

**An Information-Based Framework
for Asset Pricing:
X-Factor Theory and its Applications**

by

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Submitted to the University of London
for the degree of
Doctor of Philosophy

24 October 2006

Abstract

An Information-Based Framework for Asset Pricing: X -Factor Theory and its Applications. This thesis presents a new framework for asset pricing based on modelling the information available to market participants. Each asset is characterised by the cash flows it generates. Each cash flow is expressed as a function of one or more independent random variables called market factors or “ X -factors”. Each X -factor is associated with a “market information process”, the values of which become available to market participants. In addition to true information about the X -factor, the information process contains an independent “noise” term modelled here by a Brownian bridge. The information process thus gives partial information about the X -factor, and the value of the market factor is only revealed at the termination of the process. The market filtration is assumed to be generated by the information processes associated with the X -factors. The price of an asset is given by the risk-neutral expectation of the sum of the discounted cash flows, conditional on the information available from the filtration. The thesis develops the theory in some detail, with a variety of applications to credit risk management, share prices, interest rates, and inflation. A number of new exactly solvable models are obtained for the price processes of various types of assets and derivative securities; and a novel mechanism is proposed to account for the dynamics of stochastic volatility and correlation. If the cash flows associated with two or more assets share one or more X -factors in common, then the associated price processes are dynamically correlated in the sense that they share one or more Brownian drivers in common. A discrete-time version of the information-based framework is also developed, and is used to construct a new class of models for the real and nominal interest rate term structures, and the dynamics of the associated price index.

Acknowledgements

I am very grateful to Lane P. Hughston, my supervisor, for his support and encouragement, and for introducing me to the great intellectual world of mathematical finance. I would also like to express my gratitude to other members of the financial mathematics group (both present and former) at King's College, including I. Buckley, G. Iori, A. Lökka, M. Pistorius, W. Shaw, and M. Zervos; and also A. L. Bronstein, J. Dear, A. Jack, M. Jeannin, Z. Jiang, T. Johnson, A. Mehri, O. Precup, and A. Rafailidis; as well as G. Di Graziano, A. Elizalde, S. Galiani, G. Hałaj, M. Hinnerich, and R. Suzuki. I have benefitted from interactions with many members of the London mathematical finance community at Birkbeck College, Imperial College, and the London School of Economics, both students and faculty members—too many to name in full; but I would like to thank in particular my co-author D. Brody, and also P. Barrieu, D. Becherer, M. Davis, and L. Foldes. I have been lucky to have come to know many other members of the international mathematical finance community, and I mention T. Bielecki, M. Bruche, H. Bühlmann, P. Carr, R. Cont, F. Delbaen, P. Embrechts, W. Farkas, B. Flesaker, H. Föllmer, M. Grasselli, V. Henderson, A. Hirska, D. Hobson, H. Hulley, T. Hurd, P. Imkeller, F. Jamshidian, R. Jarrow, M. Jeanblanc, I. Karatzas, H. Korezlioglu, D. Madan, A. McNeil, F. Mercurio, M. Monoyios, D. Ocone, T. Ohmoto, M. Owen, R. Repullo, L. C. G. Rogers, W. Runggaldier, M. Rutkowski, S. Shreve, M. Schweizer, P. Schönbucher, P. Spreij, L. Stettner, D. Taylor, S. Turnbull, A. Wiese, and M. Wüthrich, where helpful remarks and critical observations have influenced this work, and also especially T. Björk and H. Geman. I would as well like to thank D. Berger, R. Curry, P. Friedlos, A. Gfeller, and I. Kouletsis. I acknowledge financial support from EPSRC, Erziehungsdirektion des Kantons Bern, Switzerland, and from the UK ORS scheme. Finally, as ever, I express my thanks to my father, my mother, and my sister for their unflagging support. Mamma, Papà, Alessia, questi studi sono stati un'impresa di famiglia. L'unione è sempre stata la nostra forza. Grazie.

The work presented in this thesis is my own.

Andrea Macrina

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Chapter 1

Introduction and summary

When confronted with the task of developing a model for asset pricing one soon faces questions of the following type: “Which asset classes are going to be considered?”, “How does one distinguish between the various asset classes?”, “What types of risk are involved?”, “In what ways do the various assets depend upon one another?”, “In what ways do the various assets depend upon the state of the economy?”, and so on. All these questions have to be kept in mind as one considers what features should be incorporated into the mathematical design of an asset pricing model. One general characteristic that we would like to include in such models is flexibility, to ensure that a suitable variety of market features can be captured. We also want tractability and efficiency, to enable us to give precise answers even when complex asset structures are being analysed. We need to give special consideration to the relation between the level of sophistication of the modelling framework and the fundamental requirements of transparency and simplicity. The modelling framework has to offer a degree of sophistication sufficiently high to ensure realistic arbitrage-free prices and risk-management policies, even for complex structures. Transparency and simplicity, on the other hand, are needed to guarantee the consistency and integrity of the models being developed, and that the results obtained are based on sound mathematical foundations. To achieve a significant element of success in satisfying these often conflicting requirements is a serious challenge for any modelling framework.

Our view in what follows will be that asset pricing models should be constructed in such a way that attention is focussed on the cash flows generated by the assets under consideration, and on the economic variables that determine these cash flows.

In particular, the approach pursued in this thesis is based on the analysis of the information regarding market factors available to market participants. We place special emphasis on the construction of the market filtration. This point of view can be contrasted with what is perhaps the more common modelling approach in mathematical finance, where the market filtration is simply “given”. For example, in many studies the market consists of a number of assets for which the associated price processes are driven, collectively, by a multi-dimensional Brownian motion. The underlying filtered probability space is then merely assumed to have a sufficiently rich structure to support this multi-dimensional Brownian motion. Alternatively, the filtration is often assumed to be that generated by the multi-dimensional Brownian motion—as described, for example, in Karatzas & Shreve 1998. However no deeper “economic foundation” for the construction of the filtration is offered. More generally, the system of asset price processes is sometimes assumed to be a collection of semi-martingales with various specified properties; but again, typically, little is said about the relevant filtration apart from the general requirement that the various asset price processes should be adapted to it. In the present approach, on the other hand, we model the filtration explicitly in terms of the information available to the market.

The contents of the thesis are adapted in part from the following research papers: (i) Brody, Hughston & Macrina (2007), henceforth BHM1; (ii) Brody, Hughston & Macrina (2006), henceforth BHM2; and (iii) Hughston & Macrina (2006), henceforth HM. In particular, Chapters 2, 3, 4, 5, and 6, in which the details of the information-based framework are developed and applied to a number of examples involving credit and equity related products, contain research associated with BHM1 and BHM2. In addition, a number of further results are presented in these chapters, including the material concerning the information-based Arrow-Debreu technique appearing in Sections 3.4, 6.9, and 6.10, the material on the Black-Scholes model in Section 6.4, the material on correlated cash flows in Section 6.6, and the material on the reduction theory for dependent cash flows appearing in Sections 6.7 and 6.8. The results presented in Chapter 7, concerning interest rates and inflation, are adapted from HM.

In the greater part of the thesis we concentrate on the continuous-time formulation of the information-based framework and the development of the associated X -factor theory. We work, in general, in an incomplete-market setting with no arbitrage, and we assume the existence of a fixed pricing kernel (or, equivalently, the existence of a fixed

pricing measure \mathbb{Q}). Explained in a nutshell, the information-based approach can be summarised as follows: First we identify the random cash flows occurring at the pre-specified dates pertinent to the particular asset or group of assets under consideration. Then we analyse the structure of the cash flows in more detail by introducing an appropriate set of X -factors, which are assumed to be independent of one another in an appropriate choice of measure, typically the preferred pricing measure, e.g., the risk-neutral measure. To each X -factor we associate an information process that consists of two terms: a signal component, and a noise component. The signal term embodies “genuine” information about the possible outcomes of the X -factor; while the “noise” component is modelled here by a Brownian bridge process, and plays the role of market speculation, inuendo, gossip, and so on. The signal term and the Brownian bridge term are assumed to be independent; in particular, the Brownian bridge term carries no useful information about the value of the relevant market factor. We assume that the information processes collectively generate the market filtration. The price of an asset is calculated by use of the standard risk-neutral valuation formula; that is to say, the asset price is given by the sum of discounted expected future cash flows, conditional on the information supplied by the market filtration.

Chapter 2 begins with a simple model for credit-risky zero-coupon bonds, where we have a single cash flow at the bond maturity. The cash flow is modelled first by a binary random variable. Then in Section 2.3 we derive the price process of a defaultable discount bond in the case for which the cash flow is modelled by a random variable with a more general discrete spectrum. This allows for a random recovery in the case of default. Here default is defined in general as a failure to fully honour a required payment, and hence as a cash flow of less than the contracted value. In this section we also explore the properties of the particular chosen form for the information process, showing that it satisfies the Markov property. This in turn facilitates the calculation of the price process of a defaultable bond, because the expectation involved need only be conditional on the current value of the information process. We are able to obtain a closed-form expression for the price process of the bond. We then proceed to analyse the dynamics of the price process of the bond and find that the driver is given by a Brownian motion that is adapted to the filtration generated by the market information process. This construction indicates the sense in which the price process of an asset can be viewed as an “emergent” phenomenon in the present framework.

Since the resulting bond price at each moment of time is given by a function of the value of the information process at that time, we are able to show in Section 2.7 that simulations for the bond price trajectories are straightforward to generate by simulating paths of the information process. The resulting plots describe various scenarios, ranging from unexpected default events to announced declines. In particular, the figures provide a means to understand better the effects on asset prices due to large or small values of the information flow rate parameter. This parameter measures the rate at which “genuine” information is leaked into the market. We show in Section 2.6 that the model presented is in a certain sense invariant in its overall form when it is updated or re-calibrated. We call this property “dynamic consistency”. In deriving these results we make use of some special orthogonality properties of the Brownian bridge process (Yor 1992, 1996), which also come into play in our analysis of the dynamics of option prices.

In Chapter 3 we compute the prices of options on credit-risky discount bonds. In particular, in the case of a defaultable binary discount bond we obtain the price of a call option in analytical form. The striking property of the resulting expression for the option price is that it is very similar to the well-known Black-Scholes price for stock options. In particular, we see that the information flow rate parameter appearing in the definition of the information process plays a role that is in many respects analogous to that of the Black-Scholes volatility parameter.

This correspondence suggests that we should undertake a sensitivity analysis of the option price with respect to different values of the information flow parameter. By analogy with the Black-Scholes greeks, in Section 3.2 we define appropriate expressions for the vega and the delta, and we show that the vega is positive. From this investigation we conclude that bond options can in principle be used to calibrate the model. In Section 3.3 we derive an expression for the price process of a bond option, and conclude that a position in the underlying bond market can serve as a hedge for a position in an associated call option.

In Section 3.4 we present an alternative technique for deriving the price processes of derivatives in the information-based framework. Instead of performing a change of measure, we introduce the concept of information-based Arrow-Debreu securities. This concept, in turn, leads to the notion of a new class of derivatives which we call “information derivatives”. The information-based Arrow-Debreu technique is considered

further, and in greater generality, in Sections 6.9 and 6.10.

Complex credit-linked structures are investigated in Chapter 4, where we begin with the consideration of defaultable coupon bonds. Such bonds are analysed in greater depth in Section 6.5, once the theory of X -factors has been developed to a higher degree of generality. We obtain an exact expression for the price process of a coupon bond, taking advantage of the analytical solution of the conditional expectations involved. In Sections 4.2, 4.3, and 4.4 we consider other complex credit products, such as credit default swaps (CDSs) and baskets of credit-risky bonds. One of the strengths of the X -factor approach is that it allows for a great deal of flexibility in the modelling of the correlation structures involved with complex credit instruments.

We then leave the field of credit risk modelling as such, and step to a more general domain of asset classes. In Chapter 5 we consider cash flows described by continuous random variables. This paves the way for the consideration of equity products. We begin in Sections 5.2 and 5.3 with a discussion on how we should model the cash flows associated with an asset that pays a sequence of dividends—e.g., a stock. After defining the information process appropriate for this type of financial instrument, in Sections 5.4 and 5.5 we derive an analytical formula for the price of a single-dividend-paying asset. We are also able to price European-style options written on assets with a single cash flow. In Section 5.6, pricing formulae are presented for the situation where the random variable associated with the single cash flow has an exponential distribution or, more generally, a gamma distribution.

The extension of this framework to assets generating multiple cash flows is established in Section 6.1. We show that once the relevant cash flows are decomposed in terms of a collection of independent market factors, then a closed-form expression for the asset price associated with a complex cash-flow structure can be obtained. Moreover, by allowing distinct assets to share one or more common market factors in the determination of one or more of their respective cash flows, we obtain a natural correlation structure for the associated asset price processes. This is described in Section 6.6. This method for introducing correlation in asset price movements contrasts with the essentially *ad hoc* approach adopted in most financial modelling. Indeed, both for portfolio risk management and for credit risk management there is a pressing need for a better understanding of correlation modelling, and one of the goals of the present work is to make a new contribution to this line of investigation.

In Section 6.3 we demonstrate that if two or more market factors affect the future cash flows associated with an asset, then the corresponding price process will exhibit unhedgeable stochastic volatility. In particular, once two or more market factors are involved, an option position cannot, in general, be hedged with a position in the underlying. In this framework there is therefore no need to introduce stochastic volatility into the price process on an artificial basis. The X -factor theory makes it possible to investigate the relationships holding between classes of assets that are different in nature from one another, and therefore have different types of risks. In Section 6.4 we show how the standard geometric Brownian motion model for asset price dynamics can be derived in an information-based approach. In this case the relevant X -factor is a Gaussian random variable.

The X -factor approach is developed further in the following sections, where we discuss the issue of how to reduce a set of dependent factors, which we call Z -factors, into a set of independent X -factors. This is carried out in Sections 6.7 and 6.8, where we focus on dependent binary random variables. In this way we illustrate a possible approach to disentangling more general discrete and dependent random factors by providing what we call a reduction algorithm. We conclude Chapter 6 with a further development of the Arrow-Debreu theory introduced in Section 3.4. In Section 6.9 we extend the Arrow-Debreu theory to the case of a continuous X -factor, and in Section 6.10 we work out the price of a bivariate intertemporal Arrow-Debreu security.

Up to this point the discussion has focussed on applications of the information-based framework to credit-risky assets and equity products in a continuous-time setting. In Chapter 7 we develop a framework for the arbitrage-free dynamics of nominal and real interest rates, and the associated price index. The goal of this chapter is the development of a scheme for the pricing and risk management of index-linked securities, in an information-based setting. We begin with a general model for discrete-time asset pricing, with the introduction in Section 7.2 of two axioms. The first axiom establishes the intertemporal relations for dividend-paying assets. The second axiom specifies the existence a positive non-dividend-paying asset with a positive return. Armed with these axioms, we derive the price process of an asset with limited liability, pinning down a transversality condition. If this condition is satisfied, then the value of the dividend-paying asset is dispersed in its dividends in the long run. In Section 7.3 we discuss the relationships that hold between the nominal pricing kernel and the positive-return asset.

Nominal discount bonds are treated in Section 7.4 where we investigate the properties that follow from the theory developed up to this stage in Chapter 7. By choosing a specific form for the pricing kernel, we are in a position to create interest rate models of the “rational” type, including discrete-time models with no immediate analogue in continuous time. We also demonstrate that any nominal interest-rate model consistent with the general scheme presented here admits a discrete-time representation of the Flesaker-Hughston type.

In Section 7.5 we embark on an analysis of the money-market account process. We find that this process is a special case of the positive-return process defined in Axiom B, if the money-market account is defined as a previsible strictly-positive non-dividend-paying asset. This property is embodied in Axiom B*, which can be used as an alternative basis of the theory, and in Proposition 7.4.1 we show that the original two Axioms A and B proposed in Section 7.2 imply Axiom B*. Returning to the information-based framework, we propose in Section 7.6 to discretise the information processes associated with the X -factors, and construct in this way the filtration to which the pricing kernel is adapted. The fact that we can generate explicit models for the pricing kernel enables us to build explicit models for the discount bond price process and for the associated money-market account process described in Section 7.5.

The resulting nominal interest rate system is then embedded in a wider system incorporating macroeconomic factors relating to the money supply, aggregate consumption, and price level. In Section 7.7 we consider a representative agent who obtains utility from real consumption and from the real liquidity benefit of the money supply. We then calculate the maximised expected utility of aggregate consumption and money supply liquidity benefit, subject to a budget constraint, where utility is discounted making use of a psychological discount factor. The fact that the utility depends on the real liquidity benefit of the money supply leads to fundamental links between the processes of aggregate consumption, money supply, price level, and the nominal liquidity benefit. The link between the nominal pricing kernel and the money supply gives rise in a natural way both to inflationary and deflationary scenarios. Using the same formulae we are then in a position to price index-linked securities, and to consider the pricing and risk management of inflation derivatives.

In concluding this introduction we make a few remarks to help the reader place the material of this thesis in the broader context of finance theory. The majority of the work

that has been carried out in mathematical finance, as represented in standard textbooks (e.g., Baxter & Rennie 1996, Bielecki & Rutkowski 2002, Björk 2004, Duffie 2001, Hunt & Kennedy 2004, Karatzas & Shreve 1998, Musiela & Rutkowski 2004) operates at what one might call a “macroscopic” level. That is to say, the price processes of the “basic” assets are regarded as “given”, and no special attempts are made to “derive” these processes from deeper principles. Instead, the emphasis is typically placed on the valuation of derivatives, and various types of optimization problems. The standard Black-Scholes theory is, for example, in this sense a “macroscopic” theory.

On the other hand, there is also a well-developed literature on so-called market microstructure which investigates how prices are formed in markets (see, e.g., O’Hara 1995 and references cited therein). We can regard this part of finance theory as referring to the “microscopic” level of the subject. Broadly speaking, models concerning interacting agents, heterogeneous preferences, asymmetric information, bid-ask spreads, statistical arbitrage, and insider trading can be regarded as operating to a significant extent at the “microscopic” level.

The work in this thesis can be situated in between the “macroscopic” and “microscopic” levels, and therefore might appropriately be called “mesoscopic”. Thus at the “mesoscopic” level we emphasize the modelling of cash flows and market information; but we assume homogeneous preferences, and symmetric information. Thus, there is a universal market filtration modelling the development of available information; but we construct the filtration, rather than taking it as given. Similarly, an asset will have a well-defined price process across the whole market; but we construct the price process, rather than taking it as given. It should be emphasized nevertheless that the “output” of our essentially mesoscopic analysis is a family of macroscopic models; thus to that extent these two levels of analysis are entirely compatible (see, e.g., Section 6.4).

Chapter 2

Credit-risky discount bonds

2.1 The need for an information-based approach for credit-risk modelling

Models for credit risk management and, in particular, for the pricing of credit derivatives are usually classified into two types: structural models and reduced-form models. For some general overviews of these approaches—see, e.g., Jeanblanc & Rutkowski 2000, Hughston & Turnbull 2001, Bielecki & Rutkowski 2002, Duffie & Singleton 2003, Schönbucher 2003, Bielecki *et al.* 2004, Giesecke 2004, Lando 2004, and Elizalde 2005.

There are differences of opinion in the literature as to the relative merits of the structural and reduced-form methodologies. Both approaches have strengths, but there are also shortcomings in each case. Structural models attempt to account at some level of detail for the events leading to default—see, e.g., Merton 1974, Black & Cox 1976, Leland & Toft 1996, Hilberink & Rogers 2002, and Hull & White 2004a,b. One of the important problems of the structural approach is its inability to deal effectively with the multiplicity of situations that can lead to failure. For example, default of a sovereign state, corporate default, and credit card default would all require quite different treatments in a structural model. For this reason the structural approach is sometimes viewed as unsatisfactory as a basis for a practical modelling framework.

Reduced-form models are more commonly used in practice on account of their tractability, and on account of the fact that, generally speaking, fewer assumptions are required about the nature of the debt obligations involved and the circumstances that

might lead to default—see, e.g., Jarrow & Turnbull 1995, Lando 1998, Flesaker *et al.* 1994, Duffie *et al.* 1996, Jarrow *et al.* 1997, Madan & Unal 1998, Duffie & Singleton 1999, Hughston & Turnbull 2000, Jarrow & Yu 2001, and Chen & Filipović 2005. By a reduced-form model we mean a model that does not address directly the issue of the “cause” of the default. Most reduced-form models are based on the introduction of a random time of default, where the default time is typically modelled as the time at which the integral of a random intensity process first hits a certain critical level, this level itself being an independent random variable. An unsatisfactory feature of such intensity models is that they do not adequately take into account the fact that defaults are typically associated with a failure in the delivery of a promised cash flow—for example, a missed coupon payment. It is true that sometimes a firm will be forced into declaration of default on a debt obligation, even though no payment has yet been missed; but this will in general often be due to the failure of some other key cash flow that is vital to the firm’s business. Another drawback of the intensity approach is that it is not well adapted to the situation where one wants to model the rise and fall of credit spreads—which can in practice be due in part to changes in the level of investor confidence.

The purpose of this chapter is to introduce a new class of reduced-form credit models in which these problems are addressed. The modelling framework that we develop is broadly speaking in the spirit of the incomplete-information approaches of Kusuoka 1999, Duffie & Lando 2001, Cetin *et al.* 2004, Gieseke 2004, Gieseke & Goldberg 2004, Jarrow & Protter 2004, and Guo *et al.* 2005.

In our approach, no attempt is made as such to bridge the gap between the structural and the intensity-based models. Rather, by abandoning the need for an intensity-based approach we are able to formulate a class of reduced-form models that exhibit a high degree of intuitively natural behaviour.

For simplicity we assume in this chapter that the underlying default-free interest rate system is deterministic. The cash flows of the debt obligation—in the case of a coupon bond, the coupon payments and the principal repayment—are modelled by a collection of random variables, and default will be identified as the event of the first such payment that fails to achieve the terms specified in the contract. We shall assume that partial information about each such cash flow is available at earlier times to market participants. However, the information of the actual value of the cash flow

will be obscured by a Gaussian noise process that is conditioned to vanish once the time of the required cash flow is reached. We proceed under these assumptions to derive an exact expression for the bond price process.

In the case of a defaultable discount bond admitting two possible payouts—e.g., either the full principal, or some partial recovery payment—we shall derive an exact expression for the value of an option on the bond. Remarkably, this turns out to be a formula of the Black-Scholes type. In particular, the parameter σ that governs the rate at which the true value of the impending cash flow is revealed to market participants against the background of the obscuring noise process turns out to play the role of a volatility parameter in the associated option pricing formula; this interpretation is reinforced with the observation that the option price can be shown to be an increasing function of this parameter, as will be shown in Section 3.2.

2.2 Simple model for defaultable discount bonds

The object in this chapter is to build an elementary modelling framework in which matters related to credit are brought to the forefront. Accordingly, we assume that the background default-free interest-rate system is deterministic. This assumption serves the purpose of allowing us to focus attention entirely on credit-related issues; it also allows us to derive explicit expressions for certain types of credit derivative prices. The general philosophy is that we should try to sort out credit-related matters first, before attempting to incorporate stochastic default-free interest rates into the picture.

As a further simplifying feature we take the view that credit events are directly associated with anomalous cash flows. Thus a default (in the sense that we use the term) is not something that happens in the abstract, but rather is associated with the failure of some agreed contractual cash flow to materialise at the required time.

Our theory will be based on modelling the flow of incomplete information to market participants about impending debt obligation payments. As usual, we model the unfolding of chance in the financial markets with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$. The probability measure \mathbb{Q} is understood to be the risk-neutral measure, and the filtration $\{\mathcal{F}_t\}$ is understood to be the market filtration. Thus all asset-price processes and other information-providing processes accessible to market participants are adapted to $\{\mathcal{F}_t\}$. Our framework is, in particular, completely

compatible with the standard arbitrage-free pricing theory as represented, for example, in Björk 2004, or Shiryaev 1999.

The real probability measure does not directly enter into the present discussion. We assume the absence of arbitrage. The First Fundamental Theorem then guarantees the existence of a (not necessarily unique) risk-neutral measure. We assume, however, that the market has chosen a fixed risk-neutral measure \mathbb{Q} for the pricing of all assets and derivatives. We assume further that the default-free discount-bond system, denoted $\{P_{tT}\}_{0 \leq t \leq T < \infty}$, can be written in the form

$$P_{tT} = \frac{P_{0T}}{P_{0t}}, \quad (2.1)$$

where the function $\{P_{0t}\}_{0 \leq t < \infty}$ is assumed to be differentiable and strictly decreasing, and to satisfy $0 < P_{0t} \leq 1$ and $\lim_{t \rightarrow \infty} P_{0t} = 0$. Under these assumptions it follows that if the integrable random variable H_T represents a cash flow occurring at T , then its value H_t at any earlier time t is given by

$$H_t = P_{tT} \mathbb{E}[H_T | \mathcal{F}_t]. \quad (2.2)$$

Now let us consider more specifically the case of a simple credit-risky discount bond that matures at time T to pay a principal of h_1 dollars, if there is no default. In the event of default, the bond pays h_0 dollars, where $h_0 < h_1$. When just two such payoffs are possible we shall call the resulting structure a “binary” discount bond. In the special case given by $h_1 = 1$ and $h_0 = 0$ we call the resulting defaultable debt obligation a “digital” bond.

We shall write p_1 for the probability that the bond will pay h_1 , and p_0 for the probability that the bond will pay h_0 . The probabilities here are risk-neutral, and hence build in any risk adjustments required in expectations needed in order to deduce appropriate prices. Thus if we write B_{0T} for the price at time 0 of the credit-risky discount bond then we have

$$B_{0T} = P_{0T}(p_1 h_1 + p_0 h_0). \quad (2.3)$$

It follows that, providing we know the market data B_{0T} and P_{0T} , we can infer the *a priori* probabilities p_1 and p_0 . In particular, we obtain

$$p_0 = \frac{1}{h_1 - h_0} \left(h_1 - \frac{B_{0T}}{P_{0T}} \right), \quad p_1 = \frac{1}{h_1 - h_0} \left(\frac{B_{0T}}{P_{0T}} - h_0 \right). \quad (2.4)$$

Given this setup we now proceed to model the bond-price process $\{B_{tT}\}_{0 \leq t \leq T}$. We suppose that the true value of H_T is not fully accessible until time T ; that is, we assume H_T is \mathcal{F}_T -measurable, but not necessarily \mathcal{F}_t -measurable for $t < T$. We shall assume, nevertheless, that *partial* information regarding the value of the principal repayment H_T is available at earlier times. This information will in general be imperfect—one is looking into a crystal ball, so to speak, but the image is cloudy and indistinct. Our model for such cloudy information will be of a simple type that allows for analytic tractability. In particular, we would like to have a model in which information about the true value of the debt repayment steadily increases over the life of the bond, while at the same time the obscuring factors at first increase in magnitude, and then eventually die away just as the bond matures. More precisely, we assume that the following $\{\mathcal{F}_t\}$ -adapted process is accessible to market participants:

$$\xi_t = \sigma H_T t + \beta_{tT}. \quad (2.5)$$

We call $\{\xi_t\}$ a market information process. The process $\{\beta_{tT}\}_{0 \leq t \leq T}$ appearing in the definition of $\{\xi_t\}$ is a standard Brownian bridge on the time interval $[0, T]$. Thus $\{\beta_{tT}\}$ is a Gaussian process satisfying $\beta_{0T} = 0$ and $\beta_{TT} = 0$, and such that $\mathbb{E}[\beta_{tT}] = 0$ and

$$\mathbb{E}[\beta_{sT}\beta_{tT}] = \frac{s(T-t)}{T} \quad (2.6)$$

for all s, t satisfying $0 \leq s \leq t \leq T$. We assume that $\{\beta_{tT}\}$ is independent of H_T , and thus represents “purely uninformative” noise. Market participants do not have direct access to $\{\beta_{tT}\}$; that is to say, $\{\beta_{tT}\}$ is not assumed to be adapted to $\{\mathcal{F}_t\}$. We can thus think of $\{\beta_{tT}\}$ as representing the rumour, speculation, misrepresentation, overreaction, and general disinformation often occurring, in one form or another, in connection with financial activity, all of which distort and obscure the information contained in $\{\xi_t\}$ concerning the value of H_T .

Clearly the choice (2.5) can be generalised to include a wider class of models enjoying similar qualitative features. In this thesis we shall primarily consider information processes of the form (2.5) for the sake of definiteness and tractability. Indeed, the ansatz $\{\xi_t\}$ defined by (2.5) has many attractive features, and can be regarded as a convenient “standard” model for an information process.

The motivation for the use of a bridge process to represent the noise is intuitively as follows. We assume that initially all available market information is taken into

account in the determination of the price; in the case of a credit-risky discount bond, the relevant information is embodied in the *a priori* probabilities. After the passage of time, however, new rumours and stories start circulating, and we model this by taking into account that the variance of the Brownian bridge steadily increases for the first half of its trajectory. Eventually, however, the variance drops and falls to zero at the maturity of the bond, when the outcome is realised.

The parameter σ in this model represents the rate at which the true value of H_T is “revealed” as time progresses. Thus, if σ is low, then the value of H_T is effectively hidden until very near the maturity date of the bond; on the other hand, if σ is high, then we can think of H_T as being revealed quickly. A rough measure for the timescale τ_D over which information is revealed is given by

$$\tau_D = \frac{1}{\sigma^2(h_1 - h_0)^2}. \quad (2.7)$$

In particular, if $\tau_D \ll T$, then the value of H_T is typically revealed rather early in the history of the bond, e.g., after the passage of a few multiples of τ_D . In this situation, if default is “destined” to occur, even though the initial value of the bond is high, then this will be signalled by a rapid decline in the value of the bond. On the other hand, if $\tau_D \gg T$, then the value of H_T will only be revealed at the last minute, so to speak, and the default will come as a surprise, for all practical purposes. It is by virtue of this feature of the present modelling framework that the use of inaccessible stopping times can be avoided.

To make a closer inspection of the default timescale we proceed as follows. For simplicity, we assume in our model that the only market information available about H_T at times earlier than T comes from observations of $\{\xi_t\}$. Let us denote by \mathcal{F}_t^ξ the subalgebra of \mathcal{F}_t generated by $\{\xi_s\}_{0 \leq s \leq t}$. Then our simplifying assumption is that

$$\mathbb{E}[H_T | \mathcal{F}_t] = \mathbb{E}[H_T | \mathcal{F}_t^\xi]. \quad (2.8)$$

With this assumption in place, we are now in a position to determine the price-process $\{B_{tT}\}_{0 \leq t \leq T}$ for a credit-risky bond with payout H_T . In particular, we wish to calculate

$$B_{tT} = P_{tT} H_{tT}, \quad (2.9)$$

where H_{tT} is the conditional expectation of the bond payout:

$$H_{tT} = \mathbb{E} \left[H_T \middle| \mathcal{F}_t \right]. \quad (2.10)$$

It turns out that H_{tT} can be worked out explicitly. The result is given by the following expression:

$$H_{tT} = \frac{p_0 h_0 \exp \left[\frac{T}{T-t} \left(\sigma h_0 \xi_t - \frac{1}{2} \sigma^2 h_0^2 t \right) \right] + p_1 h_1 \exp \left[\frac{T}{T-t} \left(\sigma h_1 \xi_t - \frac{1}{2} \sigma^2 h_1^2 t \right) \right]}{p_0 \exp \left[\frac{T}{T-t} \left(\sigma h_0 \xi_t - \frac{1}{2} \sigma^2 h_0^2 t \right) \right] + p_1 \exp \left[\frac{T}{T-t} \left(\sigma h_1 \xi_t - \frac{1}{2} \sigma^2 h_1^2 t \right) \right]}. \quad (2.11)$$

We note, in particular, that there exists a function $H(x, y)$ of two variables such that $H_{tT} = H(\xi_t, t)$. The fact that the process $\{H_{tT}\}$ converges to H_T as t approaches T follows from (2.10) and the fact that H_T is \mathcal{F}_T -measurable. The details of the derivation of the formula (2.11) will be given in the next section.

Since $\{\xi_t\}$ is given by a combination of the random bond payout and an independent Brownian bridge, it is straightforward to simulate trajectories for $\{B_{tT}\}$. Explicit examples of such simulations are presented in Section 2.7.

2.3 Defaultable discount bond price processes

Let us now consider the more general situation where the discount bond pays out the possible values

$$H_T = h_i \quad (2.12)$$

($i = 0, 1, \dots, n$) with *a priori* probabilities

$$\mathbb{Q}[H_T = h_i] = p_i. \quad (2.13)$$

For convenience we assume that

$$h_n > h_{n-1} > \dots > h_1 > h_0. \quad (2.14)$$

The case $n = 1$ corresponds to the binary bond we have just discussed. In the general situation we think of $H_T = h_n$ as the case of no default, and all the other cases as various possible degrees of recovery.

Although we consider, for simplicity, a discrete payout spectrum for H_T , the case of a continuous recovery value in the event of default can be formulated analogously. In that case we assign a fixed *a priori* probability p_1 to the case of no default, and a continuous probability distribution function

$$p_0(x) = \mathbb{Q}[H_T < x] \quad (2.15)$$

for values of x less than or equal to h , satisfying

$$p_1 + p_0(h) = 1. \quad (2.16)$$

Now defining the information process $\{\xi_t\}$ as before by (2.5), we want to find the conditional expectation (2.10). We note, first, that the conditional probability that the credit-risky bond pays out h_i is given by

$$\pi_{it} = \mathbb{Q}(H_T = h_i | \mathcal{F}_t), \quad (2.17)$$

or equivalently,

$$\pi_{it} = \mathbb{E}[\mathbf{1}_{\{H_T = h_i\}} | \mathcal{F}_t]. \quad (2.18)$$

For H_{tT} we can then write

$$H_{tT} = \sum_{i=0}^n h_i \pi_{it}. \quad (2.19)$$

It follows, however, from the Markovian property of $\{\xi_t\}$, which will be established in Proposition 2.3.1 below, that to compute (2.10) it suffices to take the conditional expectation of H_T with respect to the σ -subalgebra generated by the random variable ξ_t alone. Therefore, we have

$$H_{tT} = \mathbb{E}[H_T | \xi_t], \quad (2.20)$$

and also

$$\pi_{it} = \mathbb{Q}(H_T = h_i | \xi_t), \quad (2.21)$$

or equivalently,

$$\pi_{it} = \mathbb{E}[\mathbf{1}_{\{H_T = h_i\}} | \xi_t]. \quad (2.22)$$

Proposition 2.3.1 *The information process $\{\xi_t\}_{0 \leq t \leq T}$ satisfies the Markov property.*

Proof. We need to verify that

$$\mathbb{E}[f(\xi_t) | \mathcal{F}_s^\xi] = \mathbb{E}[f(\xi_t) | \xi_s] \quad (2.23)$$

for all s, t such that $0 \leq s \leq t \leq T$ and any measurable function $f(x)$ with $\sup_x |f(x)| < \infty$. It suffices (see, e.g., Liptser & Shiryaev 2000, theorems 1.11 and 1.12) to show that

$$\mathbb{E}[f(\xi_t) | \xi_s, \xi_{s_1}, \xi_{s_2}, \dots, \xi_{s_k}] = \mathbb{E}[f(\xi_t) | \xi_s] \quad (2.24)$$

for any collection of times $t, s, s_1, s_2, \dots, s_k$ such that

$$T \geq t \geq s \geq s_1 \geq s_2 \geq \dots \geq s_k \geq 0. \quad (2.25)$$

First, we remark that for any times t, s, s_1 satisfying $t \geq s \geq s_1$ the random variables β_{sT} and $\beta_{sT}/s - \beta_{s_1T}/s_1$ have vanishing covariance, and thus are independent. More generally, for $s \geq s_1 \geq s_2 \geq s_3$ the random variables $\beta_{sT}/s - \beta_{s_1T}/s_1$ and $\beta_{s_2T}/s_2 - \beta_{s_3T}/s_3$ are independent. Next, we note that

$$\frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1} = \frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}. \quad (2.26)$$

It follows that

$$\begin{aligned} & \mathbb{E}[f(\xi_t) \mid \xi_s, \xi_{s_1}, \xi_{s_2}, \dots, \xi_{s_k}] \\ &= \mathbb{E}\left[f(\xi_t) \mid \xi_s, \frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1}, \frac{\xi_{s_1}}{s_1} - \frac{\xi_{s_2}}{s_2}, \dots, \frac{\xi_{s_{k-1}}}{s_{k-1}} - \frac{\xi_{s_k}}{s_k}\right] \\ &= \mathbb{E}\left[f(\xi_t) \mid \xi_s, \frac{\beta_{sT}}{s} - \frac{\beta_{s_1T}}{s_1}, \frac{\beta_{s_1T}}{s_1} - \frac{\beta_{s_2T}}{s_2}, \dots, \frac{\beta_{s_{k-1}T}}{s_{k-1}} - \frac{\beta_{s_kT}}{s_k}\right]. \end{aligned} \quad (2.27)$$

However, since ξ_s and ξ_t are independent of $\beta_{sT}/s - \beta_{s_1T}/s_1, \beta_{s_1T}/s_1 - \beta_{s_2T}/s_2, \dots, \beta_{s_{k-1}T}/s_{k-1} - \beta_{s_kT}/s_k$, we see that the desired result follows immediately. \square

Next we observe that the *a priori* probability p_i and the *a posteriori* probability π_{it} at time t are related by the Bayes formula:

$$\mathbb{Q}(H_T = h_i \mid \xi_t) = \frac{p_i \rho(\xi_t \mid H_T = h_i)}{\sum_i p_i \rho(\xi_t \mid H_T = h_i)}. \quad (2.28)$$

Here the conditional density function $\rho(\xi \mid H_T = h_i), \xi \in \mathbb{R}$, for the random variable ξ_t is defined by the relation

$$\mathbb{Q}(\xi_t \leq x \mid H_T = h_i) = \int_{-\infty}^x \rho(\xi \mid H_T = h_i) d\xi, \quad (2.29)$$

and is given more explicitly by

$$\rho(\xi \mid H_T = h_i) = \frac{1}{\sqrt{2\pi t(T-t)/T}} \exp\left(-\frac{1}{2} \frac{(\xi - \sigma h_i t)^2}{t(T-t)/T}\right). \quad (2.30)$$

This expression can be deduced from the fact that conditional on $H_T = h_i$ the random variable ξ_t is normally distributed with mean $\sigma h_i t$ and variance $t(T-t)/T$. As a consequence of (2.28) and (2.30), we find that

$$\pi_{it} = \frac{p_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}. \quad (2.31)$$

It follows then, on account of (2.19), that

$$H_{tT} = \frac{\sum_i p_i h_i \exp \left[\frac{T}{T-t} \left(\sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t \right) \right]}{\sum_i p_i \exp \left[\frac{T}{T-t} \left(\sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t \right) \right]}. \quad (2.32)$$

This is the desired expression for the conditional expectation of the bond payoff. In particular, for the binary case $i = 0, 1$ we recover formula (2.11). The discount-bond price process $\{B_{tT}\}$ is therefore given by (2.9), with H_{tT} defined as in (2.32).

