TWISTED WHITTAKER MODEL AND FACTORIZABLE SHEAVES

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Introduction

0.1. **Quantum Langlands duality?** The idea behind the paper is the quest for Langlands duality for quantum groups. Let us explain what we mean by this.

Let G be a simple algebraic group over $\mathbb{C}$. Recall that the geometric Satake equivalence realizes the Langlands dual group $\check{G}$, or rather the category of its representations, explicitly in terms of G as follows. We consider the affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$ as an ind-scheme acted on (from the left) by $G[[t]]$ and let $\mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{G[[t]]}$ denote the corresponding category of D-modules.

One endows $\mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{G[[t]]}$ with a monoidal structure, and one shows that it has a natural commutativity constraint. It is a basic fact, conjectured by Drinfeld based on an earlier work of Lusztig, and proved in [Gi] and [MV], that that the resulting tensor category is equivalent to $\mathrm{Rep}(\check{G})$–the category of representations of $\check{G}$ as an algebraic group.

Now, $\mathrm{Rep}(\check{G})$, considered as a braided monoidal category, admits a one-parameter family of deformations to the category $\mathrm{Rep}(U_q(\check{G}))$, where by $U_q(\check{G})$ we denote the quantum group attached to G (when q is a root of unity, we take Lusztig's quantum group with q-divided powers). It has always been very tempting to try to realize $\mathrm{Rep}(U_q(\check{G}))$ also via Gr_G, or some closely related geometric object, as a category of D-modules or perverse sheaves with a particular equivariance property.

One has a natural candidate of how to involve the parameter q. Namely, let $\widetilde{\mathrm{Gr}}_G$ be the canonical line bundle over G, i.e., $\widetilde{\mathrm{Gr}}_G := \widehat{G}/G[[t]]$, where $\widehat{G}$ is the Kac-Moody extension of the loop group $G((t))$. The passage from $\check{G}$ to its quantum deformation should correspond to replacing D-modules (or perverse sheaves) on Gr_G by D-modules (or perverse sheaves) on $\widetilde{\mathrm{Gr}}_G$, which are monodromic along the fiber with monodromy q^2, i.e., we will consider the corresponding category of twisted D-modules on Gr_G which we will denote by $\mathfrak{D}\text{-mod}^c(\widetilde{\mathrm{Gr}}_G)$ with $q = exp(\pi i c)$.

However, the first naive attempt, i.e., to consider the category $\mathfrak{D}\text{-mod}^c(\widetilde{\mathrm{Gr}}_G)^{G[[t]]}$ leads to a wrong answer. E.g., when c is irrational (i.e.,, when q is not a root of unity) the latter category will have only one irreducible object, i.e., it cannot be a deformation of $\mathrm{Rep}(\check{G})$.

0.2. **Whittaker category.** A viable candidate for an appropriate category of twisted D-modules on Gr_G was suggested by Jacob Lurie in October 2006. To explain his idea, we will first replace the category $\mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{G[[t]]}$ by a different, but still equivalent, category, which, however, admits a natural quantum deformation.

Namely, let N be a maximal unipotent subgroup of G, and consider the corresponding loop group $N((t))$. Let $\chi : N((t)) \to \mathbb{G}_a$ be a non-degenerate character, normalized to have conductor 0. Let us consider the category of D-modules on Gr_G, equivariant with respect to $N((t))$ against

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the character χ. We shall refer to this category as that of Whittaker D-modules on Gr_G, and denote it $\mathrm{Whit}(\mathrm{Gr}_G)$, or simply Whit.

The trouble is, however, that the orbits of the group $N((t))$ on Gr_G are infinite-dimensional, and if one understands the $N((t))$-equivariance condition naively, the category $\mathrm{Whit}(\mathrm{Gr}_G)$ will be empty. There are two ways to overcome this difficulty:

One way is to try to define $\mathrm{Whit}(\mathrm{Gr}_G)$ as a triangulated category, and then extract a t-structure. This approach has not been worked out yet, but J. Lurie and the author hope that this is feasible, and leads to a manageable theory.

Another approach, which we opt for in the present paper, was developed in [FGV] and [Ga], where one replaces Gr_G with the group $N((t))$ acting on it, by a different geometric object, which is supposed to mimic its properties. This other object is the Drinfeld compactification, denoted in this paper by $\mathfrak{W}$, and it involves a choice of a projective curve X. In addition to being less natural, the category $\mathrm{Whit}(\mathrm{Gr}_G)$, defined in this way, has the main disadvantage that its local nature (we think of the ring $\mathbb{C}[[t]]$ and the field $\mathbb{C}((t))$ as associated to a formal disc of a point x on a not necessarily complete curve X) is not obvious.

We will explain the definition of $\mathrm{Whit}(\mathrm{Gr}_G)$ via $\mathfrak{W}$ in Sect. 1.2 and Sect. 2.5. For the purposes of the introduction we will pretend that the first approach mentioned above works, i.e., that we have a direct local definition of $\mathrm{Whit}(\mathrm{Gr}_G) := \mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{N((t)),\chi}$.

Having "defined" $\mathrm{Whit}(\mathrm{Gr}_G)$, the main theorem of the paper [FGV] can be rephrased as follows: there exists an equivalence of *abelian* categories $\mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{G[[t]]} \simeq \mathrm{Whit}(\mathrm{Gr}_G)$. Moreover, a naturally defined triangulated category, whose core is $\mathrm{Whit}(\mathrm{Gr}_G)$, is semi-simple, i.e., we have an equivalence of abelian categories

$$\mathrm{Whit}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(\check{G}), \tag{0.1}$$

which extends to an equivalence of the corresponding triangulated categories.

We emphasize that the latter statement regarding the original equivalence

$$\mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{G[[t]]} \simeq \mathrm{Rep}(\check{G}) \tag{0.2}$$

would be false: the $G[[t]]$-equivariant derived category of D-modules on Gr_G is not at all semi-simple. This can be viewed as another reason why (0.2) does not have a quantum deformation.

0.3. **Lurie's category.** Now we can explain Jacob Lurie's idea. By the same token as we "define" $\mathrm{Whit}(\mathrm{Gr}_G)$, we can define its twisted version

$$\mathrm{Whit}^c(\mathrm{Gr}_G) := \mathfrak{D}\text{-mod}^c(\mathrm{Gr}_G)^{N((t)),\chi}.$$

This makes sense since the Kac-Moody extension $\widehat{G}$ canonically splits over $N((t))$. His conjecture can be stated as

Conjecture 0.4. *There exists an equivalence*

$$\mathrm{Whit}^c(\mathrm{Gr}_G) \simeq \mathrm{Rep}(U_q(\check{G})),\ q = exp(\pi i c).$$

In this paper we will essentially prove this conjecture for q being not a root of unity.

Let us note that the assertion of the above conjecture is inherently transcendental, due to the appearance of the exponential function relating the parameters on both sides. We will cure this as follows:

When q is not a root of unity, i.e., when c is irrational, we will eventually replace the RHS, i.e., $\mathrm{Rep}(U_q(\check{G}))$, by another category, namely that of factorizable sheaves, denoted $\mathrm{FS}^c(\check{G})$, which will also be equivalent to $\mathrm{Rep}(U_q(\check{G}))$ by a transcendental procedure. In the present paper we will establish an equivalence

$$\mathrm{Whit}^c(\mathrm{Gr}_G) \simeq \mathrm{FS}^c(\check{G}), \tag{0.3}$$

which is algebraic; in particular, both sides and the equivalence between them are defined over an arbitrary ground field of characteristic 0.

At present we do not know how to modify the definition of $\mathrm{FS}^c(\check{G})$ to obtain a category, equivalent to $\mathrm{Rep}(U_q(\check{G}))$ for all q. [1] There is, however, another category with this property, see Sect. 0.7.

0.5. **Chiral categories.** Returning to the statement of Conjecture 0.4, we should explain what are the additional structures on both sides, which are supposed to be respected by the conjectural equivalence. (If for generic q we just kept the abelian category structure on both sides, the statement of the conjecture would not be very interesting, as both LHS and RHS are semi-simple categories with naturally identified sets of irreducible objects.)

The RHS of the equivalence, i.e., $\mathrm{Rep}(U_q(\check{G}))$, is a braided monoidal category. The LHS is supposed to have another kind of structure, that we shall call "fusion" or "chiral" category. The notion of chiral category is a subject of another work in progress of Jacob Lurie and the author.

Unfortunately, in the present global definition of $\mathrm{Whit}^c(\mathrm{Gr}_G)$, the chiral category structure on it is not evident; therefore we do not give a formal definition in the main body of the paper. Let us, nonetheless, indicate it here.

Let X be a smooth curve (not necessarily complete). First, let us recall that a chiral algebra over X (see [CHA], Sect. 3.4 for the detailed definition) is a rule that assigns to a natural number n a quasi-coherent sheaf $\mathcal{A}_n$ over X^n, with a certain factorization data. E.g., for $n = 2$, we must be given an isomorphism between $\mathcal{A}_2|_{X\times X-\Delta(X)}$ and $\mathcal{A}_1 \boxtimes \mathcal{A}_1|_{X\times X-\Delta(X)}$, and an isomorphism $\mathcal{A}_2|_{\Delta(X)} \simeq \mathcal{A}_1$. We must be given a compatible family of such isomorphisms for any partition $n = n_1 + ... + n_k$. Each $\mathcal{A}_n$ must be endowed with an equivariant structure with respect to the symmetric group Σ_n, and the factorization isomorphisms must respect this structure. Finally, the collection $\mathcal{A}_n$ must be endowed with a unit, which is a map from the collection given by $\mathcal{A}^{(n)}_{unit} := \mathcal{O}_{X^n}$ to $\mathcal{A}_n$.

The definition of a chiral category is similar. A chiral category consists of a data of categories $\mathcal{C}_n$ *over* [2] X^n defined for each n, endowed with a compatible family of equivalences such as

$$\mathcal{C}_2|_{X\times X-\Delta(X)} \simeq \mathcal{C}_1 \boxtimes \mathcal{C}_1|_{X\times X-\Delta(X)} \text{ and } \mathcal{C}_2|_{\Delta(X)} \simeq \mathcal{C}_1,$$

etc, and a family of unit objects $\mathbf{1}_n \in \mathcal{C}_n$.

For a given chiral category and a point $x \in X$ we shall denote by $\mathcal{C}_x$ the fiber of $\mathcal{C}_1$ at x. In most cases of interest, $\mathcal{C}_x$ will depend on the formal disc around x in X in a functorial way. So, one can view the notion of chiral category as that of a category $\mathcal{C}$ (thought of as $\mathcal{C}_x$) endowed with an additional structure.

Once the local definition of $\mathrm{Whit}^c(\mathrm{Gr}_G)$ becomes available, the factorization structure of the affine Grassmannian (see [MV], Sect. 5) would imply that $\mathrm{Whit}^c(\mathrm{Gr}_G)$ is a chiral category. With

[1] A candidate for a such a modification of $\mathrm{FS}^c(\check{G})$ has been found when this paper was under revision in December 2007.

[2] The notion of abelian category over a scheme or stack can be found in [Ga1]. However, for a reasonable definition of chiral categories, one needs to work at the triangulated level. We refer the reader to [Lu] or [FG], where the corresponding notions have been developed.

the current non-local definition, we will consider the corresponding categories $\mathrm{Whit}^c(\mathrm{Gr}_G)_n$ separately, without discussing their factorization properties.

Assume for a moment that our ground field is $\mathbb{C}$, the curve X is $\mathbb{A}^1$, and $\mathcal{C}$ is a braided monoidal category. In this case one expects to have a *transcendental* procedure that endows $\mathcal{C}$ with a structure of chiral category.

Thus, the equivalence of Conjecture 0.4 should be understood as an equivalence of chiral categories, where the chiral structure on the RHS is transcendental and comes from the braided monoidal structure.

0.6. **Factorizable sheaves.** Let us return to the category $\mathrm{FS}^c(\check{G})$, mentioned before. [3] This category was introduced in a series of works of M. Finkelberg and V. Schechtman in order to upgrade to the level of an equivalence of categories earlier constructions of various objects related to $U_q(\check{G})$ as cohomology of certain sheaves on configuration spaces.

By its very construction, $\mathrm{FS}^c(\check{G})$ is a chiral category. In the present paper we introduce the corresponding categories $\mathrm{FS}^c(\check{G})_n$ explicitly, but we do not discuss their factorization properties, although the latter are, in a certain sense, evident.

The main result of [BFS] can be interpreted as follows: we have an equivalence of chiral categories

$$\mathrm{FS}^c(\check{G}) \simeq \mathrm{Rep}(u_q(\check{G})),\ q = exp(\pi i c). \tag{0.4}$$

Here $u_q(\check{G})$ is the *small* quantum group, corresponding to $\check{G}$, which coincides with the big quantum group $U_q(\check{G})$, when q is not a root of unity, but is substantially different when it is. (The latter fact is responsible for our inability to pass from the category $\mathrm{Whit}^c(\mathrm{Gr}_G)$ to $\mathrm{Rep}(U_q(\check{G}))$ for all q.)

In (0.4) the RHS acquires a chiral category structure from the monoidal category structure via the procedure mentioned in Sect. 0.5.

The main result of the present paper, Theorem 3.11, is that we have an equivalence for $c \notin \mathbb{Q}$:

$$\mathrm{Whit}^c(\mathrm{Gr}_G)_n \simeq \mathrm{FS}^c(\check{G})_n$$

for every n. This should be interpreted as an equivalence of chiral categories

$$\mathrm{Whit}^c(\mathrm{Gr}_G) \simeq \mathrm{FS}^c(\check{G}), \tag{0.5}$$

thereby proving Conjecture 0.4 for irrational c.

0.7. **Kazhdan-Lusztig equivalence.** Let us now mention another equivalence of categories, namely that of [KL]. For a simple group G_1 and an invariant form $\kappa_1 : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathbb{C}$ consider the category of representations of the corresponding affine Kac-Moody algebra $\widehat{\mathfrak{g}}_1$, denoted $\widehat{\mathfrak{g}}_1\text{-mod}_{\kappa_1}$. Let us denote by $\mathrm{KL}^{\kappa_1}(G_1)$ the subcategory of $\widehat{\mathfrak{g}}_1\text{-mod}_{\kappa_1}$, consisting of representations, on which the action of the Lie subalgebra $\mathfrak{g}_1[[t]] \subset \widehat{\mathfrak{g}}_1$ integrates to an action of the group $G_1[[t]]$.

