarrow (2)$. Assume (2) and assume that E_L has highest ramification break b'. Then, since we assumed the implication $(1) \Longrightarrow (2)$, there exists a decreasing sequence $\{U'_i\}_{i\in\mathbb{N}}$ of dense open subsets in Y such that, for any separable closed point y in $\bigcap_{i\in\mathbb{N}} U'$ and its lift $\widetilde{y}$ in $\mathcal{Y}$, $E_{\widetilde{y}}$ has highest ramification break b'. Since $\bigcap_{i\in\mathbb{N}} U \cap \bigcap_{i\in\mathbb{N}} U'$ contains a separable closed point (because k is uncountable), this implies the equality $b = b'$ as desired. So it suffices to prove $(1) \Longrightarrow (2)$.

Moreover, to prove the assertion, it suffices to prove that there exists a decreasing sequence $\{V_i\}_{i\in\mathbb{N}}$ of open dense subschemes in Z such that, for any separable closed point x and its lift $\widetilde{x}$ in $\mathcal{Z}$, the restriction $E_{\widetilde{x}}$ of E to $\widetilde{x}_K \times A^1_K[\lambda, 1)$ has highest ramification break b. Indeed, if this is true, we have the assertion if we put $U_i := Y - \overline{\phi(Z - V_i)}$. So we will prove this claim.

Let us take a closed aligned intervals $I_m = [\alpha_m, \beta_m] \subseteq [\lambda, 1)\,(m \in \mathbb{N})$ such that $\alpha_m < \beta_m < \alpha_{m+1}\,(\forall m)$ and that $\lim_{m\to\infty} \alpha_m = 1$. Then, by the argument of the proof of Theorem 2.5 (in the case $b = 0$), we see that, for each m, there exists a closed aligned subinterval $I'_m \subseteq I_m$ of positive length and an open dense sub affine formal scheme $\mathcal{V}'_m \subseteq \mathcal{Z}$ such that E admits a free basis $\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_\mu)$ on $\mathcal{V}'_{m,K} \times A^1_K(I'_m)$. Then, for each m, we can define the matrices $G_{m,n} \in \mathrm{Mat}_\mu(\Gamma(\mathcal{V}'_{m,K} \times A^1_K(I'_m)))\,(n \in \mathbb{N})$ as in the proof of Theorem 2.5, by using the basis $\mathbf{e}$. Let us put $V'_m := \mathcal{V}'_m \times_{\mathcal{Z}} Z$. Then there exists a decreasing sequence $\{V'_{m,n}\}_{n\in\mathbb{N}}$ of dense open subschemes in V'_m and a decreasing sequence $\{I'_{m,n}\}_{n\in\mathbb{N}}$ of closed aligned subintervals of I'_m of positive length such that, for any separable closed point x in $V'_{m,n}$, any lift $\widetilde{x}$ of x in $\mathcal{Z}$, any $n' \le n$ and any $\rho \in I'_{m,n}$, we have the equality $|G_{m,n'}(\widetilde{x}_K)|_\rho = |G_{m,n'}|_\rho$. Now, for each m, fix $\rho_m \in \bigcap_{n\in\mathbb{N}} I'_{m,n}$ and for $i \in \mathbb{N}$, let us put $V_i := \bigcap_{m+n\le i} V_i$. Then, for any separable closed point x in $\bigcap_{i\in\mathbb{N}} V_i$ and any lift $\widetilde{x}$ of x in $\mathcal{Z}$, we have the equality $|G_{m,n}(\widetilde{x}_K)|_{\rho_m} = |G_{m,n}|_{\rho_m}$ for any m, n. Hence we have

$$\begin{aligned} R(E_{\widetilde{x}}, \rho_m) &= \min(\rho_m, \varliminf_{n\to\infty} |G_{m,n}(\widetilde{x}_K)/n!|_{\rho_m}^{-1/n}) \\ &= \min(\rho_m, \varliminf_{n\to\infty} |G_{m,n}/n!|_{\rho_m}^{-1/n}) = R(E_L, \rho_m) \end{aligned}$$

for any m. If E_L has highest ramification break b ($0 \le b < +\infty$), we have $R(E_L, \rho_m) = \rho_m^{b+1}$ for m sufficiently large, by Corollary 1.6 (1). So we have $R(E_{\widetilde{x}}, \rho_m) = \rho_m^{b+1}$ for m sufficiently large and hence $E_{\widetilde{x}}$ has also highest ramification break b, again by