2.4 Defaultable discount bond volatility

In this section we analyse the dynamics of the defaultable bond price process $\{B_{tT}\}$ determined in the previous section. The key relation we need for working out the dynamics of the bond price is that the conditional probability process $\{\pi_{it}\}$ satisfies a stochastic equation of the form

$$d\pi_{it} = \frac{\sigma T}{T-t} (h_i - H_{tT}) \pi_{it} dW_t, \quad (2.33)$$

for $0 \leq t < T$ where H_{tT} is given by equation (2.19), and the process $\{W_t\}_{0 \leq t < T}$, defined by

$$W_t = \xi_t + \int_0^t \frac{1}{T-s} \xi_s ds - \sigma T \int_0^t \frac{1}{T-s} H_{sT} ds, \quad (2.34)$$

is an $\{\mathcal{F}_t\}$ -Brownian motion.

Perhaps the most direct way of obtaining (2.33) and (2.34) is by appeal to the well-known Fujisaki-Kallianpur-Kunita (FKK) filtering theory, the main results of which we summarise below in an abbreviated form, suppressing technicalities (see, e.g., Fujisaki *et al.* 1972, or Liptser & Shiryaev 2000, chapter 8, for a more complete treatment). A probability space is given, with a background filtration $\{\mathcal{G}_t\}$, on which we specify a pair of processes $\{\xi_t\}$ (the ‘‘observed’’ process) and $\{x_t\}$ (the ‘‘unobserved’’ process). We assume that

$$\xi_t = \int_0^t \mu_s ds + Y_t \quad (2.35)$$

and

$$x_t = x_0 + \int_0^t \vartheta_s ds + M_t, \quad (2.36)$$

and that the processes $\{\xi_t\}$, $\{x_t\}$, $\{\mu_t\}$, $\{\vartheta_t\}$, $\{Y_t\}$, and $\{M_t\}$ are $\{\mathcal{G}_t\}$ -adapted. We take $\{Y_t\}$ to be a $\{\mathcal{G}_t\}$ -Brownian motion, and $\{M_t\}$ to be a $\{\mathcal{G}_t\}$ -martingale which, for

simplicity, we assume here to be independent of $\{Y_t\}$. The idea is that the values of $\{\xi_t\}$ are observable, and from this information we wish to obtain information about the unobservable process $\{x_t\}$. Let us write $\{\mathcal{F}_t\}$ for the filtration generated by the observed process, and for any process $\{X_t\}$ let us write $\hat{X}_t = \mathbb{E}[X_t | \mathcal{F}_t]$. Thus \hat{X}_t represents the best estimate of X_t , given the information of the observations up to time t . Then the basic result of the FKK theory is that the dynamics of $\{\hat{x}_t\}$ are given by

$$d\hat{x}_t = \hat{\vartheta}_t dt + [(\widehat{\mu x})_t - \hat{\mu}_t \hat{x}_t] dW_t, \quad (2.37)$$

where the so-called innovations process $\{W_t\}$, given by

$$W_t = \xi_t - \int_0^t \hat{\mu}_s ds, \quad (2.38)$$

turns out to be an $\{\mathcal{F}_t\}$ -Brownian motion.

Now let us see how the FKK theory can be used to obtain the dynamics of $\{\pi_{it}\}$. The link to the FKK theory is established by letting the market information process $\{\xi_t\}_{0 \leq t \leq T}$ be the observed process, and by letting the dynamically constant process $\mathbf{1}\{H_T = h_i\}$ be the unobserved process.

To obtain an appropriate expression for the dynamics of $\{\xi_t\}$ in the form (2.35), we recall (Karatzas & Shreve 1991) that a standard Brownian bridge $\{\beta_{tT}\}_{0 \leq t \leq T}$ satisfies a stochastic differential equation of the form

$$d\beta_{tT} = -\frac{\beta_{tT}}{T-t} dt + dY_t \quad (2.39)$$

for $0 \leq t < T$, where $\{Y_t\}$ is a standard Brownian motion. Then if we set $\xi_t = \sigma H_T t + \beta_{tT}$, a short calculation shows that

$$d\xi_t = \frac{\sigma H_T T - \xi_t}{T-t} dt + dY_t. \quad (2.40)$$

Thus in equation (2.35) we can set

$$\mu_t = \frac{\sigma H_T T - \xi_t}{T-t}, \quad (2.41)$$

and it follows that

$$\hat{\mu}_t = \frac{\sigma H_{tT} T - \xi_t}{T-t}, \quad (2.42)$$

where $H_{tT} = \mathbb{E}[H_T | \mathcal{F}_t]$. Inserting this expression for $\hat{\mu}_t$ into (2.38), we are then led to expression (2.34) for the innovation process.

To work out the dynamics of $\{\pi_{it}\}$ we need expressions for \hat{x}_t and $(\widehat{\mu x})_t$, with $x_t = \mathbf{1}\{H_T = h_i\}$ and μ_t as given by (2.41). Thus $\hat{x}_t = \pi_{it}$, and

$$(\widehat{\mu x})_t = \pi_{it} \frac{\sigma h_i T - \xi_t}{T - t}. \quad (2.43)$$

Inserting the expressions that we have obtained for $\hat{\mu}_t$, \hat{x}_t , and $(\widehat{\mu x})_t$ into (2.37), and noting that $\vartheta_t = 0$ (since the unobserved process is dynamically constant in the present context), we are then led to (2.33), as desired. The point here is that the FKK theory allows us to deduce equation (2.33) and tells us that $\{W_t\}$ is an $\{F_t\}$ -Brownian motion.

Alternatively, we can derive the dynamics of $\{\pi_{it}\}$ from (2.31). This is achieved by applying Ito's lemma and using the fact that $(d\xi_t)^2 = dt$. The fact that $\{W_t\}$, as defined by (2.34), is an $\{\mathcal{F}_t\}$ -Brownian motion can then be verified by use of the Lévy criterion. In particular one needs to show that $\{W_t\}$ is an $\{\mathcal{F}_t\}$ -martingale and that $(dW_t)^2 = dt$. To prove that $\{W_t\}$ is an $\{\mathcal{F}_t\}$ -martingale we need to show that $\mathbb{E}[W_u|\mathcal{F}_t] = W_t$, for $0 \leq t \leq u < T$. First we note that it follows from (2.34) and the Markov property of $\{\xi_t\}$ that

$$\begin{aligned} \mathbb{E}[W_u|\mathcal{F}_t] &= W_t + \mathbb{E}[(\xi_u - \xi_t)|\xi_t] + \mathbb{E}\left[\int_t^u \frac{1}{T-s} \xi_s ds \middle| \xi_t\right] \\ &\quad - \sigma T \mathbb{E}\left[\int_t^u \frac{1}{T-s} H_{sT} ds \middle| \xi_t\right]. \end{aligned} \quad (2.44)$$

This expression can be simplified if we recall that $H_{sT} = \mathbb{E}[H_T|\xi_s]$ and use the tower property in the last term on the right. Inserting the definition (2.5) into the second and third terms on the right we then have:

$$\begin{aligned} \mathbb{E}[W_u|\mathcal{F}_t] &= W_t + \mathbb{E}[\sigma H_T u + \beta_{uT}|\xi_t] - \mathbb{E}[\sigma H_T t + \beta_{tT}|\xi_t] + \sigma \mathbb{E}[H_T|\xi_t] \int_t^u \frac{s}{T-s} ds \\ &\quad + \mathbb{E}\left[\int_t^u \frac{1}{T-s} \beta_{sT} ds \middle| \xi_t\right] - \sigma \mathbb{E}[H_T|\xi_t] \int_t^u \frac{T}{T-s} ds. \end{aligned} \quad (2.45)$$

Taking into account the fact that

$$\int_t^u \frac{s}{T-s} ds = t - u + \int_t^u \frac{T}{T-s} ds, \quad (2.46)$$

we see that all terms involving the random variable H_T cancel each other in (2.45). This leads us to the following relation:

$$\mathbb{E}[W_u|\mathcal{F}_t] = W_t + \mathbb{E}[\beta_{uT}|\xi_t] - \mathbb{E}[\beta_{tT}|\xi_t] + \int_t^u \frac{1}{T-s} \mathbb{E}[\beta_{sT}|\xi_t] ds. \quad (2.47)$$

Next we use the tower property and the independence of $\{\beta_{tT}\}$ and H_T to deduce that

$$\mathbb{E}[\beta_{uT}|\xi_t] = \mathbb{E}[\mathbb{E}[\beta_{uT}|H_T, \beta_{tT}]|\xi_t] = \mathbb{E}[\mathbb{E}[\beta_{uT}|\beta_{tT}]|\xi_t]. \quad (2.48)$$

To calculate the inner expectation $\mathbb{E}[\beta_{uT}|\beta_{tT}]$ we use the fact that the random variable $\beta_{uT}/(T-u) - \beta_{tT}/(T-t)$ is independent of the random variable β_{tT} . This can be checked by calculating their covariance, and using the relation $\mathbb{E}[\beta_{uT}\beta_{tT}] = t(T-u)/T$. We conclude after a short calculation that

$$\mathbb{E}[\beta_{uT}|\beta_{tT}] = \frac{T-u}{T-t} \beta_{tT}. \quad (2.49)$$

Inserting this result into (2.48) we obtain the following formula:

$$\mathbb{E}[\beta_{uT}|\xi_t] = \frac{T-u}{T-t} \mathbb{E}[\beta_{tT}|\xi_t]. \quad (2.50)$$

Applying this formula to the second and fourth terms on the right side of (2.47), we deduce that $\mathbb{E}[W_u|\mathcal{F}_t] = W_t$. That establishes that $\{W_t\}$ is an $\{\mathcal{F}_t\}$ -martingale. Now we need to show that $(dW_t)^2 = dt$. This follows if we insert (2.5) into the definition of $\{W_t\}$ above and use again the fact that $(d\beta_{tT})^2 = dt$. Hence, by Levy's criterion $\{W_t\}$ is an $\{\mathcal{F}_t\}$ -Brownian motion.

The $\{\mathcal{F}_t\}$ -Brownian motion $\{W_t\}$, the existence of which we have established, can be regarded as part of the information accessible to market participants. We note in particular that, unlike β_{tT} , the value of W_t contains "real" information relevant to the outcome of the bond payoff. It follows from (2.19) and (2.33) that for the discount bond dynamics we have

$$dB_{tT} = r_t B_{tT} dt + \Sigma_{tT} dW_t. \quad (2.51)$$

Here the expression

$$r_t = -\frac{\partial \ln P_{0t}}{\partial t} \quad (2.52)$$

is the (deterministic) short rate at t , and the absolute bond volatility Σ_{tT} is given by

$$\Sigma_{tT} = \frac{\sigma T}{T-t} P_{tT} V_{tT}, \quad (2.53)$$

where V_{tT} is the conditional variance of the terminal payoff H_T , defined by:

$$V_{tT} = \sum_i (h_i - H_{tT})^2 \pi_{it}. \quad (2.54)$$

We thus observe that as the maturity date is approached the absolute discount bond volatility will be high unless the conditional probability has its mass concentrated around the “true” outcome; this ensures that the correct level is eventually reached.

Proposition 2.4.1 *The process $\{V_{tT}\}_{0 \leq t < T}$ for the conditional variance of the terminal payoff H_T is a supermartingale.*

Proof. This follows directly from the fact that $\{V_{tT}\}_{0 \leq t \leq T}$ can be expressed as the difference between a martingale and a submartingale:

$$V_{tT} = \mathbb{E}_t [H_T^2] - (\mathbb{E}_t [H_T])^2. \quad (2.55)$$

□

The interpretation of this result is that in the filtration $\{\mathcal{F}_t\}$ generated by $\{\xi_t\}$ one on average “gains information” about H_T . In other words, the uncertainty in the conditional estimate of H_T tends to reduce.

Given the result of Proposition 2.4.1, we shall now derive an expression for the volatility of the volatility. The “second order” volatility is of interest in connection with pricing models for options on realised volatility. In the present example it turns out that the “vol-of-vol” has a particularly simple form. Starting with (2.55), we see that the conditional variance can be written in the form

$$V_{tT} = \sum_{i=0}^n h_i^2 \pi_{it} - H_{tT}^2. \quad (2.56)$$

By use of Ito’s formula, for the dynamics of $\{V_{tT}\}$ we thus obtain

$$dV_{tT} = \sum_{i=0}^n h_i^2 d\pi_{it} - 2H_{tT} dH_{tT} - (dH_{tT})^2. \quad (2.57)$$

For the dynamics of H_{tT} , on the other hand, we have

$$dH_{tT} = \frac{\sigma T}{T-t} V_{tT} dW_t, \quad (2.58)$$

from which it follows that

$$(dH_{tT})^2 = \left(\frac{\sigma T}{T-t} \right)^2 V_{tT}^2 dt. \quad (2.59)$$

Combining these relations with (2.33) we then obtain

$$dV_{tT} = -\sigma^2 \left(\frac{T}{T-t} \right)^2 V_{tT}^2 dt + \frac{\sigma T}{T-t} K_{tT} dW_t, \quad (2.60)$$

where

$$K_{tT} = \sum_{i=0}^n (h_i - H_{tT})^3 \pi_{it} \quad (2.61)$$

is the conditional skewness (third central moment) of terminal payoff. Writing V_{0T} for the *a priori* variance of H_T , we thus have the expression

$$V_{tT} = V_{0T} - \sigma^2 \int_0^t \left(\frac{T}{T-s} \right)^2 V_{sT}^2 ds + \sigma \int_0^t \frac{T}{T-s} K_s dW_s \quad (2.62)$$

for the “risk” associated with the payoff H_T .

To calculate the vol-of-vol of the defaultable discount bond, we need to work out the dynamics of $\{\Sigma_{tT}\}$, given the dynamics of $\{V_{tT}\}$. It should be evident therefore that the second-order absolute volatility $\Sigma_{tT}^{(2)}$, i.e. the vol-of-vol of $\{B_{tT}\}$, is given by

$$\Sigma_{tT}^{(2)} = \left(\frac{\sigma T}{T-t} \right)^2 P_{tT} K_{tT}. \quad (2.63)$$

It is interesting to observe that, as a consequence of equation (2.34), the market information process $\{\xi_t\}$ satisfies the following stochastic differential equation:

$$d\xi_t = \frac{1}{T-t} (\sigma T H(\xi_t, t) - \xi_t) dt + dW_t. \quad (2.64)$$

We see that $\{\xi_t\}$ is a diffusion process; and since $H(\xi_t, t)$ is monotonic in its dependence on ξ_t , we deduce that $\{B_{tT}\}$ is also a diffusion process. To establish that $H(\xi_t, t)$ is monotonic in ξ_t , and thus that B_{tT} is increasing in ξ_t we note that

$$P_{tT} H'(\xi_t, t) = \Sigma_{tT}, \quad (2.65)$$

where $H'(\xi, t) = \partial H(\xi, t) / \partial \xi$. It follows therefore that, in principle, instead of “deducing” the dynamics of $\{B_{tT}\}$ from the arguments of the previous sections, we might have simply “proposed” on an *ad hoc* basis the one-factor diffusion process described above, noting that it leads to the correct default dynamics. This line of reasoning shows that the information formalism can be viewed, if desired, as leading to purely “classical” financial models, based on observable price processes. In that sense the information-based approach adds an additional layer of interpretation and intuition to the classical framework, without altering any of its fundamental principles.

2.5 Digital bonds, and binary bonds with partial recovery

It is interesting to ask, incidentally, whether in the case of a binary bond with partial recovery, with possible payoffs $\{h_0, h_1\}$, the price process admits the representation

$$B_{tT} = P_{tT}h_0 + D_{tT}(h_1 - h_0). \quad (2.66)$$

Here D_{tT} denotes the value of a “digital” credit-risky bond that pays at maturity a unit of currency with probability p_1 and zero with probability $p_0 = 1 - p_1$. Thus h_0 is the amount guaranteed, whereas $h_1 - h_0$ is the part that is “at risk”. It is well known that such a relation can be deduced in intensity-based models (Lando 1994, 1998). The problem is thus to find a process $\{D_{tT}\}$ consistent with our scheme such that (2.66) holds. It turns out that this can be achieved as follows. Suppose we consider a digital payoff structure $D_T \in \{0, 1\}$ for which the parameter σ is replaced by

$$\bar{\sigma} = \sigma(h_1 - h_0). \quad (2.67)$$

In other words, in establishing the appropriate dynamics for $\{D_{tT}\}$ we “renormalise” σ by replacing it with $\bar{\sigma}$. The information available to market participants in the case of the digital bond is represented by the process $\{\bar{\xi}_t\}$ defined by

$$\bar{\xi}_t = \bar{\sigma}D_T t + \beta_{tT}. \quad (2.68)$$

It follows from (2.32) that the corresponding digital bond price is given by

$$D_{tT} = P_{tT} \frac{p_1 \exp \left[\frac{T}{T-t} \left(\bar{\sigma} \bar{\xi}_t - \frac{1}{2} \bar{\sigma}^2 t \right) \right]}{p_0 + p_1 \exp \left[\frac{T}{T-t} \left(\bar{\sigma} \bar{\xi}_t - \frac{1}{2} \bar{\sigma}^2 t \right) \right]}. \quad (2.69)$$

A short calculation making use of (2.11) then allows us to confirm that (2.66) holds, where D_{tT} is given by (2.69). Thus even though at first glance the general binary bond process (2.11) does not appear to admit a decomposition of the form (2.66), in fact it does, once a suitably renormalised value for the market information parameter has been introduced.

A slightly more general result is the following. Let H_T be a random payoff and let c_T be a constant payoff. Write $\xi_t = \sigma H_T t + \beta_{tT}$ for the information process associated

with H_T , and $\xi'_t = \sigma(H_T + c_T)t + \beta_{tT}$ for the information process associated with the combined payoff $H_T + c_T$. Then a straightforward calculation shows that

$$\mathbb{E} \left[H_T + c_T \mid \mathcal{F}_t^{\xi'} \right] = \mathbb{E} \left[H_T \mid \mathcal{F}_t^\xi \right] + c_T. \quad (2.70)$$

There is no reason, on the other hand, to suppose that such “linearity” holds more generally.

2.6 Dynamic consistency and market re-calibration

The technique of “renormalising” the information flow rate has another useful application. It turns out that the model under consideration exhibits a property that might appropriately be called “dynamic consistency”.

Loosely speaking, the question is as follows: if the information process is given as described, but then we “update” or re-calibrate the model at some specified intermediate time, is it still the case that the dynamics of the model moving forward from that intermediate time can be represented by an information process?

To answer this question we proceed as follows. First, we define a standard Brownian bridge over the interval $[t, T]$ to be a Gaussian process $\{\gamma_{uT}\}_{t \leq u \leq T}$ satisfying $\gamma_{tT} = 0$, $\gamma_{TT} = 0$, $\mathbb{E}[\gamma_{uT}] = 0$ for all $u \in [t, T]$, and

$$\mathbb{E}[\gamma_{uT}\gamma_{vT}] = \frac{(u-t)(T-v)}{(T-t)} \quad (2.71)$$

for all u, v such that $t \leq u \leq v \leq T$. Then we make note of the following result.

Lemma 2.6.1 *Let $\{\beta_{tT}\}_{0 \leq t \leq T}$ be a standard Brownian bridge over the interval $[0, T]$, and define the process $\{\gamma_{uT}\}_{t \leq u \leq T}$ by*

$$\gamma_{uT} = \beta_{uT} - \frac{T-u}{T-t} \beta_{tT}. \quad (2.72)$$

Then $\{\gamma_{uT}\}_{t \leq u \leq T}$ is a standard Brownian bridge over the interval $[t, T]$, and is independent of $\{\beta_{sT}\}_{0 \leq s \leq t}$.

Proof. The lemma is easily established by use of the covariance relation

$$\mathbb{E}[\beta_{tT}\beta_{uT}] = \frac{t(T-u)}{T}. \quad (2.73)$$

We need to recall also that a necessary and sufficient condition for a pair of Gaussian random variables to be independent is that their covariance should vanish. Now let the information process $\{\xi_s\}_{0 \leq s \leq T}$ be given, and fix an intermediate time $t \in (0, T)$. Then for all $u \in [t, T]$ let us define the process $\{\eta_u\}_{0 \leq u \leq T}$ by

$$\eta_u = \xi_u - \frac{T-u}{T-t} \xi_t. \quad (2.74)$$

We claim that $\{\eta_u\}$ is an information process over the time interval $[t, T]$. In fact, a short calculation establishes that

$$\eta_u = \tilde{\sigma} H_T(u-t) + \gamma_{uT}, \quad (2.75)$$

where $\{\gamma_{uT}\}_{t \leq u \leq T}$ is a standard Brownian bridge over the interval $[t, T]$, and the new information flow rate is given by

$$\tilde{\sigma} = \frac{\sigma T}{(T-t)}. \quad (2.76)$$

□

The interpretation of these results is as follows. The “original” information process proceeds from time 0 up to time t . At that time we can re-initialise the model by taking note of the value of the random variable ξ_t , and introducing the re-initialised information process $\{\eta_u\}$. The new information process depends on H_T ; but since the value of ξ_t is supplied, the *a priori* probability distribution for H_T now changes to the appropriate *a posteriori* distribution consistent with the information gained from the knowledge of ξ_t at time t .

These interpretive remarks can be put into a more precise form as follows. Let $0 \leq t \leq u < T$. What we want is a formula for the conditional probability π_{iu} expressed in terms of the information η_u and the “new” *a priori* probability π_{it} . Such a formula indeed exists, and is given as follows:

$$\pi_{iu} = \frac{\pi_{it} \exp \left[\frac{T-t}{T-u} (\tilde{\sigma} h_i \eta_u - \frac{1}{2} \tilde{\sigma}^2 h_i^2 (u-t)) \right]}{\sum_i \pi_{it} \exp \left[\frac{T-t}{T-u} (\tilde{\sigma} h_i \eta_u - \frac{1}{2} \tilde{\sigma}^2 h_i^2 (u-t)) \right]}. \quad (2.77)$$

This relation can be verified by substituting the given expressions for π_{it} , η_u , and $\bar{\sigma}$ into the right-hand side of (2.77). But (2.77) has the same structure as the original formula (2.31) for π_{it} , and thus we see that the model exhibits dynamic consistency.

2.7 Simulation of defaultable bond-price processes

The model introduced in the previous sections allows for a simple simulation methodology for the dynamics of defaultable bonds. In the case of a defaultable discount bond all we need is to simulate the dynamics of $\{\xi_t\}$. For each “run” of the simulation we choose at random a value for H_T (in accordance with the *a priori* probabilities), and a sample path for the Brownian bridge. That is to say, each simulation corresponds to a choice of $\omega \in \Omega$, and for each such choice we simulate the path

$$\xi_t(\omega) = \sigma t H_T(\omega) + \beta_{tT}(\omega) \quad (2.78)$$

for $t \in [0, T]$. One convenient way to simulate a Brownian bridge is to write

$$\beta_{tT} = B_t - \frac{t}{T} B_T, \quad (2.79)$$

where $\{B_t\}$ is a standard Brownian motion. It is straightforward to verify that if $\{\beta_{tT}\}$ is defined this way then it has the correct auto-covariance. Since the bond price at time t is expressed as a function of the random variable ξ_t , this means that a path-wise simulation of the bond price trajectory is feasible for any number of recovery levels.

The parameter σ governs the “speed” with which the bond price converges to its terminal value. This can be seen as follows. We return to the case of a binary discount bond with the possible payoffs h_0 and h_1 . Suppose, for example in a given “run” of the simulation, the “actual” value of the payout is $H_T = h_0$. In that case we have

$$\xi_t = \sigma h_0 t + \beta_{tT}, \quad (2.80)$$

and thus by use of expression (2.11) for $\{H_{tT}\}$ we obtain

$$H_{tT} = \frac{p_0 h_0 \exp \left[\frac{T}{T-t} (\sigma h_0 \beta_{tT} + \frac{1}{2} \sigma^2 h_0^2 t) \right] + p_1 h_1 \exp \left[\frac{T}{T-t} (\sigma h_1 \beta_{tT} + \sigma h_0 h_1 t - \frac{1}{2} \sigma^2 h_1^2 t) \right]}{p_0 \exp \left[\frac{T}{T-t} (\sigma h_0 \beta_{tT} + \frac{1}{2} \sigma^2 h_0^2 t) \right] + p_1 \exp \left[\frac{T}{T-t} (\sigma h_1 \beta_{tT} + \sigma h_0 h_1 t - \frac{1}{2} \sigma^2 h_1^2 t) \right]}. \quad (2.81)$$

Next, we divide the numerator and the denominator of this formula by the coefficient of $p_0 h_0$. After some re-arrangement of terms we get

$$H_{tT} = \frac{p_0 h_0 + p_1 h_1 \exp \left[-\frac{T}{T-t} \left(\frac{1}{2} \sigma^2 (h_1 - h_0)^2 t - \sigma (h_1 - h_0) \beta_{tT} \right) \right]}{p_0 + p_1 \exp \left[-\frac{T}{T-t} \left(\frac{1}{2} \sigma^2 (h_1 - h_0)^2 t - \sigma (h_1 - h_0) \beta_{tT} \right) \right]}. \quad (2.82)$$

Inspection of this expression shows that the convergence of the bond price to the value h_0 is *exponential*. We note, further, in line with the heuristic arguments in Section 2.2 concerning τ_D , that the parameter $\sigma^2(h_1 - h_0)^2$ governs the speed at which the defaultable discount bond converges to its destined terminal value. In particular, if the *a priori* probability of no default is high (say, $p_1 \approx 1$), and if σ is very small, and if $H_T = h_0$, then it will only be when t is close to T that serious decay in the value of the bond price will set in.

In the figures that follow shortly, we present some sample trajectories of the defaultable bond price process for various values of σ (I am grateful to I. Buckley for assistance with the preparation of the figures). Each simulation is composed of ten sample trajectories where the sample of the underlying Brownian motion is the same for all paths. For all simulations we have chosen the following values: the defaultable bond's maturity is five years, the default-free interest rate system has a constant short rate of 0.05, and the *a priori* probability of default is set at 0.2. The object of these simulations is to analyse the effect on the price process of the bond when the information flow rate is increased. Each set of four figures shows the trajectories for a range of information flow rates from a low rate ($\sigma = 0.04$) up to a high rate ($\sigma = 5$).

The first four figures relate to the situation where two trajectories are destined to default ($H_T = 0$) and the other eight refer to the no-default case ($H_T = 1$). Figure 2.1 shows the case where market investors have very little information ($\sigma = 0.04$) about the future cash flow H_T until the end of the bond contract. Only in the last year or so, investors begin to obtain more and more information when the noise process dies out as the maturity is approached. In this simulation we see that default comes as a surprise, and that investors have no chance to anticipate the default.

In Figure 2.3, by way of contrast, the information flow rate is rather high ($\sigma = 1$) and already after one year the bond price process starts to react strongly to the high rate of information release. The interpretation is that investors adjust their positions in the bond market according to the amount of genuine information accessible to them and as a consequence the volatility of the price process increases until the signal term in the definition of the information process dominates the noise produced by the bridge process.

In Figures 2.5-2.8 and Figures 2.9-2.12 we separate the trajectories destined to not to default ($H_T = 1$) from those that will end in a state of default ($H_T = 0$). As long as

the information rate is kept low the price process keeps its stochasticity. If σ is high, as in Figure 2.7 and in Figure 2.8, the trajectories become increasingly deterministic.

This is an expected phenomenon if one recalls that the default-free term structure P_{tT} is assumed to be deterministic. In other words, the market participants, with much genuine information about the future cash flow H_T defining the credit-risky asset, will in this case trade the defaultable bond similarly to the credit-risk-free discount bond, making the price of the defaultable bond approach that of a credit-risk-free bond.

The simulations referring to the case where the cash flow H_T is zero at the bond maturity manifest rather interesting features and scenarios that are very much linked to episodes occurred in financial markets. For instance, Figure 2.9 can be associated with the crises at Parmalat and Swissair. Both companies had the reputation to be reliable and financially robust until, rather as a surprise, it was announced that they were not able to honour their debts. Investors had very little genuine information about the payoffs connected to the two firms, and the asset prices reacted only at the last moment with a large drop in value.

An example in which there were earlier omens that a default might be imminent is perhaps Enron's. The company seemed to be doing well for quite some time until it became apparent that a continuous and gradual deterioration in the company's finances had arisen that eventually led to a state of default. This example would correspond more closely with Figure 2.10 where the bond price is stable for the first three and a half years but then commences to fluctuate, reaching a very high volatility following the augmented amount of information related to the increased likelihood of the possibility of a payment failure.

The even more dramatic case in Figure 2.12 can be associated with the example of a new credit card holder who, very soon after receipt of his card, is not able to pay his loan back, perhaps due to irresponsibly high expenditures during the previous month.

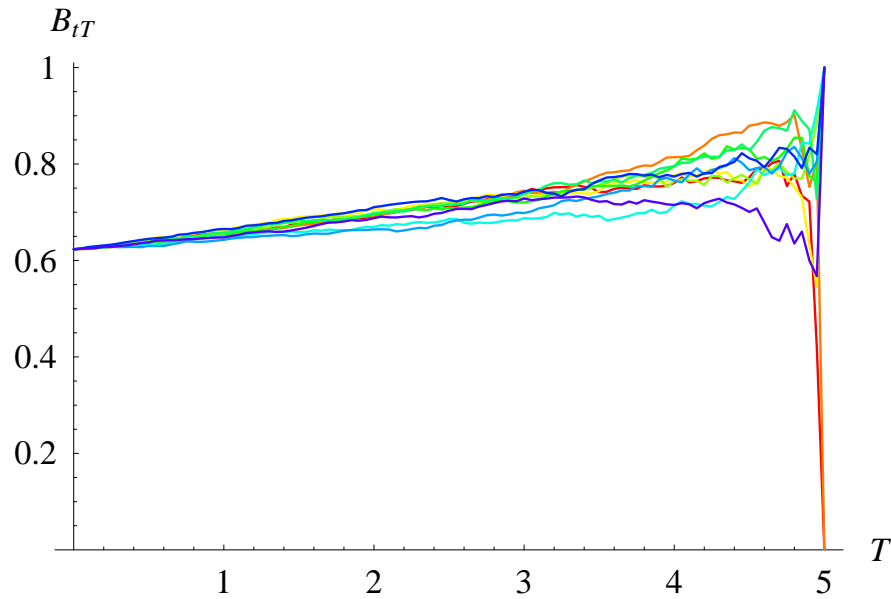


Figure 2.1: Bond price process: $\sigma = 0.04$. Two paths are conditional on default ($H_T = 0$) and eight paths are conditional on no-default ($H_T = 1$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 0.04.

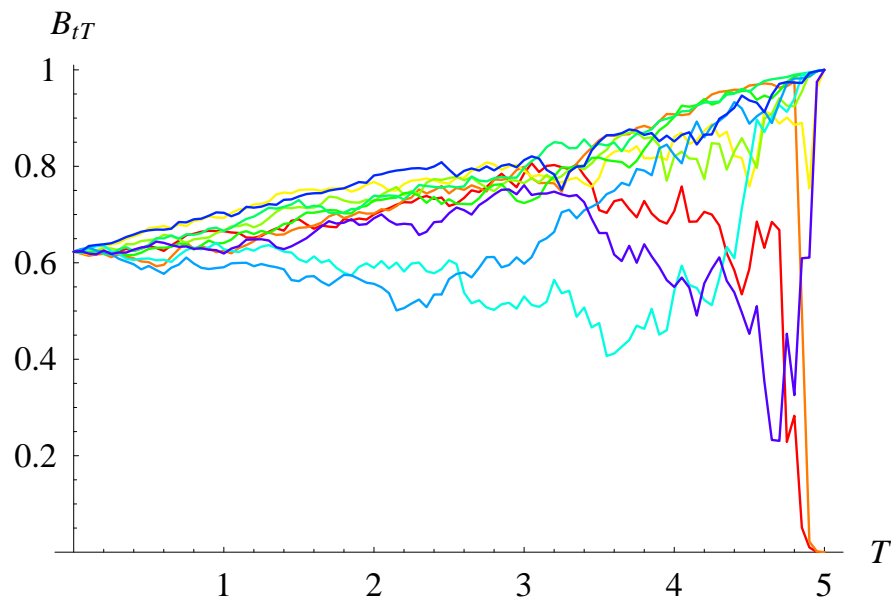


Figure 2.2: Bond price process: $\sigma = 0.2$. Two paths are conditional on default ($H_T = 0$) and eight paths are conditional on no-default ($H_T = 1$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 0.2.

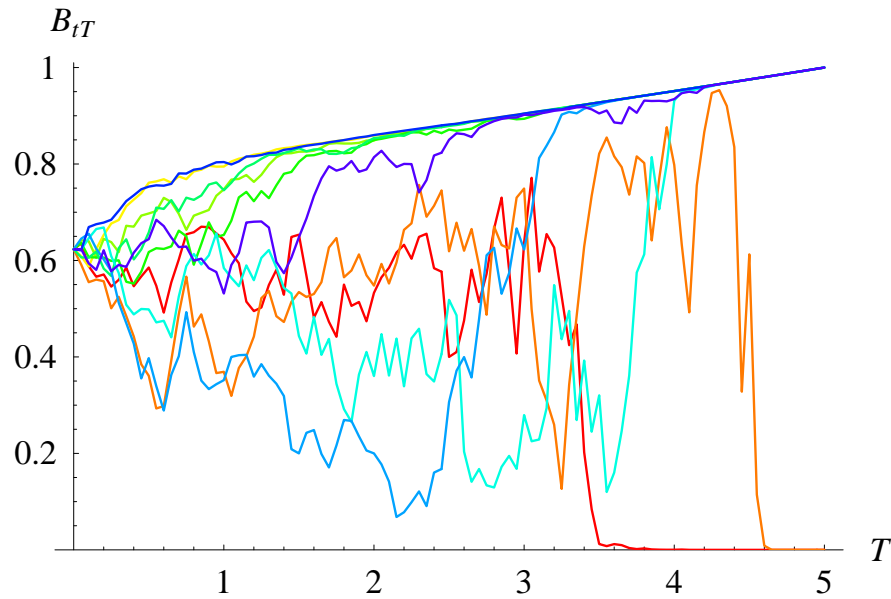


Figure 2.3: Bond price process: $\sigma = 1$. Two paths are conditional on default ($H_T = 0$) and eight paths are conditional on no-default ($H_T = 0$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 1.0.

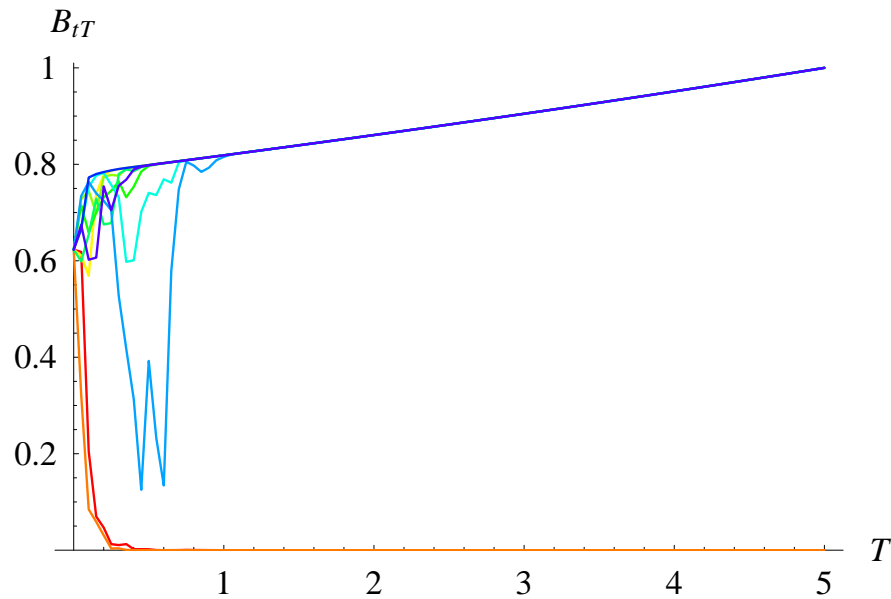


Figure 2.4: Bond price process: $\sigma = 5$. Two paths are conditional on default ($H_T = 0$) and eight paths are conditional on no-default ($H_T = 0$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 5.0.