Let us write $\kappa_1 = \frac{c_1 - \check{h}_1}{2\check{h}_1} \cdot \kappa_{Kil(\mathfrak{g}_1)}$, where $\check{h}_1$ is the dual Coxeter number of $\mathfrak{g}_1$ and $Kil(\mathfrak{g}_1)$ is the Killing form. Let as assume that $c_1 \notin \mathbb{Q}^{\geq 0}$.

In [KL] it was shown that $\mathrm{KL}^{\kappa_1}(G_1)$ can be endowed with a structure of braided monoidal category, and us such it is equivalent to the category $\mathrm{Rep}(U_q(G_1))$, where q and κ_1 are related

[3] We should note that our definition of the category $\mathrm{FS}^c(\check{G})$ differs from the original one in [BFS] in that we work with twisted D-modules on configuration spaces rather than with ordinary D-modules. This allows to introduce this category over an arbitrary curve, whereas in [BFS] one was restricted to genus 0.

by the following formula $q = exp(\frac{\pi i}{c_1 d_1})$ for c_1 as above, and where d_1 is the ratio of the squares of lengths of the shortest and longest roots in $\mathfrak{g}_1$.

This result is also natural to view in the language of chiral categories. Namely, both $\widehat{\mathfrak{g}}_1\text{-mod}_{\kappa_1}$ and $\mathrm{KL}^{\kappa_1}(G_1)$ are naturally chiral categories. We expect that the monoidal structure on $\mathrm{KL}^{\kappa_1}(\mathfrak{g}_1)$, defined in [KL], comes from a chiral category structure by the procedure mentioned in Sect. 0.5. Thus, the equivalence of [KL]

$$\mathrm{KL}^{\kappa_1}(G_1) \simeq \mathrm{Rep}(U_q(G_1)) \tag{0.6}$$

should be understood in the same framework as (0.4) and Conjecture 0.4: it is an equivalence of chiral categories, where the corresponding structure on the RHS comes from the braided monoidal structure.

Note that we can combine (0.4) and (0.6), bypassing the quantum group altogether. We propose:

Conjecture 0.8. *For $c_1 \notin \mathbb{Q}$ we have an equivalence of chiral categories*

$$\mathrm{KL}^{\kappa_1}(G_1) \simeq \mathrm{FS}^{\frac{1}{c_1 d_1}}(G_1).$$

This equivalence is algebraic, i.e., exists over an arbitrary ground field of characteristic 0.

Let us note that neither of (0.4), (0.6) or Conjecture 0.8 involves Langlands duality. In fact, it appears that Conjecture 0.8 is not so far-fetched, and is currently the subject of a work in progress.

0.9. **Combining the equivalences.** Let us now combine the discussion in Sect. 0.7 with Conjecture 0.4. As will be explained in Sect. 2.11, it is more natural to replace the parameter c in the definition of $\mathrm{Whit}^c(\mathrm{Gr}_G)$ by an invariant form $\kappa : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$, related to c by $\kappa = \frac{c-\check{h}}{2\check{h}} \cdot \kappa_{Kil(\mathfrak{g})}$.

Let κ and $\check{\kappa}$ be invariant forms on $\mathfrak{g}$ and $\check{\mathfrak{g}}$, respectively, related as follows: the forms $B_{\mathfrak{h}} := \kappa + \frac{1}{2} \cdot \kappa_{Kil(\mathfrak{g})}|_{\mathfrak{h}}$ and $B_{\check{\mathfrak{h}}} := \check{\kappa} + \frac{1}{2} \cdot \check{\kappa}_{Kil(\check{\mathfrak{g}})}|_{\check{\mathfrak{h}}}$ are non-degenerate and satisfy

$$B_{\check{\mathfrak{h}}} = B_{\mathfrak{h}}^{-1}.$$

This makes sense, since the Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and $\check{\mathfrak{h}} \subset \check{\mathfrak{g}}$ are mutually dual vector spaces. Assume, in addition, that the corresponding scalar $\check{c}$ is not in $\mathbb{Q}^{\geq 0}$. (Note that the scalars c and $\check{c}$ are related by $\check{c} = \frac{1}{cd}$.)

Combining Conjecture 0.8 and (0.5) we propose:

Conjecture 0.10. *There exists an equivalence of chiral categories*

$$\mathrm{Whit}^c(\mathrm{Gr}_G) \simeq \mathrm{KL}^{\check{\kappa}}(\check{G}).$$

Note that unlike (0.5) and Conjecture 0.8, the above Conjecture 0.5 is supposed to hold even for rational (but non-negative) values of c.

0.11. **Relation to quantum geometric Langlands correspondence.** We are now going to put the assertion of Conjecture 0.10 into the framework of quantum geometric Langlands correspondence, as was proposed by B. Feigin, E. Frenkel and A. Stoyanovsky (see [Sto]), motivated by an earlier work of Feigin and Frenkel on the duality of W-algebras.

Let κ and $\check{\kappa}$ be as above. For a global curve X, consider the stacks Bun_G and $\mathrm{Bun}_{\check{G}}$ of principal bundles on X with respect to G and $\check{G}$, respectively, along with the corresponding derived categories of twisted D-modules: $D(\mathfrak{D}\text{-mod}^c(\mathrm{Bun}_G))$ and $D(\mathfrak{D}\text{-mod}^{\check{c}}(\mathrm{Bun}_G))$. In *loc. cit.* the following equivalence was proposed:

Conjecture 0.12.
$$D(\mathfrak{D}\text{-}mod^{-c}(\mathrm{Bun}_G)) \simeq D(\mathfrak{D}\text{-}mod^{\check{c}}(\mathrm{Bun}_{\check{G}}))$$
.

It is supposed to degenerate to the "usual" geometric Langlands equivalence
$$D(\mathfrak{D}\text{-mod}(\mathrm{Bun}_G)) \simeq D(\mathrm{QCoh}(LocSys_{\check{G}}))$$
as $c \to 0$ and therefore $\check{c} \to \infty$.

We would like to propose yet one more conjecture expressing the compatibility between Conjecture 0.10 and Conjecture 0.12. Let us fix points $x_1, ..., x_n$ on the curve. We have the categories $\mathrm{Whit}^c(G)_{x_i}$ and $\mathrm{KL}^{\check{\kappa}}(\check{G})_{x_i}$ attached to each of these points.

We claim that there exists a natural functor
$$\mathrm{Poinc} : \mathrm{Whit}^c(G)_{x_1} \times ... \times \mathrm{Whit}^c(G)_{x_n} \to D(\mathfrak{D}\text{-mod}^{-c}(\mathrm{Bun}_G)).$$
This is a geometric analog of the Poincaré series operator in the theory of automorphic functions; [4] i.e., the adjoint operator to that associating to an automorphic function its Whittaker model. When we interpret the categories $\mathrm{Whit}^c(G)_{x_i}$ by the second method adopted in this paper (see Sect. 0.2), the functor Poinc corresponds to the direct image by means of the morphism of stacks $\mathfrak{W} \to \mathrm{Bun}_G$.

In addition, there exists a naturally defined functor
$$\mathrm{Loc} : \mathrm{KL}^{\check{\kappa}}(\check{G})_{x_1} \times ... \times \mathrm{KL}^{\check{\kappa}}(\check{G})_{x_n} \to D(\mathfrak{D}\text{-mod}^{\check{c}}(\mathrm{Bun}_{\check{G}})).$$
Namely, given objects $V_i \in \mathrm{KL}^{\check{\kappa}}(\check{G})_{x_i}$, the fiber of $\mathrm{Loc}(V_1, ..., V_n)$ at a $\check{G}$-bundle $\mathfrak{F}_{\check{G}}$ is given by
$$H_\bullet(\check{\mathfrak{g}}_{out}^{\mathfrak{F}_{\check{G}}}, V_1 \otimes ... \otimes V_n),$$
where $\check{\mathfrak{g}}_{out}^{\mathfrak{F}_{\check{G}}}$ is the Lie algebra of sections of the associated bundle with the adjoint representation over the punctured curve $X - \{x_1, ..., x_n\}$.

We propose:

Conjecture 0.13. *The diagram of functors*
$$\begin{array}{ccc} \mathrm{Whit}^c(G)_{x_1} \times ... \times \mathrm{Whit}^c(G)_{x_n} & \xrightarrow{\textit{Conjecture 0.10}} & \mathrm{KL}^{\check{\kappa}}(\check{G})_{x_1} \times ... \times \mathrm{KL}^{\check{\kappa}}(\check{G})_{x_n} \\ {\scriptstyle \mathrm{Poinc}}\downarrow & & {\scriptstyle \mathrm{Loc}}\downarrow \\ D(\mathfrak{D}\text{-}mod^{-c}(\mathrm{Bun}_G)) & \xrightarrow{\textit{Conjecture 0.12}} & D(\mathfrak{D}\text{-}mod^{\check{c}}(\mathrm{Bun}_{\check{G}})) \end{array}$$
commutes.

0.14. **Acknowledgments.** Jacob Lurie, whom the main idea of the present paper belongs to, decided not to sign it in the capacity of author. I would like to express my gratitude to him for numerous discussions directly and indirectly related to the contents of the present paper, as well as a lot of other work in progress.

I would also like to express my gratitude to M. Finkelberg for patient and generous explanations of the contents of [BFS], which this paper is based on.

The ideas related to quantum geometric Langlands correspondence, that center around Conjecture 0.13, have received a crucial impetus from communications with A. Stoyanovsky and from discussions with A. Braverman and E. Witten.

[4] The fact that the functor Poinc flips the sign of c is related to a certain choice we make when we define the category $\mathrm{Whit}^c(G)$, see Sect. 2.2.

I would also like to thank A. Beilinson, R. Bezrukavnikov, E. Frenkel and D. Kazhdan for useful and inspiring discussions.

1. Overview

In this section we shall explain the technical contents of this paper, section by section.

1.1. **Conventions.** Throughout the paper we work over an arbitrary algebraically closed ground field of characteristic zero; X will be a smooth projective curve. We denote by ω the canonical line bundle on X.

In the main body of the paper G will be a reductive group with $[G,G]$ simply connected. We choose a Borel subgroup $B \subset G$, its opposite $B^- \subset G$ and identify the Cartan quotient $T := B/N$ with $B \cap B^-$.

By $\check{\Lambda}$ we will denote the coweight lattice, and by Λ its dual–the weight lattice; by $\langle , \rangle$ we will denote the canonical pairing between the two. By $\check{\Lambda}^+$ (resp., Λ^+) we will denote the semi-group of dominant coweights (resp., weights). By Δ (resp., Δ^+) we shall denote the set if roots (resp., positive roots); by $\mathcal{I}$ we shall denote the set of vertices of the Dynkin diagram; for $\imath \in \mathcal{I}$ we let $\alpha_\imath$ (resp., $\check{\alpha}_\imath$) denote the corresponding simple root (resp., coroot). By $\check{\Lambda}^{pos}$ (resp., Λ^{pos}) we shall denote the positive span of simple co-roots (resp., roots). For $\check{\lambda}_1, \check{\lambda}_2 \in \check{\Lambda}$ we shall say that $\check{\lambda}_1 \geq \check{\lambda}_2$ if $\check{\lambda}_1 - \check{\lambda}_2 \in \check{\Lambda}^{pos}$.

We choose once and for all a square root $\omega^{\frac{1}{2}}$ of the canonical bundle ω. For a half-integer i, by ω^i we will mean $(\omega^{\frac{1}{2}})^{\otimes 2i}$. We let $\omega^{\check{\rho}}$ denote the T-bundle induced by means of $2\check{\rho} : \mathbb{G}_m \to T$ from $\omega^{\frac{1}{2}}$.

For an ind-scheme (or strict ind-stack) by a D-module on it we shall mean a D-module supported on some closed subscheme (or substack).

1.2. **Sect. 2.** This section is devoted to the surrogate definition of the Whittaker category Whit^c using a complete curve X. The idea is the following.

Let us think of the field $\mathbb{C}((t))$ (resp., the ring $\mathbb{C}[[t]]$) as the local field (ring) of a point $x \in X$. Let N_{out} be the group-subscheme of $N((t))$ consisting of maps $(X - x) \to N$. By construction, the character $\chi : N((t)) \to \mathbb{G}_a$ is trivial when restricted to N_{out}, so whatever $(N((t)), \chi)$-equivariant D-modules are, they should give rise to D-modules on the quotient $N_{out} \backslash \mathrm{Gr}_G$.

Although $N_{out} \backslash \mathrm{Gr}_G$ makes sense as a functor on the category of schemes, it is *not* a kind of algebraic stack, on which one can define D-modules directly. However, one can cure this pretty easily, by embedding it into another object, denoted in this paper by $\mathfrak{W}_x$, the latter being a strict ind-stack and D-modules on it make sense.

Explicitly, $N_{out} \backslash \mathrm{Gr}_G$ classifies the data of a principal G-bundle $\mathfrak{F}_G$ over X, endowed with a reduction to N over $X - x$. The stack $\mathfrak{W}_x$ replaces the word "reduction" by "generalized reduction" or "Drinfeld structure". The stacks classifying these generalized reductions are known as Drinfeld's compactifications; they are studied in detail, e.g., in [FFKM] or [BG].

Our surrogate Whittaker category Whit_x is introduced as a category of D-modules on $\mathfrak{W}_x$ that satisfy a certain equivariance condition, which is supposed to restore the $(N((t)), \chi)$-equivariance on $N_{out} \backslash \mathrm{Gr}_G$. Moreover, this equivariance condition forces objects of Whit_x to have zero stalks and co-stalks away from $N_{out} \backslash \mathrm{Gr}_G$. So, our Whit_x is the "right" replacement for $\mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{N((t)),\chi}$, the only disadvantage being that the geometric object, on which it is realized, i.e., $N_{out} \backslash \mathrm{Gr}_G$, depends on the choice of the global curve X.

The above definition generalizes in a straightforward way to the case when instead of one point x we have an n-tuple $\overline{x}$ of points $x_1, ..., x_n$. We obtain a category $\mathrm{Whit}_{\overline{x}}$. One can prove, but in a somewhat ad hoc way, that the category $\mathrm{Whit}_{\overline{x}}$ is equivalent to the tensor product $\mathrm{Whit}_{x_1} \otimes ... \otimes \mathrm{Whit}_{x_n}$. The non-triviality of the latter comparison is the expression of the non-locality of our definition of Whit_x.