Corollary 1.6 (1). If E_L is not solvable, we have $\overline{\lim_{m\to\infty}} R(E_L, \rho_m) < 1$ by Corollary 1.6 (3). So we have $\overline{\lim_{m\to\infty}} R(E_{\widetilde{x}}, \rho_m) < 1$ and hence $E_{\widetilde{x}}$ is not solvable either, again by Corollary 1.6 (3). Hence we have proved the desired assertion and so the proof of the theorem is finished. $\square$

Next we prove a kind of cut-by-curves criterion on exponent.

Theorem 2.10. *Let the situation be as in the beginning of this subsection and assume moreover that k is uncountable. Then the following are equivalent*:

(1) *E_L is solvable with highest ramification break* 0 *and* $\mathrm{Exp}(E_L) = C$.

(2) *There exists a decreasing sequence of dense open subsets $\{U_i\}_{i\in\mathbb{N}}$ in Y such that, for any separable closed point y in $\bigcap_{i\in\mathbb{N}} U_i$ and its lift $\widetilde{y}$ in $\mathcal{Y}$, $E_{\widetilde{y}}$ is solvable with highest ramification break* 0 *and* $\mathrm{Exp}(E_{\widetilde{y}}) = C$ *(on any connected component of $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$.)*

Proof. The strategy of the proof is similar to that of Theorems 2.5 and 2.9. First we prove (2) $\Longrightarrow$ (1), assuming (1) $\Longrightarrow$ (2). Assume (2) and assume that E_L has highest ramification break b' (possibly $+\infty$). Then, by Theorem 2.9, there exists a decreasing sequence $\{U'_i\}_{i\in\mathbb{N}}$ of dense open subsets in Y such that, for any separable closed point y in $\bigcap_{i\in\mathbb{N}} U'_i$ and its lift $\widetilde{y}$ in $\mathcal{Y}$, $E_{\widetilde{y}}$ has highest ramification break b'. Since $\bigcap_{i\in\mathbb{N}} U_i \cap \bigcap_{i\in\mathbb{N}} U'_i$ contains a separable closed point (because k is uncountable), this implies the equality $b' = 0$. Then, since we assumed the implication (1) $\Longrightarrow$ (2), there exists a decreasing sequence $\{U''_i\}_{i\in\mathbb{N}}$ of dense open subsets in Y such that, for any separable closed point y in $\bigcap_{i\in\mathbb{N}} U''_i$ and its lift $\widetilde{y}$ in $\mathcal{Y}$, $\mathrm{Exp}(E_{\widetilde{y}}) = \mathrm{Exp}(E_L)$. Since $\bigcap_{i\in\mathbb{N}} U_i \cap \bigcap_{i\in\mathbb{N}} U''_i$ contains a separable closed point, this implies the equality $\mathrm{Exp}(E_L) = C$, as desired. So it suffices to prove (1) $\Longrightarrow$ (2). Moreover, it suffices to prove that there exists a sequence of open dense subschemes $V_i \subseteq Z$ ($i \in \mathbb{N}$) such that, for any separable closed point x in $\bigcap_{i\in\mathbb{N}} V_i$ and its lift $\widetilde{x}$ in $\mathcal{Z}$, the restriction $E_{\widetilde{x}}$ of E to $\widetilde{x}_K \times A^1_K[\lambda, 1)$ has highest ramification break 0 with $\mathrm{Exp}(E_{\widetilde{x}}) = C$. (Indeed, if this is true, we have the assertion if we put $U_i := Y - \overline{\phi(Z - V_i)}$.) So we will prove this claim.