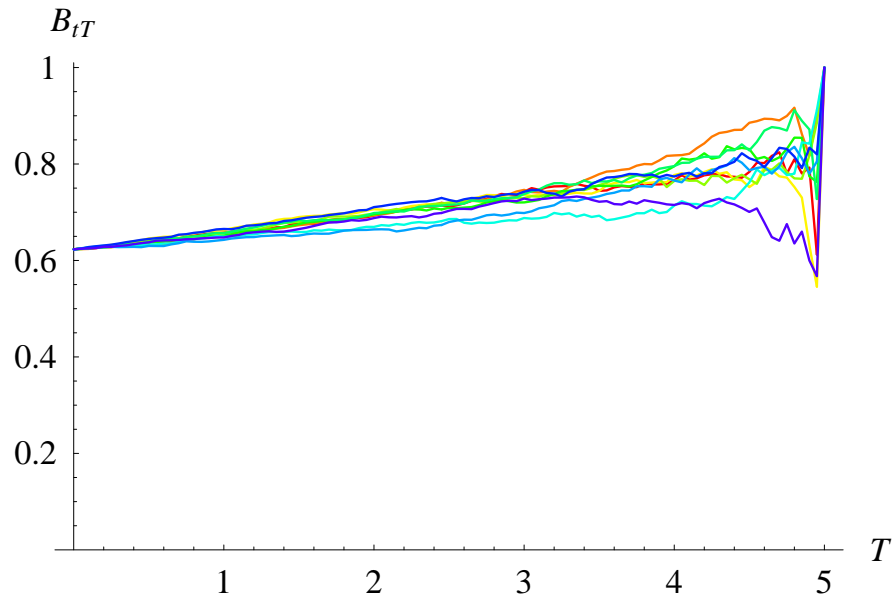


Figure 2.5: Bond price process: $\sigma = 0.04$. All paths are conditional on no-default ($H_T = 1$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 0.04.

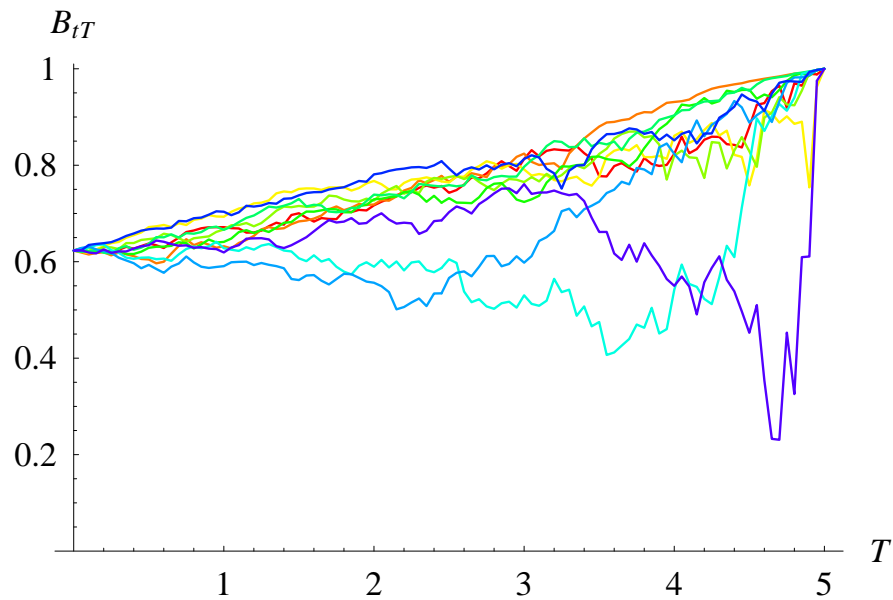


Figure 2.6: Bond price process: $\sigma = 0.2$. All paths are conditional on no-default ($H_T = 1$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 0.2.

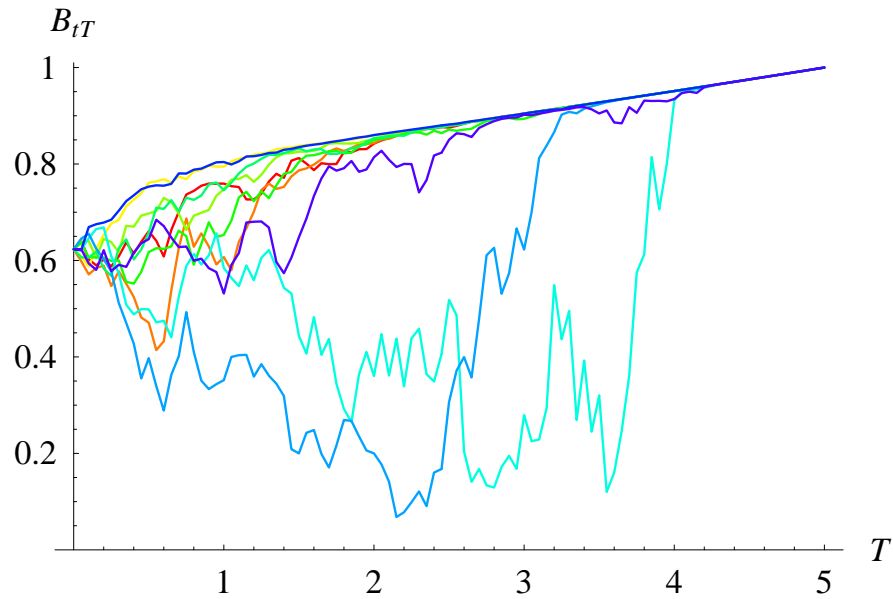


Figure 2.7: Bond price process: $\sigma = 1$. All paths are conditional on no-default ($H_T = 1$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 1.0.

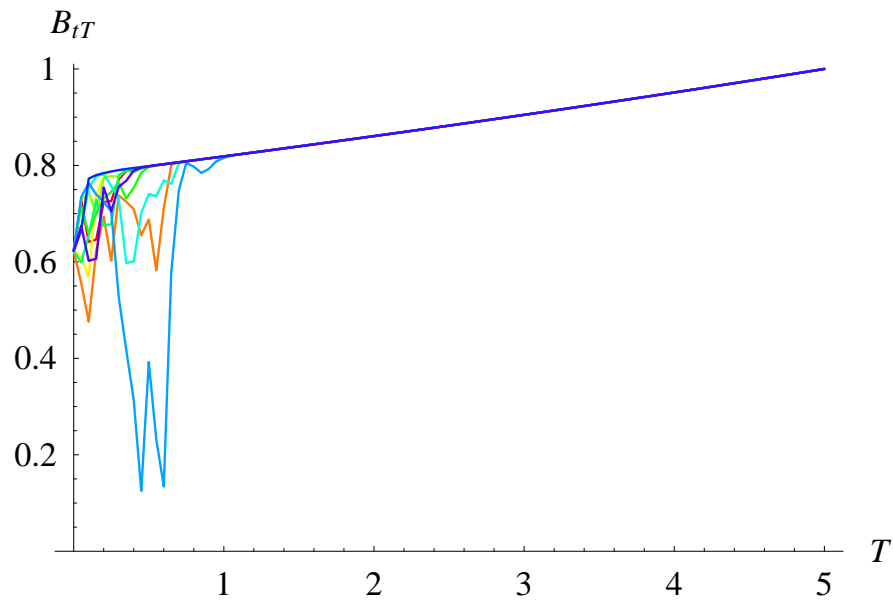


Figure 2.8: Bond price process: $\sigma = 5$. All paths are conditional on no-default ($H_T = 1$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 5.0.

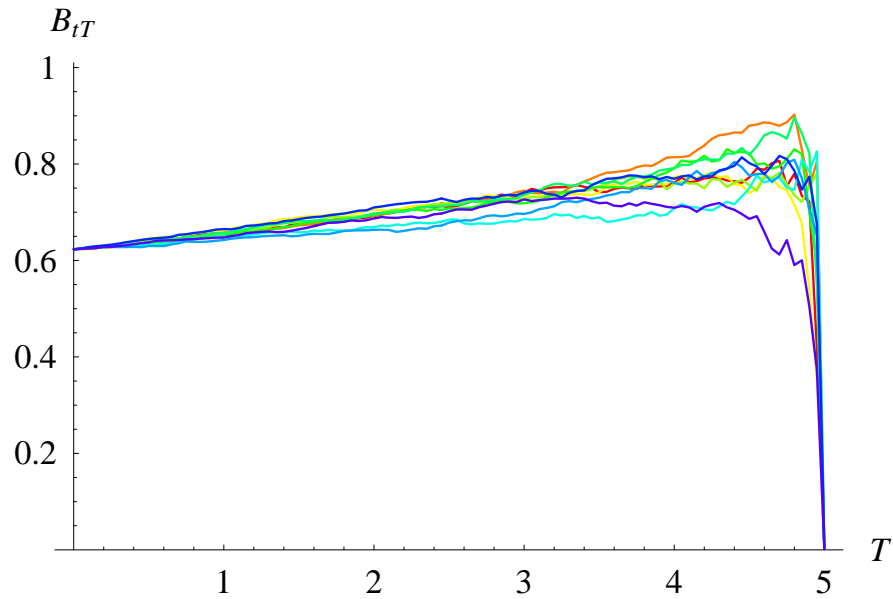


Figure 2.9: Bond price process: $\sigma = 0.04$. All paths are conditional on default ($H_T = 0$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 0.04.

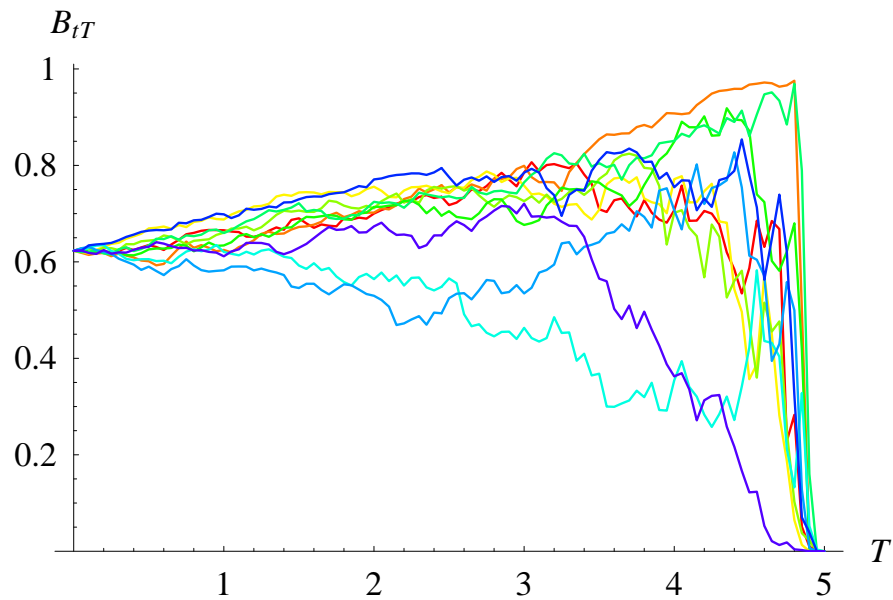


Figure 2.10: Bond price process: $\sigma = 0.2$. All paths are conditional on default ($H_T = 0$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 0.2.

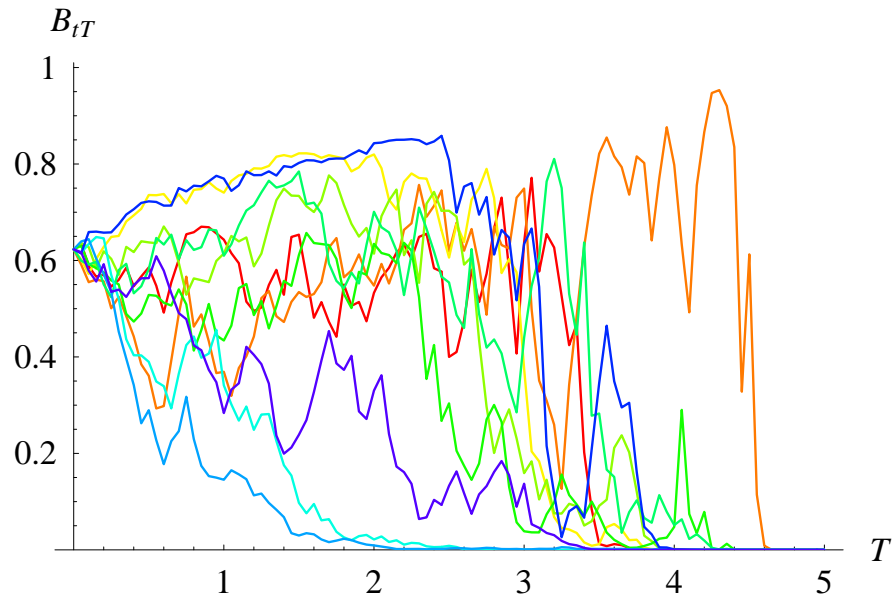


Figure 2.11: Bond price process: $\sigma = 1$. All paths are conditional on default ($H_T = 0$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 1.0.

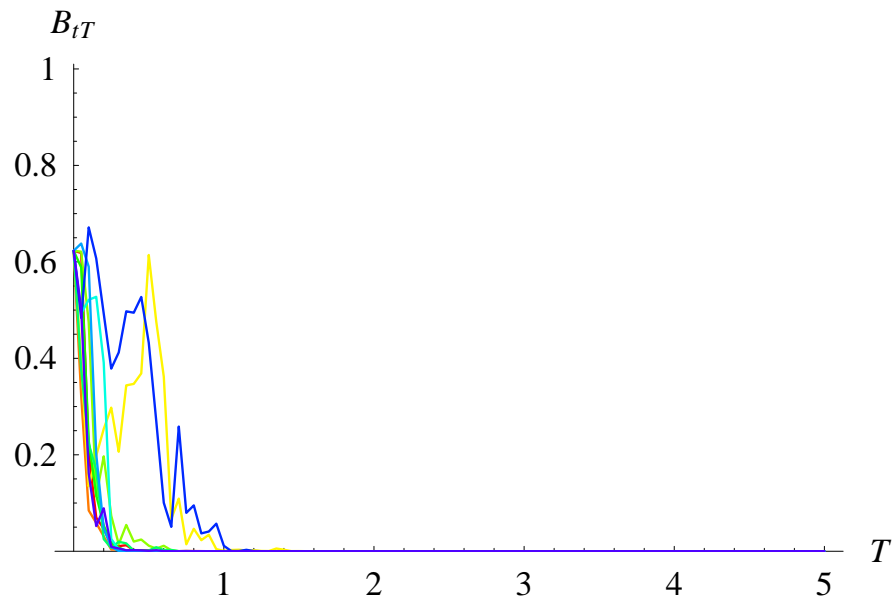


Figure 2.12: Bond price process: $\sigma = 5$. All paths are conditional on default ($H_T = 0$). The maturity of the bond is five years, the default-free interest rate is constant at 5%, the *a priori* probability of default is 20%, and the information flow rate is 5.0.

Chapter 3

Options on credit-risky discount bonds

3.1 Pricing formulae for bond options

In this chapter we consider the pricing of options on credit-risky bonds. In particular, we look at credit-risky discount bonds. As we shall demonstrate, in the case of a binary bond there is an exact solution for the valuation of European-style vanilla options. The resulting expression for the option price exhibits a structure that is strikingly analogous to that of the Black-Scholes option pricing formula.

We consider the value at time 0 of an option that is exercisable at a fixed time $t > 0$ on a credit-risky discount bond that matures at time $T > t$. The value C_0 of a call option at time 0 is given by

$$C_0 = P_{0t} \mathbb{E} [(B_{tT} - K)^+], \quad (3.1)$$

where B_{tT} is the bond price on the option maturity date and K is the strike price. Inserting formula (2.9) for B_{tT} into the valuation formula (3.1) for the option, and making use of (2.31), we obtain

$$\begin{aligned} C_0 &= P_{0t} \mathbb{E} [(P_{tT} H_{tT} - K)^+] \\ &= P_{0t} \mathbb{E} \left[\left(\sum_{i=0}^n P_{tT} \pi_{it} h_i - K \right)^+ \right], \end{aligned} \quad (3.2)$$

which after some further re-arrangement can be written in the form

$$\begin{aligned} C_0 &= P_{0t} \mathbb{E} \left[\left(\frac{1}{\Phi_t} \sum_{i=0}^n P_{tT} p_{it} h_i - K \right)^+ \right] \\ &= P_{0t} \mathbb{E} \left[\frac{1}{\Phi_t} \left(\sum_{i=0}^n (P_{tT} h_i - K) p_{it} \right)^+ \right]. \end{aligned} \quad (3.3)$$

Here the quantities p_{it} ($i = 0, 1, \dots, n$) are the “unnormalised” conditional probabilities, defined by

$$p_{it} = p_i \exp \left[\frac{T}{T-t} (\sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t) \right]. \quad (3.4)$$

Then for the “normalised” conditional probabilities we have

$$\pi_{it} = \frac{p_{it}}{\Phi_t} \quad (3.5)$$

where

$$\Phi_t = \sum_i p_{it}, \quad (3.6)$$

or, more explicitly,

$$\Phi_t = \sum_{i=0}^n p_i \exp \left[\frac{T}{T-t} (\sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t) \right]. \quad (3.7)$$

Our plan now is to use the factor Φ_t^{-1} appearing in (3.3) to make a change of probability measure on (Ω, \mathcal{F}_t) . To this end, we fix a time horizon u beyond the option expiration but before the bond maturity, so $t \leq u < T$. We define a process $\{\Phi_t\}_{0 \leq t \leq u}$ by use of the expression (3.7), where now we let t vary in the range $[0, u]$. We shall now work out the dynamics of $\{\Phi_t\}$

Let $\{p_{it}\}$, $i = 1, \dots, n$, be the un-normalised probability density processes given by (3.4). The dynamics of the probability density process $\{p_{it}\}$ can be obtained by applying Ito’s Lemma to the expression (3.4).

$$\begin{aligned} \frac{dp_{it}}{p_{it}} &= \left[\frac{T}{(T-t)^2} \left(\sigma h_i \xi_t - \frac{1}{2} \sigma^2 h_i^2 t \right) - \frac{1}{2} \frac{T}{T-t} \sigma^2 h_i^2 \right] dt \\ &+ \frac{T}{T-t} \sigma h_i d\xi_t + \frac{1}{2} \frac{T^2}{(T-t)^2} \sigma^2 h_i^2 (d\xi_t)^2. \end{aligned} \quad (3.8)$$

Recalling that for the dynamics of the information process $\{\xi_t\}$ we have

$$d\xi_t = \frac{1}{T-t} (\sigma T H_{tT} - \xi_t) dt + dW_t, \quad (3.9)$$

we see that $(d\xi_t)^2 = dt$. Inserting these expressions for $d\xi_t$ and $(d\xi_t)^2$ into the equation (3.8) we find that

$$dp_{it} = \sigma^2 \left(\frac{T}{T-t} \right)^2 h_i p_{it} H_{tT} dt + \sigma \frac{T}{T-t} h_i p_{it} dW_t. \quad (3.10)$$

Now we sum over i , recalling that

$$\sum_{i=0}^n h_i p_{it} = \Phi_t H_{tT} \quad (3.11)$$

and

$$\sum_{i=0}^n p_{it} = \Phi_t. \quad (3.12)$$

We thus obtain

$$d\Phi_t = \sigma^2 \left(\frac{T}{T-t} \right)^2 H_{tT}^2 \Phi_t dt + \sigma \frac{T}{T-t} H_{tT} \Phi_t dW_t. \quad (3.13)$$

It follows then by Ito's Lemma that

$$d\Phi_t^{-1} = -\sigma \frac{T}{T-t} H_{tT} \Phi_t^{-1} dW_t, \quad (3.14)$$

and hence that

$$\Phi_t^{-1} = \exp \left(-\sigma T \int_0^t \frac{1}{T-s} H_{sT} dW_s - \frac{1}{2} \sigma^2 T^2 \int_0^t \frac{1}{(T-s)^2} H_{sT}^2 ds \right). \quad (3.15)$$

Since $\{H_{sT}\}$ is bounded, and $t \leq u < T$, we see that the process $\{\Phi_t^{-1}\}_{0 \leq t \leq u}$ is a martingale. In particular, since $\Phi_0 = 1$, we deduce that $\mathbb{E}[\Phi_t^{-1}] = 1$, where t is the option maturity date, and hence that the factor Φ_t^{-1} appearing in (3.7) can be used to effect a change of measure on (Ω, \mathcal{F}_t) as we had earlier indicated. Writing \mathbb{B}_T for the new probability measure thus defined, we have

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{B}_T} \left[\left(\sum_{i=0}^n (P_{tT} h_i - K) p_{it} \right)^+ \right]. \quad (3.16)$$

We shall call \mathbb{B}_T the ‘‘bridge’’ measure because it has the property that it makes $\{\xi_s\}_{0 \leq s \leq t}$ a Gaussian process with mean zero and covariance

$$\mathbb{E}^{\mathbb{B}_T} [\xi_r \xi_s] = \frac{r(T-s)}{T} \quad (3.17)$$

for $0 \leq r \leq s \leq t$. In other words, with respect to the measure \mathbb{B}_T , and over the interval $[0, t]$, the information process has the law of a standard Brownian bridge over the interval $[0, T]$. Armed with this fact, we can then proceed to calculate the expectation in (3.16).

The proof that $\{\xi_s\}_{0 \leq s \leq t}$ has the claimed properties in the measure \mathbb{B}_T is as follows. For convenience we introduce a process $\{W_t^*\}_{0 \leq t \leq u}$ which we define as a Brownian motion with drift in the \mathbb{Q} -measure:

$$W_t^* = W_t + \sigma T \int_0^t \frac{1}{T-s} H_{sT} ds. \quad (3.18)$$

It is straightforward to check that on (Ω, \mathcal{F}_u) the process $\{W_t^*\}_{0 \leq t \leq u}$ is a Brownian motion with respect to the measure defined by use of the density martingale $\{\Phi_t^{-1}\}_{0 \leq t \leq u}$ given by (3.15). It then follows from the definition of $\{W_t\}$ given in equation (2.34) that

$$W_t^* = \xi_t + \int_0^t \frac{1}{T-s} \xi_s ds. \quad (3.19)$$

Taking the stochastic differential of each side of this relation, we deduce that

$$d\xi_t = -\frac{1}{T-t} \xi_t dt + dW_t^*. \quad (3.20)$$

We note, however, that (3.20) is the stochastic differential equation satisfied by a Brownian bridge (see, e.g., Karatzas & Shreve 1991, and Protter 2004) over the interval $[0, T]$. Thus we see that in the measure \mathbb{B}^T defined on (Ω, \mathcal{F}_t) the process $\{\xi_s\}_{0 \leq s \leq t}$ has the properties of a standard Brownian bridge over the interval $[0, T]$, restricted to the period $[0, t]$. For the transformation back from \mathbb{B}^T to \mathbb{Q} on (Ω, \mathcal{F}_u) , the appropriate density martingale $\{\Phi_t\}_{0 \leq t \leq u}$ with respect to \mathbb{B}^T is given by:

$$\Phi_t = \exp \left(\sigma T \int_0^t \frac{1}{T-s} H_{sT} dW_s^* - \frac{1}{2} \sigma^2 T^2 \int_0^t \frac{1}{(T-s)^2} H_{sT}^2 ds \right). \quad (3.21)$$

The crucial point that follows from this analysis is that the random variable ξ_t is \mathbb{B}_T -Gaussian. In the case of a binary discount bond, therefore, the relevant expectation for determining the option price can be carried out by standard techniques, and we are led to a formula of the Black-Scholes type. In particular, for a binary bond, equation (3.16) reads

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{B}_T} \left[\left((P_{tT} h_1 - K) p_{1t} + (P_{tT} h_0 - K) p_{0t} \right)^+ \right], \quad (3.22)$$

where p_{0t} and p_{1t} are given by

$$p_{0t} = p_0 \exp \left[\frac{T}{T-t} \left(\sigma h_0 \xi_t - \frac{1}{2} \sigma^2 h_0^2 t \right) \right], \quad (3.23)$$

and

$$p_{1t} = p_1 \exp \left[\frac{T}{T-t} \left(\sigma h_1 \xi_t - \frac{1}{2} \sigma^2 h_1^2 t \right) \right]. \quad (3.24)$$

To compute the value of (3.22) there are essentially three different cases that have to be considered:

- (1) $P_{tT} h_1 > P_{tT} h_0 > K$
- (2) $K > P_{tT} h_1 > P_{tT} h_0$
- (3) $P_{tT} h_1 > K > P_{tT} h_0$

In case (1) the option is certain to expire in the money. Thus, making use of the fact that ξ_t is \mathbb{B}_T -Gaussian with mean zero and variance $t(T-t)/T$, we see that $\mathbb{E}^{\mathbb{B}_T}[p_{it}] = p_i$; hence in case (1) we have $C_0 = B_{0T} - P_{0t}K$.

In case (2) the option expires out of the money, and thus $C_0 = 0$.

In case (3) the option can expire in or out of the money, and there is a ‘‘critical’’ value of ξ_t above which the argument of (3.22) is positive. This is obtained by setting the argument of (3.22) to zero and solving for ξ_t . Writing $\bar{\xi}_t$ for the critical value, we find that $\bar{\xi}_t$ is determined by the relation

$$\frac{T}{T-t} \sigma (h_1 - h_0) \bar{\xi}_t = \ln \left[\frac{p_0 (P_{tT} h_0 - K)}{p_1 (K - P_{tT} h_1)} \right] + \frac{1}{2} \sigma^2 (h_1^2 - h_0^2) \tau, \quad (3.25)$$

where

$$\tau = \frac{tT}{(T-t)}. \quad (3.26)$$

Next we note that since ξ_t is \mathbb{B}_T -Gaussian with mean zero and variance $t(T-t)/T$, for the purpose of computing the expectation in (3.22) we can set

$$\xi_t = Z \sqrt{\frac{t(T-t)}{T}}, \quad (3.27)$$

where Z is \mathbb{B}_T -Gaussian with zero mean and unit variance. Then writing \bar{Z} for the corresponding critical value of Z , we obtain

$$\bar{Z} = \frac{\ln \left[\frac{p_0 (K - P_{tT} h_0)}{p_1 (P_{tT} h_1 - K)} \right] + \frac{1}{2} \sigma^2 (h_1^2 - h_0^2) \tau}{\sigma \sqrt{\tau} (h_1 - h_0)}. \quad (3.28)$$

With this expression at hand, we calculate the expectation in (3.22). The result is:

$$C_0 = P_{0t} \left[p_1 (P_{tT} h_1 - K) N(d^+) - p_0 (K - P_{tT} h_0) N(d^-) \right]. \quad (3.29)$$

Here d^+ and d^- are defined by

$$d^\pm = \frac{\ln \left[\frac{p_1 (P_{tT} h_1 - K)}{p_0 (K - P_{tT} h_0)} \right] \pm \frac{1}{2} \sigma^2 (h_1 - h_0)^2 \tau}{\sigma \sqrt{\tau} (h_1 - h_0)}. \quad (3.30)$$

It is interesting to note that the information flow-rate parameter σ plays a role like that of the volatility parameter in the Black-Scholes model. The more rapidly information is “leaked out” about the “true” value of the bond repayment, the higher the volatility.

We remark that in the more general case for which there are multiple recovery levels, a semi-analytic result can be obtained that, for practical purposes, can be regarded as fully tractable. In particular, starting from (3.16) we consider the case where the strike price K lies in the range

$$P_{tT} h_{k+1} > K > P_{tT} h_k \quad (3.31)$$

for some value of $k \in \{0, 1, \dots, n\}$. It is an exercise to verify that there exists a unique critical value of ξ_t such that the summation appearing in the argument of the $\max(x, 0)$ function in (3.16) vanishes. Writing $\bar{\xi}_t$ for the critical value, which can be obtained by numerical methods, we define the scaled critical value \bar{Z} as before by setting

$$\bar{\xi}_t = \bar{Z} \sqrt{\frac{t(T-t)}{T}}. \quad (3.32)$$

A calculation then shows that the option price is given by the following formula:

$$C_0 = P_{0t} \sum_{i=0}^n p_i (P_{tT} h_i - K) N(\sigma h_i \sqrt{\tau} - \bar{Z}). \quad (3.33)$$

3.2 Model input sensitivity analysis

It is straightforward to verify that the option price has a positive “vega”, i.e. that C_0 is an increasing function of σ . This means in principle that we can use bond option prices (or, equivalently, the prices of caps and floors) to back out an implied value for σ , and hence to calibrate the model. Normally the term “vega” is used in the Black-Scholes theory to characterise the sensitivity of the option price to a change in

volatility; here we use the term analogously to denote sensitivity with respect to the information flow-rate parameter. Thus, writing

$$\mathcal{V} = \frac{\partial C_0}{\partial \sigma} \quad (3.34)$$

for the option vega, after a calculation we obtain the following positive expression:

$$\mathcal{V} = \frac{1}{\sqrt{2\pi}} e^{-rt - \frac{1}{2}A} (h_1 - h_0) \sqrt{\tau p_0 p_1 (P_{tT} h_1 - K)(K - P_{tT} h_0)}, \quad (3.35)$$

where

$$A = \frac{1}{\sigma^2 \tau (h_1 - h_0)^2} \ln^2 \left[\frac{p_1 (P_{tT} h_1 - K)}{p_0 (K - P_{tT} h_0)} \right] + \frac{1}{4} \sigma^2 \tau (h_1 - h_0)^2. \quad (3.36)$$

Another interesting and important feature of this model is the possibility to hedge an option position against moves in the underlying asset by holding a position in the credit-risky bond. The number of bond unites needed to hedge a short position in a call option is given by the option delta, which is defined by

$$\Delta = \frac{\partial C_0}{\partial B_{0T}}. \quad (3.37)$$

To calculate the option delta at time zero we need to express the initial call option value

$$C_0 = P_{0t} \left[p_1 (P_{tT} h_1 - K) N(d^+) - p_0 (K - P_{tT} h_0) N(d^-) \right] \quad (3.38)$$

in terms of the initial value of the binary bond

$$B_{0T} = P_{0T} (p_0 h_0 + p_1 h_1). \quad (3.39)$$

For this purpose we substitute the *a priori* probabilities p_0 and p_1 in the expression of the call option (3.38) by setting

$$p_0 = \frac{1}{h_1 - h_0} \left(h_1 - \frac{B_{0T}}{P_{0T}} \right), \quad p_1 = \frac{1}{h_1 - h_0} \left(\frac{B_{0T}}{P_{0T}} - h_0 \right). \quad (3.40)$$

The call price now reads:

$$\begin{aligned} C_0 = & P_{0t} \left[\frac{1}{h_1 - h_0} \left(\frac{B_{0T}}{P_{0T}} - h_0 \right) (P_{tT} h_1 - K) N(d^+) \right. \\ & \left. - \frac{1}{h_1 - h_0} \left(h_1 - \frac{B_{0T}}{P_{0T}} \right) (K - P_{tT} h_0) N(d^-) \right]. \end{aligned} \quad (3.41)$$

To obtain the option delta we need to differentiate (3.41) with respect to B_{0T} , taking into account also the dependence of d^+ and d^- on B_{0T} . The result is given as follows:

$$\Delta = \frac{(P_{tT} h_1 - K) N(d^+) + (K - P_{tT} h_0) N(d^-)}{P_{tT} (h_1 - h_0)}. \quad (3.42)$$

3.3 Bond option price processes

In the Section 3.1 we obtained the initial value C_0 of an option on a binary credit-risky bond. In the present section we extend this calculation to determine the price process of such an option. We fix the bond maturity T and the option maturity t . Then the price C_s of a call option at time $s \leq t$ is given by

$$\begin{aligned} C_s &= P_{st} \mathbb{E} [(B_{tT} - K)^+ | \mathcal{F}_s] \\ &= \frac{P_{st}}{\Phi_s} \mathbb{E}^{\mathbb{B}_T} [\Phi_t (B_{tT} - K)^+ | \mathcal{F}_s] \\ &= \frac{P_{st}}{\Phi_s} \mathbb{E}^{\mathbb{B}_T} \left[\left(\sum_{i=0}^n (P_{tT} h_i - K) p_{it} \right)^+ \middle| \mathcal{F}_s \right]. \end{aligned} \quad (3.43)$$

We recall that p_{it} , defined in (3.4), is a function of ξ_t . The calculation can thus be simplified by use of the fact that $\{\xi_t\}$ is a \mathbb{B}_T -Brownian bridge. To determine the conditional expectation (3.43) we note that the \mathbb{B}_T -Gaussian random variable Z_{st} defined by

$$Z_{st} = \frac{\xi_t}{T-t} - \frac{\xi_s}{T-s} \quad (3.44)$$

is independent of $\{\xi_u\}_{0 \leq u \leq s}$. We can then express $\{p_{it}\}$ in terms of ξ_s and Z_{st} by writing

$$p_{it} = p_i \exp \left[\frac{T}{T-s} \sigma h_i T \xi_s - \frac{1}{2} \frac{T}{T-t} \sigma^2 h_i^2 t + \sigma h_i Z_{st} T \right]. \quad (3.45)$$

Substituting (3.45) into (3.43), we find that C_s can be calculated by taking an expectation involving the random variable Z_{st} , which has mean zero and variance v_{st}^2 given by

$$v_{st}^2 = \frac{t-s}{(T-t)(T-s)}. \quad (3.46)$$

In the case of a call option on a binary discount bond that pays h_0 or h_1 , we can obtain a closed-form expression for (3.43). In that case the option price at time s is given by the following expectation:

$$C_s = \frac{P_{st}}{\Phi_s} \mathbb{E}^{\mathbb{B}_T} \left[((P_{tT} h_0 - K) p_{0t} + (P_{tT} h_1 - K) p_{1t})^+ | \mathcal{F}_s \right]. \quad (3.47)$$

Substituting (3.45) in (3.47) we find that the expression in the expectation is positive only if the inequality $Z_{st} > \bar{Z}$ is satisfied, where

$$\bar{Z} = \frac{\ln \left[\frac{\pi_{0s}(K - P_{tT} h_0)}{\pi_{1s}(P_{tT} h_1 - K)} \right] + \frac{1}{2} \sigma^2 (h_1^2 - h_0^2) v_{st}^2 T}{\sigma (h_1 - h_0) v_{st} T}. \quad (3.48)$$

It will be convenient to set

$$Z_s = v_{st}Z, \quad (3.49)$$

where Z is a \mathbb{B}_T -Gaussian random variable with zero mean and unit variance. The computation of the expectation in (3.47) then reduces to a pair of Gaussian integrals, and we obtain the following result:

Proposition 3.3.1 *Let $\{C_s\}_{0 \leq s \leq t}$ denote the price process of a European-style call option on a defaultable bond. Let t denote the option expiration date, let K denote the strike price, and let T denote the bond maturity date. Then the option price at time $s \in [0, t]$ is given by:*

$$C_s = P_{st} \left[\pi_{1s} (P_{tT}h_1 - K) N(d_s^+) - \pi_{0s} (K - P_{tT}h_0) N(d_s^-) \right], \quad (3.50)$$

where the conditional probabilities $\{\pi_{is}\}$ are as defined in (2.31), and

$$d_s^\pm = \frac{\ln \left[\frac{\pi_{1s}(P_{tT}h_1 - K)}{\pi_{0s}(K - P_{tT}h_0)} \right] \pm \frac{1}{2} \sigma^2 v_{st}^2 T^2 (h_1 - h_0)^2}{\sigma v_{st} T (h_1 - h_0)}. \quad (3.51)$$

Remark 3.3.1 *We note that $d_s^+ = d_s^- + \sigma v_{st} T (h_1 - h_0)$, and that $d_0^\pm = d^\pm$.*

One particularly attractive feature of the model worth pointing out in the present context is that delta-hedging is possible. This is because the option price process and the underlying bond price process are one-dimensional diffusions driven by the same Brownian motion. Since C_t and B_{tT} are both monotonic in their dependence on ξ_t , it follows that C_t can be expressed as a function of B_{tT} ; the delta of the option can then be defined in the conventional way as the derivative of the option price with respect to the value of the underlying. At time 0 this reduces to the expression we developed earlier.

This brings us to another interesting point. For certain types of instruments it may be desirable to model the occurrence of credit events taking place at some time that precedes a cash-flow date. In particular, we may wish to consider contingent claims based on such events. In the present framework we can regard such contingent claim as derivative structures for which the payoff is triggered by the level of ξ_t . For example, it may be that a credit event is established if B_{tT} drops below some specific level, or if the credit spread widens beyond some threshold. For that reason we see that the consideration of a barrier option becomes an important issue, where both the payoff and the barrier level are expressed in terms of the information process.

3.4 Arrow-Debreu technique and information derivatives

In this section we consider an alternative method for option pricing in the information-based framework. We concentrate on the case where the underlying asset pays a single cash flow at the maturity date T . We sketch the main ideas behind this alternative method, which is based on the concept of an Arrow-Debreu security. In section 6.9 we then present an extension of this technique to the case of an underlying asset for which the dividends can be regarded as continuous random variables.