In the main body of the paper we allow the n-tuple $\overline{x}$ vary along X^n; and we will consider the appropriate category of D-modules, denoted Whit_n. [5]

The twisted version Whit^c_x (resp., $\mathrm{Whit}^c_{\overline{x}}$, Whit^c_n) is defined as follows. By construction, the stack $\mathfrak{W}_x$ (resp., $\mathfrak{W}_{\overline{x}}$, $\mathfrak{W}_n$) is endowed with a forgetful map to the moduli stack of principal G-bundles over X, denoted Bun_G. Instead of ordinary D-modules over $\mathfrak{W}_x$ we consider D-modules, twisted by means of the $-c$-th power of the pull-back of the determinant line bundle over Bun_G. (In the main body of the paper our conventions differ from those in the introduction, in that for G simple the twisting parameter c is scaled by $2\check{h}$, i.e., we use as a basic pairing $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ the Killing form vs. the minimal integral form.)

The equivariance condition that singles the Whittaker category among all twisted D-modules on $\mathfrak{W}_x$ makes sense, just as it did in the untwisted case.

1.3. **Sect. 3.** In this section we review the category of factorizable sheaves, introduced in [BFS].

Let us fix an n-tuple of points $\overline{x}$ on our curve X. For a coweight $\check{\mu}$, let $X^{\check{\mu}}_{\overline{x}}$ denote the space of $\check{\Lambda}$-valued divisors on X of total degree $\check{\mu}$, which are required to be anti-effective away from $\mathrm{supp}(\overline{x})$. I.e., a point of $X^{\check{\mu}}_{\overline{x}}$ is an expression $\underset{k}{\Sigma}\, \check{\mu}_k \cdot y_k$, where $y_k \in X$ are pairwise distinct and $\Sigma\, \check{\mu}_k = \check{\mu}$, and such that for $y_k \neq x_i$, the corresponding coweight $\check{\mu}_k$ belongs to $-\check{\Lambda}^{pos}$, i.e., is in the span of simple coroots with coefficients in $\mathbb{Z}^{\leq 0}$.

One introduces a line bundle $\mathcal{P}_{X^{\check{\mu}}_{\overline{x}}}$ over $X^{\check{\mu}}_{\overline{x}}$, which has a local nature, i.e., the fiber of $\mathcal{P}_{X^{\check{\mu}}_{\overline{x}}}$ at a point $\underset{k}{\Sigma}\, \check{\mu}_k \cdot y_k$ as above is the tensor product of lines $(\omega^{\frac{1}{2}}_{y_k})^{\otimes N(\check{\mu}_k)}$, where $\omega^{\frac{1}{2}}_{y_k}$ denotes the fiber of $\omega^{\frac{1}{2}}$ at y_k, and $N(\check{\mu}_k) = (\check{\mu}_k, \check{\mu}_k + 2\check{\rho})_{Kil}$.

By definition, a factorizable sheaf consists of a data of a twisted D-module $\mathcal{L}^{\check{\mu}}$, *defined for each* $\check{\mu}$, where the twisting is by the c-th power of $\mathcal{P}_{X^{\check{\mu}}_{\overline{x}}}$. The twisted D-modules $\mathcal{L}^{\check{\mu}}$ for different $\check{\mu}$ are related by factorization isomorphisms. To explain what these are it would be easier to pass to the analytic topology. Thus, let ${}^1\mathbf{U}$ and ${}^2\mathbf{U}$ be two non-intersecting open subsets of X, such that $\overline{x} \subset {}^1\mathbf{U}$; let $\check{\mu} = \check{\mu}_1 + \check{\mu}_2$.

We can consider the open subset

$$ {}^1\mathbf{U}^{\check{\mu}_1}_{\overline{x}} \times {}^2\mathbf{U}^{\check{\mu}_2}_{\emptyset} \subset X^{\check{\mu}_1}_{\overline{x}} \times X^{\check{\mu}_2}_{\emptyset}. $$

It admits a natural étale map to $X^{\check{\mu}}_{\overline{x}}$. We require that the pull-back of $\mathcal{L}^{\check{\mu}}$ by means of this map decomposes as a product of $\mathcal{L}^{\check{\mu}_1}$ along the first factor, times a *standard* twisted D-module, denoted $\mathcal{L}^{\check{\mu}_1}_{\emptyset}$, along the first factor.

Thus, the "behavior" of each $\mathcal{L}^{\check{\mu}}$ near a point $\underset{k}{\Sigma}\, \check{\mu}_k \cdot y_k$ is the tensor product over $\check{\mu}_k$ of the "behaviors" of $\mathcal{L}^{\check{\mu}_k}$ near $\check{\mu}_k \cdot y_k \in X^{\check{\mu}_k}_{\overline{x}}$, and the latter is pre-determined, unless y_k coincides with one of the x_i's.

In this way one obtains a category that we denote by $\widetilde{\mathrm{FS}}^c_{\overline{x}}$. One singles out the desired subcategory $\mathrm{FS}^c_{\overline{x}} \subset \widetilde{\mathrm{FS}}^c_{\overline{x}}$ by imposing a condition on the singular support.

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[5]Deviating slightly from the notation pertaining of chiral categories introduced in Sect. 0.5, our Whit_n will not be a category over X^n, but rather the category consisting of objects in the corresponding chiral category, endowed with a connection along X^n.

In [BFS] it is shown that the category $\mathrm{FS}^c_{\overline{x}}$ of factorizable sheaves (when $\overline{x} = \{x\}$) is equivalent to that of representations of the small quantum group $u_q(\check{G})$, where the lattice $\check{\Lambda}$, being the set of coweights of G, plays the role of the weight lattice for $\check{G}$.

1.4. **Sect. 4.** In this section we show how to construct a functor from $\mathrm{Whit}^c_{\overline{x}}$ to $\mathrm{FS}^c_{\overline{x}}$, i.e., how to pass from twisted D-modules on stacks $\mathfrak{W}_{\overline{x}}$ to twisted D-modules on configurations spaces $X^{\check{\mu}}_{\overline{x}}$, $\check{\mu} \in \check{\Lambda}$.

The idea goes back to the construction of the Satake homomorphism from the spherical Hecke algebra to the algebra of functions on the Cartan subgroup of $\check{G}$, and whose geometric analog was used in [MV] to construct the fiber functor in the usual case of $\mathfrak{D}\text{-mod}(\mathrm{Gr}_G)^{G[[t]]}$. Namely, the $\check{\lambda}$-weight space is recovered as cohomology of the orbit of the group $N((t))$ (or $N^-((t))$, as the two are $G[[t]]$-conjugate) passing through the point $t^{\check{\lambda}} \in \mathrm{Gr}_G$.

We would like to do the same for $\mathrm{Whit}^c = \mathfrak{D}\text{-mod}^c(\mathrm{Gr}_G)^{N((t)),\chi}$, namely take cohomology along the orbits of $N^-((t))$. The latter makes sense (at least non-canonically), as the line bundle $\mathcal{P}_{\mathrm{Gr}_G}$ trivializes over $N^-((t))$-orbits.

However, in our context, for each $\check{\lambda}$ we want to obtain not just one vector space, but a twisted D-module over $X^{\check{\lambda}}_x$, with the factorization property. The idea is to repeat the above procedure of taking cohomology along $N^-((t))$-orbits in the family parameterized by $X^{\check{\lambda}}_x$. Such families for all $\check{\lambda}$ are provided by Zastava spaces, and they are well-adapted for our definition of Whit^c.

Zastava spaces were introduced in [FFKM]. By definition, a point of the Zastava space $\mathcal{Z}^{\check{\mu}}_{\overline{x}}$ is a point of $\mathfrak{W}_{\overline{x}}$, i.e., a G-bundle, endowed with a generalized reduction to N away from $\mathrm{supp}(\overline{x})$, and additionally, with a reduction to B^- of degree $\check{\mu}$ (up to a shift by $(2g-2)\cdot\check{\rho}$), defined everywhere, which at the generic point of X is transversal to the given reduction to N. The measure of global non-transversality of the two reductions is given by a point of $X^{\check{\mu}}_{\overline{x}}$, i.e., we have a map $\pi : \mathcal{Z}^{\check{\mu}}_{\overline{x}} \to X^{\check{\mu}}_{\overline{x}}$.

It is a basic observation of [FFKM] (see also [BFGM]) that the space $\mathcal{Z}^{\check{\mu}}_{\overline{x}}$ *factorizes* over $X^{\check{\mu}}_{\overline{x}}$. Namely, for a point $\check{\mu}_k \cdot y_k \in X^{\check{\mu}_k}_{\overline{x}}$, the fiber of π over it is a product $\prod_k \mathcal{Z}^{\check{\mu}_k}_{loc,y_k}$, where each $\mathcal{Z}^{\check{\mu}_k}_{loc,y_k}$ is a subscheme of the affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$ (here t is a local parameter at y_k) that depends only on $\check{\mu}_k$.

Another crucial observation is that the line bundle on $\mathcal{Z}^{\check{\mu}}_{\overline{x}}$, equal to the pull-back (from $\mathfrak{W}_{\overline{x}}$ of the pull back) of the determinant line bundle on Bun_G, is canonically the same as the pull-back by means of π of the line bundle $\mathcal{P}_{X^{\check{\mu}}_{\overline{x}}}$ over $X^{\check{\mu}}_{\overline{x}}$. Therefore, we have a direct image functor between the corresponding categories of twisted D-modules.

This allows us to construct the desired functor $\mathsf{G} : \mathrm{Whit}^c_{\overline{x}} \to \mathrm{FS}^c_{\overline{x}}$. Namely, starting from $\mathcal{F} \in \mathrm{Whit}^c_{\overline{x}}$, we let the $\check{\mu}$-component $\mathcal{L}^{\check{\mu}}$ of $\mathsf{G}(\mathcal{F})$ to be the direct image under π of the pull-back of $\mathcal{F}$ from $\mathfrak{W}_{\overline{x}}$ to $\mathcal{Z}^{\check{\mu}}_{\overline{x}}$. In order to show that $\mathcal{F} \mapsto \{\mathcal{L}^{\check{\mu}}\}$ defined in this way is indeed a (reasonable) functor $\mathrm{Whit}^c_{\overline{x}} \to \mathrm{FS}^c_{\overline{x}}$ we need to check several things.

First, in Proposition 4.13, we establish a factorization property of twisted D-modules on $\mathcal{Z}^{\check{\mu}}_{\overline{x}}$ obtained by pull-back from twisted D-modules on $\mathfrak{W}_{\overline{x}}$ that belong to $\mathrm{Whit}^c_{\overline{x}}$. For an object $\mathcal{F} \in \mathrm{Whit}^c_{\overline{x}}$, its restriction to the fiber of π over a point of $X^{\check{\mu}}_{\overline{x}}$, identified by the above with a product $\prod_k \mathcal{Z}^{\check{\mu}_k}_{loc,y_k}$, decomposes as an external product $\boxtimes_k \mathcal{F}^{\check{\mu}_k}_{loc,y_k}$.

This insures, among the rest, that although the forgetful map $\mathcal{Z}^{\check{\mu}}_{\overline{x}} \to \mathfrak{W}_{\overline{x}}$ is not in general smooth, the pull-back functor applied to twisted D-modules that belong to Whit^c does not produce higher or lower cohomologies.

The next step consists of analyzing the direct image with respect to the morphism $\pi : \mathcal{Z}^{\check{\mu}}_{\overline{x}} \to X^{\check{\mu}}_{\overline{x}}$. Proposition 4.10 insures that this operation does not produce higher or lower cohomology either. Moreover, Theorem 4.11 states that when $c \notin \mathbb{Q}$, the direct image coincides with the direct image with compact supports.

Finally, we need to establish that the system $\{\mathcal{L}^{\check{\mu}}\}$ satisfies the required factorization property. The fact that it satisfies *some* factorization property (i.e., one, where the standard twisted D-module $\mathcal{L}^{\check{\mu}}_{\emptyset}$ is replaced by a certain $'\mathcal{L}^{\check{\mu}}_{\emptyset}$) is an immediate corollary of Proposition 4.13, mentioned above. The fact that $'\mathcal{L}^{\check{\mu}}_{\emptyset} \simeq \mathcal{L}^{\check{\mu}}_{\emptyset}$ is one of the main technical points of the present paper and is the content of Theorem 4.16.

1.5. **Sections 5 and 6.** These two sections are devoted to the proofs of Proposition 4.10 and Theorem 4.16.

The proof of Proposition 4.10 has two ingredients. One is the estimate on the dimension of the fibers of the map $\pi : \mathcal{Z}^{\check{\mu}}_{\overline{x}} \to X^{\check{\mu}}_{\overline{x}}$, which amounts to estimating the dimensions of the schemes $\mathcal{Z}^{\check{\mu}}_{loc}$. The second step consists of showing that the lowest degree cohomology in

$$H(\mathcal{Z}^{\check{\mu}}_{loc,y}, (\mathcal{F}_{\emptyset})^{\check{\mu}}_{loc}) \tag{1.1}$$

vanishes, where $\mathcal{F}_{\emptyset}$ is the "basic" object of Whit. This amounts to a calculation that follows from [FGV], Proposition 7.1.7 coupled with [BFGM], Prop. 6.4.

Theorem 4.16 is concerned with the direct image under π of the pull-back of the basic object $\mathcal{F}_{\emptyset}$ to $\mathcal{Z}^{\check{\mu}}_{\emptyset}$ (we denote the resulting twisted D-module on $X^{\check{\mu}}_{\emptyset}$ by $'\mathcal{L}^{\check{\mu}}_{\emptyset}$). We have to identify it with the standard twisted D-module $\mathcal{L}^{\check{\mu}}_{\emptyset}$.

Part (1) of the theorem asserts that this identification exists away from the diagonal divisor on $X^{\check{\mu}}_{\emptyset}$. This amounts to identifying the cohomology $H(\mathbb{G}_m, \chi \otimes \Psi(c))$ with $\mathbb{C}$, where χ is the Artin-Schreier D-module on $\mathbb{G}_a$, and $\Psi(c)$ is the Kummer D-module on $\mathbb{G}_m$, corresponding to the scalar c.