Let K_∞ be the p-adic completion of $K(\mu_{p^\infty})$ and let O_{K_∞} be the valuation ring of K_∞. Take a closed aligned interval $I \subset [\lambda, 1)$ of positive length such that $R(E_L, \rho) = \rho$ for $\rho \in I$. By the argument of the proof of Theorem 2.5, we have the following: If we shrink $\mathcal{Z}$ and I if necessary, E admits a free basis $\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_\mu)$ on $\mathcal{Z}_K \times A^1_K(I)$ and for any separable closed point x in Z and its lift $\widetilde{x}$ in $\mathcal{Z}$, we have the equality $R(E_{\widetilde{x}}, \rho) = \rho$ for all $\rho \in I$. In particular, for $\Delta_h \in (\mathbb{Z}/p^h\mathbb{Z})^\mu$, the matrix functions $Y_{\mathbf{e}}(x, y)$ and $S_{h,\Delta_h}(x)$ associated to E_L constructed in Subsection 1.4 is in fact defined as the matrix functions with coefficients in $\Gamma(\mathcal{Z}_{K_\infty}, \mathcal{O})$, where $\mathcal{Z}_{K_\infty}$ denotes the rigid space over K_∞ associated to $\mathcal{Z}_{O_{K_\infty}} := \mathcal{Z}\widehat{\otimes}_{O_K} O_{K_\infty}$. In particular, we have $\det S_{h,\Delta_h}(x) \in \Gamma(\mathcal{Z}_{K_\infty} \times A^1_{K_\infty}(I), \mathcal{O})$. By Lemma 2.8, there exists a decreasing sequence of open dense sub affine formal schemes $\mathcal{V}'_i \subseteq \mathcal{Z}$ ($i \in \mathbb{N}$) and a decreasing sequence of aligned closed subintervals $I_i \subseteq I$ ($i \in \mathbb{N}$) such that, on any $u \in \mathcal{V}'_{i,K_\infty}$, $\rho \in I_i$ and any $h \leq i, \Delta_h \in (\mathbb{Z}/p^h\mathbb{Z})^\mu$, we have $|\det S_{h,\Delta_h}(u)|_\rho = |\det S_{h,\Delta_h}|_\rho$. Let us put $V'_i := \mathcal{V}'_i \times_{\mathcal{Z}} Z$ and let ρ_0 be any element of $\bigcap_{i\in\mathbb{N}} I_i$. Then, by definition of $\mathrm{Exp}(E_L)$, there exists an element $\Delta \in \mathrm{Exp}(E_L) = C$ such that, if we put $\Delta_h := \Delta \bmod p^h$, we have the inequalities $|\det(S_{h,\Delta_h})|_{\rho_0} \leq |\det(S_{h+1,\Delta_{h+1}})|_{\rho_0}$ for any $h \in \mathbb{N}$. Then, for any separable closed point $x \in \bigcap_{i\in\mathbb{N}} V'_i$, its lift $\widetilde{x}$ in $\mathcal{Z}$ and any point $\widetilde{x}_{K_\infty}$ of $\mathcal{Z}_{K_\infty}$ lying above $\widetilde{x}_K \in \mathcal{Z}_K$, we have $|\det(S_{h,\Delta_h}(\widetilde{x}_{K_\infty}))|_{\rho_0} \leq |\det(S_{h+1,\Delta_{h+1}}(\widetilde{x}_{K_\infty}))|_{\rho_0}$ for any $h \in \mathbb{N}$,. Also, by Theorem 2.9, there exists a decreasing sequence of open dense subschemes $V''_i \subseteq Z$ such that, for any separable closed point x in $\bigcap_{i\in\mathbb{N}} V''_i$ and its lift $\widetilde{x}$ in $\mathcal{Z}$, $E_{\widetilde{x}}$ is solvable with highest ramification break 0. Let us put $V_i := V'_i \cap V''_i$ ($i \in \mathbb{N}$). Then, for any separable closed point x in $\bigcap_{i\in\mathbb{N}} V_i$ and its lift $\widetilde{x}$ in $\mathcal{Z}$, $E_{\widetilde{x}}$ is solvable with highest ramification break 0 and its exponent $\mathrm{Exp}(E_{\widetilde{x}})$ as ∇-module on $\widetilde{x}_K \times A^1_K(I)$ is the class of Δ, which is equal to C. (See the definition of the exponent given in Subsection 1.4.) Hence $\{V_i\}_{i\in\mathbb{N}}$ satisfies the condition we need. □

Remark 2.11. If one is interested only in the proof of Theorem 0.1, Theorem 2.5 is actually unnecessary (although we need some arguments in the proof there). We included it because we think that it is interesting itself and because we think that (some generalization of) this result would be useful in the study of overconvergent isocrystals.