In the present section we introduce also a new class of contingent claims, which we call “information derivatives”. This type of security is defined by a payoff that is a function of the value of the information process. In other words the underlying of such a derivative is the information available to market participants. The price of an information derivative is given by the risk-neutral pricing formula, that is:

$$S_0 = P_{0t} \mathbb{E}^{\mathbb{Q}} [f(\xi_t)], \quad (3.52)$$

where P_{0t} is the discount function, and $f(\xi_t)$ is the payoff function. Here the derivative’s maturity is denoted t . To begin with we consider the elementary information security defined by the following payoff:

$$f(\xi_t) = \delta(\xi_t - x). \quad (3.53)$$

Let us write $A_{0t}(x)$ for the price at time 0 of such a contract. Without the introduction of a great deal of additional mathematics, the treatment of distribution-valued random variables will have to be somewhat heuristic in what follows; but in practice this causes no problems. In order to work out the price of an elementary information security we use the standard Fourier representation of the delta function, namely:

$$\delta(\xi_t - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi_t - x)\kappa} d\kappa. \quad (3.54)$$

As with all distributional expressions, this formula acquires its meaning from the context in which it is used. Here we recall that, conditional on a specific value of the random variable H_T , the information process $\{\xi_t\}$ is normally distributed. This ensures, along with the fact that the random variable H_T can take only a finite number

of different states, that the integral in (3.54) and the expectation in (3.52) can be in effect interchanged. Thus we get,

$$\begin{aligned} A_{0t}(x) &= P_{0t} \mathbb{E}^{\mathbb{Q}}[\delta(\xi_t - x)] \\ &= P_{0t} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\kappa} \mathbb{E}^{\mathbb{Q}}[e^{i\xi_t\kappa}] d\kappa. \end{aligned} \quad (3.55)$$

A standard calculation involving the computation of the moment generating function of ξ_t yields the value of the expectation in the equation above, that is:

$$\mathbb{E}^{\mathbb{Q}}[e^{i\xi_t\kappa}] = \sum_{j=0}^n p_j \exp\left[i\sigma h_j t\kappa - \frac{t(T-t)}{2T}\kappa^2\right]. \quad (3.56)$$

Inserting this intermediate result into (3.55) and completing the square, we obtain

$$\begin{aligned} &A_{0t}(x) \\ &= P_{0t} \sum_{j=0}^n p_j \frac{1}{2\pi} \sqrt{\frac{t(T-t)}{T}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\sqrt{\frac{t(T-t)}{T}}\kappa - \sqrt{\frac{(\sigma h_j t - x)^2 T}{t(T-t)}}i\right)^2\right] d\kappa. \end{aligned} \quad (3.57)$$

Carrying out the integration, we thus have:

$$A_{0t}(x) = P_{0t} \sum_{j=0}^n p_j \sqrt{\frac{T}{2\pi t(T-t)}} \exp\left[-\frac{1}{2}\frac{(\sigma h_j t - x)^2 T}{t(T-t)}\right]. \quad (3.58)$$

The price $A_{0t}(x)$ of a security paying a delta function at the derivative's maturity t can be viewed also from another angle. Let us introduce a general payoff function $g(\xi_t)$ defining an exotic payout depending on the value of the information process $\{\xi_t\}$ at time t . Then we decompose the function $g(\xi_t)$ in infinitesimal parts by use of the delta function by writing

$$g(x_0) = \int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx. \quad (3.59)$$

But this definition can be exploited to express the payoff function $g(\xi_t)$ as a superposition of elementary information securities. Thus we have:

$$g(\xi_t) = \int_{-\infty}^{\infty} \delta(\xi_t - x)g(x)dx. \quad (3.60)$$

This decomposition in terms of elementary securities is in line with the concept of a so-called Arrow-Debreu security. In fact, the price (3.58) of the elementary information

security defined by the payoff (3.53) can be regarded as an example of an Arrow-Debreu price. Thus we can now express the price of a general information derivative as a weighted integral, where the elementary information securities with the Arrow-Debreu price $A_{0t}(x)$ play the role of the weights. Following the calculation of the price of the elementary information security we hence have for the price of a general exotic information derivative the following result:

$$V_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}[g(\xi_t)] = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\int_{-\infty}^{\infty}\delta(\xi_t - x)g(x)dx\right] = \int_{-\infty}^{\infty}A_{0t}(x)g(x)dx. \quad (3.61)$$

We can now treat a European call option on a credit-risky discount bond as a further example of an exotic information derivative, and apply the Arrow-Debreu technique just shown. We refer the reader to Section 6.9 for a derivation of the call option price, using this alternative technique, in the case of an underlying asset with a continuous cash flow function.

The price of a European call option terminating at time t written on a defaultable discount bond with a discrete payoff function and maturity T is given by

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}[(B_{tT} - K)^+], \quad (3.62)$$

where K is the strike, and $\{B_{tT}\}$ is the price of the bond, given by

$$B_{tT} = P_{tT}\frac{\sum_i p_i h_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t)\right]}. \quad (3.63)$$

We note that since B_{tT} is a function of ξ_t , the payoff function

$$g(\xi_t) = (B_{tT} - K)^+ \quad (3.64)$$

can be regarded as an example of an exotic information derivative and hence we can apply the Arrow-Debreu technique. We write

$$C_0 = \int_{-\infty}^{\infty}A_{0t}(x)g(x)dx, \quad (3.65)$$

where $A_{0t}(x)$ is given as in (3.58) and the payoff function $g(x)$ by formula (3.64), with ξ_t replaced by the variable x . In the case that the defaultable bond has a binary payoff we recover, by following the calculation in Section 6.9, the same expression as in (3.29). In particular, if we use the Arrow-Debreu technique there is no need to change from the risk-neutral measure to the bridge measure. Hence we have a useful alternative technique to price derivatives, in which the fundamental step is the decomposition of the payoff function into elementary information securities.

Chapter 4

Complex credit-linked structures

4.1 Coupon bonds

The discussion so far has focused on simple structures such as discount bonds and options on discount bonds. One of the advantages of the present approach, however, is that its tractability extends to situations of a more complex nature. In this section we consider the case of a credit-risky coupon bond. One should regard a coupon bond as being a rather complicated instrument from the point of view of credit risk management, since default can occur at any of the coupon dates. The market will in general possess partial information concerning all of the future coupon payments, as well as the principal payment.

As an illustration, we consider a bond with two payments remaining—a coupon H_{T_1} at time T_1 , and a coupon plus the principal totalling H_{T_2} at time T_2 . We assume that if default occurs at T_1 , then no further payment is made at T_2 . On the other hand, if the T_1 -coupon is paid, default may still occur at T_2 . We model this by setting

$$H_{T_1} = \mathbf{c}X_{T_1}, \quad H_{T_2} = (\mathbf{c} + \mathbf{p})X_{T_1}X_{T_2}, \quad (4.1)$$

where X_{T_1} and X_{T_2} are independent random variables taking the values $\{0, 1\}$, and the constants \mathbf{c}, \mathbf{p} denote the coupon and principal payments. Let us write $\{p_0^{(1)}, p_1^{(1)}\}$ for the *a priori* probabilities that $X_{T_1} = \{0, 1\}$, and $\{p_0^{(2)}, p_1^{(2)}\}$ for the *a priori* probabilities that $X_{T_2} = \{0, 1\}$. We introduce a pair of information processes

$$\xi_t^{(1)} = \sigma_1 X_{T_1} t + \beta_{tT_1}^{(1)} \quad \text{and} \quad \xi_t^{(2)} = \sigma_2 X_{T_2} t + \beta_{tT_2}^{(2)}, \quad (4.2)$$

where $\{\beta_{tT_1}^{(1)}\}$ and $\{\beta_{tT_2}^{(2)}\}$ are independent Brownian bridges, and σ_1 and σ_2 are parameters. Then for the credit-risky coupon-bond price process we have

$$B_{tT_2} = \mathbf{c}P_{tT_1}\mathbb{E}\left[X_{T_1}|\xi_t^{(1)}\right] + (\mathbf{c} + \mathbf{p})P_{tT_2}\mathbb{E}\left[X_{T_1}|\xi_t^{(1)}\right]\mathbb{E}\left[X_{T_2}|\xi_t^{(2)}\right]. \quad (4.3)$$

The two conditional expectations appearing in this formula can be worked out explicitly using the techniques already described. The result is:

$$\mathbb{E}\left[X_{T_i}|\xi_t^{(i)}\right] = \frac{p_1^{(i)} \exp\left[\frac{T_i}{T_i-t}\left(\sigma_i\xi_t^{(i)} - \frac{1}{2}\sigma_i^2t\right)\right]}{p_0^{(i)} + p_1^{(i)} \exp\left[\frac{T_i}{T_i-t}\left(\sigma_i\xi_t^{(i)} - \frac{1}{2}\sigma_i^2t\right)\right]}, \quad (4.4)$$

for $i = 1, 2$. It should be evident that in the case of a bond with two payments remaining we obtain a “two-factor” model—the factors (i.e., the Brownian drivers) being the two innovation processes arising in connection with the information processes $\{\xi_t^{(i)}\}_{i=1,2}$.

Similarly, if there are n outstanding coupons, we model the payments by

$$H_{T_k} = \mathbf{c}X_{T_1} \cdots X_{T_k} \quad (4.5)$$

for $k \leq n - 1$ and

$$H_{T_n} = (\mathbf{c} + \mathbf{p})X_{T_1} \cdots X_{T_n}, \quad (4.6)$$

and introduce the market information processes

$$\xi_t^{(i)} = \sigma_i X_{T_i} t + \beta_{tT_i}^{(i)} \quad (i = 1, 2, \dots, n). \quad (4.7)$$

The case of n outstanding payments gives rise in general to an n -factor model. The independence of the random variables $\{X_{T_i}\}_{i=1,2,\dots,n}$ implies that the price of a credit-risky coupon bond admits a closed-form expression analogous to that obtained in (4.3).

With a slight modification of these expressions we can consider the case when there is recovery in the event of default. In the two-coupon example discussed above, for instance, we can extend the model by saying that in the event of default on the first coupon the effective recovery rate (as a percentage of coupon plus principal) is R_1 ; whereas in the case of default on the final payment the recovery rate is R_2 . Then we have

$$H_{T_1} = \mathbf{c}X_{T_1} + R_1(\mathbf{c} + \mathbf{p})(1 - X_{T_1}), \quad (4.8)$$

$$H_{T_2} = (\mathbf{c} + \mathbf{p})X_{T_1}X_{T_2} + R_2(\mathbf{c} + \mathbf{p})X_{T_1}(1 - X_{T_2}). \quad (4.9)$$

A further extension of this line of reasoning allows for the introduction of random recovery rates.

4.2 Credit default swaps

Swap-like structures can be treated in a similar way. For example, in the case of a basic credit default swap we have a series of premium payments, each of the amount \mathbf{g} , made to the seller of protection. The payments continue until the failure of a coupon payment in the reference bond, at which point a lump-sum payment \mathbf{n} is made to the buyer of protection.

As an illustration, suppose we consider two reference coupons, letting X_{T_1} and X_{T_2} be the associated independent random variables, following the pattern of the previous example. We assume for simplicity that the default-swap premium payments are made immediately after the bond coupon dates. Then the value of the default swap, from the point of view of the seller of protection, is given by the following expression:

$$\begin{aligned} V_t &= \mathbf{g}P_{tT_1}\mathbb{E}\left[X_{T_1}|\xi_t^{(1)}\right] - \mathbf{n}P_{tT_1}\mathbb{E}\left[1 - X_{T_1}|\xi_t^{(1)}\right] \\ &+ \mathbf{g}P_{tT_2}\mathbb{E}\left[X_{T_1}|\xi_t^{(1)}\right]\mathbb{E}\left[X_{T_2}|\xi_t^{(2)}\right] - \mathbf{n}P_{tT_2}\mathbb{E}\left[X_{T_1}|\xi_t^{(1)}\right]\mathbb{E}\left[1 - X_{T_2}|\xi_t^{(2)}\right] \end{aligned} \quad (4.10)$$

After some rearrangement of terms, this can be expressed more compactly as follows:

Proposition 4.2.1 *Let $\{V_t\}_{0 \leq t \leq T_2}$ be the price process of a credit default swap. Let \mathbf{g} denote the premium payment, and let \mathbf{n} denote the payment made to the buyer of the protection in the event of default. Then the price of a default swap written on a reference defaultable two-coupon bond is given by*

$$\begin{aligned} V_t &= -\mathbf{n}P_{tT_1} + [(\mathbf{g} + \mathbf{n})P_{tT_1} - \mathbf{n}P_{tT_2}]\mathbb{E}\left[X_{T_1}|\xi_t^{(1)}\right] \\ &+ (\mathbf{g} + \mathbf{n})P_{tT_2}\mathbb{E}\left[X_{T_1}|\xi_t^{(1)}\right]\mathbb{E}\left[X_{T_2}|\xi_t^{(2)}\right]. \end{aligned} \quad (4.11)$$

A similar approach can be adapted in the multi-name credit situation. The importance of multi-credit correlation modelling has been emphasised by many authors—see e.g. Davis & Lo 2001, Duffie & Garleanu 2001, Frey & McNeil 2003, and Hull & White 2004a. The point that we would like to emphasise here is that in the information-based framework there is a good deal of flexibility available in the manner in which the various cash-flows can be modelled to depend on one another, and in many situations tractable expressions emerge that can be used as the basis for the modelling of complex multi-name credit instruments.

4.3 Baskets of credit-risky bonds

We consider now the valuation problem for a basket of bonds in the situation for which there are correlations in the payoffs. We shall demonstrate how to obtain a closed-form expression for the value of a basket of defaultable bonds with various different maturities.

For definiteness we consider a set of digital bonds each with two possible payoffs $\{0, 1\}$. It will be convenient to label the bonds in chronological order with respect to their maturities. Therefore, we let H_{T_1} denote the payoff of the bond that expires first; we let H_{T_2} ($T_2 \geq T_1$) denote the payoff of the first bond that matures after T_1 ; and so on. In general the various bond payouts will not be independent.

We propose to model this set of dependent random variables in terms of an underlying set of independent random variables. To achieve this we let X denote the random variable associated with the payoff of the first bond: $H_{T_1} = X$. The random variable X takes on the values $\{1, 0\}$ with *a priori* probabilities $\{p, 1 - p\}$. The payoff of the second bond H_{T_2} can then be represented in terms of three independent random variables: $H_{T_2} = XX_1 + (1 - X)X_0$. Here X_0 takes the values $\{1, 0\}$ with the probabilities $\{p_0, 1 - p_0\}$, and X_1 takes the values $\{1, 0\}$ with the probabilities $\{p_1, 1 - p_1\}$. Clearly, the payoff of the second bond is unity if and only if the random variables (X, X_0, X_1) take the values $(0, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$, or $(1, 1, 1)$. Since these random variables are independent, the *a priori* probability that the second bond does not default is $p_0 + p(p_1 - p_0)$, where p is the *a priori* probability that the first bond does not default.

To represent the payoff of the third bond we introduce four additional independent random variables:

$$H_{T_3} = XX_1X_{11} + X(1 - X_1)X_{10} + (1 - X)X_0X_{01} + (1 - X)(1 - X_0)X_{00}. \quad (4.12)$$

Here the random variables $\{X_{ij}\}_{i,j=0,1}$ take on the values $\{1, 0\}$ with the probabilities $\{p_{ij}, 1 - p_{ij}\}$. It is a matter of combinatorics to determine the *a priori* probability that $H_{T_3} = 1$ in terms of p , $\{p_i\}$, and $\{p_{ij}\}$.

The scheme above can be extended to represent the payoff of a generic bond in the basket with an expression of the following form:

$$H_{T_{n+1}} = \sum_{\{k_j\}=1,0} X^{\omega(k_1)} X_{k_1}^{\omega(k_2)} X_{k_1 k_2}^{\omega(k_3)} \dots X_{k_1 k_2 \dots k_{n-1}}^{\omega(k_n)} X_{k_1 k_2 \dots k_{n-1} k_n}. \quad (4.13)$$

Here, for any random variable X we define $X^{\omega(0)} = 1 - X$ and $X^{\omega(1)} = X$. The point is that if we have a basket of N digital bonds with arbitrary *a priori* default probabilities and arbitrary *a priori* correlation, then we can introduce $2^N - 1$ independent digital random variables to represent the N correlated random variables associated with the bond payoffs. The scheme above provides a convenient way of achieving this.

One advantage of the decomposition into independent random variables is that we retain analytical tractability for the pricing of the basket. In particular, since the random variables $\{X_{k_1 k_2 \dots k_n}\}$ are independent, it is natural to introduce a set of $2^N - 1$ independent Brownian bridges to represent the noise that hides the values of the independent random variables:

$$\xi_t^{k_1 k_2 \dots k_n} = \sigma_{k_1 k_2 \dots k_n} X_{k_1 k_2 \dots k_n} t + \beta_{t T_{n+1}}^{k_1 k_2 \dots k_n}. \quad (4.14)$$

The number of independent factors in general grows rapidly with the number of bonds in the portfolio. As a consequence, a market that consists of correlated bonds is in general highly incomplete. This, in turn, provides a justification for the creation of products such as CDSs and CDOs that enhance the “hedgeability” of such portfolios.

4.4 Homogeneous baskets

In the case of a “homogeneous” basket the number of independent random variables characterising the payoff of the portfolio can be reduced. We assume for simplicity that the basket contains n defaultable discount bonds, each maturing at time T , and each paying 0 or 1, with the same *a priori* probability of default. This is an artificial situation, but is of interest as a first step in the analysis of the more general setup.

The goal is to model default correlations in the portfolio, and in particular to model the flow of market information concerning default correlation. Let us write H_T for the payoff at time T of the homogeneous portfolio, and set

$$H_T = n - Z_1 - Z_1 Z_2 - Z_1 Z_2 Z_3 - \dots - Z_1 Z_2 \dots Z_n, \quad (4.15)$$

where the random variables $\{Z_j\}_{j=1,2,\dots,n}$, each taking the values $\{0, 1\}$, are assumed to be independent. Thus if $Z_1 = 0$, then $H_T = n$; if $Z_1 = 1$ and $Z_2 = 0$, then $H_T = n - 1$; if $Z_1 = 1$, $Z_2 = 1$, and $Z_3 = 0$, then $H_T = n - 2$; and so on.

Now suppose we write $p_j = \mathbb{Q}(Z_j = 1)$ and $q_j = \mathbb{Q}(Z_j = 0)$ for $j = 1, 2, \dots, n$. Then $\mathbb{Q}(H_T = n) = q_1$, $\mathbb{Q}(H_T = n - 1) = p_1 q_2$, $\mathbb{Q}(H_T = n - 2) = p_1 p_2 q_3$, and so

on. More generally, we have $\mathbb{Q}(H_T = n - k) = p_1 p_2 \dots p_k q_{k+1}$. Thus if $p_1 \ll 1$ but p_2, p_3, \dots, p_k are large, then we are in a situation of low default probability and high default correlation; that is to say, the probability of a default occurring in the portfolio is small, but conditional on at least one default occurring, the probability of several defaults is high.

The market will take a view on the likelihood of various numbers of defaults occurring in the portfolio. We model this by introducing a set of independent market information processes $\{\xi_t^j\}$ defined by

$$\xi_t^j = \sigma_j X_j t + \beta_{tT}^j, \quad (4.16)$$

where $\{\sigma_j\}_{j=1,2,\dots,n}$ are parameters, and $\{\beta_{tT}^j\}_{j=1,2,\dots,n}$ are independent Brownian bridges. The market filtration $\{\mathcal{F}_t\}$ is taken to be that generated collectively by $\{\xi_t^j\}_{j=1,2,\dots,n}$, and for the portfolio value $H_t = P_{tT} \mathbb{E}[H_T | \mathcal{F}_t]$ we have

$$H_t = P_{tT} \left[n - \mathbb{E}_t[X_1] - \mathbb{E}_t[X_1] \mathbb{E}_t[X_2] - \dots - \mathbb{E}_t[X_1] \mathbb{E}_t[X_2] \dots \mathbb{E}_t[X_n] \right]. \quad (4.17)$$

The conditional expectations appearing here can be calculated by means of formulae established earlier in the paper. The resulting dynamics for $\{H_t\}$ can thus be used to describe the evolution of correlations in the portfolio.

For example, if $\mathbb{E}_t[X_1]$ is low and $\mathbb{E}_t[X_2]$ is high, then the conditional probability at time t of a default at time T is small; whereas if $\mathbb{E}_t[X_1]$ were to increase suddenly, then the conditional probability of two or more defaults at T would rise as a consequence. Thus, the model is sufficiently rich to admit a detailed account of the correlation dynamics of the portfolio. The losses associated with individual tranches can be identified, and derivative structures associated with such tranches can be defined.

For example, a digital option that pays out in the event that there are three or more defaults has the payoff structure $H_T^{(3)} = X_1 X_2 X_3$. The homogeneous portfolio model has the property that the dynamics of equity-level and mezzanine-level tranches involve a relatively small number of factors. The market prices of tranches can be used to determine the *a priori* probabilities, and the market prices of options on tranches can be used to fix the information-flow parameters.

In summary, we see that the information-based framework for default dynamics introduced in this work is applicable to the analysis of both single-name and multi-name credit products.

Chapter 5

Assets with general cash-flow structures

5.1 Asset pricing: general overview of information-based framework

In the pricing of derivative securities, the starting point is usually the specification of a model for the price process of the underlying asset. Such models tend to be of an *ad hoc* nature. For example, in the Black-Scholes theory, the underlying asset has a geometric Brownian motion as its price process which, although very useful as a mathematical model is nevertheless widely agreed to be in some respects artificial. More generally, but more or less equally arbitrarily, the economy is often modelled by a probability space equipped with the filtration generated by a multi-dimensional Brownian motion, and it is assumed that asset prices are Ito processes that are adapted to this filtration. This particular example is of course the “standard” model within which a great deal of financial engineering has been carried out.

The basic methodological problem with the standard model (and the same applies to various generalisations thereof) is that the market filtration is fixed once and for all, and little or no comment is offered on the issue of “where it comes from”. In other words, the filtration, which represents the unfolding of information available to market participants, is modelled first, in an *ad hoc* manner, and then it is assumed that the asset price processes are adapted to it. But no indication is given about the nature of

this “information”; and it is not at all obvious, *a priori*, why the Brownian filtration, for example, should be regarded as providing information rather than simply noise.

In a complete market there is a sense in which the Brownian filtration provides all of the relevant information, and no irrelevant information. That is, in a complete market based on a Brownian filtration the asset price movements precisely reflect the information content of the filtration. Nevertheless, the notion that the market filtration should be “prespecified” is an unsatisfactory one in financial modelling.

The usual intuition behind the “prespecified-filtration” approach is to imagine that the filtration represents the unfolding in time of a succession of random events that “influence” the markets, thus causing prices to change. For example, a spell of bad weather in South America results in a decrease in the supply of coffee beans and hence an increase in the price of coffee.

The idea is that one then “abstractifies” these various influences in the form of a prespecified background filtration to which asset price processes are adapted. What is unsatisfactory about this is that so little structure is given to the filtration: price movements behave as though they were spontaneous. In reality, we expect the price-formation process to exhibit more structure. It would be out of place, in this thesis, to attempt a complete account of the process of price formation or to address the literature of market microstructure in a systematic way. Nevertheless, we can try to improve on the “prespecified” approach. In that spirit we proceed as follows.

We note that price changes arise from two rather distinct sources. The first source of price change is that resulting from changes in market-agent preferences—that is to say, changes in the pricing kernel. Movements in the pricing kernel are associated with (a) changes in investor attitudes towards risk, and (b) changes in investor “impatience”, i.e., the subjective discounting of future cash flows. But equally important, if not more so, are those changes in price resulting from the revelation to market agents of information about the future cash flows derivable from possession of a given asset.

When a market agent decides to buy or sell an asset, the decision is made in accordance with the information available to the agent concerning the likely future cash flows associated with the asset. A change in the information available to the market agent about a future cash flow will typically have an effect on the price at which they are willing to buy or sell, even if the agent’s preferences remain unchanged.

Let us consider, for example, the situation where one is thinking of purchasing an

item at a price that seems attractive. But then, by chance, one reads a newspaper article pointing out some undesirable feature of the product. After some reflection, one decides that the price is not so attractive. As a result, one decides not to buy, and eventually—possibly because other individuals have read the same report—the price drops.

The movements of the price of an asset should, therefore, be regarded as constituting an emergent phenomenon. To put the matter another way, the price process of an asset should be viewed as the output of the various decisions made relating to possible transactions in the asset, and these decisions in turn should be understood as being induced primarily by the flow of information to market participants.

Taking into account these elementary observations, we are now in a position in this chapter to propose the outlines of a general framework for asset pricing based on modelling of the flow of market information. The information will be that concerning the values of the future cash flows associated with the given assets. For example, if the asset represents a share in a firm that will make a single distribution at some pre-agreed date, then there is a single cash flow corresponding to the random amount of the distribution. If the asset is a credit-risky discount bond, then the future cash flow is the payout of the bond at the maturity date. In each case, based on the information available relating to the likely payouts of the given financial instrument, market participants determine, as best as they can, estimates for the value of the right to the impending cash flows. These estimates, in turn, lead to decisions concerning transactions, which then trigger movements in the price.

In this chapter we present a simple class of models capturing the essence of the scenario described above. As we remarked in the introduction of the thesis, in building this framework we have several criteria in mind that we would like to see satisfied:

- The first of these is that our model for the flow of market information should be intuitively appealing, and should allow for a reasonably sophisticated account of aggregate investor behaviour.
- At the same time, the model should be simple enough to allow one to derive explicit expressions for the asset price processes thus induced, in a suitably rich range of examples, as well as for various associated derivative price processes.
- The framework should also be flexible enough to allow for the modelling of assets

having complex cash-flow structures.

- Furthermore, it should be suitable for practical implementation, with the property that calibration and pricing can be carried out swiftly and robustly, at least for more elementary structures.
- We would like the framework to be mathematically sound, and to be manifestly arbitrage-free.

In what follows we shall attempt to make some headway with these diverse criteria.

5.2 The three ingredients

In asset pricing we require three basic ingredients, namely, (a) the cash flows, (b) the investor preferences, and (c) the flow of information available to market participants. Translated into somewhat more mathematical language, these ingredients amount to the following: (a') cash flows are modelled as random variables; (b') investor preferences are modelled with the determination of a pricing kernel; and (c') the market information flow is modelled with the specification of a filtration. As we have indicated above, asset pricing theory conventionally attaches more weight to (a) and (b) than to (c). In this paper, however, we emphasise the importance of ingredient (c).

Our theory will be based on modelling the flow of information accessible to market participants concerning the future cash flows associated with the possession of an asset, or with a position in a financial contract. The idea that information should play a foundational role in asset pricing has been long appreciated—see, e.g., Back 1992, Back & Baruch 2004, and references cited therein. Our contribution to this area will involve an explicit technique for modelling the filtration. We start by setting the notation and introducing the assumptions employed in this paper. We model the financial markets with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on which a filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ will be constructed. The probability measure \mathbb{Q} is understood to be the risk-neutral measure, and the filtration $\{\mathcal{F}_t\}$ is understood to be the market filtration. All asset-price processes and other information-providing processes accessible to market participants will be adapted to $\{\mathcal{F}_t\}$. We do not regard $\{\mathcal{F}_t\}$ as something handed to us on a platter. Instead, it will be modelled explicitly.

Several simplifying assumptions will be made, so that we can concentrate our efforts on the problems associated with the flow of market information. The first of these assumptions is the use of the risk-neutral measure. The “real” probability measure does not enter into the present investigation.

We leap over that part of the economic analysis that determines the pricing measure. More specifically, we assume the absence of arbitrage and the existence of an established pricing kernel (see, e.g., Cochrane 2005, and references cited therein). With these conditions the existence of a unique risk-neutral pricing measure \mathbb{Q} is ensured, even though the markets we consider will, in general, be incomplete. For a discussion of the issues associated with pricing in incomplete markets see, e.g., Carr *et al.* 2001.

The second assumption is that we take the default-free system of interest rates to be deterministic. The view is that we should first develop our framework in a simplified setting, where certain essentially macroeconomic issues are put to one side; then, once we are satisfied with the tentative framework, we can attempt to generalise it in such a way as to address these issues. We therefore assume a deterministic default-free discount bond system. The absence of arbitrage implies that the corresponding system of discount functions $\{P_{tT}\}_{0 \leq t \leq T < \infty}$ can be written in the form $P_{tT} = P_{0T}/P_{0t}$ for $t \leq T$, where $\{P_{0t}\}_{0 \leq t < \infty}$ is the initial discount function, which we take to be part of the initial data of the model. The function $\{P_{0t}\}_{0 \leq t < \infty}$ is assumed to be differentiable and strictly decreasing, and to satisfy $0 < P_{0t} \leq 1$ and $\lim_{t \rightarrow \infty} P_{0t} = 0$. These conditions can be relaxed somewhat for certain applications. A method for extending in the information-based framework to a background stochastic interest rate environment is considered in Rutkowski & Yu 2005, using a forward measure technique (Geman *et al.* 1995).

We also assume, for simplicity, that all cash flows occur at pre-determined dates. Now clearly for some purposes we would like to allow for cash flows occurring effectively at random times—in particular, at stopping times associated with the market filtration. But in the present exposition we want to avoid the idea of a “prespecified” filtration with respect to which stopping times are defined. We take the view that the market filtration is a “derived” notion, generated by information about impending cash flows, and by the actual values of cash flows when they occur. In the present paper we regard a “randomly-timed” cash flow as being a set of random cash flows occurring at various times—and with a joint distribution function that ensures only one of these flows is non-zero. Hence in our view the ontological status of a cash flow is that its timing is

definite, only the amount is random—and that cash flows occurring at different times are, by their nature, different cash flows.

5.3 Modelling the cash flows

First we consider the case of a single isolated cash flow occurring at time T , represented by a random variable D_T . We assume that $D_T \geq 0$. The value S_t of the cash flow at any earlier time t in the interval $0 \leq t < T$ is then given by the discounted conditional expectation of D_T :

$$S_t = P_{tT} \mathbb{E}^{\mathbb{Q}} [D_T | \mathcal{F}_t]. \quad (5.1)$$

In this way we model the price process $\{S_t\}_{0 \leq t < T}$ of a limited-liability asset that pays the single dividend D_T at time T . The construction of the price process here is carried out in such a way as to guarantee an arbitrage-free market if other assets are priced by the same method—see Davis 2004 for a related point of view. With a slight abuse of terminology we shall use the terms “cash flow” and “dividend” more or less interchangeably. If a more specific use of one of these terms is needed, then this will be evident from the context. We adopt the convention that when the dividend is paid the asset price goes “ex-dividend” immediately. Hence in the example above we have $\lim_{t \rightarrow T} S_t = D_T$ and $S_T = 0$.

In the case that the asset pays a sequence of dividends D_{T_k} ($k = 1, 2, \dots, n$) on the dates T_k the price (for values of t earlier than the time of the first dividend) is

$$S_t = \sum_{k=1}^n P_{tT_k} \mathbb{E}^{\mathbb{Q}} [D_{T_k} | \mathcal{F}_t]. \quad (5.2)$$

More generally, taking into account the ex-dividend behaviour, we have

$$S_t = \sum_{k=1}^n \mathbf{1}_{\{t < T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} [D_{T_k} | \mathcal{F}_t]. \quad (5.3)$$

It turns out to be useful if we adopt the convention that a discount bond also goes ex-dividend on its maturity date. In the case of a discount bond we assume that the price of the bond is given, for dates earlier than the maturity date, by the product of the principal and the relevant discount factor. But at maturity (when the principal is paid) the value of the bond drops to zero. In the case of a coupon bond, there is

a downward jump in the price of the bond at the time a coupon is paid (the value lost may be captured back in the form of an “accrued interest” payment). In this way we obtain a consistent treatment of the “ex-dividend” behaviour of the asset price processes under consideration. With this convention it follows that all price processes have the property that they are right continuous with left limits.

5.4 Construction of the market information flow

Now we present a simple model for the flow of market information. We consider first the case of a single distribution, occurring at time T , and assume that market participants have only partial information about the upcoming cash flow D_T . The information available in the market about the cash flow is assumed to be contained in a process $\{\xi_t\}_{0 \leq t \leq T}$ defined by:

$$\xi_t = \sigma D_T t + \beta_{tT}. \quad (5.4)$$

We call $\{\xi_t\}$ the market information process. The information process is composed of two parts. The term $\sigma D_T t$ contains the “true information” about the upcoming dividend. This term grows in magnitude as t increases.

The process $\{\beta_{tT}\}_{0 \leq t \leq T}$ is a standard Brownian bridge over the time interval $[0, T]$. Thus $\beta_{0T} = 0$, $\beta_{TT} = 0$, and at time t the random variable β_{tT} has mean zero and variance $t(T-t)/T$; the covariance of β_{sT} and β_{tT} for $s \leq t$ is $s(T-t)/T$. We assume that D_T and $\{\beta_{tT}\}$ are independent. Thus the information contained in the bridge process is “pure noise”. The information contained in $\{\xi_t\}$ is clearly unchanged if we multiply $\{\xi_t\}$ by some overall scale factor.

An earlier well-known example of the use of a Brownian bridge process in the context of interest rate modelling can be found in Ball & Torous 1983; they are concerned, however, with the default-free interest rate term structure, and their model is unrelated to the approach presented in this thesis.

We assume that the market filtration $\{\mathcal{F}_t\}$ is generated by the market information process. That is to say, we assume that $\{\mathcal{F}_t\} = \{\mathcal{F}_t^\xi\}$, where $\{\mathcal{F}_t^\xi\}$ is the filtration generated by $\{\xi_t\}$. The dividend D_T is therefore \mathcal{F}_T -measurable, but is not \mathcal{F}_t -measurable for $t < T$. Thus the value of D_T becomes “known” at time T , but not earlier. The bridge process $\{\beta_{tT}\}$ is not adapted to $\{\mathcal{F}_t\}$ and thus is not directly accessible to

market participants. This reflects the fact that until the dividend is paid the market participants cannot distinguish the “true information” from the “noise” in the market.

The introduction of the Brownian bridge models the fact that market perceptions, whether valid or not, play a role in determining asset prices. Initially, all available information is used to determine the *a priori* risk-neutral probability distribution for D_T . Then after the passage of time rumours, speculations, and general disinformation start circulating, reflected in the steady increase in the variance of the Brownian bridge. Eventually the variance drops and falls to zero at the time the distribution to the shareholders is made. The parameter σ represents the rate at which information about the true value of D_T is revealed as time progresses. If σ is low, the value of D_T is effectively hidden until very near the time of the dividend payment; whereas if σ is high, then the value of the cash flow is for all practical purposes revealed very quickly.

In the example under consideration we have made some simplifying assumptions concerning our choice for the market information structure. For instance, we assume that σ is constant. We have also assumed that the random dividend D_T enters directly into the structure of the information process, and enters linearly. As we shall indicate later, a more general and in some respects more natural setup is to let the information process depend on a random variable X_T which we call a “market factor”; then the dividend is regarded as a function of the market factor. This arrangement has the advantage that it easily generalises to the situation where a cash flow might depend on several independent market factors, or indeed where cash flows associated with different financial instruments have one or more market factors in common. But for the moment we regard the single cash flow D_T as being the relevant market factor, and we assume the information-flow rate to be constant.

With the market information structure described above for a single cash flow in place, we proceed to construct the associated price dynamics. The price process $\{S_t\}$ for a share in the firm paying the specified dividend is given by formula (5.1). It is assumed that the *a priori* probability distribution of the dividend D_T is known. This distribution is regarded as part of the initial data of the problem, which in some cases can be calibrated from knowledge of the initial price of the asset, possibly along with other price data.

The general problem of how the *a priori* distribution is obtained is an important one—any asset pricing model has to confront some version of this issue—which we

defer for later consideration. The main point is that the initial distribution is not to be understood as being “absolutely” determined, but rather represents the “best estimate” for the distribution given the data available at that time, in accordance with what one might call a Bayesian point of view. We recall the fact that the information process $\{\xi_t\}$ is Markovian, which we showed earlier. Making use of this property of the information process together with the fact that D_T is \mathcal{F}_T -measurable we deduce that

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \mathbb{E}^{\mathbb{Q}} [D_T | \xi_t]. \quad (5.5)$$

If the random variable D_T that represents the payoff has a continuous distribution, then the conditional expectation in (5.5) can be expressed in the form

$$\mathbb{E}^{\mathbb{Q}} [D_T | \xi_t] = \int_0^\infty x \pi_t(x) dx. \quad (5.6)$$

Here $\pi_t(x)$ is the conditional probability density for the random variable D_T :

$$\pi_t(x) = \frac{d}{dx} \mathbb{Q}(D_T \leq x | \xi_t). \quad (5.7)$$

We implicitly assume appropriate technical conditions on the distribution of the dividend that will suffice to ensure the existence of the expressions under consideration. Also, for convenience we use a notation appropriate for continuous distributions, though corresponding results can easily be inferred for discrete distributions, or more general distributions, by slightly modifying the stated assumptions and conclusions.

Bearing in mind these points, we note that the conditional probability density process for the dividend can be worked out by use of a form of the Bayes formula:

$$\pi_t(x) = \frac{p(x) \rho(\xi_t | D_T = x)}{\int_0^\infty p(x) \rho(\xi_t | D_T = x) dx}. \quad (5.8)$$

Here $p(x)$ denotes the *a priori* probability density for D_T , which we assume is known as an initial condition, and $\rho(\xi_t | D_T = x)$ denotes the conditional density function for the random variable ξ_t given that $D_T = x$. Since β_{tT} is a Gaussian random variable with variance $t(T-t)/T$, the conditional probability density for ξ_t is

$$\rho(\xi_t | D_T = x) = \sqrt{\frac{T}{2\pi t(T-t)}} \exp\left(-\frac{(\xi_t - \sigma t x)^2 T}{2t(T-t)}\right). \quad (5.9)$$

Inserting this expression into the Bayes formula, we get

$$\pi_t(x) = \frac{p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right]}{\int_0^\infty p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx}. \quad (5.10)$$

We thus obtain the following result for the asset price:

Proposition 5.4.1 *The information-based price process $\{S_t\}_{0 \leq t \leq T}$ of a limited-liability asset that pays a single dividend D_T at time T with distribution*

$$\mathbb{Q}(D_T \leq y) = \int_0^y p(x) dx \quad (5.11)$$

is given by

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \frac{\int_0^\infty xp(x) \exp \left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2} \sigma^2 x^2 t) \right] dx}{\int_0^\infty p(x) \exp \left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2} \sigma^2 x^2 t) \right] dx}, \quad (5.12)$$

where $\xi_t = \sigma D_T t + \beta_{tT}$ is the market information.