Part (2) of the theorem asserts that $'\mathcal{L}^{\check{\mu}}_{\emptyset}$ is the Goresky-MacPherson extension of its restriction to the complement of the diagonal divisor. This is true only under the assumption that c is irrational. The latter amounts to two things: one is the essential self-duality of $'\mathcal{L}^{\check{\mu}}_{\emptyset}$, which follows from Theorem 4.11; the other is the assertion that for $\check{\mu}$, not equal to the negative of one of the simple co-roots, the cohomology (1.1) vanishes also in the sub-minimal degree, allowed by dimension considerations. This is done by a direct analysis.

1.6. **Sect. 7.** In this section we prove Theorem 4.11, which states that the functor $\mathsf{G} : \mathrm{Whit}^c_n \to \mathrm{FS}^c_n$ essentially commutes with Verdier duality. More precisely, we prove that for a twisted D-module on $\mathcal{Z}^{\check{\mu}}_n$, obtained as a pull-back of an object $\mathcal{F} \in \mathrm{Whit}^c_n \subset \mathfrak{D}\text{-mod}^c(\mathfrak{W}_n)$, its direct image onto $X^{\check{\mu}}_n$ under π equals the direct image with compact supports.

To prove this fact we introduce a compactification $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$ of the stack $\mathrm{Bun}^{\check{\mu}}_{B^-}$ along the fibers of the projection $\mathrm{Bun}^{\check{\mu}}_{B^-} \to \mathrm{Bun}_G$, by allowing the reduction of a G-bundle to B^- to degenerate to a Drinfeld structure.

Whereas $\mathcal{Z}^{\check{\mu}}_n$ was an open sub-stack of the fiber product $\mathfrak{W}_n \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\check{\mu}}_{B^-}$, we define $\overline{\mathcal{Z}}^{\check{\mu}}_n$ to be the corresponding open sub-stack of $\mathfrak{W}_n \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$. The map $\pi : \mathcal{Z}^{\check{\mu}}_n \to X^{\check{\mu}}_n$ extends to a map $\overline{\pi} : \overline{\mathcal{Z}}^{\check{\mu}}_n \to X^{\check{\mu}}_n$. The main observation is that the map $\overline{\pi}$ is proper.

Let $\jmath^-$ denote the open embedding $\mathrm{Bun}^{\check{\mu}}_{B^-} \hookrightarrow \overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$, and denote by $'\jmath^-$ the base-changed map $\mathcal{Z}^{\check{\mu}}_n \hookrightarrow \overline{\mathcal{Z}}^{\check{\mu}}_n$. Theorem 4.11 is an easy corollary of another result, Theorem 7.3, that states that for $\mathcal{F} \in \mathrm{Whit}^c_n$, its pull-back to $\mathcal{Z}^{\check{\mu}}_n$ is *clean* with respect to $'\jmath^-$, i.e., the direct image $'\jmath^-_*$ equals $'\jmath^-_!$.

We deduce Theorem 7.3 from Theorem 7.6 that states the "constant" twisted D-module on $\mathrm{Bun}^{\check{\mu}}_{B^-}$ is clean with respect to $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$. The latter theorem is proved by a word-for-word repetition of the calculation of the intersection cohomology sheaf on $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$, performed in [BFGM].

1.7. **Sect. 8.** In this section we finish the proof of the fact that the functor $\mathsf{G} : \mathrm{Whit}^c_n \to \mathrm{FS}^c_n$ is an equivalence.

We first show that the functor G induces an equivalence for a fixed set of pole points $\overline{x} = x_1, ..., x_n$:

$$\mathrm{Whit}^c_{\overline{x}} \to \mathrm{FS}^c_{\overline{x}}. \tag{1.2}$$

This essentially reduces to the fact that the category $\mathrm{Whit}^c_{\overline{x}}$ is semi-simple for c irrational.

The final step is to show that the equivalences (1.2) glue together as $\overline{x}$ moves along X^n. The essential ingredient here is Theorem 4.11 that asserts that the functor G is essentially Verdier self-dual.

2. The twisted Whittaker category

2.1. For $n \in \mathbb{Z}^{\geq 0}$ we consider the n-th power of X, and the corresponding version of the Drinfeld compactification, denoted $\mathfrak{W}_n$ over X^n. By definition, $\mathfrak{W}_n$ is the ind-stack that classifies the following data:

- An n-tuple of points $x_1, ..., x_n$ of X.
- A G-bundle $\mathfrak{F}_G$ on X (For a dominant weight λ we shall denote by $\mathcal{V}^\lambda_{\mathfrak{F}_G}$ the vector bundle associated with the corresponding highest weight representation.)
- For each dominant weight λ a non-zero map
$$\kappa^\lambda : \omega^{\langle \lambda, \check{\rho} \rangle} \to \mathcal{V}^\lambda_{\mathfrak{F}_G},$$
which is allowed to have poles at $x_1, ..., x_n$. The maps κ^λ are required to satisfy the Plücker relations (see [BG], Sect. 1.2.1).

Let $\mathfrak{p}$ denote the natural forgetful map $\mathfrak{W}_n \to \mathrm{Bun}_G$. When $n = 0$ we shall use the notation $\mathfrak{W}_\emptyset$, or sometimes simply $\mathfrak{W}$.

2.2. Let $\mathcal{P}_{\mathrm{Bun}_G}$ be the determinant line bundle on Bun_G. We normalize it so that the fiber over $\mathfrak{F}_G \in \mathrm{Bun}_G$ is

$$\det R\Gamma(X, \mathfrak{g}_{\mathfrak{F}_G}) \otimes \left(\bigotimes_{\alpha \in \Delta} \det R\Gamma\left(X, \omega^{\langle \alpha, \check{\rho} \rangle}\right) \right)^{\otimes -1} \bigotimes \Big(\det R\Gamma\left(X, \mathcal{O}\right) \Big)^{-\dim(\mathfrak{t})},$$

where $\mathfrak{t}$ is the Cartan subalgebra of G. (The second and third factors are lines that do not depend on the point of Bun_G; the reason for introducing them will become clear later.)

Let $\mathcal{P}_{\mathfrak{W}_n}$ denote the *inverse* of the pull-back of $\mathcal{P}_{\mathrm{Bun}_G}$ to $\mathfrak{W}_n$ by means of $\mathfrak{p}$. For a scalar we shall denote by $\mathfrak{D}\text{-mod}^c(\mathfrak{W}_n)$ the category of "$(\mathcal{P}_{\mathfrak{W}_n})^{\otimes c}$"-twisted D-modules on $\mathfrak{W}_n$. When $c = 0$ (or, more generally, when c is an integer) this category is canonically equivalent to $\mathfrak{D}\text{-mod}(\mathfrak{W}_n)$.

2.3. We are now going to introduce a full subcategory of $\mathfrak{D}^c\text{-mod}(\mathfrak{W}_n)$, denoted Whit_n^c. When $c=0$, this is the Whittaker category of [FGV]; for an arbitrary c the definition is not much different. As the definition follows closely [FGV], Sect. 6.2 and [Ga], Sect. 4 we shall omit most of the proofs and the refer the reader to *loc. cit.*.

Fix a point $y \in X$ let $\mathcal{B}_y^{\text{reg}}$ (resp., $\mathcal{B}_y^{\text{mer}}$) be the group (resp., group ind-scheme) of automorphisms of the B-bundle induced by means of $T \to B$ from $\omega^{\check{\rho}}$ over the formal disc $\mathcal{D}_y$ (resp., formal punctured disc $\mathcal{D}_y^\times$) around y. This group is non-canonically isomorphic to $B(\mathcal{O}_y)$ (resp., $B(\mathcal{K}_y)$), where $\mathcal{O}_y$ (resp., $\mathcal{K}_y$) is the completed local ring (resp., field) at y.

Let $\mathcal{N}_y^{\text{reg}} \subset \mathcal{B}_y^{\text{reg}}$ (resp., $\mathcal{N}_y^{\text{mer}} \subset \mathcal{B}_y^{\text{mer}}$) be the kernel of the natural homomorphism $\mathcal{B}_y^{\text{reg}} \to T(\mathcal{O}_y)$ (resp., $\mathcal{B}_y^{\text{mer}} \to T(\mathcal{K}_y)$). Note that

$$\mathcal{N}_y^{\text{mer}}/[\mathcal{N}_y^{\text{mer}}, \mathcal{N}_y^{\text{mer}}] \simeq \underbrace{\omega|_{\mathcal{D}_y^\times} \times ... \times \omega|_{\mathcal{D}_y^\times}}_{r \text{ times}},$$

where r is the semi-simple rank of G. Taking the residue along each component we obtain a canonical homomorphism $\chi_y : \mathcal{N}_y^{\text{mer}} \to \mathbb{G}_a$.

Fix a non-empty collection of distinct points $\overline{y} := y_1, ..., y_m$, and set $\mathcal{N}_{\overline{y}}^{\text{reg}}$ (resp., $\mathcal{N}_{\overline{y}}^{\text{mer}}$) to be the product of the corresponding groups $\mathcal{N}_{y_j}^{\text{reg}}$ (resp., $\mathcal{N}_{y_j}^{\text{mer}}$). We shall denote by $\chi_{\overline{y}}$ the corresponding homomorphism $\mathcal{N}_{\overline{y}}^{\text{mer}} \to \mathbb{G}_a$.

Consider an open substack $(\mathfrak{W}_n)_{\text{good at } \overline{y}}$ of $\mathfrak{W}_n$ corresponding to the condition that the points $x_1, ..., x_n$ stay away from $\overline{y}$, and the maps κ^λ are injective on the fibers over y_j, $j = 1, ..., m$ (this is equivalent to asking that κ^λ be an injective bundle map on a neighborhood of these points).

Note that a point of this substack defines a B-bundle over each $\mathcal{D}_{y_j}$, such that the induced T-bundle is $\omega^{\check{\rho}}|_{\mathcal{D}_{y_j}}$. We define a $\mathcal{N}_{\overline{y}}^{\text{reg}}$-torsor over $(\mathfrak{W}_n)_{\text{good at } \overline{y}}$ that classifies the data as above plus an additional choice of identification β_{y_j} of this B-bundle with $B \overset{T}{\times} \omega^{\check{\rho}}|_{\mathcal{D}_{y_j}}$, which is compatible with the existing identification of the corresponding T-bundles.

Let us denote the resulting stack by ${}_{\overline{y}}\mathfrak{W}_n$. The standard re-gluing construction equips ${}_{\overline{y}}\mathfrak{W}_n$ with an action of the group ind-scheme $\mathcal{N}_{\overline{y}}^{\text{mer}}$ (see [FGV], Sect. 3.2 or [Ga], Sect. 4.3).

Let $\mathcal{P}_{{}_{\overline{y}}\mathfrak{W}_n}$ be the pull-back of the line bundle $\mathcal{P}_{\mathfrak{W}_n}$ to ${}_{\overline{y}}\widetilde{\mathfrak{W}}_n$.

Lemma 2.4. *The action of $\mathcal{N}_{\overline{y}}^{\text{mer}}$ on ${}_{\overline{y}}\mathfrak{W}_n$ naturally lifts to an action on $\mathcal{P}_{{}_{\overline{y}}\mathfrak{W}_n}$.*

Proof. For a point

$$\{(x_1, ..., x_n), \mathfrak{F}_G, \kappa^\lambda, \beta_j\} \in {}_{\overline{y}}\widetilde{\mathfrak{W}}_n \text{ and } \{\mathbf{n}_j \in \mathcal{N}_{y_j}^{\text{mer}}\},$$

let $\{(x_1, ..., x_n), \mathcal{F}'_G, \kappa'^\lambda, \beta'_j\}$ be the corresponding new point of ${}_{\overline{y}}\widetilde{\mathfrak{W}}_n$. We have to show that the lines

$$(\mathcal{P}_{\text{Bun}_G})_{\mathfrak{F}_G} \text{ and } (\mathcal{P}_{\text{Bun}_G})_{\mathcal{F}'_G}$$

are canonically isomorphic.

However, the ratio of these two lines is canonically isomorphic to the product over $j = 1, ..., m$ of relative determinants of the G-bundles $\mathfrak{F}_G|_{\mathcal{D}_{y_j}}$ and $\mathcal{F}'_G|_{\mathcal{D}_{y_j}}$, which by definition are identified over the corresponding punctured discs $\mathcal{D}_{y_j}^\times$. Both these bundles are equipped with reductions to B that coincide over $\mathcal{D}_{y_j}^\times$ and such that the induced isomorphism of T-bundles is regular over the non-punctured disc. This establishes the required isomorphism between the lines. $\square$

2.5. We define $(\mathrm{Whit}^c_n)_{\text{good at } \overline{y}}$ to be the full subcategory of $\mathfrak{D}\text{-mod}^c\left((\mathfrak{W}_n)_{\text{good at } \overline{y}}\right)$ consisting of D-modules on ${}_{\overline{y}}\widetilde{\mathfrak{W}}_n$ that are $(\mathcal{N}^{\text{mer}}_{\overline{y}}, \chi_{\overline{y}})$-equivariant [6] (see [FGV], Sect. 6.2.6 or [Ga], Sect. 4.7).

Note that the $\mathcal{N}^{\text{reg}}_{\overline{y}}$-equivariance condition canonically descends any such D-module from ${}_{\overline{y}}\widetilde{\mathfrak{W}}_n$ to $(\mathfrak{W}_n)_{\text{good at } \overline{y}}$.

Let now $\overline{y}'$ and $\overline{y}''$ be two collections of points, and set $\overline{y} = \overline{y}' \cup \overline{y}''$. Note that

$$(\mathfrak{W}_n)_{\text{good at } \overline{y}'} \cap (\mathfrak{W}_n)_{\text{good at } \overline{y}''} = (\mathfrak{W}_n)_{\text{good at } \overline{y}}.$$

In particular, we can consider the corresponding groups $\mathcal{N}^{\text{reg}}_{\overline{y}'}$, $\mathcal{N}^{\text{reg}}_{\overline{y}''}$ and $\mathcal{N}^{\text{reg}}_{\overline{y}}$ and torsors with respect to them over $\left(\widetilde{\mathfrak{W}}_n\right)_{\text{good at } \overline{y}}$. Let us consider the corresponding three subcategories of $\mathfrak{D}\text{-mod}^c\left(\left(\widetilde{\mathfrak{W}}_n\right)_{\text{good at } \overline{y}}\right)$. As in [Ga], Corollary 4.14 one shows that, as long as $\overline{y}'$ and $\overline{y}''$ are non-empty, the above three subcategories coincide.