2.3 Proof of the main theorem and a variant

In this subsection, we give a proof of Theorem 0.1 by using the results in the previous subsection, and give also a variant of Theorem 0.1 as a corollary, which treats the case where k is not necessarily uncountable. First we give a proof of Theorem 0.1:

Proof of Theorem 0.1. First we prove (1) $\Longrightarrow$ (2). Since $\mathcal{E}$ is Σ-unipotent by assumption, there exists an isocrystal $\overline{\mathcal{E}}$ on $((\overline{X}, M_{\overline{X}})/O_K)_{\mathrm{conv}}$ with exponents in Σ which extends $\mathcal{E}$, by Theorem 1.23. Then, $\iota^*\overline{\mathcal{E}}$ gives an isocrystal $\overline{\mathcal{E}}$ on $((\overline{C}, M_{\overline{C}})/O_K)_{\mathrm{conv}}$ with exponents in $\iota^*\Sigma$ which extends $\iota^*\mathcal{E}$. Hence $\iota^*\mathcal{E}$ is $\iota^*\Sigma$-unipotent again by Theorem 1.23.

Next we prove (2) $\Longrightarrow$ (1). By definition of Σ-unipotence given in Definition 1.19, it suffices to prove the assertion locally on $\overline{X} - Z_{\mathrm{sing}}$. Hence we may assume that $\overline{X}$ is connected and affine, Z is a non-empty connected smooth divisor in $\overline{X}$ and that $Z \hookrightarrow \overline{X}$ admits a coordinatized tubular neighborhood, that is, we may assume that we are in the situation of Hypothesis 2.4. Then, for each separable closed point y in Y, $\mathcal{E}$ restricts to the overconvergent isocrystal $\mathcal{E}_y$ on $(X_y, \overline{X}_y)$ having Σ-unipotent monodromy. Let $\widetilde{y}$ be any lift of y in $\mathcal{Y}$. By Propositions 1.15, 1.18 and 1.20, the associated ∇-module $E_{\mathcal{E},\widetilde{y}}$ on $\mathcal{Z}_{\widetilde{y},K} \times A^1_K[\lambda, 1)$ is solvable with highest ramification 0 such that $\mathrm{Exp}(E_{\mathcal{E},\widetilde{y}}) \in \overline{\Sigma}$. Then, by Theorem 2.10, the ∇-module $E_{\mathcal{E},L}$ on $A^1_L[\lambda, 1)$ associated to $\mathcal{E}$ is solvable with highest ramification break 0 such that $\mathrm{Exp}(E_{\mathcal{E},L}) \in \overline{\Sigma}$. By Propositions 1.17 and 1.18, it implies that the ∇-module $E_{\mathcal{E}}$ on

$\mathcal{Z}_K \times A^1_K[\lambda, 1)$ associated to $\mathcal{E}$ is Σ-unipotent for some λ. Hence $\mathcal{E}$ is Σ-unipotent, as desired. $\square$

Next we introduce several notations which we need to describe a variant of Theorem 0.1. Let K, O_K, k be as in Convention (with k not necessarily uncountable) and let $X, \overline{X}, Z = \overline{X} - X = \bigcup_{i=1}^r Z_i, M_{\overline{X}}, \Sigma = \prod_{i=1}^r \Sigma_i$ be as in Introduction. For a field extension $k \subseteq k'$, let us put $O_{K'} := O_K \otimes_{W(k)} W(k')$, $K' := \mathrm{Frac}\, O_K$, $X' := X \otimes_k k', \overline{X}' := \overline{X} \otimes_k k'$, $Z' := Z \otimes_k k' = \bigcup_{i=1}^{r'} Z'_i$ and let $M_{\overline{X}'}$ be the log structure on $\overline{X}'$ associated to Z' and denote the projections $(\overline{X}', M_{\overline{X}'}) \longrightarrow (\overline{X}, M_{\overline{X}}), (X', \overline{X}') \longrightarrow (X, \overline{X})$ by π. Then π induces a well-defined morphism of sets $\{1, ..., r\} \longrightarrow \{1, ..., r'\}$ (which we denote also by π) by the rule $\pi(Z'_i) \subseteq Z_{\pi(i)}$. Then we put $\Sigma' := \prod_{i=1}^{r'} \Sigma_{\pi(i)} \subseteq \mathbb{Z}_p^{r'}$.