5.5 Asset price dynamics in the case of a single random cash flow

In order to analyse the properties of the price process deduced above, and to be able to compare it with other models, we need to work out the dynamics of $\{S_t\}$. One of the advantages of the model under consideration is that we have a completely explicit expression for the price process at our disposal. Thus in obtaining the dynamics we need to find the stochastic differential equation of which $\{S_t\}$ is the solution. This turns out to be an interesting exercise because it offers some insights into what we mean by the assertion that market price dynamics should be regarded as constituting an “emergent phenomenon”. The basic mathematical tool that we make use of here is nonlinear filtering theory—see, e.g., Bucy & Joseph 1968, Kallianpur & Striebel 1968, Davis & Marcus 1981, and Liptser & Shiryaev 2000. The specific applications that we make of the theory here are original.

To obtain the dynamics associated with the price process $\{S_t\}$ of a single-dividend-paying asset, let us write

$$D_{tT} = \mathbb{E}^{\mathbb{Q}}[D_T | \xi_t] \quad (5.13)$$

for the conditional expectation of D_T with respect to the market information ξ_t . Evidently, D_{tT} can be expressed in the form $D_{tT} = D(\xi_t, t)$, where the function $D(\xi, t)$ is defined by

$$D(\xi, t) = \frac{\int_0^\infty xp(x) \exp \left[\frac{T}{T-t} (\sigma x \xi - \frac{1}{2} \sigma^2 x^2 t) \right] dx}{\int_0^\infty p(x) \exp \left[\frac{T}{T-t} (\sigma x \xi - \frac{1}{2} \sigma^2 x^2 t) \right] dx}. \quad (5.14)$$

A straightforward calculation making use of the Ito rules shows that the dynamical equation for the conditional expectation $\{D_{tT}\}$ is given by

$$dD_{tT} = \frac{\sigma T}{T-t} V_t \left[\frac{1}{T-t} (\xi_t - \sigma T D_{tT}) dt + d\xi_t \right]. \quad (5.15)$$

Here V_t is the conditional variance of the dividend:

$$V_t = \int_0^\infty x^2 \pi_t(x) dx - \left(\int_0^\infty x \pi_t(x) dx \right)^2. \quad (5.16)$$

Therefore, if we define a new process $\{W_t\}_{0 \leq t < T}$ by setting

$$W_t = \xi_t - \int_0^t \frac{1}{T-s} (\sigma T D_{sT} - \xi_s) ds, \quad (5.17)$$

we find, after some rearrangement of terms, that

$$dD_{tT} = \frac{\sigma T}{T-t} V_t dW_t. \quad (5.18)$$

For the dynamics of the asset price process we thus have

$$dS_t = r_t S_t dt + \Gamma_{tT} dW_t, \quad (5.19)$$

where the short rate r_t is given by $r_t = -d \ln P_{0t}/dt$, and the absolute price volatility Γ_{tT} is given by

$$\Gamma_{tT} = P_{tT} \frac{\sigma T}{T-t} V_t. \quad (5.20)$$

A slightly different way of arriving at this result is as follows. We start with the conditional probability process $\pi_t(x)$. Then, using the same notation as above, for the dynamics of $\pi_t(x)$ we obtain

$$d\pi_t(x) = \frac{\sigma T}{T-t} (x - D_{tT}) \pi_t(x) dW_t. \quad (5.21)$$

Since the asset price is given by

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \int_0^\infty x \pi_t(x) dx, \quad (5.22)$$

we are thus able to infer the dynamics of the price $\{S_t\}$ from the dynamics of the conditional probability $\{\pi_t(x)\}$, once we take into account the formula for the conditional variance.

As we have demonstrated earlier, in the context of a discrete payment, the process $\{W_t\}$ defined in (5.17) is an $\{\mathcal{F}_t\}$ -Brownian motion. Hence *from the point of view of the market it is the process $\{W_t\}$ that drives the asset price dynamics*. In this way our framework resolves the somewhat paradoxical point of view usually adopted in financial modelling in which $\{W_t\}$ is regarded as “noise”, and yet also generates the market information flow. And thus, instead of hypothesising the existence of a driving process for the dynamics of the markets, we are able from the information-based perspective to deduce the existence of such a process.

The information-flow parameter σ determines the overall magnitude of the volatility. In fact, as we have remarked earlier, the parameter σ plays a role that is in many respects analogous to the similarly-labelled parameter in the Black-Scholes theory. Thus, we can say that the rate at which information is revealed in the market determines the overall magnitude of the market volatility. In other words, everything else being the same, if we increase the information flow rate, then the market volatility will increase as well. It is ironic that, according to this point of view, those mechanisms that one might have thought were designed to make markets more efficient—e.g., globalisation of the financial markets, reduction of trade barriers, improved communications, a more robust regulatory environment, and so on—can have the effect of increasing market volatility, and hence market risk, rather than reducing it.

5.6 European-style options on a single-dividend paying asset

Before we turn to the consideration of more general cash flows and more general market information structures, let us consider the problem of pricing a derivative on an asset for which the price process is governed by the dynamics (5.19). We shall look at the valuation problem for a European-style call option on such an asset, with strike price K , and exercisable at a fixed maturity date t . The option is written on an asset that pays a single dividend D_T at time $T > t$. The value of the option at time 0 is clearly

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{Q}} [(S_t - K)^+]. \quad (5.23)$$

Inserting the information-based expression for the price S_t derived in the previous section into this formula, we obtain

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{Q}} \left[\left(P_{tT} \int_0^\infty x \pi_t(x) dx - K \right)^+ \right]. \quad (5.24)$$

For convenience we write the conditional probability $\pi_t(x)$ in the form

$$\pi_t(x) = \frac{p_t(x)}{\int_0^\infty p_t(x) dx}, \quad (5.25)$$

where the “unnormalised” density process $\{p_t(x)\}$ is defined by

$$p_t(x) = p(x) \exp \left[\frac{T}{T-t} \left(\sigma x \xi_t - \frac{1}{2} \sigma^2 x^2 t \right) \right]. \quad (5.26)$$

Substituting (5.26) into (5.24) we find that the initial value of the option is given by

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{\Phi_t} \left(\int_0^\infty (P_{tT} x - K) p_t(x) dx \right)^+ \right], \quad (5.27)$$

where

$$\Phi_t = \int_0^\infty p_t(x) dx. \quad (5.28)$$

The random variable Φ_t can be used to introduce a measure \mathbb{B}_T applicable over the time horizon $[0, t]$, which as before we call the “bridge measure”. The call option price can thus be written:

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{B}_T} \left[\left(\int_0^\infty (P_{tT} x - K) p_t(x) dx \right)^+ \right]. \quad (5.29)$$

The special feature of the bridge measure is that the random variable ξ_t is Gaussian under \mathbb{B}_T . In particular, under the measure \mathbb{B}_T we find that $\{\xi_t\}$ has mean 0 and variance $t(T-t)/T$. Since $p_t(x)$ can be expressed as a function of ξ_t , when we carry out the expectation above we are led to a tractable formula for C_0 .

To obtain the value of the option we define a constant ξ^* (the critical value) by the following condition:

$$\int_0^\infty (P_{tT} x - K) p(x) \exp \left[\frac{T}{T-t} \left(\sigma x \xi^* - \frac{1}{2} \sigma^2 x^2 t \right) \right] dx = 0. \quad (5.30)$$

Then the option price is given by:

$$C_0 = P_{0T} \int_0^\infty x p(x) N(-z^* + \sigma x \sqrt{\tau}) dx - P_{0t} K \int_0^\infty p(x) N(-z^* + \sigma x \sqrt{\tau}) dx, \quad (5.31)$$

where

$$\tau = \frac{tT}{T-t}, \quad z^* = \xi^* \sqrt{\frac{T}{t(T-t)}}, \quad (5.32)$$

and $N(x)$ denotes the standard normal distribution function. We see that a tractable expression is obtained, and that it is of the Black-Scholes type. The option pricing problem, even for general $p(x)$, reduces to an elementary numerical problem. It is interesting to note that although the probability distribution for the price S_t at time t is not of a “standard” type, nevertheless the option valuation problem remains a solvable one.

5.7 Dividend structures: specific examples

In this section we consider the dynamics of assets with various specific continuous dividend structures. First we look at a simple asset for which the cash flow is exponentially distributed. The *a priori* probability density for D_T is thus of the form

$$p(x) = \frac{1}{\delta} \exp\left(-\frac{x}{\delta}\right), \quad (5.33)$$

where δ is a constant. The idea of an exponentially distributed payout is of course somewhat artificial; nevertheless we can regard this as a useful model for the situation where little is known about the probability distribution of the dividend, apart from its mean. Then from formula (5.12) we find that the corresponding asset price is:

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \frac{\int_0^\infty x \exp(-x/\delta) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx}{\int_0^\infty \exp(-x/\delta) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx}. \quad (5.34)$$

We note that $S_0 = P_{0T}\delta$, so we can calibrate the choice of δ by use of the initial price. The integrals in the numerator and denominator in the expression above can be worked out explicitly. Hence, we obtain a closed-form expression for the price in the case of an asset with an exponentially-distributed terminal cash flow. This is given by:

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \left[\frac{\exp\left(-\frac{1}{2}B_t^2/A_t\right)}{\sqrt{2\pi A_t} N(B_t/\sqrt{A_t})} + \frac{B_t}{A_t} \right], \quad (5.35)$$

where

$$A_t = \sigma^2 \frac{tT}{(T-t)}, \quad (5.36)$$

and

$$B_t = \frac{\sigma T}{(T-t)} \xi_t - \delta^{-1}. \quad (5.37)$$

Next we consider the case of an asset for which the single dividend paid at time T has a gamma distribution. More specifically, we assume the probability density is of the form

$$p(x) = \frac{\delta^n}{(n-1)!} x^{n-1} \exp(-\delta x), \quad (5.38)$$

where δ is a positive real number and n is a positive integer. This choice for the probability density also leads to a closed-form expression. We find that

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \frac{\sum_{k=0}^n \binom{n}{k} A_t^{\frac{1}{2}k-n} B_t^{n-k} F_k(-B_t/\sqrt{A_t})}{\sum_{k=0}^{n-1} \binom{n-1}{k} A_t^{\frac{1}{2}k-n+1} B_t^{n-k-1} F_k(-B_t/\sqrt{A_t})}, \quad (5.39)$$

where A_t and B_t are as above, and

$$F_k(x) = \int_x^\infty z^k \exp\left(-\frac{1}{2}z^2\right) dz. \quad (5.40)$$

A recursion formula can be worked out for the function $F_k(x)$. This is given by

$$(k+1)F_k(x) = F_{k+2}(x) - x^{k+1} \exp\left(-\frac{1}{2}x^2\right), \quad (5.41)$$

from which it follows that

$$\begin{aligned} F_0(x) &= \sqrt{2\pi} N(-x), \\ F_1(x) &= e^{-\frac{1}{2}x^2}, \\ F_2(x) &= xe^{-\frac{1}{2}x^2} + \sqrt{2\pi} N(-x), \\ F_3(x) &= (x^2 + 2)e^{-\frac{1}{2}x^2}, \end{aligned} \quad (5.42)$$

and so on. In general, the polynomial parts of $\{F_k(x)\}_{k=0,1,2,\dots}$ are related to the Legendre polynomials.

Chapter 6

X -factor analysis and applications

6.1 Multiple cash flows

In this chapter we generalise the preceding material to the situation where the asset pays multiple dividends. This allows us to consider a wider range of financial instruments. Let us write D_{T_k} ($k = 1, \dots, n$) for a set of random cash flows paid at the pre-designated dates T_k ($k = 1, \dots, n$). Possession of the asset at time t entitles the bearer to the cash flows occurring at times $T_k > t$. For simplicity we assume n is finite.

For each value of k we introduce a set of independent random variables $X_{T_k}^\alpha$ ($\alpha = 1, \dots, m_k$), which again we call market factors or X -factors. For each value of α we assume that the factor $X_{T_k}^\alpha$ is \mathcal{F}_{T_k} -measurable, where $\{\mathcal{F}_t\}$ is the market filtration.

Intuitively speaking, for each value of k the market factors $\{X_{T_j}^\alpha\}_{j \leq k}$ represent the independent elements that determine the cash flow occurring at time T_k . Thus for each value of k the cash flow D_{T_k} is assumed to have the following structure:

$$D_{T_k} = \Delta_{T_k}(X_{T_1}^\alpha, X_{T_2}^\alpha, \dots, X_{T_k}^\alpha), \quad (6.1)$$

where $\Delta_{T_k}(X_{T_1}^\alpha, X_{T_2}^\alpha, \dots, X_{T_k}^\alpha)$ is a function of $\sum_{j=1}^k m_j$ variables. For each cash flow it is, so to speak, the job of the financial analyst (or actuary) to determine the relevant independent market factors, and the form of the cash-flow function Δ_{T_k} for each cash flow. With each market factor $X_{T_k}^\alpha$ we associate an information process $\{\xi_{tT_k}^\alpha\}_{0 \leq t \leq T_k}$ of the form

$$\xi_{tT_k}^\alpha = \sigma_{T_k}^\alpha X_{T_k}^\alpha t + \beta_{tT_k}^\alpha. \quad (6.2)$$

Here $\sigma_{T_k}^\alpha$ is an information flux parameter, and $\{\beta_{tT_k}^\alpha\}$ is a standard Brownian bridge process over the interval $[0, T_k]$. We assume that the X -factors and the Brownian bridge processes are all independent. The parameter $\sigma_{T_k}^\alpha$ determines the rate at which information about the value of the market factor $X_{T_k}^\alpha$ is revealed. The Brownian bridge $\beta_{tT_k}^\alpha$ represents the associated noise. We assume that the market filtration $\{\mathcal{F}_t\}$ is generated by the totality of the independent information processes $\{\xi_{tT_k}^\alpha\}_{0 \leq t \leq T_k}$ for $k = 1, 2, \dots, n$ and $\alpha = 1, 2, \dots, m_k$. Hence, the price process of the asset is given by

$$S_t = \sum_{k=1}^n \mathbf{1}_{\{t < T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} \left[D_{T_k} \middle| \mathcal{F}_t \right]. \quad (6.3)$$

Again, here \mathbb{Q} represents the risk-neutral measure, and the default-free interest rate term structure is assumed to be deterministic.

6.2 Simple model for dividend growth

As an elementary example of a multi-dividend structure, we shall look at a simple growth model for dividends in the equity markets. We consider an asset that pays a sequence of dividends D_{T_k} , where each dividend date has an associated X -factor. Let $\{X_{T_k}\}_{k=1, \dots, n}$ be a set of independent, identically-distributed X -factors, each with mean $1 + g$. The dividend structure is assumed to be of the form

$$D_{T_k} = D_0 \prod_{j=1}^k X_{T_j}, \quad (6.4)$$

where D_0 is a constant. The parameter g can be interpreted as the dividend growth factor, and D_0 can be understood as representing the most recent dividend before time zero. For the price process of the asset we have:

$$S_t = D_0 \sum_{k=1}^n \mathbf{1}_{\{t < T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} \left[\prod_{j=1}^k X_{T_j} \middle| \mathcal{F}_t \right]. \quad (6.5)$$

Since the X -factors are independent of one another, the conditional expectation of the product appearing in this expression factorises into a product of conditional expectations, and each such conditional expectation can be written in the form of an expression of the type we have already considered. As a consequence we are led to a completely tractable family of dividend growth models.

6.3 A natural class of stochastic volatility models

Based on the general model introduced in the previous section, we are now in a position to make an observation concerning the nature of stochastic volatility. In particular, we shall show how stochastic volatility arises in the information-based framework. This is achieved without the need for any *ad hoc* assumptions concerning the dynamics of the stochastic volatility. In fact, a very specific dynamical model for stochastic volatility is obtained—thus leading in principal to a possible means by which the theory proposed here might be tested.

We shall work out the volatility associated with the dynamics of the general asset price process $\{S_t\}$ given by equation (6.3). The result is given in Proposition 6.3.1 below.

First, as an example, we consider the dynamics of an asset that pays a single dividend D_T at time T . We assume that the dividend depends on a set of market factors $\{X_T^\alpha\}_{\alpha=1,\dots,m}$. For $t < T$ we then have:

$$\begin{aligned} S_t &= P_{tT} \mathbb{E}^{\mathbb{Q}} [\Delta_T (X_T^1, \dots, X_T^m) | \xi_{tT}^1, \dots, \xi_{tT}^m] \\ &= P_{tT} \int \cdots \int \Delta_T(x^1, \dots, x^m) \pi_{tT}^1(x_1) \cdots \pi_{tT}^m(x_m) dx_1 \cdots dx_m. \end{aligned} \quad (6.6)$$

Here the various conditional probability density functions $\pi_{tT}^\alpha(x)$ for $\alpha = 1, \dots, m$ are

$$\pi_{tT}^\alpha(x) = \frac{p^\alpha(x) \exp \left[\frac{T}{T-t} (\sigma^\alpha x \xi_{tT}^\alpha - \frac{1}{2} (\sigma^\alpha)^2 x^2 t) \right]}{\int_0^\infty p^\alpha(x) \exp \left[\frac{T}{T-t} (\sigma^\alpha x \xi_{tT}^\alpha - \frac{1}{2} (\sigma^\alpha)^2 x^2 t) \right] dx}, \quad (6.7)$$

where $p^\alpha(x)$ denotes the *a priori* probability density function for the market factor X_T^α . The drift of $\{S_t\}_{0 \leq t < T}$ is given by the short rate of interest. This is because \mathbb{Q} is the risk-neutral measure, and no dividend is paid before T .

Thus, we are left with the problem of determining the volatility of $\{S_t\}$. We find that for $t < T$ the dynamical equation of $\{S_t\}$ assumes the following form:

$$dS_t = r_t S_t dt + \sum_{\alpha=1}^m \Gamma_{tT}^\alpha dW_t^\alpha. \quad (6.8)$$

Here the volatility term associated with factor number α is given by

$$\Gamma_{tT}^\alpha = \sigma^\alpha \frac{T}{T-t} P_{tT} \text{Cov} [\Delta_T (X_T^1, \dots, X_T^m), X_T^\alpha | \mathcal{F}_t], \quad (6.9)$$

and $\{W_t^\alpha\}$ denotes the Brownian motion associated with the information process $\{\xi_t^\alpha\}$, as defined in (5.17). The absolute volatility of $\{S_t\}$ is evidently of the form

$$\Gamma_t = \left(\sum_{\alpha=1}^m (\Gamma_{tT}^\alpha)^2 \right)^{1/2}. \quad (6.10)$$

For the dynamics of a multi-factor single-dividend-paying asset we can thus write

$$dS_t = r_t S_t dt + \Gamma_t dZ_t, \quad (6.11)$$

where the $\{\mathcal{F}_t\}$ -Brownian motion $\{Z_t\}$ that drives the asset-price process is defined by

$$Z_t = \int_0^t \frac{1}{\Gamma_s} \sum_{\alpha=1}^m \Gamma_{sT}^\alpha dW_s^\alpha. \quad (6.12)$$

The key point is that in the case of a multi-factor model we obtain an unhedgeable stochastic volatility. That is, although the asset price is in effect driven by a single Brownian motion, its volatility depends on a multiplicity of Brownian motions. This means that in general an option position cannot be hedged with a position in the underlying asset. The components of the volatility vector are given by the covariances of the terminal cash flow with the independent market factors. Unhedgeable stochastic volatility emerges from the multiplicity of uncertain elements in the market that affect the value of the future cash flow. As a consequence we see that *in this framework we obtain a possible explanation for the origin of stochastic volatility.*

This result can be contrasted with, say, the Heston model (Heston 1993), which despite its popularity suffers somewhat from the fact that it is essentially *ad hoc* in nature. Much the same has to be said for the various generalisations of the Heston model that have been so widely used in commercial applications. The approach to stochastic volatility proposed in this thesis is thus of a new character.

Of course, stochastic volatility is the “rule” rather than the exception in asset price modelling—that is to say, the “generic” model will have stochastic volatility; the problem is, to select on a rational basis a natural class of stochastic volatility models from the myriad of possible such models. In the present analysis, what we mean by “rational basis” is that the model is deduced from a set of specific, simple assumptions concerning the underlying cash flows and market filtration. It is an open question whether some of the well-known stochastic volatility models can be re-derived from an information-based perspective.

Expression (6.8) generalises naturally to the case in which the asset pays a set of dividends D_{T_k} ($k = 1, \dots, n$), and for each k the dividend depends on the X -factors $\{\{X_{T_j}^\alpha\}_{j=1, \dots, k}^{\alpha=1, \dots, m_j}\}$. The result can be summarised as below.

Proposition 6.3.1 *The price process of a multi-dividend asset has the following dynamics:*

$$\begin{aligned} dS_t &= r_t S_t dt \\ &+ \sum_{k=1}^n \mathbf{1}_{\{t < T_k\}} P_{tT_k} \sum_{j=1}^k \sum_{\alpha=1}^{m_j} \frac{\sigma_j^\alpha T_j}{T_j - t} \text{Cov}_t \left[\Delta_{T_k}, X_{T_j}^\alpha \right] dW^{\alpha j} \\ &+ \sum_{k=1}^n \Delta_{T_k} d\mathbf{1}_{\{t < T_k\}}, \end{aligned} \quad (6.13)$$

where $\Delta_{T_k} = \Delta_{T_k}(X_{T_1}^\alpha, X_{T_2}^\alpha, \dots, X_{T_k}^\alpha)$ is the dividend at time T_k ($k = 1, 2, \dots, n$).

We conclude that the multi-factor, multi-dividend situation is also fully tractable. A straightforward extension of Proposition 6.3.1 then allows us to formulate the joint price dynamics of a system of assets, the associated dividend flows of which may depend on common market factors. As a consequence, it follows that a rather specific model for stochastic volatility and correlation emerges for such a system of assets, and it is one of the principal conclusions of this work that such a model, which is entirely natural in character, can indeed be formulated.

The information-based X -factor approach presented here thus offers new insights into the nature of volatility and correlation, and as such may find applications in a number of different areas of financial risk analysis. We have in mind, in particular, applications to equity portfolios, credit portfolios, and insurance, all of which exhibit important intertemporal market correlation effects. We also have in mind the problem of firm-wide risk management and optimal capital allocation for banking institutions. A further application of the X -factor method may arise in connection with the modelling of asymmetric information flows and insider trading, i.e. in stratified markets, where some participants have better access to information than others, but all agents act optimally (cf. Föllmer *et al.* 1999).

6.4 Black-Scholes model from an information-based perspective

In the section above we have derived the dynamics of the price process of a multi-dividend-paying asset in the case where the dividends depend on a set of market factors, and we have seen how a model for stochastic volatility emerges in this context. An interesting question that can be asked now is whether it is possible to recover the standard Black-Scholes geometric Brownian motion asset-price model by a particular choice of the dividend structure and market factors. It is arguable that any asset pricing framework with a claim of generality should include the Black-Scholes asset price model as a special case. The set-up we consider is the following.

We consider a limited-liability asset that pays no interim dividends, and that at time T is sold off for the value S_T . Thus in the present example S_T plays the role of the “single” cash flow Δ_T . Our goal is to find the price process $\{S_t\}_{0 \leq t \leq T}$ of such an asset. In particular, we look at the case when S_T is log-normally distributed and is of the form

$$S_T = S_0 \exp \left(rT - \frac{1}{2} \nu^2 T + \nu \sqrt{T} X_T \right), \quad (6.14)$$

where S_0 , r , and ν are given constants and X_T is a standard normally distributed random variable. The corresponding information process is given by

$$\xi_t = \sigma X_T t + \beta_{tT}, \quad (6.15)$$

and the price process $\{S_t\}_{0 \leq t \leq T}$ is then obtained by using (6.6). The result is:

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} S_0 \exp \left[rT - \frac{1}{2} \nu^2 T + \frac{1}{2} \frac{\nu \sqrt{T}}{\sigma^2 \tau + 1} + \frac{\nu \sqrt{T} \sigma \tau}{t(\sigma^2 \tau + 1)} \xi_t \right], \quad (6.16)$$

where we set $\tau = tT/(T - t)$.

The dynamics of a single-dividend paying asset, in the case that the dividend is a function of a single random variable, are given by the following stochastic differential equation, which is a special case of Proposition 6.3.1:

$$dS_t = rS_t dt + \frac{\sigma T}{T - t} \text{Cov}_t [S_T, X_T] dW_t. \quad (6.17)$$

The conditional covariance $\text{Cov}_t [S_T, X_T]$ between the random variables S_T and X_T can be written as follows:

$$\text{Cov}_t [S_T, X_T] = \mathbb{E}_t [S_T X_T] - \mathbb{E}_t [S_T] \mathbb{E}_t [X_T]. \quad (6.18)$$

The conditional expectation, with respect to ξ_t , of a function $f(X_T)$ is

$$\mathbb{E}_t [f(X_T)] = \frac{\int_{-\infty}^{\infty} f(x)p(x) \exp \left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2} \sigma^2 x^2 t) \right] dx}{\int_{-\infty}^{\infty} p(x) \exp \left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2} \sigma^2 x^2 t) \right] dx}, \quad (6.19)$$

where $p(x)$ is the *a priori* probability density function associated with the random variable X_T . Since we assume that X_T is standard normally distributed we have,

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} x^2 \right). \quad (6.20)$$

The computation of the conditional covariance requires the results of four integrals of the form

$$\int_{-\infty}^{\infty} f(x)p(x) \exp \left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2} \sigma^2 x^2 t) \right] dx, \quad (6.21)$$

with $f(x) = 1$, $f(x) = x$, $f(x) = S(x)$, and $f(x) = xS(x)$, where

$$S(x) = S_0 \exp \left(rT + \nu \sqrt{T} x - \frac{1}{2} \nu^2 T \right). \quad (6.22)$$

To proceed it will be useful to have at our disposal the following two well-known Gaussian integrals, with parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$. These are:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2} x^2 \right) \exp (ax - bx^2) dx = \frac{1}{\sqrt{2b+1}} \exp \left(\frac{1}{2} \frac{a^2}{2b+1} \right), \quad (6.23)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2} x^2 \right) x \exp (ax - bx^2) dx = \frac{a}{(2b+1)^{3/2}} \exp \left(\frac{1}{2} \frac{a^2}{2b+1} \right). \quad (6.24)$$

Armed with these results we can now proceed to calculate the four integrals involved in the computation of the conditional covariance (6.18). The first of these integrals, for which $f(x) = 1$, gives:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2} x^2 \right) \exp \left[\frac{T}{T-t} (\sigma x \xi_t - \frac{1}{2} \sigma^2 x^2 t) \right] dx \\ = \frac{1}{\sqrt{\sigma^2 \tau + 1}} \exp \left(\frac{1}{2} \frac{\sigma^2 \tau^2 \xi_t^2}{t^2 (\sigma^2 \tau + 1)} \right), \end{aligned} \quad (6.25)$$

where for the parameters a and b in (6.23) we have

$$a = \frac{\sigma T}{T-t} \xi_t, \quad b = \frac{1}{2} \sigma^2 \tau. \quad (6.26)$$

The second integral, with $f(x) = x$, gives:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2}x^2\right) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx \\ = \frac{\sigma \tau \xi_t}{t(\sigma^2 \tau + 1)^{3/2}} \exp\left(\frac{1}{2} \frac{\sigma^2 \tau^2 \xi_t^2}{t^2 (\sigma^2 \tau + 1)}\right), \end{aligned} \quad (6.27)$$

where here we use the result (6.24) with

$$a = \frac{\sigma T}{T-t} \xi_t, \quad b = \frac{1}{2} \sigma^2 \tau. \quad (6.28)$$

The third integral, with $f(x) = S(x)$, gives:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2\right) S_0 \exp\left(rT + \nu\sqrt{T}x - \frac{1}{2}\nu^2 T\right) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx \\ = S_0 \exp\left(rT - \frac{1}{2}\nu^2 T\right) \frac{1}{\sqrt{\sigma^2 \tau + 1}} \exp\left[\frac{\left(\nu\sqrt{T} + \frac{\sigma \tau}{t} \xi_t\right)^2}{2(\sigma^2 \tau + 1)}\right], \end{aligned} \quad (6.29)$$

where in (6.23) we insert

$$a = \nu\sqrt{T} + \frac{\sigma T}{T-t} \xi_t, \quad b = \frac{1}{2} \sigma^2 \tau. \quad (6.30)$$

Finally we calculate that the fourth integral, with $f(x) = xS(x)$, gives:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2\right) S_0 \exp\left(rT + \nu\sqrt{T}x - \frac{1}{2}\nu^2 T\right) x \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] dx \\ = S_0 \exp\left(rT - \frac{1}{2}\nu^2 T\right) \frac{\nu\sqrt{T} + \frac{\sigma T}{T-t} \xi_t}{(\sigma^2 \tau + 1)^{3/2}} \exp\left[\frac{\left(\nu\sqrt{T} + \frac{\sigma \tau}{t} \xi_t\right)^2}{2(\sigma^2 \tau + 1)}\right], \end{aligned} \quad (6.31)$$

where in this case we have

$$a = \nu\sqrt{T} + \frac{\sigma T}{T-t} \xi_t, \quad b = \frac{1}{2} \sigma^2 \tau. \quad (6.32)$$

The results of the integrals above enable us to calculate the conditional expectations composing the conditional covariance (6.18). In particular, we have:

$$\mathbb{E}_t [X_T S_T] = S_0 \exp\left(rT - \frac{1}{2}\nu^2 T\right) \frac{\nu\sqrt{T} + \frac{\sigma T}{T-t} \xi_t}{\sigma^2 \tau + 1} \exp\left[\frac{1}{2} \frac{\nu\sqrt{T}}{\sigma^2 \tau + 1} + \frac{\nu\sqrt{T}\sigma\tau}{t(\sigma^2 \tau + 1)} \xi_T\right]. \quad (6.33)$$

The second term in the conditional covariance is given by

$$\mathbb{E}_t [S_T] \mathbb{E}_t [X_T] = S_0 \exp \left(rT - \frac{1}{2} \nu^2 T \right) \frac{\sigma \tau \xi_t}{t(\sigma^2 \tau + 1)} \exp \left[\frac{1}{2} \frac{\nu \sqrt{T}}{\sigma^2 \tau + 1} + \frac{\nu \sqrt{T} \sigma \tau}{t(\sigma^2 \tau + 1)} \xi_T \right]. \quad (6.34)$$

We now subtract the term (6.34) from the first term (6.33) and after some cancellations we obtain the following formula for the conditional covariance:

$$P_{tT} \text{Cov}_t [X_T, S_T] = \frac{\nu \sigma T^{3/2}}{T + (\sigma^2 T - 1)t} S_t, \quad (6.35)$$

where the price S_t of the asset at time t is given by equation (6.16). This expression for S_t is consistent with the formula

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \mathbb{E}_t [S_T | \xi_t]. \quad (6.36)$$

In particular, the conditional expectation in (6.36) can be computed by use of the integrals (6.25) and (6.29). Thus, for the dynamics of the asset price (6.36) we have the following relation:

$$\frac{dS_t}{S_t} = r dt + \frac{\nu \sigma T^{3/2}}{T + (\sigma^2 T - 1)t} dW_t. \quad (6.37)$$

We see therefore that the volatility is a deterministic function. But . . . if! . . .

$$\sigma^2 T = 1, \quad (6.38)$$

then the volatility is *constant*, and for the asset price dynamics we obtain

$$\frac{dS_t}{S_t} = r dt + \nu dW_t. \quad (6.39)$$

In other words, we have a geometric Brownian motion.

An alternative way of deriving this result is as follows. We begin with the expression (6.16) and impose the condition (6.38). It may not be immediately evident, but one can easily verify in this case that equation (6.16) reduces to

$$S_t = S_0 \exp \left(rt + \nu \xi_t - \frac{1}{2} \nu^2 t \right). \quad (6.40)$$

Now, in the situation where X_T is a standard Gaussian random variable, and where $\sigma^2 T = 1$, the information process $\{\xi_t\}$ takes the form

$$\xi_t = X_T \frac{t}{\sqrt{T}} + \beta_{tT}, \quad (6.41)$$

and, in particular, is a Gaussian process. A short calculation, making use of the fact that X_T and $\{\beta_{tT}\}$ are independent, shows that (6.41) implies that

$$\mathbb{E}[\xi_s \xi_t] = s \quad (6.42)$$

for $s \leq t$. This shows that $\{\xi_t\}_{0 \leq t \leq T}$ is a standard Brownian motion. We note further that $\xi_T = X_T \sqrt{T}$, and thus that

$$\xi_t - \frac{t}{T} \xi_T = \beta_{tT}. \quad (6.43)$$

But this demonstrates that $\{\beta_{tT}\}$ is the natural Brownian bridge associated with $\{\xi_t\}_{0 \leq t \leq T}$. Thus we have shown that in the case of a Gaussian X -factor and an information flow rate $\sigma = 1/\sqrt{T}$, the information process coincides with the innovation process, and the noise $\{\beta_{tT}\}$ is the associated Brownian bridge.

6.5 Defaultable n -coupon bonds with multiple recovery levels

Now we consider the case of a defaultable coupon bond where default can occur at any of the n coupon payment dates. Let us introduce a sequence of coupon dates T_k , $k = 1, 2, \dots, n$. At each date T_k a cash-flow H_{T_k} occurs. We introduce a set of independent binary random variables X_{T_j} , $j = 1, \dots, k$ (the X -factors), where they take either the value 0 (default) or 1 (no default) with *a priori* probability $p_{X_j} = \mathbb{Q}(X_{T_j} = 0)$ and $1 - p_{X_j} = \mathbb{Q}(X_{T_j} = 1)$ respectively.

Proposition 6.5.1 *Let T_k , $k = 1, 2, \dots, n$, be the pre-specified payment dates. The coupon is denoted by \mathbf{c} and the principal by \mathbf{p} . In case of default at the date T_k we have the recovery payment $R_k(\mathbf{c} + \mathbf{p})$ instead, where R_k is a percentage of the owed coupon and principal payment. At each date T_k the cash-flow structure is given by:*

For $k = 1, 2, \dots, n - 1$

$$H_{T_k} = \mathbf{c} \prod_{j=1}^k X_{T_j} + R_k(\mathbf{c} + \mathbf{p}) \prod_{j=1}^{k-1} X_{T_j} (1 - X_{T_k}), \quad (6.44)$$

and for $k = n$

$$H_{T_n} = (\mathbf{c} + \mathbf{p}) \prod_{j=1}^n X_{T_j} + R_n(\mathbf{c} + \mathbf{p}) \prod_{j=1}^{n-1} X_{T_j}(1 - X_{T_n}). \quad (6.45)$$

Now we introduce a set of market information processes, given by

$$\xi_{tT_j} = \sigma_j X_{T_j} t + \beta_{tT_j}, \quad (6.46)$$

and we assume that the market filtration $\{\mathcal{F}_t^\xi\}$ is generated collectively by the market information processes.

Proposition 6.5.2 *Let the default-free discount bond system $\{P_{tT}\}$ be deterministic. Then the information-based price process of a binary defaultable coupon bond is given by*

$$S_t = \sum_{k=1}^{n-1} \mathbf{1}_{\{t < T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} \left[H_{T_k} \middle| \mathcal{F}_t^\xi \right] + P_{tT_n} \mathbf{1}_{\{t < T_n\}} \mathbb{E}^{\mathbb{Q}} \left[H_{T_n} \middle| \mathcal{F}_t^\xi \right], \quad (6.47)$$

where H_{T_k} and H_{T_n} are defined by equations (6.44) and (6.45).

Example. The price process of a binary defaultable bond paying a coupon \mathbf{c} at the dates T_1 and T_2 , as well as a third coupon \mathbf{c} plus the principal \mathbf{p} at the date T_3 , is given for $t < T_1$ by:

$$S_t = P_{tT_1} \mathbb{E}^{\mathbb{Q}} \left[H_{T_1} \middle| \mathcal{F}_t^\xi \right] + P_{tT_2} \mathbb{E}^{\mathbb{Q}} \left[H_{T_2} \middle| \mathcal{F}_t^\xi \right] + P_{tT_3} \mathbb{E}^{\mathbb{Q}} \left[H_{T_3} \middle| \mathcal{F}_t^\xi \right], \quad (6.48)$$

where

$$H_{T_1} = \mathbf{c}X_{T_1} + R_1(\mathbf{c} + \mathbf{p})(1 - X_{T_1}), \quad (6.49)$$

$$H_{T_2} = \mathbf{c}X_{T_1}X_{T_2} + R_2(\mathbf{c} + \mathbf{p})X_{T_1}(1 - X_{T_2}), \quad (6.50)$$

$$H_{T_3} = (\mathbf{c} + \mathbf{p})X_{T_1}X_{T_2}X_{T_3} + R_3(\mathbf{c} + \mathbf{p})X_{T_1}X_{T_2}(1 - X_{T_3}). \quad (6.51)$$

6.6 Correlated cash flows

The multiple-dividend asset pricing model introduced in this chapter can be extended in a natural way to the situation where two or more assets are being priced. In this case we consider a collection of N assets with price processes $\{S_t^{(i)}\}_{i=1,2,\dots,N}$. With asset number (i) we associate the cash flows $\{D_{T_k}^{(i)}\}$ paid at the dates $\{T_k\}_{k=1,2,\dots,n}$. We note

that the dates $\{T_k\}_{k=1,2,\dots,n}$ are not tied to any specific asset, but rather represent the totality of possible cash-flow dates of any of the given assets. If a particular asset has no cash flow on one of the given dates, then it is simply assigned a zero cash-flow for that date. From this point, the theory proceeds in the single asset case. That is to say, with each value of k we associate a set of X -factors $X_{T_k}^\alpha$ ($\alpha = 1, 2, \dots, m_k$), and a corresponding system of market information processes, $\{\xi_{tT_k}^\alpha\}$. The X -factors and the information processes are not tied to any particular asset. The cash flow $D_{T_k}^{(i)}$ occurring at time T_k for asset (i) is assumed to be given by a cash flow function of the form

$$D_{T_k}^{(i)} = \Delta_{T_k}^{(i)}(X_{T_1}^\alpha, X_{T_2}^\alpha, \dots, X_{T_k}^\alpha). \quad (6.52)$$

In other words, for each asset each cash flow can depend on all of the X -factors that have been “activated” so far.