This shows that we have a well-defined full-subcategory $\mathrm{Whit}^c_n \subset \mathfrak{D}^c\text{-mod}(\mathfrak{W}_n)$: an object $\mathcal{F}$ belongs to Whit^c_n if for any $\overline{y}$ as above, its restriction to any $\left(\widetilde{\mathfrak{W}}_n\right)_{\text{good at } \overline{y}}$ belongs to $(\mathrm{Whit}^c_n)_{\text{good at } \overline{y}}$.

2.6. Let us fix points $\overline{x} := x_1, ., , x_n$ and denote by $\mathfrak{W}_{\overline{x}}$ (resp., $\widetilde{\mathfrak{W}}_{\overline{x}}$) the fiber over the corresponding point of X^n. With no restriction of generality we can assume that all the points x_i are distinct. Let $\mathrm{Whit}^c_{\overline{x}}$ denote the corresponding category of twisted D-modules on $\mathfrak{W}_{\overline{x}}$.

The same analysis as in [FGV], Lemma 6.2.4 or [Ga], Prop. 4.14, shows that every object of $\mathrm{Whit}^c_{\overline{x}}$ is holonomic, and one obtains the following explicit description of the irreducibles (and some other standard objects) in this category.

Let $\overline{\check{\lambda}} = \check{\lambda}_1, ..., \check{\lambda}_n$ be an n-tuple of dominant coweights of G. Let $\mathfrak{W}_{\overline{x},\overline{\check{\lambda}}}$ be a locally closed substack of $\mathfrak{W}_{\overline{x}}$ consisting of points $\{\mathfrak{F}_G, \kappa^\lambda\}$, where each κ^λ has a pole of order $\langle \lambda, \check{\lambda}_i \rangle$ at x_i and no zeroes anywhere else. Let $\jmath_{\overline{x},\overline{\check{\lambda}}}$ denote the corresponding locally closed embedding; by [FGV], Prop. 3.3.1, this map is affine.

Proceeding as above, for every such $\overline{\check{\lambda}}$, one can introduce the category $\mathrm{Whit}^c_{\overline{x},\overline{\check{\lambda}}}$. The following is shown as [FGV], Lemma 6.2.4 or [Ga], Prop. 4.13:

Lemma 2.7. *The category* $\mathrm{Whit}^c_{\overline{x},\overline{\check{\lambda}}}$ *is (non-canonically) equivalent to that of vector spaces.*

Let $\mathcal{F}_{\overline{x},\overline{\check{\lambda}}}$ denote the unique irreducible object of the above category (it is a priori defined up to a non-canonical scalar automorphism). Let $\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!}$ (resp., $\mathcal{F}_{\overline{x},\overline{\check{\lambda}},*}$, $\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!*}$) denote its extension by means of $\jmath_{\overline{x},\overline{\check{\lambda}},!}$ (resp., $\jmath_{\overline{x},\overline{\check{\lambda}},*}$, $\jmath_{\overline{x},\overline{\check{\lambda}},!*}$) on the entire $\mathfrak{W}_{\overline{x}}$. All of the above objects are D-modules since the map $\jmath_{\overline{x},\overline{\check{\lambda}}}$ is affine (see [FGV], Prop. 3.3.1 or [BFG], Theorem 11.6 for an alternative proof). As in [FGV], Prop. 6.2.1 or [Ga], Lemma 4.11, one shows that all three are objects of $\mathrm{Whit}^c_{\overline{x}}$.

Lemma 2.8.
(a) *Every irreducible in* $\mathrm{Whit}^c_{\overline{x}}$ *is of the form* $\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!*}$ *for some* $\overline{\check{\lambda}}$.

[6]The role of the Artin-Schreier sheaf in the world of D-modules in plaid by the exponential D-module *exp* on $\mathbb{G}_a$. We recall that the D-module *exp* on $\mathbb{G}_a$ is generated by one section "e^z" that satisfies the relation $\partial_z \cdot$ "e^z" $=$ "e^z", where z is a coordinate on $\mathbb{G}_a$.

(b) *The cones of the canonical maps*

$$\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!} \to \mathcal{F}_{\overline{x},\overline{\check{\lambda}},!*} \to \mathcal{F}_{\overline{x},\overline{\check{\lambda}},*} \tag{2.1}$$

are extensions of objects $\mathcal{F}_{\overline{x},\overline{\check{\lambda}}',!*}$ *for* $\overline{\check{\lambda}}' < \overline{\check{\lambda}}$.

It is a basic fact (which is the main theorem of [FGV]) that for $c = 0$ the canonical maps in (2.1) are isomorphisms. This will no longer be true for an arbitrary c, but as we shall show, it will still be true for $c \notin \mathbb{Q}$.

2.9. Let us describe more explicitly the basic object of the category $\mathrm{Whit}^c_{\emptyset}$, which we shall denote by $\mathcal{F}_{\emptyset}$. Consider the open substack $\mathfrak{W}_{\emptyset,0} \subset \mathfrak{W}_{\emptyset}$. From the definition of the line bundle $\mathcal{P}_{\mathrm{Bun}_G}$ we obtain:

Lemma 2.10. *The restriction of* $\mathcal{P}_{\mathfrak{W}_{\emptyset}}$ *to* $\mathfrak{W}_{\emptyset,0}$ *admits a canonical trivialization.*

Thus, the category $\mathrm{Whit}^c_{\emptyset,0}$ is the same as $\mathrm{Whit}_{\emptyset,0}$.

In addition, the sum of residues gives rise to a map $\mathfrak{W}_{\emptyset,0} \to \mathbb{G}_a$. We define $\mathcal{F}_{\emptyset,0}$ as the object of $\mathrm{Whit}^c_{\emptyset,0}$, corresponding to the pull-back of the D-module *exp* from $\mathbb{G}_a$ to $\mathfrak{W}_{\emptyset,0}$ under this morphism.

Since there are no dominant weights ≤ 0, from Lemma 2.8(b) we obtain:

$$\jmath_{\emptyset,0,!}(\mathcal{F}_{\emptyset,0}) \simeq \jmath_{\emptyset,0,!*}(\mathcal{F}_{\emptyset,0}) \simeq \jmath_{\emptyset,0,*}(\mathcal{F}_{\emptyset,0}).$$

We set $\mathcal{F}_{\emptyset}$ to be the above object of $\mathrm{Whit}^c_{\emptyset}$.

2.11. **The parameter "c".** Note that when the adjoint group, corresponding to G, is semi-simple (and not simple), the line bundle $\mathcal{P}_{\mathrm{Bun}_G}$ is naturally a product of lines bundles corresponding to the simple factors of G_{ad}. Therefore, when defining the categories Whit^c_n, instead of one scalar c one can work with a k-tuple of scalars, where k is the number of simple factors.

More invariantly, from now on we shall understand c as an ad-invariant symmetric bilinear form $\mathfrak{g}_{ad} \otimes \mathfrak{g}_{ad} \to \mathbb{C}$, or equivalently, a Weyl group invariant symmetric bilinear form $(,)_c$: $\check{\Lambda}_{G_{ad}} \otimes \check{\Lambda}_{G_{ad}} \to \mathbb{C}$.

We will say that c is "integral" if the latter form takes integral values. In this case it is known that "$(\mathcal{P}_{\mathrm{Bun}_G})^{\otimes c}$" is defined as a line bundle. Hence, the categories $\mathrm{Whit}^{c'}_n$ and $\mathrm{Whit}^{c''}_n$ with $c'' - c'$ integral are equivalent.

We will say that c is non-integral if $(\check{\alpha}_\imath, \check{\alpha}_\imath)_c \notin \mathbb{Z}$ for any $\imath \in \mathcal{I}$. Note, that unless G is simple, not "integral" is not the same as "non-integral".

We shall say that c is irrational if the restriction of c to each of the simple factors is an irrational multiple of the Killing form. Equivalently, this means that $(\check{\alpha}_\imath, \check{\alpha}_\imath)_c \notin \mathbb{Q}$ for any $\imath \in \mathcal{I}$.

3. The FS category

From now on in the paper we will assume that c is non-integral.

3.1. For $n \in \mathbb{Z}^{\geq 0}$ and $\check{\mu} \in \check{\Lambda}$ we introduce the ind-scheme $X_n^{\check{\mu}}$, fibered over X^n to classify pairs $\{(x_1, ..., x_n) \in X^n, D\}$, where D is a $\check{\Lambda}$-valued divisor on X of total degree $\check{\mu}$ with the condition that for every dominant weight λ, the $\mathbb{Z}$-valued divisor $\langle \lambda, D \rangle$ is anti-effective away from $x_1, ..., x_n$.

When $n = 0$, we shall use the notation $X_\emptyset^{\check{\mu}}$, or sometimes simply $X^{\check{\mu}}$. For this scheme to be non-empty we need that $\check{\mu} \in -\check{\Lambda}^{pos}$. If $\check{\mu} = -\Sigma\, m_\imath \cdot \check{\alpha}_\imath$, we have

$$X_\emptyset^{\check{\mu}} = \prod_\imath X^{(m_\imath)}. \tag{3.1}$$

We can represent $X_n^{\check{\mu}}$ explicitly as a union of schemes as follows. Fix an n-tuple $\overline{\check{\lambda}} = \check{\lambda}_1, ..., \check{\lambda}_n$ of elements of $\check{\Lambda}$. We define a closed subscheme $X^{\check{\mu}}_{n, \leq \overline{\check{\lambda}}} \subset X_n^{\check{\mu}}$ by the condition that the divisor $D' := D - \sum_{i=1,...,n} \check{\lambda}_i \cdot x_i$ is such that $\langle \lambda, D' \rangle$ is anti-effective. By adding the divisor $\sum_{i=1,...,n} \check{\lambda}_i \cdot x_i$, we identify the scheme $X^{\check{\mu}}_{n, \leq \overline{\check{\lambda}}}$ with $X_\emptyset^{\check{\mu} - \check{\lambda}_1 - ... - \check{\lambda}_n}$.

For another n-tuple $\overline{\check{\lambda}}' = \check{\lambda}'_1, ..., \check{\lambda}'_n$ with $\check{\lambda}'_i \geq \check{\lambda}_i$ we have a natural closed embedding $X^{\check{\mu}}_{n, \leq \overline{\check{\lambda}}} \hookrightarrow X^{\check{\mu}}_{n, \leq \overline{\check{\lambda}}'}$. It is clear that

$$X_n^{\check{\mu}} = \varinjlim_{\overline{\check{\lambda}}} X^{\check{\mu}}_{n, \leq \overline{\check{\lambda}}}.$$

3.2. We are now going to introduce a certain canonical line bundle $\mathcal{P}_{X_n^{\check{\mu}}}$ over $X_n^{\check{\mu}}$. Consider the stack $\mathrm{Bun}_T \simeq \mathrm{Pic}(X) \underset{\mathbb{Z}}{\otimes} \check{\Lambda}$. On it we consider the following line bundle $\mathfrak{F}_{\mathrm{Bun}_T}$: its fiber at $\mathcal{P}_T \in \mathrm{Bun}_T$ is the line

$$\left(\bigotimes_{\alpha \in \Delta} \det R\Gamma\left(X, \alpha(\mathfrak{F}_T)\right) \right) \otimes \left(\bigotimes_{\alpha \in \Delta} \det R\Gamma\left(X, \omega^{\langle \alpha, \check{\rho} \rangle}\right) \right)^{\otimes -1}.$$

Consider the Abel-Jacobi map

$$AJ : X_n^{\check{\mu}} \to \mathrm{Bun}_T$$

that sends a point $\{(x_1, ..., x_n) \in X^n, D\}$ to the T-bundle $\omega^{\check{\rho}}(-D)$. We set

$$\mathcal{P}_{X_n^{\check{\mu}}} := AJ^*(\mathcal{P}_{\mathrm{Bun}_T}^{\otimes -1}).$$

The main property of the line bundle $\mathcal{P}_{X_n^{\check{\mu}}}$ is that it has a local nature in X:

Lemma 3.3. *The fiber of $\mathcal{P}_{X_n^{\check{\mu}}}$ at point $\{(x_1, ..., x_n) \in X^n, D\} \in X_n^{\check{\mu}}$ with $D = \Sigma\, \check{\mu}_k \cdot y_k$, is canonically isomorphic to*

$$\bigotimes_k (\omega_{y_k}^{\frac{1}{2}})^{\otimes (\check{\mu}_k, \check{\mu}_k + 2\check{\rho})_{Kil}},$$

where $(\cdot, \cdot)_{Kil}$ is the Killing form on $\check{\Lambda}$.

This lemma implies in particular that the line bundle $\mathcal{P}_{X_n^{\check{\mu}}}$ can be defined over $X_n^{\check{\mu}}$ for a not necessarily complete curve X. [7]

For a partition $n = n_1 + n_2$, $\check{\mu} = \check{\mu}_1 + \check{\mu}_2$, consider the natural addition map $\mathrm{add}_{\check{\mu}_1, \check{\mu}_2} : X_{n_1}^{\check{\mu}_1} \times X_{n_2}^{\check{\mu}_1} \to X_n^{\check{\mu}}$.

[7] The construction of the line bundle $\mathcal{P}_{X_n^{\check{\mu}}}$ starting from the form $(\cdot, \cdot)_{Kil}$ is a particular case of a θ-data, see [CHA], Sect. 3.10.3.

For an n_1-tuple and an n_2-tuple of elements of $\check{\Lambda}$, $\overline{\overline{\lambda}}_1$ and $\overline{\overline{\lambda}}_2$, respectively, let $X^{\check{\mu}_1}_{n_1,\leq\overline{\overline{\lambda}}_1} \subset X^{\check{\mu}_1}_{n_1}$ and $X^{\check{\mu}_2}_{n_2,\leq\overline{\overline{\lambda}}_2} \subset X^{\check{\mu}_2}_{n_2}$ be the corresponding closed subschemes. Let

$$\left(X^{\check{\mu}_1}_{n_1,\leq\overline{\overline{\lambda}}_1} \times X^{\check{\mu}_1}_{n_2,\leq\overline{\overline{\lambda}}_2}\right)_{disj}$$

be the open part of the product, corresponding to the condition that the supports of the corresponding divisors are disjoint. Note that the restriction of the map $\operatorname{add}_{\check{\mu}_1,\check{\mu}_2}$ to this open subset is an étale map to the subscheme $X^{\check{\mu}}_{n,\leq\overline{\overline{\lambda}}_1\cup\overline{\overline{\lambda}}_2}$.