Assume we are given an open immersion of smooth k'-curves $C' \hookrightarrow \overline{C}'$ such that $P' := \overline{C}' - C' = \coprod_{i=1}^s P'_i$ is a simple normal crossing divisor, and denote the log structure on $\overline{C}'$ assciated to P' by $M_{\overline{C}}$. Assume also that we are given a commutative diagram

$$(2.5)\qquad \begin{array}{ccc} (\overline{C}', M_{\overline{C}'}) & \xrightarrow{\iota} & (\overline{X}, M_{\overline{X}}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k' & \longrightarrow & \mathrm{Spec}\, k \end{array}$$

such that ι induces an exact locally closed immersion $\iota' : (\overline{C}', M_{\overline{C}'}) \hookrightarrow (\overline{X}', M_{\overline{X}'})$. Then we can define $\iota^*\Sigma$ by $\iota^*\Sigma := \iota'^*\Sigma'$, where $\iota'^*\Sigma'$ is 'the pull-back of Σ' by the exact locally closed immersion ι'' defined in Introduction. Also, ι defines a morphism of pairs $(C', \overline{C}') \longrightarrow (X, \overline{X})$ (which we also denote by ι) and hence, for an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$, we can define the pull-back $\iota^*\mathcal{E}$, which is an overconvergent isocrystal on $(C', \overline{C}')/K'$. With this notation, we can state a variant of our main theorem as follows:

Corollary 2.12. *Let $K, k, X, \overline{X}, M_{\overline{X}}, \Sigma$ be as above (k is not necessarily uncountable). Then, for an overconvergent isocrystal $\mathcal{E}$ on $(X, \overline{X})/K$, the following are equivalent*:

(1) *$\mathcal{E}$ has Σ-unipotent monodromy.*

(2) *For any field extension $k \subseteq k'$, for any $(C', \overline{C}'), M_{\overline{C}'}$ over k' as above and for any diagram* (2.5) *as above, $\iota^*\mathcal{E}$ has $\iota^*\Sigma$-unipotent monodromy.*

Proof. The proof of the implication (1) $\Longrightarrow$ (2) is the same as the proof in Theorem 0.1. So we omit it and only give a proof of (2) $\Longrightarrow$ (1). Fix a field extension $k \hookrightarrow k'$ such that k' is uncountable and define $X', \overline{X}', Z' = \bigcup_{i=1}^{r'} Z'_i, M_{\overline{X}'}, \pi : (X', \overline{X}') \longrightarrow (X, \overline{X})$ as above. Then, for $C', \overline{C}', M_{\overline{C}'}$ over k' as above and for any exact locally closed immersion $\iota' : (\overline{C}', M_{\overline{C}'}) \longrightarrow (\overline{X}', M_{\overline{X}'})$, $\iota^*\mathcal{E} = \iota'^*\pi^*\mathcal{E}$ has $\iota^*\Sigma = \iota'^*\Sigma'$-unipotent monodromy (with Σ' defined as above). Hence, by Theorem 0.1, $\pi^*\mathcal{E}$ has

Σ'-unipotent monodromy. So, to prove the corollary, it suffices to prove the following claim:

claim. With the above notation, if $\pi^*\mathcal{E}$ has Σ'-unipotent monodromy, $\mathcal{E}$ has Σ-unipotent monodromy.