In particular, it is possible for two or more assets to “share” an X -factor associated with one or more of the cash flows of each of the assets. This in turn implies that the various assets will have at least one Brownian motion in common in the dynamics of their price processes. As a consequence we obtain a natural model for the existence of correlation structures in the prices of these assets. The intuition is that as new information comes in (whether “true” or “bogus”) there will be several different assets affected by the news, and as a consequence there will be a correlated movement in their prices. Thus for the general multi-asset model we have the following price process system:

$$S_t^{(i)} = \sum_{k=1}^n \mathbf{1}_{\{t < T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} \left[D_{T_k}^{(i)} \mid \mathcal{F}_t \right]. \quad (6.53)$$

As an illustration we imagine the following situation: A big factory has an outstanding debt that needs to be honoured at time T_1 . Across the street there is a restaurant that has also an outstanding loan that has to be paid back at time T_2 , where we assume that $T_1 < T_2$.

Let us suppose that the main source of income of the restaurant comes from the workers of the factory, who regularly have their lunch at the restaurant. However, the factory has been encompassing a long period of decreasing profits and as a result fails to pay its debt. The factory decides to send home almost all of its workers thus putting, as a side effect, the restaurant in a financially distressed state. As a consequence, despite its overall good management, the restaurant fails to repay the loan.

The second scenario is that the factory is financially robust and honours its debt, but the restaurant has bad management. In this case the workers continue having their lunch at the restaurant, but the restaurant is unable to repay its debt.

The third scenario is the worst-case picture, in which the factory fails to repay its debt and then the restaurant fails to repay its debt as a result both of its bad management and the factory's decision to reduce the number of its workers.

These various situations can be modelled as follows in the X -factor theory: The two debts can be viewed as a pair two zero-coupon bonds. The first discount bond is defined by a cash flow H_{T_1} at time T_1 . The second discount bond associated with the restaurant's debt is defined by a cash flow H_{T_2} at time T_2 . We introduce two independent market factors X_{T_1} and X_{T_2} that can take the values zero and one. The three default scenarios described above can be reproduced by defining the cash flows H_{T_1} and H_{T_2} in such a way that the dependence between the two cash flows is captured. This is equivalent to describing the economic microstructure intertwining the factory and the restaurant, and is thus a natural way of constructing the dependence between the businesses by analysing how the relevant cash flows are linked to each other. For the particular example presented above the cash flow structure is given by:

$$H_{T_1} = \mathbf{n}_1 X_{T_1} + R_1 \mathbf{n}_1 (1 - X_{T_1}) \quad (6.54)$$

$$\begin{aligned} H_{T_2} = & \mathbf{n}_2 X_{T_1} X_{T_2} + R_2^a \mathbf{n}_2 (1 - X_{T_1}) X_{T_2} \\ & + R_2^b \mathbf{n}_2 X_{T_1} (1 - X_{T_2}) + R_2^c \mathbf{n}_2 (1 - X_{T_1}) (1 - X_{T_2}). \end{aligned} \quad (6.55)$$

Here \mathbf{n}_1 and \mathbf{n}_2 denote the amounts of the two outstanding debts (or, equivalently, the bond principals). We also introduce recovery rates R_1 , R_2^a , R_2^b , and R_2^c for the cases when the factory and/or the restaurant are not able to pay back the loans. The recovery rates take into account the salvage values that can be extracted in the various scenarios. It is natural, for example to suppose that $R_2^b > R_2^a > R_2^c$. In this example, the price of the factory bond will be driven by a single Brownian motion, whereas the restaurant will have a pair of Brownian drivers. Since one of these coincides with the driver of the factory bond, the dynamics of the two bonds will be correlated; analytic formulae can be derived for the resulting volatilities and correlations.

6.7 From Z -factors to X -factors

So far we have always assumed that the relevant market factors (X -factors) are independent random variables. In reality, what one often loosely refers to as “market factors” tend to be dependent quantities and it may be a difficult task to find a suitable set of independent X -factors. In this section we illustrate a method to express a set of dependent factors, which we shall call Z -factors, in terms of a set of X -factors. In particular, the scheme described below allows one to reduce a set of binary Z -factors, i.e. random variables that can take two possible values, into a set of binary X -factors.

Let us introduce a set of n dependent binary Z -factors denoted $\{Z_j\}_{j=1,\dots,n}$ for which reduction to a set $\{X_k\}_{k=1,\dots,2^n-1}$ of $2^n - 1$ independent binary X -factors is required. We shall establish a system of reduction equations of the form $Z_j = Z_j(X_1, \dots, X_{2^j-1})$ for $j = 1, \dots, n$. For example, in the case $n = 2$ the set $\{Z_1, Z_2\}$ admits a reduction of the form

$$Z_1 = Z_1(X_1) \tag{6.56}$$

$$Z_2 = Z_2(X_1, X_2, X_3). \tag{6.57}$$

in terms of a set of three independent X -factors.

The idea is to construct an algorithm for the reduction scheme for any number of dependent binary Z -factors. For the two possible values of the binary random variable Z_j let us write $\{z_j, \bar{z}_j\}$. The independent X -factors will be assumed to take values in $\{0, 1\}$. Since we deal with binary random variables, one can guess that behind the reduction method a binary tree structure must play a fundamental role.

Indeed, this is the case, and the reduction for each Z -factor builds upon an embedding scheme. For instance the reduction equation for three dependent Z -factors $\{Z_1, Z_2, Z_3\}$ embeds the reduction system for a set of two Z -factors, which in turn is based on the reduction for one dependent factor. Before constructing the reduction algorithm we shall as an example write down the reduction equations for a set of five Z -factors $\{Z_1, Z_2, \dots, Z_5\}$. This gives us an opportunity to recognise the above-mentioned embedding property of the reduction scheme, and also the pattern of the same which is needed in order to produce the general reduction algorithm. The reduction equations for the set $\{Z_1, Z_2, \dots, Z_5\}$ read as follows. For each market factor X let us define $\bar{X} = 1 - X$, and call \bar{X} the co-factor of X . Then the reduction of the

dependent variables $\{Z_1, \dots, Z_5\}$ in terms of the independent variables $\{X_1, \dots, X_{31}\}$ is given explicitly by the following scheme:

$$Z_1 = z_1 X_1 + \bar{z}_1 \bar{X}_1 \quad (6.58)$$

$$Z_2 = X_1(z_2 X_2 + \bar{z}_2 \bar{X}_2) + \bar{X}_1(z_2 X_3 + \bar{z}_2 \bar{X}_3) \quad (6.59)$$

$$\begin{aligned} Z_3 = & X_1 X_2(z_3 X_4 + \bar{z}_3 \bar{X}_4) + X_1 \bar{X}_2(z_3 X_5 + \bar{z}_3 \bar{X}_5) \\ & + \bar{X}_1 X_3(z_3 X_6 + \bar{z}_3 \bar{X}_6) + \bar{X}_1 \bar{X}_3(z_3 X_7 + \bar{z}_3 \bar{X}_7) \end{aligned} \quad (6.60)$$

$$\begin{aligned} Z_4 = & X_1 X_2 X_4(z_4 X_8 + \bar{z}_4 \bar{X}_8) + X_1 X_2 \bar{X}_4(z_4 X_9 + \bar{z}_4 \bar{X}_9) \\ & + X_1 \bar{X}_2 X_5(z_4 X_{10} + \bar{z}_4 \bar{X}_{10}) + X_1 \bar{X}_2 \bar{X}_5(z_4 X_{11} + \bar{z}_4 \bar{X}_{11}) \\ & + \bar{X}_1 X_3 X_6(z_4 X_{12} + \bar{z}_4 \bar{X}_{12}) + \bar{X}_1 X_3 \bar{X}_6(z_4 X_{13} + \bar{z}_4 \bar{X}_{13}) \\ & + \bar{X}_1 \bar{X}_3 X_7(z_4 X_{14} + \bar{z}_4 \bar{X}_{14}) + \bar{X}_1 \bar{X}_3 \bar{X}_7(z_4 X_{15} + \bar{z}_4 \bar{X}_{15}) \end{aligned} \quad (6.61)$$

$$\begin{aligned} Z_5 = & X_1 X_2 X_4 X_8(z_5 X_{16} + \bar{z}_5 \bar{X}_{16}) + X_1 X_2 X_4 \bar{X}_8(z_5 X_{17} + \bar{z}_5 \bar{X}_{17}) \\ & + X_1 X_2 \bar{X}_4 X_9(z_5 X_{18} + \bar{z}_5 \bar{X}_{18}) + X_1 X_2 \bar{X}_4 \bar{X}_9(z_5 X_{19} + \bar{z}_5 \bar{X}_{19}) \\ & + X_1 \bar{X}_2 X_5 X_{10}(z_5 X_{20} + \bar{z}_5 \bar{X}_{20}) + X_1 \bar{X}_2 X_5 \bar{X}_{10}(z_5 X_{21} + \bar{z}_5 \bar{X}_{21}) \\ & + X_1 \bar{X}_2 \bar{X}_5 X_{11}(z_5 X_{22} + \bar{z}_5 \bar{X}_{22}) + X_1 \bar{X}_2 \bar{X}_5 \bar{X}_{11}(z_5 X_{23} + \bar{z}_5 \bar{X}_{23}) \\ & + \bar{X}_1 X_3 X_6 X_{12}(z_5 X_{24} + \bar{z}_5 \bar{X}_{24}) + \bar{X}_1 X_3 X_6 \bar{X}_{12}(z_5 X_{25} + \bar{z}_5 \bar{X}_{25}) \\ & + \bar{X}_1 X_3 \bar{X}_6 X_{13}(z_5 X_{26} + \bar{z}_5 \bar{X}_{26}) + \bar{X}_1 X_3 \bar{X}_6 \bar{X}_{13}(z_5 X_{27} + \bar{z}_5 \bar{X}_{27}) \\ & + \bar{X}_1 \bar{X}_3 X_7 X_{14}(z_5 X_{28} + \bar{z}_5 \bar{X}_{28}) + \bar{X}_1 \bar{X}_3 X_7 \bar{X}_{14}(z_5 X_{29} + \bar{z}_5 \bar{X}_{29}) \\ & + \bar{X}_1 \bar{X}_3 \bar{X}_7 X_{15}(z_5 X_{30} + \bar{z}_5 \bar{X}_{30}) + \bar{X}_1 \bar{X}_3 \bar{X}_7 \bar{X}_{15}(z_5 X_{31} + \bar{z}_5 \bar{X}_{31}) \end{aligned} \quad (6.62)$$

Before we discuss the issue of how to produce an algorithm that gives the reduction system for any number of dependent market factors, we address the question of how the *a priori* probability distributions of the independent X -factors can be expressed in terms of the *a priori* joint probability distribution of the dependent Z -factors. We will shortly see that the derivation of the *a priori* probability distribution of the X -factors in terms of the *a priori* joint probability distributions of the Z -factors is closely related to the reduction system of the considered set of dependent random variables.

As an example we investigate a set of three Z -factors $\{Z_1, Z_2, Z_3\}$, for which the corresponding reduction in terms of X -factors is given by equations (6.58), (6.59), and (6.60) above, and with which the following *a priori* joint probability distribution are associated:

$$q_{z_1 z_2 z_3} := \mathbb{Q}[Z_1 = z_1, Z_2 = z_2, Z_3 = z_3] = p_{X_1} p_{X_2} p_{X_4} \quad (6.63)$$

$$q_{z_1 z_2 \bar{z}_3} := \mathbb{Q}[Z_1 = z_1, Z_2 = z_2, Z_3 = \bar{z}_3] = p_{X_1} p_{X_2} (1 - p_{X_4}) \quad (6.64)$$

$$q_{z_1 \bar{z}_2 z_3} := \mathbb{Q}[Z_1 = z_1, Z_2 = \bar{z}_2, Z_3 = z_3] = p_{X_1} (1 - p_{X_2}) p_{X_5} \quad (6.65)$$

$$q_{z_1 \bar{z}_2 \bar{z}_3} := \mathbb{Q}[Z_1 = z_1, Z_2 = \bar{z}_2, Z_3 = \bar{z}_3] = p_{X_1} (1 - p_{X_2}) (1 - p_{X_5}) \quad (6.66)$$

$$q_{\bar{z}_1 z_2 z_3} := \mathbb{Q}[Z_1 = \bar{z}_1, Z_2 = z_2, Z_3 = z_3] = (1 - p_{X_1}) p_{X_3} p_{X_6} \quad (6.67)$$

$$q_{\bar{z}_1 z_2 \bar{z}_3} := \mathbb{Q}[Z_1 = \bar{z}_1, Z_2 = z_2, Z_3 = \bar{z}_3] = (1 - p_{X_1}) p_{X_3} (1 - p_{X_6}) \quad (6.68)$$

$$q_{\bar{z}_1 \bar{z}_2 z_3} := \mathbb{Q}[Z_1 = \bar{z}_1, Z_2 = \bar{z}_2, Z_3 = z_3] = (1 - p_{X_1}) (1 - p_{X_3}) p_{X_7} \quad (6.69)$$

$$q_{\bar{z}_1 \bar{z}_2 \bar{z}_3} := \mathbb{Q}[Z_1 = \bar{z}_1, Z_2 = \bar{z}_2, Z_3 = \bar{z}_3] = (1 - p_{X_1}) (1 - p_{X_3}) (1 - p_{X_7}) \quad (6.70)$$

Recalling that $\bar{X}_k = 1 - X_k$ and hence $p_{\bar{X}_k} = 1 - p_{X_k}$, we recognise that each joint probability distribution is associated with the corresponding term in the reduction equation for the set $\{Z_1, Z_2, Z_3\}$. For instance $q_{z_1 z_2 z_3}$ is the probability distribution connected with the first term in (6.60), i.e. $X_1 X_2 X_4 z_3$.

The relations above can be inverted to give the corresponding univariate probability distributions for the independent X -factors in terms of the joint distributions of the Z -factors:

$$\begin{aligned} p_{X_1} &= q_{z_1 z_2 z_3} + q_{z_1 z_2 \bar{z}_3} + q_{z_1 \bar{z}_2 z_3} + q_{z_1 \bar{z}_2 \bar{z}_3} \\ p_{X_2} &= \frac{q_{z_1 z_2 z_3} + q_{z_1 z_2 \bar{z}_3}}{q_{z_1 z_2 z_3} + q_{z_1 z_2 \bar{z}_3} + q_{z_1 \bar{z}_2 z_3} + q_{z_1 \bar{z}_2 \bar{z}_3}} & p_{X_3} &= \frac{q_{\bar{z}_1 z_2 z_3} + q_{\bar{z}_1 z_2 \bar{z}_3}}{q_{\bar{z}_1 z_2 z_3} + q_{\bar{z}_1 z_2 \bar{z}_3} + q_{\bar{z}_1 \bar{z}_2 z_3} + q_{\bar{z}_1 \bar{z}_2 \bar{z}_3}} \\ p_{X_4} &= \frac{q_{z_1 z_2 z_3}}{q_{z_1 z_2 z_3} + q_{z_1 z_2 \bar{z}_3}} & p_{X_5} &= \frac{q_{z_1 \bar{z}_2 z_3}}{q_{z_1 \bar{z}_2 z_3} + q_{z_1 \bar{z}_2 \bar{z}_3}} \\ p_{X_6} &= \frac{q_{\bar{z}_1 z_2 z_3}}{q_{\bar{z}_1 z_2 z_3} + q_{\bar{z}_1 z_2 \bar{z}_3}} & p_{X_7} &= \frac{q_{\bar{z}_1 \bar{z}_2 z_3}}{q_{\bar{z}_1 \bar{z}_2 z_3} + q_{\bar{z}_1 \bar{z}_2 \bar{z}_3}}. \end{aligned} \quad (6.71)$$

It is a short calculation to verify that the resulting system of univariate probabilities is consistent in the sense that these probabilities all lie in the range $[0, 1]$.

6.8 Reduction algorithm

In this section we show how to produce the reduction system for a specific set of dependent random variables once a fixed j has been chosen; that is to say we show how to determine the function

$$Z_j = Z_j(X_1, X_2, \dots, X_{2^{j-1}}). \quad (6.72)$$

In particular, we aim at a reduction algorithm that allows us to find the reduction equation for Z_j directly without needing to produce all the reduction equations for all other Z -factors $\{Z_1, \dots, Z_{j-1}\}$ implicitly involved due to the embedding property.

To give an example, we would like for the reduction algorithm to give us the reduction system (6.61) without us needing to write down explicitly the equations (6.58), (6.59), and (6.60) for Z_1 , Z_2 , and Z_3 respectively.

A possible application of a reduction scheme could be the situation where one has economic quantities which are clearly dependent on each other and can be represented by a system of dependent binary random variables. In this case one may be interested in the dependence structure linking the various economic factors, and in being able to describe the dependence in terms of a set of independent X -factors. These would be viewed as the “fundamental” quantities in the economy producing the dependence among the economic quantities modelled by the set of Z -factors.

Let us assume a set of n Z -factors $\{Z_j\}_{j=1, \dots, n}$. We now write down the reduction algorithm for Z_j for a number $j \in \{1, \dots, n\}$ in a series of steps.

Step 1. We observe first that the reduction equation for Z_j contains 2^{j-1} so-called X -terms, where an X -term is a summand of the form

$$X \cdots X(z_j X + \bar{z}_j \bar{X}). \quad (6.73)$$

For example, Z_3 is composed by four X -terms, as can be verified in formula (6.60).

Step 2. In order to describe the building blocks of the reduction procedure we introduce some further terminology. An X -term, for fixed $k < j$, of the form

$$X \cdots X_k(\bar{z}_j X_{2k} + \bar{z}_j \bar{X}_{2k}) \quad (6.74)$$

is called branch number k , and an X -term of the form

$$X \cdots \bar{X}_k (\bar{z}_j X_{2k+1} + \bar{z}_j \bar{X}_{2k+1}) \quad (6.75)$$

is called co-branch number k . For example, the X -term $X_1 X_2 (z_3 X_4 + \bar{z}_3 \bar{X}_4)$ in (6.60) is branch number two, and the X -term $\bar{X}_1 \bar{X}_3 (z_3 X_7 + \bar{z}_3 \bar{X}_7)$ is the co-branch number three. We also introduce the concept of a node in the reduction system. The branch k leads to the “node” $2k$, i.e. the term given by:

$$X \cdots X_k \underbrace{(z_j X_{2k} + \bar{z}_j \bar{X}_{2k})}_{\text{node number } 2k}. \quad (6.76)$$

The co-branch k leads, instead, to the node $2k + 1$, that is:

$$X \cdots \bar{X}_k \underbrace{(z_j X_{2k+1} + \bar{z}_j \bar{X}_{2k+1})}_{\text{node number } 2k+1}. \quad (6.77)$$

For instance, co-branch number three in (6.60), $\bar{X}_1 \bar{X}_3 (z_3 X_7 + \bar{z}_3 \bar{X}_7)$, leads to node number seven.

Step 3. The first branch of the reduction system for Z_j is given by:

$$Z_j = X_1 \cdots X_{2^j-2} (z_j X_{2^j-1} + \bar{z}_j \bar{X}_{2^j-1}) + \dots \quad (6.78)$$

To the first branch we then add the corresponding co-branch, i.e.

$$Z_j = X_1 \cdots X_{2^j-2} (z_j X_{2^j-1} + \bar{z}_j \bar{X}_{2^j-1}) + X_1 \cdots \bar{X}_{2^j-2} (z_j X_{2^j-1+1} + \bar{z}_j \bar{X}_{2^j-1+1}) \dots \quad (6.79)$$

and declare branch number 2^j-2 complete. Here we adopt the convention, for convenience, that $2^{j-2} = 0$ for all $j < 2$. We set $X_0 = 1$.

Step 4. Before we continue with the remaining terms in the reduction equation for Z_j , we need to establish two rules, which we call “connection rules”. These rules tell us how to “hop” from one branch or co-branch to the next.

(i) If the preceding independent random variable is an X -factor, then the next market factor is obtained by doubling the index of the predecessor, that is to say:

$$X \cdots X_k \longrightarrow X \cdots X_k \cdot \begin{cases} X_{2k} \\ \bar{X}_{2k} \end{cases}. \quad (6.80)$$

Example: In the reduction equation (6.61) we have in the first term the expression $X_1 X_2 X_4 (z_4 X_8 + \bar{z}_4 \bar{X}_8)$. We see that all X -factor indexes are doubles of the predecessors.

(ii) If the preceding factor is an X -co-factor, i.e. of the form $\bar{X}_k = 1 - X_k$, then the next factor is obtained by doubling the index of the predecessor and adding one; that is to say:

$$X \cdots \bar{X}_k \longrightarrow X \cdots \bar{X}_k \cdot \begin{cases} X_{2k+1} \\ \bar{X}_{2k+1} \end{cases} \quad (6.81)$$

Example: In the reduction equation (6.61) we have, respectively, in the third and in the fourth terms, $X_1 \bar{X}_2 X_5 (z_4 X_{10} + \bar{z}_4 \bar{X}_{10})$ and $X_1 \bar{X}_2 \bar{X}_5 (z_4 X_{11} + \bar{z}_4 \bar{X}_{11})$, where the X -co-factor \bar{X}_2 leads, respectively, to branch number 5 and co-branch number 5.

Now we return to the task of reducing the dependent factor Z_j into its independent constituents. We started with the first X -term (or branch, in this case) given by

$$X_1 \cdots X_{2^{j-2}} (z_j X_{2^{j-1}} + \bar{z}_j \bar{X}_{2^{j-1}}), \quad (6.82)$$

and added its complementary co-branch

$$X_1 \cdots \bar{X}_{2^{j-2}} (z_j X_{2^{j-1}+1} + \bar{z}_j \bar{X}_{2^{j-1}+1}). \quad (6.83)$$

Once we have added to a particular branch its complementary co-branch, we then say that the specific branch/co-branch pair is complete. We then work back through all preceding branches using the connection rules and completing the various branches. For the case of Z_j this is carried out as follows: To the first branch and its complementary co-branch

$$X_1 \cdots X_{2^{j-2}} (z_j X_{2^{j-1}} + \bar{z}_j \bar{X}_{2^{j-1}}) + X_1 \cdots \bar{X}_{2^{j-2}} (z_j X_{2^{j-1}+1} + \bar{z}_j \bar{X}_{2^{j-1}+1}) \quad (6.84)$$

we add the term

$$\begin{aligned} & X_1 \cdots \bar{X}_{2^{j-3}} X_{2^{j-2}+1} (z_j X_{2^{(j-2)+1}} + \bar{z}_j \bar{X}_{2^{(j-2)+1}}) \\ + & X_1 \cdots \bar{X}_{2^{j-3}} \bar{X}_{2^{j-2}+1} (z_j X_{2^{(j-2)+1}+1} + \bar{z}_j \bar{X}_{2^{(j-2)+1}+1}). \end{aligned} \quad (6.85)$$

Hence branch number 2^{j-3} is complete as well. The next step is to add branch number 2^{j-4} and then to complete it with its complementary co-branch number 2^{j-4} . This

procedure stops once all branches are added up and completed, where the last pair is composed of the two final X -terms, namely the final branch and final co-branch:

$$\begin{aligned} & \bar{X}_1 \bar{X}_3 \bar{X}_7 \cdots X_{2^j-1} (z_j X_{2(2^j-1)} + \bar{z}_j \bar{X}_{2(2^j-1)}) \\ + & \bar{X}_1 \bar{X}_3 \bar{X}_7 \cdots \bar{X}_{2^j-1} (z_j X_{2(2^j-1)+1} + \bar{z}_j \bar{X}_{2(2^j-1)+1}). \end{aligned} \quad (6.86)$$

At this point the reduction procedure is complete and we obtain the full reduction of the Z -factor Z_j . Recapping, the reduction for Z_j is given by:

$$\begin{aligned} Z_j = & X_1 \cdots \bar{X}_{2^{j-2}} (z_j X_{2^{j-1}+1} + \bar{z}_j \bar{X}_{2^{j-1}+1}) \\ + & X_1 \cdots \bar{X}_{2^{j-3}} X_{2^{j-2}+1} (z_j X_{2(2^{j-2}+1)} + \bar{z}_j \bar{X}_{2(2^{j-2}+1)}) \\ & + X_1 \cdots \bar{X}_{2^{j-3}} \bar{X}_{2^{j-2}+1} (z_j X_{2(2^{j-2}+1)+1} + \bar{z}_j \bar{X}_{2(2^{j-2}+1)+1}) \\ + & \dots \\ & \vdots \\ + & \bar{X}_1 \bar{X}_3 \bar{X}_7 \cdots X_{2^j-1} (z_j X_{2(2^j-1)} + \bar{z}_j \bar{X}_{2(2^j-1)}) \\ & + \bar{X}_1 \bar{X}_3 \bar{X}_7 \cdots \bar{X}_{2^j-1} (z_j X_{2(2^j-1)+1} + \bar{z}_j \bar{X}_{2(2^j-1)+1}). \end{aligned} \quad (6.87)$$

This completes the derivation of the reduction equation for the dependent Z -factor Z_j , where $j \in \{1, 2, \dots, n\}$, and also the derivation of a general reduction scheme for the disentanglement of a set of dependent binary random variables into to a set of binary X -factors.

We have confined the discussion to the case of a set of dependent binary Z -factors, and we have shown how such a set can be “reduced” to a corresponding set of independent X -factors. In practice, of course, we would like to have a similar collection of results for wider classes of dependent random variables. For the moment, the binary case offers a useful heuristic motivation for the notion that in a financial context we may assume the existence, as a basis for our modelling framework, of a set of underlying independent X -factors upon which the observed Z -factors depend.

6.9 Information-based Arrow-Debreu securities and option pricing

In Section 3.4 we presented the concept of the information-based Arrow-Debreu technique applied to an asset with cash flows modelled by discrete random variables. In this

section we make a further step forward and present the technique for the case that cash flows are expressed in terms of continuous random variables. We shall construct the information-based price of an Arrow-Debreu security having in mind, as an example, a single-dividend-paying asset defined in terms of a single cash flow D_T occurring at time T . The price process of such an asset is given in the information-based approach by:

$$S_t = \mathbf{1}_{\{t < T\}} P_{tT} \frac{\int_0^\infty zp(z) \exp \left[\frac{T}{T-t} (\sigma z \xi_t - \frac{1}{2} \sigma^2 z^2 t) \right] dz}{\int_0^\infty p(z) \exp \left[\frac{T}{T-t} (\sigma z \xi_t - \frac{1}{2} \sigma^2 z^2 t) \right] dz}, \quad (6.88)$$

where $p(z)$ is the *a priori* probability density for a continuous random variable D_T that takes values in the range $z \in [0, \infty)$. We remark, incidentally, that in the case of an X -factor with a more general distribution, not necessarily continuous, then appropriate formulae can be obtained by replacing $p(z)dz$ with $\mu(dz)$ in the formula above, and elsewhere. For simplicity we will refer here to the continuous case. The information process $\{\xi_t\}$ associated with the cash flow D_T is given by

$$\xi_t = \sigma D_T t + \beta_{tT}. \quad (6.89)$$

We observe that, since the maturity date T is a fixed pre-specified date, the price S_t at time t is given by a function of the value of the information process at time t . That is to say $S_t = S(\xi_t, t)$. Since the asset price S_t is a positive strictly-increasing function of $\{\xi_t\}$, it is possible to invert the function $S(t, \xi_t)$ such that the information process at time t can be given in a unique way in terms of the asset price S_t at the time t . In the special case that the cash flow D_T is given in terms of a binary random variable (reverting briefly to the discrete case), it is possible to express the information process $\{\xi_t\}$ in the following form:

$$\xi_t = \frac{T-t}{\sigma(h_1 - h_0)T} \ln \left[\frac{p_0(P_{tT}d_0 - S_t)}{p_1(S_t - P_{tT}d_1)} \right] + \frac{1}{2} \sigma (h_1 + h_0) \frac{t(T-t)}{T}, \quad (6.90)$$

where p_0 and p_1 are the probabilities that D_T takes the values h_0 and h_1 . This shows that the price of an Arrow-Debreu security can in principal be calibrated by use of other assets with which the information process is associated.

Following the notation of Section 3.4, we now define the payoff of an Arrow-Debreu security where the underlying is given by the information process $\{\xi_t\}$. Thus for the payoff we have:

$$f(\xi_t) = \delta(\xi_t - x), \quad (6.91)$$

where $\delta(\xi_t)$ is the delta function. As before, the value of the Arrow-Debreu security is given by the discounted risk-neutral expectation of the payoff above:

$$A_{0t}(x) = P_{0t} \mathbb{E}^{\mathbb{Q}}[\delta(\xi_t - x)]. \quad (6.92)$$

The only difference between the expressions (6.92) and (3.55), is that the random variable D_T with which the information process $\{\xi_t\}$ in 6.92 is associated is assumed to be continuous.

Following through the same steps as in Section (3.4), we now calculate explicitly the Arrow-Debreu price A_{0t} . For the delta function we use the representation

$$\delta(\xi_t - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi_t - x)\kappa} d\kappa. \quad (6.93)$$

Inserting (6.93) into the expectation in equation (6.92), we now calculate as follows. We shall assume that D_T has properties sufficient to ensure that

$$\mathbb{E}^{\mathbb{Q}}[\delta(\xi_t - x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\kappa} \mathbb{E}^{\mathbb{Q}}[e^{i\kappa\xi_t}] d\kappa. \quad (6.94)$$

We insert the definition of the information process (6.89) into the expectation under the integral, and recall that D_T and $\{\beta_{tT}\}$ are independent, and the fact that $\{\beta_{tT}\}$ is normally distributed with mean zero and variance $t(T-t)/T$. This leads us to the relation

$$\mathbb{E}^{\mathbb{Q}} [e^{i\kappa\xi_t}] = \int_0^{\infty} p(z) e^{i\kappa\sigma z t - \frac{1}{2}\kappa^2 \frac{t(T-t)}{T}} dz. \quad (6.95)$$

where $p(z)$ is the *a priori* density for D_T . Hence, the expectation in (6.92) turns out to be

$$\mathbb{E}^{\mathbb{Q}}[\delta(\xi_t - x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\kappa} \int_0^{\infty} p(z) e^{i\kappa\sigma z t - \frac{1}{2}\kappa^2 \frac{t(T-t)}{T}} dz d\kappa. \quad (6.96)$$

If we swap the integration order and re-arrange slightly the terms in the exponent we obtain

$$\mathbb{E}^{\mathbb{Q}}[\delta(\xi_t - x)] = \frac{1}{2\pi} \int_0^{\infty} p(z) \int_{-\infty}^{\infty} e^{i(\sigma z t - x)\kappa - \frac{1}{2}\frac{t(T-t)}{T}\kappa^2} d\kappa dz. \quad (6.97)$$

Working out the inner integral, we eventually obtain

$$A_{0t}(x) := P_{0t} \mathbb{E}^{\mathbb{Q}}[\delta(\xi_t - x)] = P_{0t} \int_0^{\infty} p(z) \sqrt{\frac{T}{2\pi t(T-t)}} \exp\left[-\frac{1}{2} \frac{(\sigma z t - x)^2 T}{t(T-t)}\right] dz. \quad (6.98)$$

We call $A_{0t}(x)$ the information-based price of an Arrow-Debreu security for the case that the underlying information process is associated with a continuous random variable. The price $A_{0t}(x)$ is the present value of a security paying at time t a delta distribution centered at $\xi_t = x$.

The calculation above serves the purpose of illustrating an application of the information-based approach to the pricing of an Arrow-Debreu security. However the derived expressions, especially equation (6.98), can be used to price a series of more complex derivatives. In Section 5.6 the price of a European-style call option written on a single-dividend-paying asset with continuous payoff is calculated by use of a change-of-measure technique. In what follows we show the derivation of the price of a vanilla call within the information-based approach, avoiding changing the measure to the bridge measure used in Sections 3.1 and 5.6.

The price C_0 of a European-style call option written on a single-dividend-paying asset with price process (6.88) is given by

$$C_0 = P_{0t} \mathbb{E}^{\mathbb{Q}} [(S_t - K)^+], \quad (6.99)$$

where K is the strike price, and the option matures at time $t \leq T$. Let us recall that T is the time at which the single dividend D_T is paid. The idea now is to view the option payoff as a “continuum” of delta functions. In more detail, the value of the payoff

$$C_t = (S(\xi_t, t) - K)^+ \quad (6.100)$$

can be regarded as being a continuous “superposition” of delta distributions, each of which depends on the value of the information process at time t .

Treating the call option pricing formula no differently than the formula for an Arrow-Debreu security (6.92), we now express the value of a call given by (6.99) in terms of the function

$$A(x) = \frac{A_{0t}(x)}{P_{0t}}, \quad (6.101)$$

which we call the (non-discounted) Arrow-Debreu density. The price of a call in terms of the information-based Arrow-Debreu density is thus:

$$C_0 = P_{0t} \int_{-\infty}^{\infty} (S(x) - K)^+ A(x) dx. \quad (6.102)$$

Then we substitute $S(x)$ with the according expression in (6.88) to obtain:

$$C_0 = P_{0t} \int_{-\infty}^{\infty} \left(P_{tT} \frac{\int_0^{\infty} zp(z) \exp \left[\frac{T}{T-t} (\sigma zx - \frac{1}{2} \sigma^2 z^2 t) \right] dz}{\int_0^{\infty} p(z) \exp \left[\frac{T}{T-t} (\sigma zx - \frac{1}{2} \sigma^2 z^2 t) \right] dz} - K \right)^+ A(x) dx. \quad (6.103)$$

We observe that the Arrow-Debreu density $A(x)$ is a positive function, which enables us to take it inside the maximum function. That is to say:

$$C_0 = P_{0t} \int_{-\infty}^{\infty} \left(P_{tT} \frac{\int_0^{\infty} zp(z) \exp \left[\frac{T}{T-t} (\sigma zx - \frac{1}{2} \sigma^2 z^2 t) \right] dz}{\int_0^{\infty} p(z) \exp \left[\frac{T}{T-t} (\sigma zx - \frac{1}{2} \sigma^2 z^2 t) \right] dz} A(x) - K A(x) \right)^+ dx. \quad (6.104)$$

Next, we rewrite the density $A(x)$ in the form

$$A(x) = \sqrt{\frac{T}{2\pi t(T-t)}} \exp \left[-\frac{T}{2t(T-t)} x^2 \right] \int_0^{\infty} p(z) \exp \left[\frac{T}{T-t} (\sigma zx - \frac{1}{2} \sigma^2 z^2 t) \right] dz. \quad (6.105)$$

We see that the equation (6.104) simplifies due the cancellation of the denominator in the first term. Thus, we obtain

$$C_0 = P_{0t} \sqrt{\frac{T}{2\pi t(T-t)}} \int_{-\infty}^{\infty} \left(P_{tT} \exp \left[-\frac{T}{2t(T-t)} x^2 \right] \int_0^{\infty} p_t(z) z dz - K \exp \left[-\frac{T}{2t(T-t)} x^2 \right] \int_0^{\infty} p_t(z) dz \right)^+ dx. \quad (6.106)$$

where $p_t(z)$ is the “unnormalised” conditional probability density, given by

$$p_t(z) = p(z) \exp \left[\frac{T}{T-t} (\sigma zx - \sigma^2 z^2 t) \right]. \quad (6.107)$$

We observe that the argument of the maximum function (6.106) vanishes when x takes the value x^* —the critical value—that solves the following equation:

$$\begin{aligned} & P_{tT} \int_0^{\infty} p(z) z \exp \left[\frac{T}{2t(T-t)} (x^* - \sigma z t)^2 \right] dz - K \int_0^{\infty} p(z) \exp \left[\frac{T}{2t(T-t)} (x^* - \sigma z t)^2 \right] dz \\ & = 0. \end{aligned} \quad (6.108)$$

Now we define a random variable

$$\eta^*(z) = x^* - \sigma z t, \quad (6.109)$$

and re-scale it by the variance of the standard Brownian bridge, to give us a new variable

$$\nu^*(z) = \frac{\eta^*(z)}{\sqrt{\frac{t(T-t)}{T}}}. \quad (6.110)$$

The random variable ν^* is normally distributed with zero mean and unit variance. Then the equation (6.106) reads

$$C_0 = P_{0t} \left[P_{tT} \int_0^\infty p(z) z \left(\frac{1}{\sqrt{2\pi}} \int_{\nu^*}^\infty \exp\left(-\frac{1}{2}\nu^2\right) d\nu \right) dz \right. \quad (6.111)$$

$$\left. - K \int_0^\infty p(z) \left(\frac{1}{\sqrt{2\pi}} \int_{\nu^*}^\infty \exp\left(-\frac{1}{2}\nu^2\right) d\nu \right) dz \right]. \quad (6.112)$$

We now make use of the identity

$$\frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{1}{2}\eta^2\right) d\eta = N(-x), \quad (6.113)$$

where $N(x)$ is the standard normal distribution function. And thus, we obtain the price of European-style call option on a single-dividend-paying asset:

$$C_0 = P_{0t} \left[P_{tT} \int_0^\infty p(z) z N[-\nu^*(z)] dz - K \int_0^\infty p(z) N[-\nu^*(z)] dz \right]. \quad (6.114)$$

Here we note that the above expression coincides with the formula (5.31) derived using the change-of-measure technique presented in Section 5.5, once this has been adapted to the case of an asset with a continuous payoff function.