From Lemma 3.3 we obtain the following *factorization property*:

$$\operatorname{add}^*_{\check{\mu}_1,\check{\mu}_2}(\mathcal{P}_{X^{\check{\mu}}_n})|_{\left(X^{\check{\mu}_1}_{n_1,\leq\overline{\overline{\lambda}}_1}\times X^{\check{\mu}_1}_{n_2,\leq\overline{\overline{\lambda}}_2}\right)_{disj}} \simeq \mathcal{P}_{X^{\check{\mu}_1}_{n_1}} \boxtimes \mathcal{P}_{X^{\check{\mu}_2}_{n_2}}|_{\left(X^{\check{\mu}_1}_{n_1,\leq\overline{\overline{\lambda}}_1}\times X^{\check{\mu}_1}_{n_2,\leq\overline{\overline{\lambda}}_2}\right)_{disj}}, \tag{3.2}$$

compatible with refinements of partitions.

Let us also denote by $(X^{\check{\mu}_1}_{\emptyset} \times X^{\check{\mu}_2}_n)_{disj}$ ind-subscheme of $X^{\check{\mu}_1}_{\emptyset} \times X^{\check{\mu}_2}_n$ consisting of points

$$\{D_1 \in X^{\check{\mu}_1}_{\emptyset}, \overline{x} \in X^n, D_2 \in X^{\check{\mu}_2}_n\},$$

such that D_1 is disjoint from both $\overline{x}$ and D_2. Let us denote by $\operatorname{add}_{\check{\mu}_1,\check{\mu}_2,disj}$ the restriction of the map $\operatorname{add}_{\check{\mu}_1,\check{\mu}_2}$ to this open subset; it is also étale. We have an isomorphism

$$\operatorname{add}^*_{\check{\mu}_1,\check{\mu}_2,disj}(\mathcal{P}_{X^{\check{\mu}}_n}) \simeq \mathcal{P}_{X^{\check{\mu}_1}_{\emptyset}} \boxtimes \mathcal{P}_{X^{\check{\mu}_2}_n}. \tag{3.3}$$

3.4. Let $\mathfrak{D}\text{-mod}^c(X^{\check{\mu}}_n)$ denote the category of "$\mathcal{P}^{\otimes c}_{X^{\check{\mu}}_n}$"-twisted D-modules on $X^{\check{\mu}}_n$.

As in Sect. 2.11, if the group G_{ad} is not simple, the line bundle $\mathcal{P}_{X^{\check{\mu}}_n}$ is naturally a product of several line bundles, one for each simple factor. Hence, also in the present context we will interpret c as an invariant symmetric bilinear form $\check{\Lambda} \otimes \check{\Lambda} \to \mathbb{C}$. As in *loc. cit.*, the categories $\mathfrak{D}\text{-mod}^{c'}(X^{\check{\mu}}_n)$ and $\mathfrak{D}\text{-mod}^{c''}(X^{\check{\mu}}_n)$ are equivalent if $c'' - c'$ is integral.

In order to introduce the category of factorizable sheaves, we will need to define a particular object of the category $\mathfrak{D}\text{-mod}^c(X^{\check{\mu}}_{\emptyset})$, denoted $\mathcal{L}^{\check{\mu}}_{\emptyset}$.

Let $\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset} \subset X^{\check{\mu}}_{\emptyset}$ be the open subscheme, corresponding to divisors of the form $\sum\limits_k \check{\mu}_k \cdot y_k$ with all y_k distinct and each $\check{\mu}_k$ being the negative of one of the simple co-roots. Let j^{Diag} denote the corresponding open embedding.

By Lemma 3.3, the line bundle $\mathcal{P}_{X^{\check{\mu}}_{\emptyset}}|_{\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset}}$ canonically trivializes. Indeed, $(\check{\lambda}, \check{\lambda} + 2\check{\rho})_{Kil} = 0$ whenever $\check{\lambda}$ is of the form $w(\check{\rho}) - \check{\rho}$ for some $w \in W$; in particular for $\check{\lambda}$ being a simple co-root.

Hence, the category $\mathfrak{D}\text{-mod}^c(\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset})$ is canonically the same as $\mathfrak{D}\text{-mod}(\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset})$.

We let $\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\emptyset} \in \mathfrak{D}\text{-mod}^c(\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset})$ to be the following object. It corresponds via the above equivalence

$$\mathfrak{D}\text{-mod}^c(\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset}) \simeq \mathfrak{D}\text{-mod}(\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset})$$

to the *sign* local system on $\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset}$. The latter is, by definition, the product of sign local systems on each $\overset{\circ}{X}{}^{(m_i)}$ when we write $X^{\check{\mu}}_{\emptyset}$ as in (3.1).

We define

$$\mathcal{L}^{\check{\mu}}_{\emptyset} := j^{Diag}_{!*}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\emptyset}).$$

Note that the Goresky-MacPherson extension is taken in the category $\mathfrak{D}\text{-mod}^c(\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset})$ (and *not* in $\mathfrak{D}\text{-mod}(\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset})$).

Example. Let $G = SL_2$ and c be irrational. Then the fibers and co-fibers of $\mathcal{L}^{\check{\mu}}_{\emptyset}$ on the closed sub-variety $X^{\check{\mu}}_{\emptyset} - \overset{\circ}{X}{}^{\check{\mu}}_{\emptyset}$ are zero.

By construction, the system of objects $\check{\mu} \mapsto \mathcal{L}^{\check{\mu}}_{\emptyset}$ has the following factorization property with respect to (3.3): for $\check{\mu} = \check{\mu}_1 + \check{\mu}_2$,

$$\mathrm{add}^*_{\check{\mu}_1,\check{\mu}_2,disj}(\mathcal{L}^{\check{\mu}}_{\emptyset}) \simeq \mathcal{L}^{\check{\mu}_1}_{\emptyset} \boxtimes \mathcal{L}^{\check{\mu}_2}_{\emptyset}. \tag{3.4}$$

These isomorphisms are compatible with refinements of partitions.

3.5. We are now ready to introduce the sought-for category of factorizable sheaves. We define $\widetilde{\mathrm{FS}}{}^c_n$ to have as objects (twisted) D-modules $\mathcal{L}^{\check{\mu}}_n \in \mathfrak{D}\text{-mod}^c(X^{\check{\mu}}_n)$, defined for each $\check{\mu} \in \check{\Lambda}$, equipped with *factorization isomorphisms*:

For any partition $\check{\mu} = \check{\mu}_1 + \check{\mu}_2$ and the corresponding map

$$\mathrm{add}_{\check{\mu}_1,\check{\mu}_2,disj} : \left(X^{\check{\mu}_1}_{\emptyset} \times X^{\check{\mu}_2}_n\right)_{disj} \to X^{\check{\mu}}_n,$$

we must be given an isomorphism

$$\mathrm{add}^*_{\check{\mu}_1,\check{\mu}_2,disj}(\mathcal{L}^{\check{\mu}}_n) \simeq \mathcal{L}^{\check{\mu}_1}_{\emptyset} \boxtimes \mathcal{L}^{\check{\mu}_2}_n, \tag{3.5}$$

compatible with refinements of partitions with respect to the isomorphism (3.4).

A morphisms between two factorizable sheaves ${}^1\mathcal{L}_n = \{{}^1\mathcal{L}^{\check{\mu}}_n\}$ and ${}^2\mathcal{L}_n = \{{}^2\mathcal{L}^{\check{\mu}}_n\}$ is a collection of maps ${}^1\mathcal{L}^{\check{\mu}}_n \to {}^2\mathcal{L}^{\check{\mu}}_n$, compatible with the isomorphisms (3.5).

Let $\overset{\circ}{X}{}^n \overset{j^{poles}}{\hookrightarrow} X^n$ be the complement to the diagonal divisor. By the same token, we define the category $\widetilde{\mathrm{FS}}{}^c_{\overset{\circ}{n}}$. We have a natural restriction functor $(j^{poles})^* : \widetilde{\mathrm{FS}}{}^c_n \to \widetilde{\mathrm{FS}}{}^c_{\overset{\circ}{n}}$ and its right adjoint

$$(j^{poles})_* : \widetilde{\mathrm{FS}}{}^c_{\overset{\circ}{n}} \to \widetilde{\mathrm{FS}}{}^c_n.$$

Let now $\overline{n}$ be a partition $n = n_1 + ... + n_k$, and let $X^k \overset{\Delta_{\overline{n}}}{\to} X^n$ and $\overset{\circ}{X}{}^k \overset{\overset{\circ}{\Delta}_{\overline{n}}}{\to} X^n$ be the corresponding subschemes. We have the natural functors

$$(\Delta_{\overline{n}})_* : \widetilde{\mathrm{FS}}{}^c_k \to \widetilde{\mathrm{FS}}{}^c_n \text{ and } (\overset{\circ}{\Delta}_{\overline{n}})_* : \widetilde{\mathrm{FS}}{}^c_{\overset{\circ}{k}} \to \widetilde{\mathrm{FS}}{}^c_n.$$

The right adjoint functors are easily seen to be defined on the level of derived categories (the latter are understood simply as the derived categories of the corresponding abelian categories)

$$(\Delta_{\overline{n}})^! : D^+(\widetilde{\mathrm{FS}}{}^c_n) \to D^+(\widetilde{\mathrm{FS}}{}^c_k) \text{ and } (\overset{\circ}{\Delta}_{\overline{n}})^! : D^+(\widetilde{\mathrm{FS}}{}^c_n) \to D^+(\widetilde{\mathrm{FS}}{}^c_{\overset{\circ}{k}}),$$

and coincide with the same-named functors on the level of underlying twisted D-modules.

3.6. We shall now introduce a (full, abelian) subcategory $\mathrm{FS}^c_n \subset \widetilde{\mathrm{FS}}{}^c_n$, which will be our main object of study. An object $\mathcal{L}_n \in \widetilde{\mathrm{FS}}{}^c_n$ belongs to FS^c_n if the following two conditions are satisfied:

(i) Finiteness of support: $\mathcal{L}^{\check{\mu}}_n$ is non-zero only for $\check{\mu}$ belonging to finitely many cosets $\check{\Lambda}/\operatorname{Span}(\check{\Delta})$. For each such coset, there exists $\overline{\check{\nu}} = \check{\nu}_1, ..., \check{\nu}_n \in \check{\Lambda}^n$, such that for each $\check{\mu}$ belonging to the above coset, the support of $\mathcal{L}^{\check{\mu}}_n$ is contained in the subscheme $X^{\check{\mu}}_{n,\leq\overline{\check{\nu}}}$.

(ii) To state the second condition, we shall first do it "over" $\overset{\circ}{X}{}^n$, i.e., we will single out the subcategory $\mathrm{FS}^c_{\overset{\circ}{n}}$ inside $\widetilde{\mathrm{FS}}^c_{\overset{\circ}{n}}$.

Our requirement is that there are *only finitely many* collections $(\check{\mu}_1, ..., \check{\mu}_n)$, such that for $\check{\mu} = \underset{i=1,...,n}{\Sigma} \check{\mu}_i$, the singular support of $\mathcal{L}^{\check{\mu}}_{\overset{\circ}{n}}$, viewed as a twisted D-module on

$$X^{\check{\mu}}_{n,\leq\overline{\check{\nu}}} \underset{X^n}{\times} \overset{\circ}{X}{}^n \tag{3.6}$$

(for some/any choice of $\overline{\check{\nu}}$ such that the support condition is satisfied), contains the conormal to the sub-scheme $\overset{\circ}{X}{}^n$, where the latter is embedded into (3.6) by means of $(x_1, ..., x_n) \mapsto \Sigma\, \check{\mu}_i \cdot x_i$. (Note that the above condition is actually a condition on $\check{\mu}$: when the latter is fixed, there are only finitely many partitions $\check{\mu} = \underset{i=1,...,n}{\Sigma} \check{\mu}_i$ with $\check{\mu}_i \leq \check{\nu}_i$ where $\overline{\check{\nu}}$ bounds the support of our sheaf.)

Now, condition (ii) over X^n is that for any partition $n = n_1 + ... + n_k$ each of the cohomologies of $(\overset{\circ}{\Delta}_{\overline{n}})^!(\mathcal{L}_n)$, which is a priori an object of $\widetilde{\mathrm{FS}}^c_{\overset{\circ}{k}}$, belongs in fact to $\mathrm{FS}^c_{\overset{\circ}{k}}$.

3.7. Let us fix the pole points $\overline{x} = (x_1, ..., x_n) \in X^n$, and let $X^{\check{\mu}}_{\overline{x}}$ denote the fiber of $X^{\check{\mu}}_n$ over this configuration. Proceeding as above, we can introduce the categories $\widetilde{\mathrm{FS}}^c_{\overline{x}}$ and $\mathrm{FS}^c_{\overline{x}}$. We shall now describe some special objects in them. With no restriction of generality we can assume that the x_i's are distinct.

Let $\overline{\check{\lambda}} = \check{\lambda}_1, ..., \check{\lambda}_n$ be an n-tuple of elements of $\check{\Lambda}$. For each $\check{\mu}$ consider the corresponding closed subscheme $X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}} := X^{\check{\mu}}_{\overline{x}} \cap X^{\check{\mu}}_{n,\leq\overline{\check{\lambda}}}$. If $\check{\mu} = \underset{i=1,...,n}{\Sigma} \check{\lambda}_i - \underset{\imath\in\mathcal{I}}{\Sigma} m'_\imath \cdot \check{\alpha}_\imath$, then

$$X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}} \simeq \underset{\imath\in\mathcal{I}}{\Pi} X^{(m'_\imath)}. \tag{3.7}$$

Let

$$\overset{\circ}{X}{}^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}} \overset{j^{Diag,poles}}{\hookrightarrow} X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}}$$

be the open subscheme corresponding to divisors of the form

$$\Sigma\, \check{\mu}_k \cdot y_k + \underset{i=1,...,n}{\Sigma} \check{\lambda}_i \cdot x_i, \text{ with } \Sigma\, \check{\mu}_k + \underset{i=1,...,n}{\Sigma} \check{\lambda}_i = \check{\mu}$$

where all the y_k's are pairwise distinct and different from the x_i's, and each $\check{\mu}_k$ is the negative of a simple coroot. I.e., $\overset{\circ}{X}{}^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}}$ is the complement of the diagonal divisor in the product (3.7). Note that the restriction to this subscheme of the line bundle $\mathcal{P}_{X^{\check{\mu}}_n}$ is constant with fiber

$$\bigotimes_{i=1,...,n} \omega_{x_i}^{(\check{\lambda}_i, \check{\lambda}_i + 2\check{\rho})\frac{Kil}{2}}.$$

We define a local system on $\overset{\circ}{X}{}^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}}$, denoted $\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}}}$, as in the case of $\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\emptyset}$, using the product of sign local systems on the factors in (3.7).