We prove the claim. Let us take an affine open covering $\overline{X} - Z_{\mathrm{sing}} = \bigcup_{\alpha \in \Delta} \overline{U}_\alpha$, put $U_\alpha := X \cap \overline{U}_\alpha$ and take charted smooth standard small frames $((U_\alpha, \overline{U}_\alpha, \mathcal{P}_\alpha, i_\alpha, j_\alpha), t_\alpha)$ enclosing $(U_\alpha, \overline{U}_\alpha)$ $(\alpha \in \Delta)$. Let us put $\Delta' := \{\alpha \in \Delta \,|\, Z \cap \overline{U}_\alpha \neq \emptyset\}$. For $\alpha \in \Delta'$, let $\mathcal{Q}_\alpha \subseteq \mathcal{P}_\alpha$ be the zero locus of t_α, denote the ∇-module on $\mathcal{Q}_{\alpha,K} \times A^1_K[\lambda, 1)$ (for some λ) induced by $\mathcal{E}$ by $E_{\mathcal{E},\alpha}$ and let a_α be the unique index satisfying $\overline{U}_\alpha \cap Z \subseteq Z_{a_\alpha}$. Let $((U'_\alpha, \overline{U}'_\alpha, \mathcal{P}'_\alpha, i'_\alpha, j'_\alpha), t'_\alpha), \mathcal{Q}'_\alpha$ be the base change of $((U_\alpha, \overline{U}_\alpha, \mathcal{P}_\alpha, i_\alpha, j_\alpha), t_\alpha), \mathcal{Q}_\alpha$ by $\mathrm{Spf}\, O_{K'} \longrightarrow \mathrm{Spf}\, O_K$ respectively, let $\mathcal{P}''_\alpha \subseteq \mathcal{P}'_\alpha$ be a non-empty affine open formal subscheme such that $\mathcal{Q}''_\alpha := \mathcal{Q}'_\alpha \cap \mathcal{P}''_\alpha$ is irreducible and let us put

$$((U''_\alpha, \overline{U}''_\alpha, \mathcal{P}''_\alpha, i''_\alpha, j''_\alpha), t''_\alpha) := ((U'_\alpha \cap \mathcal{P}''_\alpha, \overline{U}'_\alpha \cap \mathcal{P}''_\alpha, \mathcal{P}''_\alpha, i'_\alpha|_{\overline{U}''_\alpha}, j'_\alpha|_{U''_\alpha}), t'_\alpha|_{\mathcal{P}''_\alpha}).$$

Let $E'_{\mathcal{E},\alpha}$ be the restriction of $E_{\mathcal{E},\alpha}$ to $\mathcal{Q}''_{\alpha,K} \times A^1_K[\lambda, 1)$ (which is nothing but the ∇-module induced by $\pi^*\mathcal{E}$) and let a'_α be the unique index satisfying $\overline{U}''_\alpha \cap Z' \subseteq Z'_{a'_\alpha}$. (Then we have $\pi(a'_\alpha) = a_\alpha$.) Let us take an injection $\Gamma(\mathcal{Q}''_{\alpha,K}, \mathcal{O}) \hookrightarrow L$ into a field complete with respect to a norm which restricts to the supremum norm on $\mathcal{Q}''_{\alpha,K}$. Then the composite

$$\Gamma(\mathcal{Q}_{\alpha,K}, \mathcal{O}) \longrightarrow \Gamma(\mathcal{Q}'_{\alpha,K}, \mathcal{O}) \longrightarrow \Gamma(\mathcal{Q}''_{\alpha,K}, \mathcal{O}) \hookrightarrow L$$

is also an injection such that the norm on L restricts to the supremum norm. Now let us assume that $\pi^*\mathcal{E}$ is Σ'-unipotent. Then, by Proposition 1.20, $E'_{\mathcal{E},\alpha}$ is $\Sigma_{\pi(a'_\alpha)} = \Sigma_{a_\alpha}$-unipotent. So the restriction of $E'_{\mathcal{E},\alpha}$ to $A^1_L[\lambda, 1)$ is Σ_{a_α}-unipotent. In other words, the restriction of $E_{\mathcal{E},\alpha}$ to $A^1_L[\lambda, 1)$ is Σ_{a_α}-unipotent. Then, by Proposition 1.17, $E_{\mathcal{E},\alpha}$ is Σ_{a_α}-unipotent on $\mathcal{Q}_{\alpha,K} \times A^1_K[\lambda', 1)$ for $\lambda' \in (\lambda, 1) \cap \Gamma^*$. Since this is true for any α, we can conclude that $\mathcal{E}$ has Σ-unipotent monodromy, as desired. So we are done. □

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