6.10 Intertemporal Arrow-Debreu densities

In this section we develop the Arrow-Debreu price for the case where we consider a single information process at two distinct fixed times, t_1 and t_2 where $t_1 \leq t_2$. Thus, we write

$$\xi_{t_1} = \sigma H_T t_1 + \beta_{t_1 T} \quad (6.115)$$

$$\xi_{t_2} = \sigma H_T t_2 + \beta_{t_2 T}. \quad (6.116)$$

The random variable H_T is assumed to be discrete. Then we follow the same scheme as in Section 4.2 to compute the bivariate Arrow-Debreu density

$$A(x_1, x_2) = \mathbb{E}[\delta(\xi_{t_1} - x_1)\delta(\xi_{t_2} - x_2)]. \quad (6.117)$$

It should be evident from properties of the delta function that an equivalent way of writing (6.117) is given by

$$A(x_1, x_2) = \mathbb{E} \left[\delta \left(\xi_{t_1} - \frac{t_1}{t_2} \xi_{t_2} + \frac{t_1}{t_2} x_2 - x_1 \right) \delta(\xi_{t_2} - x_2) \right]. \quad (6.118)$$

Lemma 6.10.1 *The random variable*

$$\xi_{t_1} - \frac{t_1}{t_2}\xi_{t_2} \quad (6.119)$$

is independent of ξ_{t_2} .

Proof. A short calculation shows that

$$\xi_{t_1} - \frac{t_1}{t_2}\xi_{t_2} = \beta_{t_1 T} - \frac{t_1}{t_2}\beta_{t_2 T}. \quad (6.120)$$

Then we show that

$$\beta_{t_1 T} - \frac{t_1}{t_2}\beta_{t_2 T} \quad (6.121)$$

is independent of ξ_{t_2} due to the fact that (6.121) is independent of H_T , and $\beta_{t_2 T}$. Expression (6.121) is by definition independent of H_T and independence of $\beta_{t_2 T}$ is shown by computing the following covariance:

$$\begin{aligned} \mathbb{E} \left[\left(\beta_{t_1 T} - \frac{t_1}{t_2}\beta_{t_2 T} \right) \beta_{t_2 T} \right] &= \mathbb{E}[\beta_{t_1 T}\beta_{t_2 T}] - \frac{t_1}{t_2}\mathbb{E}[\beta_{t_2 T}\beta_{t_2 T}] \\ &= \frac{t_1(T-t_2)}{T} - \frac{t_1 t_2(T-t_2)}{t_2 T} \\ &= 0. \end{aligned} \quad (6.122)$$

□

Making use of Lemma 6.10.1, we thus obtain

$$A(x_1, x_2) = \mathbb{E} \left[\delta \left(\xi_{t_1} - \frac{t_1}{t_2}\xi_{t_2} + \frac{t_1}{t_2}x_2 - x_1 \right) \right] \mathbb{E}[\delta(\xi_{t_2} - x_2)]. \quad (6.123)$$

Now we use the Fourier representation of the delta function and rewrite the Arrow-Debreu density in the form

$$\begin{aligned} A(x_1, x_2) &= \mathbb{E} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i(\xi_{t_2} - x_2)y_2] dy_2 \right] \\ &\times \mathbb{E} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[i \left(\xi_{t_1} - \frac{t_1}{t_2}\xi_{t_2} + \frac{t_1}{t_2}x_2 - x_1 \right) y_1 \right] dy_1 \right]. \end{aligned} \quad (6.124)$$

After swapping the integral with the expectation we then obtain

$$\begin{aligned} A(x_1, x_2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dy_1 \frac{1}{2\pi} \int_{-\infty}^{\infty} dy_2 \exp \left[ix_2 y_2 - i \left(x_1 - \frac{t_1}{t_2}x_2 \right) y_1 \right] \\ &\times \mathbb{E}[\exp(iy_2 \xi_{t_2})] \mathbb{E} \left[\exp \left(iy_1 \left(\xi_{t_1} - \frac{t_1}{t_2}\xi_{t_2} \right) \right) \right]. \end{aligned} \quad (6.125)$$

We now compute both expectations in the equation above, and observe that

$$\mathbb{E} \left[\exp \left(\kappa_1 \left(\xi_{t_1} - \frac{t_1}{t_2} \xi_{t_2} \right) \right) \right] = \mathbb{E} \left[\exp \left(\kappa_1 \left(\beta_{t_1 T} - \frac{t_1}{t_2} \beta_{t_2 T} \right) \right) \right], \quad (6.126)$$

where $\kappa_1 = iy_1$.

Lemma 6.10.2 *Let $\beta_{t_1 T}$ and $\beta_{t_2 T}$ be values of a Brownian bridge process over the interval $[0, T]$, with $0 \leq t_1 \leq t_2 \leq T$. Then the random variable*

$$\beta_{t_1 T} - \frac{t_1}{t_2} \beta_{t_2 T} \quad (6.127)$$

is normally distributed with mean zero and variance

$$\frac{t_1(t_2 - t_1)}{t_2}. \quad (6.128)$$

Proof. The mean of a linear combination of standard Brownian bridge variables is zero. The variance of expression (6.127) is computed as follows:

$$\begin{aligned} \text{Var} \left[\beta_{t_1 T} - \frac{t_1}{t_2} \beta_{t_2 T} \right] &= \mathbb{E} \left[\left(\beta_{t_1 T} - \frac{t_1}{t_2} \beta_{t_2 T} \right)^2 \right] \\ &= \mathbb{E} [\beta_{t_1 T}^2] - 2 \frac{t_1}{t_2} \mathbb{E} [\beta_{t_1 T} \beta_{t_2 T}] + \frac{t_1^2}{t_2^2} \mathbb{E} [\beta_{t_2 T}^2] \\ &= \frac{t_1(T - t_1)}{T} - 2 \frac{t_1^2(T - t_2)}{t_2 T} + \frac{t_1^2(T - t_2)}{t_2 T} \\ &= \frac{t_1(t_2 - t_1)}{t_2}. \end{aligned} \quad (6.129)$$

□

From Lemma 6.10.2 we conclude that

$$\mathbb{E} \left[\exp \left(\kappa_1 \left(\xi_{t_1} - \frac{t_1}{t_2} \xi_{t_2} \right) \right) \right] = \exp \left[-\frac{1}{2} \frac{t_1(t_2 - t_1)}{t_2} \right]. \quad (6.130)$$

Analogously, using the fact that the random variable $\beta_{t_2 T}$ is normally distributed with zero mean and variance $t_2(T - t_2)/T$ and is independent of the random variable H_T , we have:

$$\mathbb{E} [\exp (iy_2 \xi_{t_2})] = \sum_{j=0}^n p_j \exp \left[\kappa \sigma h_j t_2 - \frac{1}{2} \frac{t_2(T - t_2)}{T} y_2^2 \right], \quad (6.131)$$

where p_j is the *a priori* probability that the random variable H_T takes the value h_j .

Thus, the term involving the two expectations in the equation (6.125) yields

$$\begin{aligned} & \mathbb{E} [\exp (i y_2 \xi_{t_2})] \mathbb{E} \left[\exp \left(i y_1 \left(\xi_{t_1} - \frac{t_1}{t_2} \xi_{t_2} \right) \right) \right] \\ &= \sum_{j=0}^n p_j \exp (i \sigma h_j t_2 y_2) \exp \left(-\frac{t_2(T-t_2)}{2T} y_2^2 - \frac{t_1(t_2-t_1)}{2t_2} y_1^2 \right). \end{aligned} \quad (6.132)$$

Hence, the Arrow-Debreu density (6.125) can be written in the form

$$\begin{aligned} A(x_1, x_2) &= \sum_{j=0}^n p_j \frac{1}{2\pi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[-i \left(x_1 - \frac{t_1}{t_2} x_2 \right) y_1 - \frac{t_1(t_2-t_1)}{2t_2} y_1^2 \right] \\ &\times \exp \left[-i(x_2 - \sigma h_j t_2) y_2 - \frac{t_2(T-t_2)}{2T} y_2^2 \right] dy_1 dy_2. \end{aligned} \quad (6.133)$$

Here we have interchanged the sum with the integrals. Carrying out the integration we are then led to the desired result.

Proposition 6.10.1 *Let t_1 , t_2 , and T be fixed times, where $t_1 \leq t_2 \leq T$. Consider the values of the information process $\{\xi_t\}$ at t_1 and t_2 . The associated bivariate intertemporal Arrow-Debreu density is given by:*

$$\begin{aligned} A(x_1, x_2) &= \frac{1}{2\pi} \sqrt{\frac{T}{t_1(T-t_2)(t_2-t_1)}} \exp \left[-\frac{t_2}{2t_1(t_2-t_1)} \left(x_1 - \frac{t_1}{t_2} x_2 \right)^2 \right] \\ &\times \sum_{j=0}^n p_j \exp \left[-\frac{T}{2t_2(T-t_2)} (x_2 - \sigma h_j t_2)^2 \right]. \end{aligned} \quad (6.134)$$

The bivariate intertemporal Arrow-Debreu price enables us to calculate the prices of options that depend on the values of the information process at two distinct times. A multivariate intertemporal Arrow-Debreu price can be similarly constructed.

Chapter 7

Information-based approach to interest rates and inflation

7.1 Overview

In this chapter we apply the information-based framework to introduce a class of discrete-time models for interest rates and inflation. The key idea is that market participants have at any time partial information about the future values of macro-economic factors that influence consumption, money supply, and other variables that determine interest rates and price levels. We present a model for such partial information, and show how it leads to a consistent framework for the arbitrage-free dynamics of real and nominal interest rates, price-indices, and index-linked securities.

We begin with a general model for discrete-time asset pricing. We take a pricing kernel approach, which has the effect of building in the arbitrage-free property, and providing the desired link to economic equilibrium. We require that the pricing kernel should be consistent with a pair of axioms, one giving the general intertemporal relations for dividend-paying assets, and the other relating to the existence of a money market asset. Instead of directly assuming the existence of a previsible money-market account, we make a somewhat weaker assumption, namely the existence of an asset that offers a positive rate of return. It can be deduced, however, that the assumption of the existence of a positive-return asset is sufficient to imply the existence of a previsible money-market account, once the intertemporal relations implicit in the first axiom are taken into account.

The main result of Section 7.2 is the derivation of a general expression for the price process of a limited-liability asset. This expression includes two terms, one being the familiar discounted and risk-adjusted value of the dividend stream, and the other characterising retained earnings. The vanishing of the latter is shown in Proposition 7.2.1 to be given by a transversality condition, equation (7.10). In particular, we are able to show (under the conditions of Axioms A and B) that in the case of a limited-liability asset with no permanently retained earnings, the general form of the price process is given by the ratio of a potential and the pricing kernel, as expressed in equation (7.21). In Section 7.3 we consider the per-period rate of return $\{\bar{r}_i\}$ offered by the positive return asset, and we show in Proposition 7.3.1 that there exists a constant-value asset with limited liability such that the associated dividend flow is given by $\{\bar{r}_i\}$. This result is then used in Proposition 7.3.2 to establish that the pricing kernel admits a decomposition of the form (7.32). In Proposition 7.3.3 we prove what then might be interpreted as a converse to this result, thus giving us a procedure for constructing examples of systems satisfying Axioms A and B. The method involves the introduction of an increasing sequence that converges to an integrable random variable. Given the sequence we then construct an associated pricing kernel and positive-return asset satisfying the intertemporal relations.

In Section 7.4 we introduce the nominal discount bond system arising with the specification of a given pricing kernel, and in Proposition 7.4.1 we show that the discount bond system admits a representation of the Flesaker-Hughston type. In Section 7.5 we consider the case when the positive-return asset has a previsible price process, and hence can be consistently interpreted (in a standard way) as a money-market account, or “risk-free” asset. The results of the previous sections do not depend on this additional assumption. A previsible money-market account has the structure of a series of one-period discount-bond investments. Then in Proposition 7.5.1 we show under the conditions of Axioms A and B that there exists a unique previsible money-market account. In other words, although we only assume the existence of a positive-return asset, we can then establish the existence of a money-market account asset.

In Section 7.6 we outline a general approach to interest rate modelling in the information-based framework. In Section 7.7 we are then able to propose a class of stochastic models for the pricing of inflation-linked assets. The nominal and real pricing kernels, in terms of which the consumer price index can be expressed, are modelled

by introducing a bivariate utility function depending on (a) aggregate consumption, and (b) the aggregate real liquidity benefit conferred by the money supply. Consumption and money supply policies are chosen such that the expected joint utility obtained over a specified time horizon is maximised, subject to a budget constraint that takes into account the “value” of the liquidity benefit associated with the money supply. For any choice of the bivariate utility function, the resulting model determines a relation between the rate of consumption, the price level, and the money supply. The model also produces explicit expressions for the real and nominal pricing kernels, and hence establishes a basis for the valuation of inflation-linked securities.

7.2 Asset pricing in discrete time

The development of asset-pricing theory in discrete time has been pursued by many authors. In the context of interest rate modelling, it is worth recalling that the first example of a fully developed term-structure model where the initial discount function is freely specifiable is that of Ho & Lee 1986, in a discrete-time setting. For our purposes it will be useful to develop a general discrete-time scheme from first principles, taking an axiomatic approach in the spirit of Hughston & Rafailidis (2005).

Let $\{t_i\}_{i=0,1,2,\dots}$ denote a sequence of discrete times, where t_0 represents the present and $t_{i+1} > t_i$ for all $i \in \mathbb{N}_0$. We assume that the sequence $\{t_i\}$ is unbounded in the sense that for any given time T there exists a value of i such that $t_i > T$. We do not assume that the elements of $\{t_i\}$ are equally spaced; for some applications, however, we can consider the case where $t_n = n\tau$ for all $n \in \mathbb{N}_0$ and for some unit of time τ .

Each asset is characterised by a pair of processes $\{S_{t_i}\}_{i \geq 0}$ and $\{D_{t_i}\}_{i \geq 0}$ which we refer to as the “value process” and the “dividend process”, respectively. We interpret D_{t_i} as a random cash flow or dividend paid to the owner of the asset at time t_i . Then S_{t_i} denotes the “ex-dividend” value of the asset at t_i . We can think of S_{t_i} as the cash flow that would result if the owner were to dispose of the asset at time t_i .

For simplicity, we shall frequently use an abbreviated notation, and write $S_i = S_{t_i}$ and $D_i = D_{t_i}$. Thus S_i denotes the value of the asset at time t_i , and D_i denotes the dividend paid at time t_i . Both S_i and D_i are expressed in nominal terms, i.e. in units of a fixed base currency. We use the term “asset” in the broad sense here—the scheme is thus applicable to any liquid financial position for which the values and cash flows

are well defined, and for which the principles of no arbitrage are applicable.

The unfolding of random events in the economy will be represented with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_i\}_{i \geq 0}$ which we call the “market filtration”. For the moment we regard the market filtration as given, but later we shall construct it explicitly. For each asset we assume that the associated value and dividend processes are adapted to $\{\mathcal{F}_i\}$. In what follows \mathbb{P} is taken to be the “physical” or “objective” probability measure; all equalities and inequalities between random variables are to be understood as holding almost surely with respect to \mathbb{P} . For convenience we often write $\mathbb{E}_i[-]$ for $\mathbb{E}[-|\mathcal{F}_i]$.

In order to ensure the absence of arbitrage in the financial markets and to establish intertemporal pricing relations, we assume the existence of a strictly positive pricing kernel $\{\pi_i\}_{i \geq 0}$, and make the following assumptions:

Axiom A. *For any asset with associated value process $\{S_i\}_{i \geq 0}$ and dividend process $\{D_i\}_{i \geq 0}$, the process $\{M_i\}_{i \geq 0}$ defined by*

$$M_i = \pi_i S_i + \sum_{n=0}^i \pi_n D_n \quad (7.1)$$

is a martingale, i.e. $\mathbb{E}[|M_i|] < \infty$ for all $i \in \mathbb{N}_0$, and $\mathbb{E}[M_j|\mathcal{F}_i] = M_i$ for all $i \leq j$.

Axiom B. *There exists a strictly positive non-dividend-paying asset, with value process $\{\bar{B}_i\}_{i \geq 0}$, that offers a strictly positive return, i.e. such that $\bar{B}_{i+1} > \bar{B}_i$ for all $i \in \mathbb{N}_0$. We assume that the process $\{\bar{B}_i\}$ is unbounded in the sense that for any $b \in \mathbb{R}$ there exists a time t_i such that $\bar{B}_i > b$.*

Given this axiomatic scheme, we proceed to explore its consequences. The notation $\{\bar{B}_i\}$ is used in Axiom B to distinguish the positive return asset from the previsible money-market account asset $\{B_i\}$ that will be introduced later; in particular, in Proposition 7.5.1 it will be shown that Axioms A and B imply the existence of a unique money-market account asset. We note that since the positive-return asset is non-dividend paying, it follows from Axiom A that $\{\pi_i \bar{B}_i\}$ is a martingale. Writing $\bar{\rho}_i = \pi_i \bar{B}_i$, we have $\pi_i = \bar{\rho}_i / \bar{B}_i$. Since $\{\bar{B}_i\}$ is assumed to be strictly increasing, we see that $\{\pi_i\}$ is a supermartingale. In fact, we have the somewhat stronger relation $\mathbb{E}_i[\pi_j] < \pi_i$. Indeed, we note that

$$\mathbb{E}_i[\pi_j] = \mathbb{E}_i \left[\frac{\bar{\rho}_j}{\bar{B}_j} \right] < E_i \left[\frac{\bar{\rho}_j}{\bar{B}_i} \right] = \frac{E_i[\bar{\rho}_j]}{\bar{B}_i} = \frac{\bar{\rho}_i}{\bar{B}_i} = \pi_i. \quad (7.2)$$

The significance of $\{\bar{\rho}_i\}$ is that it has the interpretation of being the likelihood ratio appropriate for a change of measure from the objective measure \mathbb{P} to the equivalent martingale measure \mathbb{Q} characterised by the property that non-dividend-paying assets when expressed in units of the numeraire $\{\bar{B}_i\}$ are martingales.

We recall the definition of a potential. An adapted process $\{x_i\}_{0 \leq i < \infty}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_i\}$ is said to be a potential if $\{x_i\}$ is a non-negative supermartingale and $\lim_{i \rightarrow \infty} \mathbb{E}[x_i] = 0$. It is straightforward to show that $\{\pi_i\}$ is a potential. We need to demonstrate that given any $\epsilon > 0$ we can find a time t_j such $\mathbb{E}[\pi_n] < \epsilon$ for all $n \geq j$. This follows from the assumption that the positive-return asset price process $\{\bar{B}_i\}$ is unbounded in the sense specified in Axiom B. Thus given ϵ let us set $b = \bar{\rho}_0/\epsilon$. Now given b we can find a time t_j such that $\bar{B}_{t_n} > b$ for all $n \geq j$. But for that value of t_j we have

$$\mathbb{E}[\pi_j] = \mathbb{E}\left[\frac{\bar{\rho}_j}{\bar{B}_j}\right] < \frac{\mathbb{E}[\bar{\rho}_j]}{b} = \frac{\bar{\rho}_0}{b} = \epsilon, \quad (7.3)$$

and hence $\mathbb{E}[\pi_n] < \epsilon$ for all $n \geq j$. It follows that

$$\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0. \quad (7.4)$$

Next we recall the Doob decomposition for discrete-time supermartingales (see, e.g., Meyer 1966, chapter 7). If $\{x_i\}$ is a supermartingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_i\}$, then there exists a martingale $\{y_i\}$ and a previsible increasing process $\{a_i\}$ such that $x_i = y_i - a_i$ for all $i \geq 0$. By previsible, we mean that a_i is \mathcal{F}_{i-1} -measurable. The decomposition is given explicitly by $a_0 = 0$ and $a_i = a_{i-1} + x_{i-1} - \mathbb{E}_{i-1}[x_i]$ for $i \geq 1$.

It follows that the pricing kernel admits a decomposition of this form, and that one can write

$$\pi_i = Y_i - A_i, \quad (7.5)$$

where $A_0 = 0$ and

$$A_i = \sum_{n=0}^{i-1} (\pi_n - \mathbb{E}_n[\pi_{n+1}]) \quad (7.6)$$

for $i \geq 1$; and where $Y_0 = \pi_0$ and

$$Y_i = \sum_{n=0}^{i-1} (\pi_{n+1} - \mathbb{E}_n[\pi_{n+1}]) + \pi_0 \quad (7.7)$$

for $i \geq 1$. The Doob decomposition for $\{\pi_i\}$ has an interesting expression in terms of discount bonds, which we shall mention later, in Section 7.5.

In the case of a potential $\{x_i\}$ it can be shown (see, e.g., Gihman & Skorohod 1979, chapter 1) that the limit $a_\infty = \lim_{i \rightarrow \infty} a_i$ exists, and that $x_i = \mathbb{E}_i[a_\infty] - a_i$. As a consequence, we conclude that the pricing kernel admits a decomposition of the form

$$\pi_i = \mathbb{E}_i[A_\infty] - A_i, \quad (7.8)$$

where $\{A_i\}$ is the previsible process defined by (7.6). With these facts in hand, we shall establish a useful result concerning the pricing of limited-liability assets. By a limited-liability asset we mean an asset such that $S_i \geq 0$ and $D_i \geq 0$ for all $i \in \mathbb{N}$.

Proposition 7.2.1 *Let $\{S_i\}_{i \geq 0}$ and $\{D_i\}_{i \geq 0}$ be the value and dividend processes associated with a limited-liability asset. Then $\{S_i\}$ is of the form*

$$S_i = \frac{m_i}{\pi_i} + \frac{1}{\pi_i} \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n D_n \right], \quad (7.9)$$

where $\{m_i\}$ is a non-negative martingale that vanishes if and only if the following transversality condition holds:

$$\lim_{j \rightarrow \infty} \mathbb{E}[\pi_j S_j] = 0. \quad (7.10)$$

Proof. It follows from Axiom A, as a consequence of the martingale property, that

$$\pi_i S_i + \sum_{n=0}^i \pi_n D_n = \mathbb{E}_i \left[\pi_j S_j + \sum_{n=0}^j \pi_n D_n \right] \quad (7.11)$$

for all $i \leq j$. Taking the limit $j \rightarrow \infty$ on the right-hand side of this relation we have

$$\pi_i S_i + \sum_{n=0}^i \pi_n D_n = \lim_{j \rightarrow \infty} \mathbb{E}_i[\pi_j S_j] + \lim_{j \rightarrow \infty} \mathbb{E}_i \left[\sum_{n=0}^j \pi_n D_n \right]. \quad (7.12)$$

Since $\pi_i D_i \geq 0$ for all $i \in \mathbb{N}_0$, it follows from the conditional form of the monotone convergence theorem—see, e.g., Steele 2001, Williams 1991—that

$$\lim_{j \rightarrow \infty} \mathbb{E}_i \left[\sum_{n=0}^j \pi_n D_n \right] = \mathbb{E}_i \left[\lim_{j \rightarrow \infty} \sum_{n=0}^j \pi_n D_n \right], \quad (7.13)$$

and hence that

$$\pi_i S_i + \sum_{n=0}^i \pi_n D_n = \lim_{j \rightarrow \infty} \mathbb{E}_i[\pi_j S_j] + \mathbb{E}_i \left[\sum_{n=0}^{\infty} \pi_n D_n \right]. \quad (7.14)$$

Now let us define

$$m_i = \lim_{j \rightarrow \infty} \mathbb{E}_i[\pi_j S_j]. \quad (7.15)$$

Then clearly $m_i \geq 0$ for all $i \in \mathbb{N}_0$. We see, moreover, that $\{m_i\}_{i \geq 0}$ is a martingale, since $m_i = M_i - \mathbb{E}_i[F_\infty]$, where M_i is defined as in equation (7.1), and

$$F_\infty = \sum_{n=0}^{\infty} \pi_n D_n. \quad (7.16)$$

It is implicit in the axiomatic scheme that the sum $\sum_{n=0}^{\infty} \pi_n D_n$ converges in the case of a limited-liability asset. This follows as a consequence of the martingale convergence theorem and Axiom A. Thus, writing equation (7.14) in the form

$$\pi_i S_i + \sum_{n=0}^i \pi_n D_n = m_i + \mathbb{E}_i \left[\sum_{n=0}^{\infty} \pi_n D_n \right], \quad (7.17)$$

after some re-arrangement of terms we obtain

$$\pi_i S_i = m_i + \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n D_n \right], \quad (7.18)$$

and hence (7.9), as required. On the other hand, by the martingale property of $\{m_i\}$ we have $\mathbb{E}[m_i] = m_0$ and hence

$$\mathbb{E}[m_i] = \lim_{j \rightarrow \infty} \mathbb{E}[\pi_j S_j] \quad (7.19)$$

for all $i \in \mathbb{N}$. Thus since $m_i \geq 0$ we see that the transversality condition (7.10) holds if and only if $\{m_i\} = 0$. \square

The interpretation of the transversality condition is as follows. For each $j \in \mathbb{N}_0$ the expectation $V_j = \mathbb{E}[\pi_j S_j]$ measures the present value of an instrument that pays at t_j an amount equal to the proceeds of a liquidation of the asset with price process $\{S_i\}_{i \geq 0}$. If $\lim_{j \rightarrow \infty} V_j = 0$ then one can say that in the long term all of the value of the asset will be dispersed in its dividends. On the other hand, if some or all of the

dividends are “retained” indefinitely, then $\{V_j\}$ will retain some value, even in the limit as t_j goes to infinity.

The following example may clarify this interpretation. Suppose investors put \$100m of capital into a new company. The management of the company deposits \$10m into a money market account. The remaining \$90m is invested in an ordinary risky line of business, the entire proceeds of which, after costs, are paid to share-holders as dividends. Thus at time t_i we have $S_i = B_i + H_i$, where B_i is a position in the money market account initialised at \$10m, and where H_i is the value of the remaining dividend flow. Now $\{\pi_i B_i\}$ is a martingale, and thus $\mathbb{E}[\pi_i B_i] = \10m for all $i \in \mathbb{N}_0$, and therefore $\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i B_i] = \10m . On the other hand $\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i H_i] = 0$; this means that given any value h we can find a time T such that $\mathbb{E}[\pi_i H_i] < h$ for all $t_i \geq T$.

There are other ways of “retaining” funds than putting them into a domestic money market account. For example, one could put the \$10m into a foreign bank account; or one could invest it in shares in a securities account, with a standing order that all dividends should be immediately re-invested in further shares. Thus if the investment is in a general “dividend-retaining” asset (such as a foreign bank account), then $\{m_i\}$ can in principal be any non-negative martingale. The content of Proposition 7.2.1 is that any limited-liability investment can be separated in a unique way into a growth component and an income component.

In the case of a “pure income” investment, i.e. in an asset for which the transversality condition is satisfied, the price is directly related to the future dividend flow, and we have

$$S_i = \frac{1}{\pi_i} \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n D_n \right]. \quad (7.20)$$

This is the so-called “fundamental equation” which some authors use directly as a basis for asset pricing theory—see, e.g., Cochrane 2005. Alternatively we can write (7.20) in the form

$$S_i = \frac{1}{\pi_i} (\mathbb{E}_i[F_\infty] - F_i), \quad (7.21)$$

where

$$F_i = \sum_{n=0}^i \pi_n D_n, \quad \text{and} \quad F_\infty = \lim_{i \rightarrow \infty} F_i. \quad (7.22)$$

It is a straightforward exercise to show that the process $\{\pi_i S_i\}$ is a potential. Clearly, $\{\mathbb{E}_i[F_\infty] - F_i\}$ is a positive supermartingale, since $\{F_i\}$ is increasing; and by the tower

property and the monotone convergence theorem we have $\lim_{i \rightarrow \infty} \mathbb{E}[\mathbb{E}_i[F_\infty] - F_i] = \mathbb{E}[F_\infty] - \lim_{i \rightarrow \infty} \mathbb{E}[F_i] = \mathbb{E}[F_\infty] - \mathbb{E}[\lim_{i \rightarrow \infty} F_i] = 0$. On the other hand, $\{\pi_i\}$ is also a potential, so we reach the conclusion that in the case of an income generating asset the price process can be expressed as a ratio of potentials, thus giving us a discrete-time analogue of a result obtained by Rogers 1997. Indeed, the role of the concept of a potential as it appears here is consistent with the continuous-time theories developed by Flesaker & Hughston 1996, Rogers 1997, Rutkowski 1997, Jin & Glasserman 2001, Hughston & Rafailidis 2005, and others, where essentially the same mathematical structures appear.

7.3 Nominal pricing kernel

To proceed further we need to say more about the relation between the pricing kernel $\{\pi_i\}$ and the positive-return asset $\{\bar{B}_i\}$. To this end let us write

$$\bar{r}_i = \frac{\bar{B}_i - \bar{B}_{i-1}}{\bar{B}_{i-1}} \quad (7.23)$$

for the rate of return on the positive-return asset realised at time t_i on an investment made at time t_{i-1} . Since the time interval $t_i - t_{i-1}$ is not necessarily small, there is no specific reason to presume that the rate of return \bar{r}_i is already known at time t_{i-1} . This is consistent with the fact that we have assumed that $\{\bar{B}_i\}$ is $\{\mathcal{F}_i\}$ -adapted. The notation \bar{r}_i is used here to distinguish the rate of return on the positive-return asset from the rate of return r_i on the money market account, which will be introduced in Section 7.5.

Next we present a simple argument to motivate the idea that there should exist an asset with constant value unity that pays a dividend stream given by $\{\bar{r}_i\}$. We consider the following portfolio strategy. The portfolio consists at any time of a certain number of units of the positive-return asset. Let ϕ_i denote the number of units, so that at time t_i the (ex-dividend) value of the portfolio is given by $V_i = \phi_i \bar{B}_i$. Then in order to have $V_i = 1$ for all $i \geq 0$ we set $\phi_i = 1/\bar{B}_i$. Let D_i denote the dividend paid out by the portfolio at time t_i . Then clearly if the portfolio value is to remain constant we must have $D_i = \phi_{i-1} \bar{B}_i - \phi_{i-1} \bar{B}_{i-1}$ for all $i \geq 1$. It follows immediately that $D_i = \bar{r}_i$, where \bar{r}_i is given by (7.23).

This shows that we can construct a portfolio with a constant value and with the desired cash flows. Now we need to show that such a system satisfies Axiom A.

Proposition 7.3.1 *There exists an asset with constant nominal value $S_i = 1$ for all $i \in \mathbb{N}_0$, for which the associated cash flows are given by $\{\bar{r}_i\}_{i \geq 1}$.*

Proof. We need to verify that the conditions of Axiom A are satisfied in the case for which $S_i = 1$ and $D_i = \bar{r}_i$ for $i \in \mathbb{N}_0$. In other words we need to show that

$$\pi_i = \mathbb{E}_i[\pi_j] + \mathbb{E}_i \left[\sum_{n=i+1}^j \pi_n \bar{r}_n \right] \quad (7.24)$$

for all $i \leq j$. The calculation proceeds as follows. We observe that

$$\begin{aligned} \mathbb{E}_i \left[\sum_{n=i+1}^j \pi_n \bar{r}_n \right] &= \mathbb{E}_i \left[\sum_{n=i+1}^j \pi_n \frac{\bar{B}_n - \bar{B}_{n-1}}{\bar{B}_{n-1}} \right] \\ &= \mathbb{E}_i \left[\sum_{n=i+1}^j \frac{\bar{\rho}_n}{\bar{B}_n} \frac{\bar{B}_n - \bar{B}_{n-1}}{\bar{B}_{n-1}} \right] \\ &= \mathbb{E}_i \left[\sum_{n=i+1}^j \left(\frac{\bar{\rho}_n}{\bar{B}_{n-1}} - \frac{\bar{\rho}_n}{\bar{B}_n} \right) \right] \\ &= \mathbb{E}_i \left[\sum_{n=i+1}^j \left(\mathbb{E}_{n-1} \left[\frac{\bar{\rho}_n}{\bar{B}_{n-1}} \right] - \frac{\bar{\rho}_n}{\bar{B}_n} \right) \right], \end{aligned} \quad (7.25)$$

the last step being achieved by use of the tower property. It follows then by use of the martingale property of $\{\rho_n\}$ that:

$$\begin{aligned} \mathbb{E}_i \left[\sum_{n=i+1}^j \pi_n \bar{r}_n \right] &= \mathbb{E}_i \left[\sum_{n=i+1}^j \left(\frac{1}{\bar{B}_{n-1}} \mathbb{E}_{n-1}[\bar{\rho}_n] - \frac{\bar{\rho}_n}{\bar{B}_n} \right) \right] \\ &= \mathbb{E}_i \left[\sum_{n=i+1}^j \left(\frac{\bar{\rho}_{n-1}}{\bar{B}_{n-1}} - \frac{\bar{\rho}_n}{\bar{B}_n} \right) \right] \\ &= \mathbb{E}_i \left[\left(\frac{\bar{\rho}_i}{\bar{B}_i} - \frac{\bar{\rho}_{i+1}}{\bar{B}_{i+1}} \right) + \left(\frac{\bar{\rho}_{i+1}}{\bar{B}_{i+1}} - \frac{\bar{\rho}_{i+2}}{\bar{B}_{i+2}} \right) + \dots + \left(\frac{\bar{\rho}_{j-1}}{\bar{B}_{j-1}} - \frac{\bar{\rho}_j}{\bar{B}_j} \right) \right] \\ &= \mathbb{E}_i \left[\frac{\bar{\rho}_i}{\bar{B}_i} \right] - \mathbb{E}_i \left[\frac{\bar{\rho}_j}{\bar{B}_j} \right] \\ &= \pi_i - \mathbb{E}_i[\pi_j]. \end{aligned} \quad (7.26)$$

But that gives us (7.24). □

The existence of the constant-value asset leads to an alternative decomposition of the pricing kernel, which can be described as follows.

Proposition 7.3.2 *Let $\{\bar{B}_i\}$ be a positive-return asset satisfying the conditions of Axiom B, and let $\{\bar{r}_i\}$ be its rate-of-return process. Then the pricing kernel can be expressed in the form $\pi_i = \mathbb{E}_i[G_\infty] - G_i$, where $G_i = \sum_{n=1}^i \pi_n \bar{r}_n$ and $G_\infty = \lim_{i \rightarrow \infty} G_i$.*

Proof. First we remark that if an asset has constant value then it satisfies the transversality condition (7.10). In particular, letting the constant be unity, we see that the transversality condition reduces to

$$\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0, \quad (7.27)$$

which is satisfied since $\{\pi_i\}$ is a potential. Next we show that

$$\lim_{j \rightarrow \infty} \mathbb{E}_i[\pi_j] = 0 \quad (7.28)$$

for all $i \in \mathbb{N}_0$. In particular, fixing i , we have $\mathbb{E}[\mathbb{E}_i[\pi_j]] = \mathbb{E}[\pi_j]$ by the tower property, and thus

$$\lim_{j \rightarrow \infty} \mathbb{E}[\mathbb{E}_i[\pi_j]] = 0 \quad (7.29)$$

by virtue of (7.27). But $\mathbb{E}_i[\pi_j] < \pi_i$ for all $j > i$, and $\mathbb{E}[\pi_i] < \infty$; hence by the dominated convergence theorem we have

$$\lim_{j \rightarrow \infty} \mathbb{E}[\mathbb{E}_i[\pi_j]] = \mathbb{E}[\lim_{j \rightarrow \infty} \mathbb{E}_i[\pi_j]], \quad (7.30)$$

from which the desired result (7.28) follows, since the argument of the expectation is non-negative. As a consequence of (7.28) it follows from (7.24) that

$$\pi_i = \lim_{j \rightarrow \infty} \mathbb{E}_i \left[\sum_{n=i+1}^j \pi_n \bar{r}_n \right], \quad (7.31)$$

and thus by the monotone convergence theorem we have

$$\begin{aligned} \pi_i &= \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n \bar{r}_n \right] \\ &= \mathbb{E}_i \left[\sum_{n=1}^{\infty} \pi_n \bar{r}_n \right] - \sum_{n=1}^i \pi_n \bar{r}_n \\ &= \mathbb{E}_i [G_\infty] - G_i, \end{aligned} \quad (7.32)$$

and that gives us the result of the proposition. \square

We shall establish a converse to this result, which allows us to construct a system satisfying Axioms A and B from any strictly-increasing non-negative adapted process that converges, providing a certain integrability condition holds.