Let $X^{\check{\mu}}_{\overline{x},=\overline{\check{\lambda}}}$ be the open subset of $X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}}$, corresponding to divisors of the form $\Sigma\, \check{\mu}_k \cdot y_k + \underset{i=1,...,n}{\Sigma} \check{\lambda}_i \cdot x_i$ with $y_k \neq x_i$. We have the corresponding open embeddings:

$$\overset{\circ}{X}{}^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}} \overset{'j^{Diag,poles}}{\hookrightarrow} X^{\check{\mu}}_{\overline{x},=\overline{\check{\lambda}}} \overset{''j^{Diag,poles}}{\hookrightarrow} X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\lambda}}}$$

with $j^{Diag,poles} = {}'' j^{Diag,poles} \circ {}' j^{Diag,poles}$.

We let

$$\mathcal{L}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}},!} := {}'' j^{Diag,poles}_{!} \circ {}' j^{Diag,poles}_{!*}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}}}), \quad \mathcal{L}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}},*} := {}'' j^{Diag,poles}_{*} \circ {}' j^{Diag,poles}_{!*}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}}})$$

$$\text{and } \mathcal{L}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}},!*} := {}'' j^{Diag,poles}_{!*} \circ {}' j^{Diag,poles}_{!*}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}}}) \simeq j^{Diag,poles}_{!*}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}}}).$$

The collections $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!} := \{\mathcal{L}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}},!}\}$, $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!*} := \{\mathcal{L}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}},!*}\}$, $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},*} := \{\mathcal{L}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}},*}\}$ are naturally objects of $\widetilde{\mathrm{FS}}^c_{\overline{x}}$.

Points (a) and (c) of the following lemma essentially results from the definitions, whereas point (b) follows from [BFS] (see Sect. 3.13 below).

Lemma 3.8.
(a) *The objects $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!*}$ are the irreducibles of $\widetilde{\mathrm{FS}}^c_{\overline{x}}$.*
(b) *The objects $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!}$ and $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},*}$ are of finite length.*
(c) *The cones of the natural maps*

$$\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!} \to \mathcal{L}_{\overline{x},\overline{\check{\lambda}},*!} \to \mathcal{L}_{\overline{x},\overline{\check{\lambda}},*}$$

are extensions of objects of the form $\mathcal{L}_{\overline{x},\overline{\check{\lambda}}',!*}$ for $\overline{\check{\lambda}}' \leq \overline{\check{\lambda}}$.

Finally, we will use the following result, which also follows from [BFS] (see Sect. 3.13 below):

Theorem 3.9.
(a) Assume that c is rational. Then all the objects $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!}, \mathcal{L}_{\overline{x},\overline{\check{\lambda}},*!}$ and $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},*}$ belong to $\mathrm{FS}^c_{\overline{x}}$.
(b) Assume that c is irrational. Then

- *(i) The objects $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!}$ and $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},*}$ never belong to $\mathrm{FS}^c_{\overline{x}}$.*
- *(ii) An object $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!*}$ belongs to $\mathrm{FS}^c_{\overline{x}}$ if and only if all $\check{\lambda}_i$ are dominant.*
- *(iii) The category $\mathrm{FS}^c_{\overline{x}}$ is semi-simple.*

3.10. We can now formulate our main theorem:

Theorem 3.11. *Let c be irrational. Then there exists an equivalence of abelian categories*

$$\mathrm{Whit}^c_n \to \mathrm{FS}^c_n\,.$$

3.12. Let us now explain the relation between our set-up and that of [BFS]. The difference is that in *loc. cit.* the authors work with D-modules on the spaces $X^{\check{\mu}}_{n,\leq\overline{\check{\nu}}}$, rather than with twisted D-modules.

Let us again fix an n-tuple of distinct points $x_1, ..., x_n$ and identify the line bundle $\mathcal{P}_{X^{\check{\mu}}_n}|_{X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\nu}}}}$ explicitly. Let

$$\check{\mu} - \sum_{i=1,\dots,n} \check{\nu}_i = -\sum_{\imath} m_\imath \cdot \check{\alpha}_\imath$$

The scheme $X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\nu}}}$ can be identified with the corresponding product of symmetric powers, $\prod_{\imath} X^{(m_\imath)}$.

The line bundle $\mathcal{P}_{X^{\check{\mu}}}|_{X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\nu}}}}$ is then

$$\mathcal{O}_{X^{\check{\mu}}_{\overline{x},\leq\overline{\check{\nu}}}}\left(-\sum_{\imath} d_\imath \cdot \Delta_\imath - \sum_{\imath_1\neq\imath_2} d_{\imath_1,\imath_2} \cdot \Delta_{\imath_1,\imath_2} - \sum_{\imath,j=1,\dots,n} d_{\imath,j} \cdot \Delta_{\imath,j}\right),$$

where $\Delta_\imath$ is the diagonal divisor on $X^{(m_\imath)}$, $\Delta_{\imath_1,\imath_2}$ is the incidence divisor of $X^{(m_{\imath_1})}\times X^{(m_{\imath_2})}$, $\Delta_{\imath,j}$ the incidence divisor on $X^{(m_\imath)}\times x_j$, and

$$d_\imath=(\check{\alpha}_\imath,\check{\alpha}_\imath)_{\frac{Kil}{2}},\ d_{\imath_1,\imath_2}=(\alpha_{\imath_1},\alpha_{\imath_2})_{Kil},\ d_{\imath,j}=(\alpha_\imath,\check{\nu}_j)_{Kil}.$$

Let us assume now that our curve X is $\mathbb{A}^1$ (as in [BFS]), with coordinate t. We will denote by $t^\imath_1,...,t^\imath_{m_\imath}$ the corresponding functions on $X^{m_\imath}$. The function

$$\mathsf{f}^{\check{\mu}}_{\overline{\nu}}:=\prod_\imath\left(\prod_{1\le k_1,k_2\le m_\imath}(t^\imath_{k_1}-t^\imath_{k_2})\right)^{2d_\imath}\cdot\prod_{\imath_1\neq\imath_2}\left(\prod_{1\le k_1\le m_{\imath_1},1\le k_2\le m_{\imath_2}}(t^{\imath_1}_{k_1}-t^{\imath_2}_{k_2})\right)^{d_{\imath_1,\imath_2}}\cdot$$
$$\cdot\prod_{\imath,j=1,\dots,n}\left(\prod_{1\le k\le m_\imath}(t^k_\imath-x_j)\right)^{d_{\imath,j}}$$

on $X^{\check{\mu}}_{\overline{x},\le\overline{\nu}}$ trivializes the line bundle in question. This allows to view the twisted D-modules $\mathcal{L}^{\check{\mu}}_{\overline{x}}$ comprising an object $\mathcal{L}_{\overline{x}}\in\widetilde{\mathrm{FS}}^c$ as plain D-modules.

However, the definition of the standard object, such as $\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\overline{x},\overline{\lambda}}$ is more complicated. The latter equals the product of the sign local system and the pull-back by means of the map

$$\overset{\circ}{X}{}^{\check{\mu}}_{\overline{x},\le\overline{\nu}}\overset{\mathsf{f}^{\check{\mu}}_{\overline{\nu}}}{\longrightarrow}\mathbb{G}_m$$

of the D-module $\Psi(c)$ on $\mathbb{G}_m$. Here $\Psi(c)$ is the Kummer D-module, generated by one section "z^c" and satisfying the relation

$$z\partial_z\cdot\text{"}z^c\text{"}=c\cdot\text{"}z^c\text{"},\tag{3.8}$$

where z is a coordinate on $\mathbb{G}_m$.

3.13. This subsection is included in order to navigate the reader in the structure of the proofs of results from [BFS] that are used in this paper.

To simplify the notation, let as assume that $n=1$, i.e., $\overline{x}=\{x\}$. The main hard result that we use is that the the objects $\mathcal{L}_{x,\check{\lambda},!*}$ for $\check{\lambda}\in\check{\Lambda}^+$ belong to FS^c_x, i.e., they satisfy the singular support condition. This is established in Theorem II.8.18 of *loc. cit.* This theorem amounts to an explicit calculation of vanishing cycles.

In fact, that theorem says that for any $\check{\lambda}\in\check{\Lambda}$ and c, the cotangent space to the point $\check{\mu}\cdot x\in X^{\check{\mu}}_x$ belongs to the singular support of $\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*}$ (resp., $\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!}$, $\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},*}$) if and only if $\check{\mu}$ appears as a weight of the irreducible module (resp., Verma module, dual Verma module) of highest weight $\check{\lambda}$ over the corresponding quantum group.

This implies point (b) of Lemma 3.8, as well as points (a), (b,i), (b,ii) of Theorem 3.9.

Let us now comment on how to deduce that FS^c_x is semi-simple for c irrational (we will just copy the argument from loc. cit. Sect. III.18). The proof relies on the following (Lemma III.5.3 of loc. cit.):

Lemma 3.14. *Assume that both $\mathcal{L}_1,\mathcal{L}_2,\in\mathrm{FS}^c_x$ are supported on $X^{\check{\mu}}_x$ for $\check{\mu}$ belonging to a single coset in $\check{\Lambda}/\operatorname{Span}(\check{\Delta})$. Then there exists $\check{\mu}_0$ such that for all $\check{\mu}\ge\check{\mu}_0$ the map*

$$\mathrm{Hom}_{\mathrm{FS}^c_x}(\mathcal{L}_1,\mathcal{L}_2)\to\mathrm{Hom}_{\mathfrak{D}\text{-}mod^c(X^{\check{\mu}}_x)}(\mathcal{L}^{\check{\mu}}_1,\mathcal{L}^{\check{\mu}}_2)$$

is an isomorphism.

The proof of this lemma is not difficult, but we emphasize that it uses condition (ii) that singles out FS^c_x inside $\widetilde{\mathrm{FS}}^c_x$ in an essential way.

To prove the semi-simplicity of FS^c_x we have to show that for any $\check{\lambda}_1, \check{\lambda}_2 \in \check{\Lambda}^+$,

$$Ext^1_{\mathrm{FS}^c_x}(\mathcal{L}_{x,\check{\lambda}_1,!*}, \mathcal{L}_{x,\check{\lambda}_2,!*}) = 0.$$

We distinguish two cases.

Case 1: $\check{\lambda}_1 = \check{\lambda}_2 =: \check{\lambda}$. An extension class gives rise to an extension of twisted D-modules over $X^{\check{\mu}}_{x \leq \check{\lambda}}$.

$$0 \to \mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*} \to \mathcal{M}^{\check{\mu}} \to \mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*} \to 0.$$

Applying Lemma 3.14, we obtain that for all $\check{\mu}$ that are sufficiently large, this extension is *non-split*. Restricting this extension to the open subscheme $\overset{\circ}{X}{}^{\check{\mu}}_{x,\leq\check{\lambda}}$, we obtain a non-trivial extension of the corresponding local systems. This, however, contradicts the factorization property.

Case 2: $\check{\lambda}_1 \neq \check{\lambda}_2$. We will show that

$$Ext^1_{\mathfrak{D}\text{-mod}^c(X^{\check{\mu}}_x)}(\mathcal{L}^{\check{\mu}}_{x,\check{\lambda}_1,!*}, \mathcal{L}^{\check{\mu}}_{x,\check{\lambda}_2,!*}) = 0,$$

which would imply our assertion in view of Lemma 3.14.

Since the situation is essentially Verdier self-dual, we can assume that $\check{\mu}_2 \geq \check{\mu}_1$. The object $\mathcal{L}^{\check{\mu}}_{x,\check{\lambda}_1,!*}$ is supported on the closed sub-scheme $X^{\check{\mu}}_{x,\leq\check{\lambda}_1}$. We claim that both !- and $*$- restrictions of $\mathcal{L}^{\check{\mu}}_{x,\check{\lambda}_2,!*}$ to this subscheme are zero.

By factorization, it is enough to prove the latter assertion for $\check{\mu} = \check{\lambda}_1$, in which case $X^{\check{\mu}}_{x,\leq\check{\lambda}_1} =$ pt. The assertion is local, so we can assume that $(x \in X) \simeq (0 \in \mathbb{A}^1)$, and pass from twisted D-modules to usual D-modules, as in Sect. 3.12. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^1$, and hence on $(\mathbb{A}_1)^{\check{\mu}_1}_{0,\leq\check{\lambda}_2}$ We obtain that $\mathcal{L}^{\check{\mu}}_{x,\check{\lambda}_2,!*}$ is monodromic against the character sheaf $\Psi(c')$, where

$$c' = c \cdot \left((\check{\lambda}_2, \check{\lambda}_2 + 2\check{\rho})_{\frac{Kil}{2}} - (\check{\lambda}_1, \check{\lambda}_1 + 2\check{\rho})_{\frac{Kil}{2}} \right).$$

However, the integer $(\check{\lambda}_2, \check{\lambda}_2 + 2\check{\rho})_{\frac{Kil}{2}} - (\check{\lambda}_1, \check{\lambda}_1 + 2\check{\rho})_{\frac{Kil}{2}}$ is non-zero, and since $c \notin \mathbb{Q}$, we obtain that $c' \notin \mathbb{Z}$. Hence, the stalk and co-stalk of this D-module at 0 is zero.

4. Zastava spaces

4.1. In order to prove Theorem 3.11 we have to be able to pass from (twisted) D-modules on $\mathfrak{W}_n$ to (twisted) D-modules on $X^{\check{\mu}}_n$. This will be done using ind-schemes $\mathcal{Z}^{\check{\mu}}_n$ (defined for each $\check{\mu} \in \check{\Lambda}$), that map to both $\mathfrak{W}_n$ and $X^{\check{\mu}}_n$, and that are called "Zastava spaces".