Proposition 7.3.3 *Let $\{G_i\}_{i \geq 0}$ be a strictly increasing adapted process satisfying $G_0 = 0$, and $\mathbb{E}[G_\infty] < \infty$, where $G_\infty = \lim_{i \rightarrow \infty} G_i$. Let the processes $\{\pi_i\}$, $\{\bar{r}_i\}$, and $\{\bar{B}_i\}$, be defined by $\pi_i = \mathbb{E}_i[G_\infty] - G_i$ for $i \geq 0$; $\bar{r}_i = (G_i - G_{i-1})/\pi_i$ for $i \geq 1$; $\bar{B}_i = \prod_{n=1}^i (1 + \bar{r}_n)$ for $i \geq 1$, with $\bar{B}_0 = 1$. Let the process $\{\bar{\rho}_i\}$ be defined by $\bar{\rho}_i = \pi_i \bar{B}_i$ for $i \geq 0$. Then $\{\bar{\rho}_i\}$ is a martingale, and $\lim_{j \rightarrow \infty} \bar{B}_j = \infty$. Thus $\{\pi_i\}$ and $\{\bar{B}_i\}$, as constructed, satisfy Axioms A and B.*

Proof. Writing $g_i = G_i - G_{i-1}$ for $i \geq 1$ we have

$$\pi_i = \mathbb{E}_i[G_\infty] - G_i = \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} g_n \right], \quad (7.33)$$

and

$$\bar{B}_i = \prod_{n=1}^i (1 + \bar{r}_n) = \prod_{n=1}^i \left(1 + \frac{g_n}{\pi_n} \right) = \prod_{n=1}^i \left(\frac{\pi_n + g_n}{\pi_n} \right). \quad (7.34)$$

Hence, writing $\bar{\rho}_i = \pi_i \bar{B}_i$, we have

$$\begin{aligned} \bar{\rho}_i &= \pi_i \prod_{n=1}^i \left(\frac{\pi_n + g_n}{\pi_n} \right) \\ &= (\pi_i + g_i) \prod_{n=1}^{i-1} \left(\frac{\pi_n + g_n}{\pi_n} \right), \end{aligned} \quad (7.35)$$

and thus

$$\bar{\rho}_i = (\pi_i + g_i) \bar{B}_{i-1} = \frac{\pi_i + g_i}{\pi_{i-1}} \bar{\rho}_{i-1}. \quad (7.36)$$

To show that $\{\bar{\rho}_i\}$ is a martingale it suffices to verify for all $i \geq 1$ that $\mathbb{E}[\bar{\rho}_i] < \infty$ and that $\mathbb{E}_{i-1}[\bar{\rho}_i] = \bar{\rho}_{i-1}$. In particular, if $\mathbb{E}[\bar{\rho}_i] < \infty$ then the ‘‘take out what is known

rule” applies, and by (7.33) and (7.35) we have

$$\begin{aligned}
\mathbb{E}_{i-1}[\bar{\rho}_i] &= \mathbb{E}_{i-1} \left[\frac{\pi_i + g_i}{\pi_{i-1}} \bar{\rho}_{i-1} \right] \\
&= \frac{\bar{\rho}_{i-1}}{\pi_{i-1}} \mathbb{E}_{i-1} [\pi_i + g_i] \\
&= \frac{\bar{\rho}_{i-1}}{\pi_{i-1}} \mathbb{E}_{i-1} \left[\sum_{n=i}^{\infty} g_n \right] \\
&= \frac{\bar{\rho}_{i-1}}{\pi_{i-1}} (\mathbb{E}_{i-1} [G_{\infty}] - G_{i-1}) \\
&= \bar{\rho}_{i-1}.
\end{aligned} \tag{7.37}$$

Here, in going from the first to the second line we have used the fact that $\mathbb{E}[\pi_i + g_i] < \infty$, together with the assumption that $\mathbb{E}[\bar{\rho}_i] < \infty$. To verify that $\mathbb{E}[\bar{\rho}_i] < \infty$ let us write

$$J_{i-1}^{\alpha} = \min \left[\frac{\bar{\rho}_{i-1}}{\pi_{i-1}}, \alpha \right] \tag{7.38}$$

for $\alpha \in \mathbb{N}_0$. Then by use of monotone convergence and the tower property we have

$$\begin{aligned}
\mathbb{E}[\bar{\rho}_i] &= \mathbb{E} \left[(\pi_i + g_i) \lim_{\alpha \rightarrow \infty} J_{i-1}^{\alpha} \right] \\
&= \lim_{\alpha \rightarrow \infty} \mathbb{E} [(\pi_i + g_i) J_{i-1}^{\alpha}] \\
&= \lim_{\alpha \rightarrow \infty} \mathbb{E} [\mathbb{E}_{i-1} [(\pi_i + g_i) J_{i-1}^{\alpha}]] \\
&= \lim_{\alpha \rightarrow \infty} \mathbb{E} [J_{i-1}^{\alpha} \mathbb{E}_{i-1} [(\pi_i + g_i)]] \\
&\leq \mathbb{E} \left[\frac{\bar{\rho}_{i-1}}{\pi_{i-1}} \mathbb{E}_{i-1} [\pi_i + g_i] \right] \\
&= \mathbb{E}[\bar{\rho}_{i-1}],
\end{aligned} \tag{7.39}$$

since

$$J_{i-1}^{\alpha} \leq \frac{\bar{\rho}_{i-1}}{\pi_{i-1}}. \tag{7.40}$$

Thus we see for all $i \geq 1$ that if $\mathbb{E}[\bar{\rho}_{i-1}] < \infty$ then $\mathbb{E}[\bar{\rho}_i] < \infty$. But $\bar{\rho}_0 < \infty$ by construction; hence by induction we deduce that $\mathbb{E}[\bar{\rho}_i] < \infty$ for all $i \geq 0$.

To show that $\lim_{j \rightarrow \infty} \{\bar{B}_j\} = \infty$ let us assume the contrary and show that this leads to a contradiction. Suppose, in particular, that there were to exist a number b such that $\bar{B}_i < b$ for all $i \in \mathbb{N}_0$. Then for all $i \in \mathbb{N}_0$ we would have

$$\mathbb{E} \left[\frac{\bar{\rho}_i}{\bar{B}_i} \right] > \frac{1}{b} \mathbb{E}[\bar{\rho}_i] = \frac{\bar{\rho}_0}{b}. \tag{7.41}$$

But by construction we know that $\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0$ and hence

$$\lim_{i \rightarrow \infty} \mathbb{E} \left[\frac{\bar{\rho}_i}{\bar{B}_i} \right] = 0. \quad (7.42)$$

Thus given any $\epsilon > 0$ we can find a time t_i such that

$$\mathbb{E} \left[\frac{\bar{\rho}_i}{\bar{B}_i} \right] < \epsilon. \quad (7.43)$$

But this is inconsistent with (7.41); and thus we conclude that $\lim_{j \rightarrow \infty} \bar{B}_j = \infty$. That completes the proof of Proposition 7.3.3. \square

7.4 Nominal discount bonds

Now we proceed to consider the properties of nominal discount bonds. By such an instrument we mean an asset that pays a single dividend consisting of one unit of domestic currency at some designated time t_j . For the price P_{ij} at time t_i ($i < j$) of a discount bond that matures at time t_j we thus have

$$P_{ij} = \frac{1}{\pi_i} \mathbb{E}_i[\pi_j]. \quad (7.44)$$

Since $\pi_i > 0$ for all $i \in \mathbb{N}$, and $\mathbb{E}_i[\pi_j] < \pi_i$ for all $i < j$, it follows that $0 < P_{ij} < 1$ for all $i < j$. We observe, in particular, that the associated interest rate R_{ij} defined by

$$P_{ij} = \frac{1}{1 + R_{ij}} \quad (7.45)$$

is strictly positive. Note that in our theory we regard a discount bond as a “dividend-paying” asset. Thus in the case of a discount bond with maturity t_j we have $P_{jj} = 0$ and $D_j = 1$. Usually discount bonds are defined by setting $P_{jj} = 1$ at maturity, with $D_j = 0$; but it is perhaps more logical to regard the bonds as giving rise to a unit cash flow at maturity. We also note that the definition of the discount bond system does not involve the specific choice of the positive-return asset.

It is important to point out that in the present framework there is no reason or need to model the dynamics of $\{P_{ij}\}$, or to model the volatility structure of the discount bonds. Indeed, from the present point of view this would be a little artificial. The important issue, rather, is how to model the pricing kernel. Thus, our scheme differs

somewhat in spirit from the discrete-time models discussed, e.g., in Heath *et al.* 1990, and Filipović & Zabczyk 2002.

As a simple example of a family of discrete-time interest rate models admitting tractable formulae for the associated discount bond price processes, suppose we set

$$\pi_i = \alpha_i + \beta_i N_i \quad (7.46)$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are strictly-positive, strictly-decreasing deterministic sequences, satisfying $\lim_{i \rightarrow \infty} \alpha_i = 0$ and $\lim_{i \rightarrow \infty} \beta_i = 0$, and where $\{N_i\}$ is a strictly positive martingale. Then by (7.44) we have

$$P_{ij} = \frac{\alpha_j + \beta_j N_i}{\alpha_i + \beta_i N_i}, \quad (7.47)$$

thus giving a family of “rational” interest rate models. Note that in a discrete-time setting we can produce classes of models that have no immediate analogues in continuous time—for example, we can let $\{N_i\}$ be the natural martingale associated with a branching process.

Now we shall demonstrate that any discount bond system consistent with our general scheme admits a representation of the Flesaker-Hughston type. For accounts of the Flesaker-Hughston theory see, e.g., Flesaker & Hughston 1996, Rutkowski 1997, Hunt & Kennedy 2000, or Jin & Glasserman 2001.

Proposition 7.4.1 *Let $\{\pi_i\}$, $\{\bar{B}_i\}$, $\{P_{ij}\}$ satisfy the conditions of Axioms A and B. Then there exists a family of positive martingales $\{m_{in}\}_{0 \leq i \leq n}$ indexed by $n \in \mathbb{N}$ such that*

$$P_{ij} = \frac{\sum_{n=j+1}^{\infty} m_{in}}{\sum_{n=i+1}^{\infty} m_{in}}. \quad (7.48)$$

Proof. We shall use the fact that π_i can be written in the form

$$\begin{aligned} \pi_i &= \mathbb{E}_i[G_\infty] - G_i \\ &= \mathbb{E}_i \left[\sum_{n=1}^{\infty} g_n \right] - \sum_{n=1}^i g_n \\ &= \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} g_n \right], \end{aligned} \quad (7.49)$$

where $g_i = G_i - G_{i-1}$ for each $i \geq 1$. Then $g_i > 0$ for all $i \geq 1$ since $\{G_i\}$ is a strictly increasing sequence. By the monotone convergence theorem we have

$$\pi_i = \sum_{n=i+1}^{\infty} \mathbb{E}_i[g_n] \quad (7.50)$$

and

$$\mathbb{E}_i[\pi_j] = \sum_{n=j+1}^{\infty} \mathbb{E}_i[g_n]. \quad (7.51)$$

For each $n \geq 1$ we define $m_{in} = \mathbb{E}_i[X_n]$. Then for each $n \in \mathbb{N}$ we see that $\{m_{in}\}_{0 \leq i \leq n}$ is a strictly positive martingale, and (7.48) follows immediately. \square

7.5 Nominal money-market account

In the analysis presented so far we have assumed that the positive-return process $\{\bar{B}_i\}$ is $\{F_i\}$ -adapted, but is not necessarily previsible. The point is that many of our conclusions are valid under the weaker hypothesis of mere adaptedness, as we have seen. There are also economic motivations behind the use of the more general assumption. One can imagine that the time sequence $\{t_i\}$ is in reality a “course graining” of a finer time sequence that includes the original sequence as a sub-sequence. Then likewise one can imagine that $\{\bar{B}_i\}$ is a sub-sequence of a finer process that assigns a value to the positive-return asset at each time in the finer time sequence. Finally, we can imagine that $\{\mathcal{F}_i\}$ is a sub-filtration of a finer filtration based on the finer sequence. In the case of a money market account, where the rate of interest is set at the beginning of each short deposit period (say, one day), we would like to regard the relevant value process as being previsible with respect to the finer filtration, but merely adapted with respect to the course-grained filtration.

Do positive-return assets, other than the standard previsible money market account, actually exist in a discrete-time setting? The following example gives an affirmative answer. In the setting of the standard binomial model, in the case of a single period, let S_0 denote the value at time 0 of a risky asset, and let $\{U, D\}$ denote its possible values at time 1. Let B_0 and B_1 denote the values at times 0 and 1 of a deterministic money-market account. We assume that $B_1 > B_0$ and $U > S_0 B_1 / B_0 > D$. A standard calculation shows that the risk-neutral probabilities for $S_0 \rightarrow U$ and $S_0 \rightarrow D$ are given by p^* and $1 - p^*$, where $p^* = (S_0 B_1 / B_0 - D) / (U - D)$. We shall now construct a

“positive-return” asset, i.e. an asset with initial value \bar{S}_0 and with possible values $\{\bar{U}, \bar{D}\}$ at time 1 such that $\bar{U} > \bar{S}_0$ and $\bar{D} > \bar{S}_0$. Risk-neutral valuation implies that $\bar{S}_0 = (B_0/B_1)[p^*\bar{U} + (1-p^*)\bar{D}]$. Thus, given \bar{S}_0 , we can determine \bar{U} in terms of \bar{D} . A calculation then shows that if $(B_1/B_0 - p^*)/(1-p^*) > \bar{D}/\bar{S}_0 > 1$, then $\bar{U} > \bar{S}_0$ and $\bar{D} > \bar{S}_0$, as desired. Thus, in the one-period binomial model, for the given initial value \bar{S}_0 , we obtain a one-parameter family of positive-return assets.

Let us consider now the special case where the positive-return asset is previsible. Thus for $i \geq 1$ we assume that B_i is \mathcal{F}_{i-1} -measurable and we drop the “bar” over B_i to signify the fact that we are now considering a money-market account. In that case we have

$$\begin{aligned} P_{i-1,i} &= \frac{1}{\pi_{i-1}} \mathbb{E}_{i-1}[\pi_i] \\ &= \frac{B_{i-1}}{\rho_{i-1}} \mathbb{E}_{i-1} \left[\frac{\rho_i}{B_i} \right] \\ &= \frac{B_{i-1}}{B_i}, \end{aligned} \tag{7.52}$$

by virtue of the martingale property of $\{\rho_i\}$. Thus, in the case of a money-market account we see that

$$P_{i-1,i} = \frac{1}{1+r_i}. \tag{7.53}$$

where $r_i = R_{i-1,i}$. In other words, the rate of return on the money-market account is previsible, and is given by the one-period simple discount factor associated with the discount bond that matures at time t_i .

Reverting now to the general situation, it follows that if we are given a pricing kernel $\{\pi_i\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_i\}$, and a system of assets satisfying Axioms A and B, then we can construct a plausible candidate for an associated previsible money market account by setting $B_0 = 1$ and defining

$$B_i = (1+r_i)(1+r_{i-1}) \cdots (1+r_1), \tag{7.54}$$

for $i \geq 1$, where

$$r_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} - 1. \tag{7.55}$$

We shall refer to the process $\{B_i\}$ thus constructed as the “natural” money market account associated with the pricing kernel $\{\pi_i\}$.

To justify this nomenclature, we need to verify that $\{B_i\}$, so constructed, satisfies the conditions of Axioms A and B. To this end, we make note of the following decomposition. Let $\{\pi_i\}$ be a positive supermartingale satisfying $\mathbb{E}_i[\pi_j] < \pi_i$ for all $i < j$ and $\lim_{j \rightarrow \infty} \pi_j = 0$. Then as an identity we can write

$$\pi_i = \frac{\rho_i}{B_i}, \quad (7.56)$$

where

$$\rho_i = \frac{\pi_i}{\mathbb{E}_{i-1}[\pi_i]} \frac{\pi_{i-1}}{\mathbb{E}_{i-2}[\pi_{i-1}]} \cdots \frac{\pi_1}{\mathbb{E}_0[\pi_1]} \pi_0 \quad (7.57)$$

for $i \geq 0$, and

$$B_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} \frac{\pi_{i-2}}{\mathbb{E}_{i-2}[\pi_{i-1}]} \cdots \frac{\pi_1}{\mathbb{E}_1[\pi_2]} \frac{\pi_0}{\mathbb{E}_0[\pi_1]} \quad (7.58)$$

for $i \geq 1$, with $B_0 = 1$. Thus, in this scheme we have

$$\rho_i = \frac{\pi_i}{\mathbb{E}_{i-1}[\pi_i]} \rho_{i-1}, \quad (7.59)$$

with the initial condition $\rho_0 = \pi_0$; and

$$B_i = \frac{\pi_{i-1}}{\mathbb{E}_{i-1}[\pi_i]} B_{i-1}, \quad (7.60)$$

with the initial condition $B_0 = 1$. It is evident that $\{\rho_i\}$ as thus defined is $\{\mathcal{F}_i\}$ -adapted, and that $\{B_i\}$ is previsible and strictly increasing. Making use of the identity (7.60) we are now in a position to establish the following:

Proposition 7.5.1 *Let $\{\pi_i\}$ be a non-negative supermartingale satisfying $\mathbb{E}_i[\pi_j] < \pi_i$ for all $i < j \in \mathbb{N}_0$, and $\lim_{i \rightarrow \infty} \mathbb{E}[\pi_i] = 0$. Let $\{B_i\}$ be defined by $B_0 = 1$ and $B_i = \prod_{n=1}^i (1 + r_n)$ for $i \geq 1$, where $1 + r_i = \pi_{i-1}/\mathbb{E}_{i-1}[\pi_i]$, and set $\rho_i = \pi_i B_i$ for $i \geq 0$. Then $\{\rho_i\}$ is a martingale, and the interest rate system defined by $\{\pi_i\}$, $\{B_i\}$, and $\{P_{ij}\}$ satisfies Axioms A and B.*

Proof. To show that $\{\rho_i\}$ is a martingale it suffices to verify for all $i \geq 1$ that $\mathbb{E}[\rho_i] < \infty$ and that $\mathbb{E}_{i-1}[\rho_i] = \rho_{i-1}$. In particular, if $\mathbb{E}[\rho_i] < \infty$ then the “take out what is known rule” is applicable, and by (7.59) we have

$$\mathbb{E}_{i-1}[\rho_i] = \mathbb{E}_{i-1} \left[\frac{\pi_i}{\mathbb{E}_{i-1}[\pi_i]} \rho_{i-1} \right] = \rho_{i-1}. \quad (7.61)$$

Thus to show that $\{\rho_i\}$ is a martingale all that remains is to verify that $\mathbb{E}[\rho_i] < \infty$.

Let us write

$$J_{i-1}^\alpha = \min \left[\frac{\rho_{i-1}}{\mathbb{E}_{i-1}[\pi_i]}, \alpha \right] \quad (7.62)$$

for $\alpha \in \mathbb{N}_0$. Then by monotone convergence and the tower property we have

$$\mathbb{E}[\rho_i] = \mathbb{E} \left[\pi_i \lim_{\alpha \rightarrow \infty} J_{i-1}^\alpha \right] \quad (7.63)$$

$$= \lim_{\alpha \rightarrow \infty} \mathbb{E} [\pi_i J_{i-1}^\alpha] \quad (7.64)$$

$$= \lim_{\alpha \rightarrow \infty} \mathbb{E} [\mathbb{E}_{i-1}[\pi_i J_{i-1}^\alpha]]. \quad (7.65)$$

But since J_{i-1}^α is bounded we can move this term outside the inner conditional expectation to give

$$\mathbb{E}[\rho_i] = \lim_{\alpha \rightarrow \infty} \mathbb{E} [J_{i-1}^\alpha \mathbb{E}_{1-i}[\pi_i]] \leq \mathbb{E}[\rho_{i-1}], \quad (7.66)$$

since

$$J_{i-1}^\alpha \leq \frac{\rho_{i-1}}{\mathbb{E}_{i-1}[\pi_i]}. \quad (7.67)$$

Thus we see for all $i \geq 1$ that if $\mathbb{E}[\rho_{i-1}] < \infty$ then $\mathbb{E}[\rho_i] < \infty$. But $\rho_0 < \infty$ by construction, and hence by induction we deduce that $\mathbb{E}[\rho_i] < \infty$ for all $i \geq 0$. \square

The martingale $\{\rho_i\}$ is the likelihood ratio process appropriate for a change of measure from the objective measure \mathbb{P} to the equivalent martingale measure \mathbb{Q} characterised by the property that non-dividend-paying assets are martingales when expressed in units of the money-market account. An interesting feature of Proposition 7.5.1 is that no integrability condition is required on $\{\rho_i\}$. In other words, the natural previsible money market account defined by (7.58) “automatically” satisfies the conditions of Axiom A. For some purposes it may therefore be advantageous to incorporate the existence of the natural money market account directly into the axioms. Then instead of Axiom B we would have:

Axiom B*. *There exists a strictly-positive non-dividend paying asset, the money-market account, with value process $\{B_i\}_{i \geq 0}$, having the properties that $B_{i+1} > B_i$ for all $i \in \mathbb{N}_0$ and that B_i is \mathcal{F}_{i-1} -measurable for all $i \in \mathbb{N}$. We assume that $\{B_i\}$ is unbounded in the sense that for any $b \in \mathbb{R}$ there exists a time t_i such that $B_i > b$.*

The content of Proposition 7.5.1 is that Axioms A and B together imply Axiom B*. As an exercise we shall establish that the class of interest rate models satisfying

Axioms A and B* is non-vacuous. In particular, suppose we consider the “rational” models defined by equations (7.46) and (7.47) for some choice of the martingale $\{N_i\}$. It is straightforward to see that the unique previsible money market account in this model is given by $B_0 = 1$ and

$$B_i = \prod_{n=1}^i \frac{\alpha_{n-1} + \beta_{n-1}N_{n-1}}{\alpha_n + \beta_n N_{n-1}} \quad (7.68)$$

for $i \geq 1$. For $\{\rho_i\}$ we then have

$$\rho_i = \rho_0 \prod_{n=1}^i \frac{\alpha_n + \beta_n N_n}{\alpha_n + \beta_n N_{n-1}}, \quad (7.69)$$

where $\rho_0 = \alpha_0 + \beta_0 N_0$. But it is easy to check that for each $i \geq 0$ the random variable ρ_i is bounded; therefore $\{\rho_i\}$ is a martingale, and the money market account process $\{B_i\}$ satisfies the conditions of Axioms A and B*.

Now let us return to the Doob decomposition for $\{\pi_i\}$ given in formula (7.5). Evidently, we have $\pi_i = \mathbb{E}_i[A_\infty] - A_i$, with

$$\begin{aligned} A_i &= \sum_{n=0}^{i-1} (\pi_n - \mathbb{E}_n[\pi_{n+1}]) \\ &= \sum_{n=0}^{i-1} \pi_n \left(1 - \frac{\mathbb{E}_n[\pi_{n+1}]}{\pi_n}\right) \\ &= \sum_{n=0}^{i-1} \pi_n (1 - P_{n,n+1}) \\ &= \sum_{n=0}^{i-1} \pi_n r_{n+1} P_{n,n+1}, \end{aligned} \quad (7.70)$$

where $\{r_i\}$ is the previsible short rate process defined by (7.53). The pricing kernel can therefore be put in the form

$$\pi_i = \mathbb{E}_i \left[\sum_{n=i}^{\infty} \pi_n r_{n+1} P_{n,n+1} \right]. \quad (7.71)$$

Comparing the Doob decomposition (7.71) with the alternative decomposition given by (7.32), we thus deduce that if we set

$$\bar{r}_i = \frac{r_i \pi_{i-1} P_{i-1,i}}{\pi_i} \quad (7.72)$$

then we obtain a positive-return asset for which the corresponding decomposition of the pricing kernel, as given by (7.32), is the Doob decomposition. On the other hand, since the money-market account is a positive-return asset, by Proposition 7.3.2 we can also write

$$\pi_i = \mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n r_n \right]. \quad (7.73)$$

As a consequence, we see that the price process of a pure income asset can be written in the symmetrical form

$$S_i = \frac{\mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n D_n \right]}{\mathbb{E}_i \left[\sum_{n=i+1}^{\infty} \pi_n r_n \right]}, \quad (7.74)$$

where $\{D_n\}$ is the dividend process, and $\{r_n\}$ is the short rate process.

7.6 Information-based interest rate models

So far in the discussion we have regarded the pricing kernel $\{\pi_i\}$ and the filtration $\{\mathcal{F}_i\}$ as being subject to an exogenous specification. In order to develop the framework further we need to make a more specific indication of how the pricing kernel might be determined, and how information is made available to market participants. To obtain a realistic model for $\{\pi_i\}$ we need to develop the model in conjunction with a theory of consumption, money supply, price level, inflation, real interest rates, and information. We shall proceed in two steps. First we consider a general “reduced-form” model for nominal interest rates, in which we model the filtration explicitly; then in the next section we consider a more general “structural” model in which both the nominal and the real interest rate systems are characterised.

Our reduced-form model for interest rates will be based on the theory of X -factors, following the general line of the previous chapters. Associated with each time t_i we introduce a collection of one or more random variables X_i^α ($\alpha = 1, \dots, m_i$), where m_i denotes the number of random variables associated with time t_i . For each value of n , we assume that the various random variables $X_1^\alpha, X_2^\alpha, \dots, X_n^\alpha$ are independent. We regard the random variables X_n^α as being “revealed” at time t_n , and hence \mathcal{F}_n -measurable. More precisely, we shall construct the filtration $\{\mathcal{F}_i\}$ in such a way that this property holds. Intuitively, we can think of $X_1^\alpha, X_2^\alpha, \dots, X_n^\alpha$ as being the various independent macroeconomic “market factors” that determine cash flows at time t_n .

Now let us consider how the filtration will be modelled. For each $j \in \mathbb{N}_0$, at any time t_i before t_j only partial information about the market factors X_j^α will be available to market participants. We model this partial information for each market factor X_j^α by defining a discrete-time information process $\{\xi_{t_i t_j}^\alpha\}_{0 \leq t_i \leq t_j}$, setting

$$\xi_{t_i t_j}^\alpha = \sigma t_i X_j^\alpha + \beta_{t_i t_j}^\alpha. \quad (7.75)$$

Here $\{\beta_{t_i t_j}^\alpha\}_{0 \leq t_i \leq t_j}$ can, for each value of α , be thought of as an independent discretised Brownian bridge. Thus, we consider a standard Brownian motion starting at time zero and ending at time t_j , and sample its values at the times $\{t_i\}_{i=0, \dots, j}$. Let us write $\xi_{i j}^\alpha = \xi_{t_i t_j}^\alpha$ and $\beta_{i j}^\alpha = \beta_{t_i t_j}^\alpha$, in keeping with our usual shorthand conventions for discrete-time modelling. Then for each value of α we have $\mathbb{E}[\beta_{i j}^\alpha] = 0$ and

$$\text{Cov}[\beta_{i k}^\alpha, \beta_{j k}^\alpha] = \frac{t_i(t_k - t_j)}{t_k} \quad (7.76)$$

for $i \leq j \leq k$. We assume that the bridge processes are independent of the X -factors (i.e., the macroeconomic factors); and hence that the various information processes are independent of one another. Finally, we assume that the market filtration is generated collectively by the various information processes. For each value of k the sigma-algebra \mathcal{F}_k is generated by the random variables $\{\xi_{i j}^\alpha\}_{0 \leq i \leq j \leq k}$.

Thus, as in the earlier chapters, the filtration is not simply “given”, but rather is modelled explicitly. It is a straightforward exercise to verify that, for each value of α , the process $\{\xi_{i j}^\alpha\}$ has the Markov property. The proof follows the pattern of the continuous-time argument. This has the implication that the conditional expectation of a function of the market factors X_j^α , taken with respect to \mathcal{F}_i , can be reduced to a conditional expectation with respect to the sigma-algebra $\sigma(\xi_{i j}^\alpha)$. That is to say, the history of the process $\{\xi_{n j}^\alpha\}_{n=0, 1, \dots, i}$ can be neglected, and only the most “recent” information, $\xi_{i j}^\alpha$, needs to be considered in taking the conditional expectation.

For example, in the case of a function of a single \mathcal{F}_j -measurable market factor X_j , with the associated information process $\{\xi_{n j}\}_{n=0, 1, \dots, j}$, we obtain:

$$\mathbb{E}[f(X_j)|\mathcal{F}_i] = \frac{\int_0^\infty p(x) f(x) \exp\left[\frac{t_j}{t_j - t_i} (\sigma x \xi_{i j} - \frac{1}{2} \sigma^2 x^2 t_i)\right] dx}{\int_0^\infty p(x) \exp\left[\frac{t_j}{t_j - t_i} (\sigma x \xi_{i j} - \frac{1}{2} \sigma^2 x^2 t_i)\right] dx}, \quad (7.77)$$

for $i \leq j$, where $p(x)$ denotes the *a priori* probability density function for the random variable X_j .

In the formula above we have for convenience presented the result in the case of a single X -factor represented by a continuous random variable taking non-negative values; the extension to other classes of random variables, and to collections of random variables, is straightforward.

Now we are in a position to state how we propose to model the pricing kernel. First, we shall assume that $\{\pi_i\}$ is adapted to the market filtration $\{\mathcal{F}_i\}$. This is clearly a natural assumption from an economic point of view, and is necessary for the general consistency of the theory. This means that the random variable π_j , for any fixed value of j , can be expressed as a function of the totality of the available market information at time j . In other words, π_j is a function of the values taken, between times 0 and j , of the information processes associated with the various market factors.

Next we make the simplifying assumption that π_j (for any fixed j) depends on the values of only a finite number of information processes. This corresponds to the intuitive idea that when we are pricing a contingent claim, there is a limit to the amount of information we can consider.

But this implies that expectations of the form $\mathbb{E}_i[\pi_j]$, for $i \leq j$, can be computed explicitly. The point is that since π_j can be expressed as a function of a collection of intertemporal information variables, the relevant conditional expectations can be worked out in closed form by use of the methods of Chapter 6. As a consequence, we are led to a system of essentially tractable expressions for the resulting discount bond prices and the previsible money market account. Thus we are left only with the question of what is the correct functional form for $\{\pi_i\}$, given the relevant market factors. If we simply “propose” or “guess” a form for $\{\pi_i\}$, then we have a “reduced-form” or “ad hoc” model. If we provide an economic argument that leads to a specific form for $\{\pi_i\}$, then we say that we have a “structural” model.

7.7 Models for inflation and index-linked securities

For a more complete picture we must regard the nominal interest rate system as embedded in a larger system that takes into account the various macroeconomic factors that inter-relate the money supply, aggregate consumption, and the price level. We shall present a simple model in this spirit that is consistent with the information-based approach that we have been taking.

To this end we introduce the following quantities. We envisage a closed economy with aggregate consumption $\{k_i\}_{i \geq 1}$. This consumption takes place at discrete times, and k_i denotes the aggregate level of consumption, in units of goods and services, taking place at time t_i . Let us write $\{M_i\}_{i \geq 0}$ for the process corresponding to the nominal money supply, and $\{C_i\}_{i \geq 0}$ for the process of the consumer price index (the “price level”). For convenience we can think of $\{k_i\}$ and $\{M_i\}$ both as being expressed on a *per capita* basis. Hence these quantities can be regarded, respectively, as the consumption and money balance associated with a representative agent. We can therefore formulate the optimisation problem from the perspective of the representative agent; but the role of the agent here is to characterise the structure of the economy as a whole.

We shall assume that at each time t_i the agent receives a benefit or service from the money balance maintained in the economy; this will be given in nominal terms by $\lambda_i M_i$, where λ_i is the nominal liquidity benefit conferred to the agent per unit of money “carried” by the agent, and M_i is the money supply, expressed on a *per capita* basis, at that time. The corresponding “real” benefit (in units of goods and services) provided by the money supply at time t_i is defined by the quantity

$$l_i = \frac{\lambda_i M_i}{C_i}. \quad (7.78)$$

It follows from these definitions that we can think of $\{\lambda_i\}$ as a kind of “convenience yield” process associated with the money supply. Rather in the way a country will obtain a convenience yield (per barrel) from its oil reserves, which can be expressed on a *per capita* basis, likewise an economy derives a convenience yield (per unit of money) from its money supply.

It is important to note that what matters in reality is the *real benefit* of the money supply, which can be thought of effectively as a flow of goods and services emanating from the presence of the money supply. It is quite possible that the “wealth” attributable to the face value of the money may in totality be insignificant. For example, if the money supply consists exclusively of notes issued by the government, and hence takes the form of government debt, then the *per capita* wealth associated with the face value of the notes is essentially null, since the representative agent is also responsible (ultimately) for a share of the government debt. Nevertheless, the presence of the money supply confers an overall positive flow of benefit to the agent. On the other hand, if the money supply consists, say, of gold coins, or units of some other valuable

commodity, then the face value of the money supply will make a positive contribution to overall wealth, as well as providing a liquidity benefit.

Our goal is to obtain a consistent structural model for the pricing kernel $\{\pi_i\}_{i \geq 0}$. We assume that the representative agent gets utility both from consumption and from the real benefit of the money supply in the spirit of Sidrauski 1969. Let $U(x, y)$ be a standard bivariate utility function $U : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$, satisfying $U_x > 0$, $U_y > 0$, $U_{xx} < 0$, $U_{yy} < 0$, and $U_{xx}U_{yy} > (U_{xy})^2$. Then the objective of the representative agent is to maximise an expression of the form

$$J = \mathbb{E} \left[\sum_{n=0}^N e^{-\gamma t_n} U(k_n, l_n) \right] \quad (7.79)$$

over the time horizon $[t_0, t_1, \dots, t_N]$, where γ is the appropriate discount rate applicable to delayed gains in utility. For simplicity of exposition we assume a constant discount rate. The optimisation problem faced by the agent is subject to the budget constraint

$$W = \mathbb{E} \left[\sum_{n=0}^N \pi_n (C_n k_n + \lambda_n M_n) \right]. \quad (7.80)$$

Here W represent the total *per capita* wealth, in nominal terms, available for consumption related expenditure over the given time horizon. The agent can maintain a position in money, and “consume” the benefit of the money; or the money position can be liquidated (in part, or in whole) to purchase consumption goods. In any case, we must include the value of the benefit of the money supply in the budget for the relevant period. In other words, since the presence of the money supply “adds value”, we need to recognise this value as a constituent of the budget. The budget includes also any net initial funds available, together with the value of any expected income (e.g., derivable from labour or natural resources) over the relevant period.

The fact that the utility depends on the real benefit of the money supply, whereas the budget depends on the nominal value of the money supply, leads to a fundamental relationship between the processes $\{k_i\}$, $\{M_i\}$, $\{C_i\}$, and $\{\lambda_i\}$. Introducing a Lagrange multiplier μ , after some re-arrangement we obtain the associated unconstrained optimisation problem, for which the objective is to maximise the following expression:

$$\mathbb{E} \left[\sum_{n=0}^N e^{-\gamma t_n} U(k_n, l_n) - \mu \sum_{n=0}^N \pi_n C_n (k_n + l_n) \right]. \quad (7.81)$$

A straightforward argument then shows that the solution for the optimal policy (if it exists) satisfies the first order conditions

$$U_x(k_n, l_n) = \mu e^{\gamma t_n} \pi_n C_n, \quad (7.82)$$

and

$$U_y(k_n, l_n) = \mu e^{\gamma t_n} \pi_n C_n, \quad (7.83)$$

for each value of n in the relevant time frame, where μ is determined by the budget constraint. As a consequence we obtain the fundamental relation

$$U_x(k_n, \lambda_n M_n / C_n) = U_y(k_n, \lambda_n M_n / C_n), \quad (7.84)$$

which allows us to eliminate any one of the variables k_n , M_n , λ_n , and C_n in terms of the other three. In this way, for a given level of consumption, money supply, and liquidity benefit, we can work out the associated price level. Then by use of (7.82), or equivalently (7.83), we can deduce the form taken by the nominal pricing kernel, and hence the corresponding interest rate system. We also obtain thereby an expression for the “real” pricing kernel $\{\pi_i C_i\}$.

We shall take the view that aggregate consumption, the liquidity benefit rate, and the money supply level are all determined exogenously. In particular, in the information-based framework we take these processes to be adapted to the market filtration, and hence determined, at any given time, by the values of the information variables upon which they depend. The theory outlined above then shows how the values of the real and nominal pricing kernels can be obtained, at each time, as functions of the relevant information variables.

It will be useful to have an explicit example in mind, so let us consider a standard “log-separable” utility function of the form

$$U(x, y) = A \ln(x) + B \ln(y), \quad (7.85)$$

where A and B are non-negative constants. From the fundamental relation (7.84) we immediately obtain

$$\frac{A}{k_n} = \frac{B}{l_n}, \quad (7.86)$$

and hence the equality

$$k_n C_n = \frac{A}{B} \lambda_n M_n. \quad (7.87)$$

Thus, in the case of log-separable utility we see that the level of consumption, in nominal terms, is always given by a fixed proportion of the nominal liquidity benefit obtained from the money supply. For any fixed values of λ_n and k_n , we note, for example, that an increase in the money supply leads to an increase in the price level.

One observes that in the present framework we *derive* an expression for the consumer price index process. This contrasts somewhat with current well-known methodologies for pricing inflation-linked securities—see, e.g., Hughston 1998, and Jarrow & Yildirim 2003—where the form of the consumer price index is specified on an exogenous, essentially *ad hoc* basis.

The quantity $k_n C_n / M_n$ is commonly referred to as the “velocity” of money. It measures, roughly speaking, the rate at which money changes hands, as a percentage of the total money supply, as a consequence of consumption. Evidently, in the case of a log-separable utility (7.85), the velocity has a fixed ratio to the liquidity benefit. This is a satisfying conclusion, which shows that even with a relatively simple assumption about the nature of the utility we are able to obtain an intuitively natural relation between the velocity of money and the liquidity benefit. In particular, if liquidity is increased, then a lower money supply will be required to sustain a given level of nominal consumption, and hence the velocity will be increased as well. The situation when the velocity is constant leads to the so-called “quantity” theory of money, which in the present approach arises in the case of a representative agent with log-separable utility and a constant liquidity benefit.

It is interesting to note that the results mentioned so far, in connection with log-separable utility, are not too sensitive to the choice of the discount rate γ , which does not enter into the fundamental relation (7.84). On the other hand, γ does enter into the expression for the nominal pricing kernel; in particular, in the log-separable case we obtain the following expression for the pricing kernel:

$$\pi_n = \frac{B e^{-\gamma t_n}}{\mu \lambda_n M_n}. \quad (7.88)$$

Hence, in the log-separable utility theory we can see explicitly the relation between the nominal money supply and the term structure of interest rates.

Let us consider now a contingent claim with the random nominal payoff H_j at time t_j . Then the value of the claim at time t_0 in the log-separable utility model is given by

the following formula:

$$H_0 = \lambda_0 M_0 e^{-\gamma t_j} \mathbb{E} \left[\frac{H_j}{\lambda_j M_j} \right]. \quad (7.89)$$

One can evidently see two different influences on the value of H_0 . First one has the discount factor; but equally importantly one sees the effect of the money supply. For a given level of the liquidity (i.e., for constant λ_j), an increase in the likely money supply at time t_j will reduce the value of H_0 . This example illustrates how market perceptions of the direction of future monetary policy can potentially affect the valuation of contingent claims in a fundamental way. In particular, the value of the money supply M_j at time t_j will be given as a function of the best available information at that time concerning future random factors affecting the economy. The question of how best to model the money supply process $\{M_i\}$ takes us, to some extent, outside of the realm of pure mathematical finance, and more into the territory of macroeconomics and, ultimately, political economics. Nevertheless, it is gratifying and perhaps surprising that we can have come as far as we have.

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