For a coweight $\check{\mu}$ let $\mathrm{Bun}^{\check{\mu}}_{B^-}$ denote the stack of B^--bundles of degree $(2g-2)\check{\rho} - \check{\mu}$. We will think of a point of $\mathrm{Bun}^{\check{\mu}}_{B^-}$ as a triple:

- A G-bundle $\mathfrak{F}_G$.
- A T-bundle $\mathfrak{F}_T$ such that for any $\lambda \in \Lambda$ the degree of the corresponding line bundle $\lambda(\mathfrak{F}_T)$ is $\langle \lambda, (2g-2)\check{\rho} - \check{\mu} \rangle$.
- A collection of surjective bundle maps

$$\kappa^{\lambda,-} : \mathcal{V}^\lambda_{\mathfrak{F}_G} \to \lambda(\mathfrak{F}_T),$$

which satisfy the Plücker equations.

Let $\mathfrak{p}^-$ and $\mathfrak{q}^-$ denote the natural maps

$$\mathrm{Bun}_G \leftarrow \mathrm{Bun}^{\check{\mu}}_{B^-} \to \mathrm{Bun}_T,$$

respectively. Let $\mathcal{P}_{\mathrm{Bun}^{\check{\mu}}_{B^-}}$ denote the line bundle $\mathfrak{p}^{-*}(\mathcal{P}_{\mathrm{Bun}_G})$. Almost by definition we have:

Lemma 4.2. *The line bundle $\mathcal{P}_{\mathrm{Bun}^{\check{\mu}}_{B^-}}$ is isomorphic to the pull-back under $\mathfrak{q}^-$ of the line bundle $\mathcal{P}_{\mathrm{Bun}_T}$ over Bun_T (see Sect. 3.2).*

4.3. We let $\mathcal{Z}^{\check{\mu}}_n$ denote the open sub-stack in the product $\mathfrak{W}_n \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}^{\check{\mu}}_{B^-}$ that corresponds to the condition that the composed (meromorphic) maps

$$\omega^{\langle \lambda, \check{\rho}\rangle} \overset{\kappa^\lambda}{\to} \mathcal{V}^\lambda_{\mathfrak{F}_G} \overset{\kappa^{\lambda,-}}{\to} \lambda(\mathfrak{F}_T) \tag{4.1}$$

are non-zero. Let $'\mathfrak{p}^-$, $'\mathfrak{p}$ denote the projections $\mathcal{Z}^{\check{\mu}}_n \to \mathfrak{W}_n$ and $\mathcal{Z}^{\check{\mu}}_n \to \mathrm{Bun}^{\check{\mu}}_{B^-}$, respectively.

Taking the zeroes/poles of the maps (4.1), we obtain a natural map

$$\pi^{\check{\mu}} : \mathcal{Z}^{\check{\mu}}_n \to X^{\check{\mu}}_n.$$

4.4. The next three assertions repeat [BFG], Sect. 2.16 (see also [BFGM], Sect. 2 for a less abstract treatment):

Proposition 4.5. *Let $\{\overline{x}, \mathfrak{F}_G, \mathfrak{F}_T, (\kappa^\lambda), (\kappa^{\lambda,-})\}$ be a point of $\mathcal{Z}^{\check{\mu}}_n$, and let $D \in X^{\check{\mu}}_n$ be its image under $\pi^{\check{\mu}}$. Then the restriction of $\mathfrak{F}_G$ to the open curve $X - \mathrm{supp}(D)$ is canonically isomorphic to $\mathfrak{F}_G = \omega^{\check{\rho}} \overset{T}{\times} G$, with the tautological maps $\kappa^\lambda, \kappa^{\lambda,-}$.*

Proof. This is just the fact that the stack $N\backslash \overset{\circ}{G}/B^-$ is isomorphic to pt, where $\overset{\circ}{G}$ denotes the open Bruhat cell in G. □

Corollary 4.6. *The (ind)-stack $Z^{\check{\mu}}_n$ is in fact an (ind)-scheme.*

Proposition 4.7. *For $\check{\mu} = \check{\mu}' + \check{\mu}''$ there exists a canonical isomorphism of stacks*

$$\left(X^{\check{\mu}'}_\emptyset \times X^{\check{\mu}''}_n\right)_{disj} \underset{X^{\check{\mu}}_n}{\times} \mathcal{Z}^{\check{\mu}}_n \simeq \left(X^{\check{\mu}'}_\emptyset \times X^{\check{\mu}''}_n\right)_{disj} \underset{(X^{\check{\mu}'}_\emptyset \times X^{\check{\mu}''}_n)}{\times} \left(\mathcal{Z}^{\check{\mu}'}_\emptyset \times \mathcal{Z}^{\check{\mu}''}_n\right). \tag{4.2}$$

Proof. Let D' and D'' be points of $X^{\check{\mu}'}_\emptyset$ and $X^{\check{\mu}''}_n$, respectively, with disjoint supports.

Objects classified by both the LHS and the RHS in (4.2) are local in X. Thus, we can think of the LHS as defining a certain data on $X - \mathrm{supp}(D')$ and $X - \mathrm{supp}(D'')$ separately, with a gluing datum over $X - (\mathrm{supp}(D') \cup \mathrm{supp}(D''))$. We have to show that the gluing datum in question is in fact superfluous, but this follows immediately from Proposition 4.5.

□

4.8. Let us make the following observation:

Lemma 4.9. *The line bundle $'\mathfrak{p}^{-*}(\mathcal{P}_{\mathfrak{W}_n}) \simeq {'\mathfrak{p}}^*(\mathcal{P}^{\otimes -1}_{\mathrm{Bun}^{\check{\mu}}_{B^-}})$ identifies canonically with $\pi^{\check{\mu}*}(\mathcal{P}_{X^{\check{\mu}}_n})$.*

Proof. This follows from the fact that the diagram

$$\begin{array}{ccc} \mathcal{Z}^{\check{\mu}}_n & \xrightarrow{'\mathfrak{p}} & \mathrm{Bun}^{\check{\mu}}_{B^-} \\ {\scriptstyle \pi^{\check{\mu}}} \downarrow & & \downarrow {\scriptstyle \mathfrak{q}^-} \\ X^{\check{\mu}}_n & \xrightarrow{AJ} & \mathrm{Bun}_T \end{array}$$

commutes, where the lower horizontal arrow is the Abel-Jacobi map of Sect. 3.2, i.e, it sends a divisor D to the T-bundle $\omega^{\check{\rho}}(-D)$. □

This allows us to define the functors

$$D(\mathfrak{D}^c\text{-mod}(\mathfrak{W}_n)) \to D(\mathfrak{D}\text{-mod}^c(X_n^{\check{\mu}})).$$

Let us denote by $'\mathfrak{p}^{-,\check{\mu},\cdot}$ the functor $D(\mathfrak{D}\text{-mod}^c(\mathfrak{W}_n)) \to D(\mathfrak{D}\text{-mod}^c(\mathcal{Z}_n^{\check{\mu}}))$ given by

$$\mathcal{F} \mapsto ('\mathfrak{p}^-)^!(\mathcal{F})[-\dim.\mathrm{rel.}(\mathrm{Bun}_{B^-}^{\check{\mu}}, \mathrm{Bun}_G)].$$

(We note that this functor *essentially* commutes with Verdier duality, since the morphism $\mathfrak{p}^- : \mathrm{Bun}_{B^-}^{\check{\mu}} \to \mathrm{Bun}_G$ is smooth for $\check{\mu}$ such $\langle \alpha, \check{\mu} \rangle < -(2g-2)$, which is what we will be able to assume in practice.)

Thus, we can consider the functor

$$\pi_*^{\check{\mu}} \circ {'\mathfrak{p}^{-,\check{\mu},\cdot}} : D(\mathfrak{D}\text{-mod}^c(\mathfrak{W}_n)) \to D(\mathfrak{D}\text{-mod}^c(X_n^{\check{\mu}})).$$

We will prove:

Proposition 4.10. *For all c and $\mathcal{F} \in \mathrm{Whit}_n^c$ the object $\pi_*^{\check{\mu}}({'\mathfrak{p}^{-,\check{\mu},\cdot}})(\mathcal{F}))$ is concentrated in the cohomological degree 0.*

In addition, we will prove the following assertion:

Theorem 4.11. *For c irrational and any $\mathcal{F} \in \mathrm{Whit}_n^c$, the object*

$$\pi_!^{\check{\mu}}({'\mathfrak{p}^{-,\check{\mu},\cdot}})(\mathcal{F})) \in D(\mathfrak{D}\text{-}mod^c(X_n^{\check{\mu}}))$$

is well-defined, [8] *and the natural morphism*

$$\pi_!^{\check{\mu}}({'\mathfrak{p}^{-,\check{\mu},\cdot}})(\mathcal{F})) \to \pi_*^{\check{\mu}}({'\mathfrak{p}^{-,\check{\mu},\cdot}}(\mathcal{F}))$$

is an isomorphism.

We emphasize that assertion of Theorem 4.11 is false without the assumption that c be irrational.

Remark. The morphism $\pi^{\check{\mu}}$ is affine, so for a (twisted) D-module $\mathcal{F}'$ on $\mathcal{Z}_n^{\check{\mu}}$, the object of the derived category given by $\pi_*^{\check{\mu}}(\mathcal{F}')$ lives in non-positive cohomological degrees, and the object $\pi_!^{\check{\mu}}(\mathcal{F}')$ lives in non-negative cohomological degrees. Hence, Theorem 4.11 formally implies Proposition 4.10. Nonetheless, we will give an independent proof of this proposition, because it holds without the assumption that c be irrational.

4.12. Our present goal is to establish a key factorization property of the D-modules on $\mathcal{Z}_n^{\check{\mu}}$ that are obtained from objects of Whit_n^c by means of $'\mathfrak{p}^{-,\check{\mu},\cdot}$:

Proposition 4.13. *For $\mathcal{F} \in \mathrm{Whit}_n^c$ and $\check{\mu} = \check{\mu}_1 + \check{\mu}_2$, under the isomorphism of (4.2), the D-module*

$$\mathrm{add}^*_{\check{\mu}_1,\check{\mu}_2,disj}\left({'\mathfrak{p}^{-,\check{\mu},\cdot}}(\mathcal{F})\right) \in \mathfrak{D}\text{-}mod^c\left(\left(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2}\right)_{disj} \underset{X_n^{\check{\mu}}}{\times} \mathcal{Z}_n^{\check{\mu}}\right)$$

goes over to

$$'\mathfrak{p}^{-,\check{\mu}_1,\cdot}(\mathcal{F}_\emptyset) \boxtimes {'\mathfrak{p}^{-,\check{\mu}_2,\cdot}}(\mathcal{F}) \in \mathfrak{D}\text{-}mod^c\left(\left(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2}\right)_{disj} \underset{(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2})}{\times} \left(\mathcal{Z}_\emptyset^{\check{\mu}_1} \times \mathcal{Z}_n^{\check{\mu}_2}\right)\right).$$

These isomorphisms are compatible with refinements of factorizations.

[8] The issue here is that the direct image with compact supports in not a priori defined on non-holonomic D-modules.

Proof. Let us consider the following relative version of the stack $(\mathfrak{W}_n)_{\text{good at } \overline{y}}$, introduced in Sect. 2.3. Namely, let $(\mathfrak{W}_n)_{\text{good at } \check{\mu}_1}$ be the open substack of $X_\emptyset^{\check{\mu}_1} \times \mathfrak{W}_n$, where a divisor $D \in X_\emptyset^{\check{\mu}_1}$ is forbidden to hit the pole points $(x_1, ..., x_n) \in X^n$, and the κ^λ's are bundle maps on a neighbourhood of supp(D).

Over $X_\emptyset^{\check{\mu}_1}$ we have a group-scheme, denoted $\mathcal{N}_{\check{\mu}_1}^{\text{reg}}$, and a group ind-scheme $\mathcal{N}_{\check{\mu}_1}^{\text{mer}}$; the latter is endowed with a character $\chi_{\check{\mu}_1} : \mathcal{N}_{\check{\mu}_1}^{\text{mer}} \to \mathbb{G}_a$. Over $(\mathfrak{W}_n)_{\text{good at } \check{\mu}_1}$ there is a $\mathcal{N}_{\check{\mu}_1}^{\text{reg}}$-torsor, denoted ${}_{\check{\mu}_1}\mathfrak{W}_n$. The total space of this torsor is acted on by $\mathcal{N}_{\check{\mu}_1}^{\text{mer}}$.

Consider the action map

$$\text{act}_{\check{\mu}_1} : \mathcal{N}_{\check{\mu}_1}^{\text{mer}} \overset{\mathcal{N}_{\check{\mu}_1}^{\text{reg}}}{\times} {}_{\check{\mu}_1}\mathfrak{W}_n \to (\mathfrak{W}_n)_{\text{good at } \check{\mu}_1}.$$

From the definition of the Whittaker category it follows that for any $\mathcal{F} \in \text{Whit}_n$, we have:

$$\text{act}_{\check{\mu}_1}^*(\mathcal{F}) \simeq \chi_{\check{\mu}_1}^*(exp) \boxtimes \mathcal{F}.$$

The pre-image of $(\mathfrak{W}_n)_{\text{good at } \check{\mu}_1}$ under the map

$$\left(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2}\right)_{disj} \underset{X_n^{\check{\mu}}}{\times} \mathcal{Z}_n^{\check{\mu}} \overset{'\mathfrak{p}^-}{\longrightarrow} X_\emptyset^{\check{\mu}_1} \times \mathfrak{W}_n$$

goes over under the isomorphism (4.2) to the substack

$$\left(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2}\right)_{disj} \underset{(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2})}{\times} \left(\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check{\mu}_1} \times \mathcal{Z}_n^{\check{\mu}_1}\right), \tag{4.3}$$

where $\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check{\mu}_1} = \mathcal{Z}_\emptyset^{\check{\mu}_1} \underset{\mathfrak{W}_\emptyset}{\times} \mathfrak{W}_{\emptyset,0}$.

Note that by construction, we have a locally closed embedding of schemes over $X_\emptyset^{\check{\mu}_1}$

$$\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check{\mu}_1} \to \mathcal{N}_{\check{\mu}_1}^{\text{mer}} / \mathcal{N}_{\check{\mu}_1}^{\text{reg}},$$

such that the pull-back of $\chi_{\check{\mu}_1}^*(exp)$ identifies with the restriction of $'\mathfrak{p}^{-,\check{\mu}_1,\cdot}(\mathcal{F}_\emptyset)$ to this subscheme.

For $\mathcal{F} \in \text{Whit}_n^c$, its pull-back onto the product

$$(X_\emptyset^{\check{\mu}_1} \times \mathfrak{W}_n) \underset{X_\emptyset^{\check{\mu}_1} \times X^n}{\times} (X_\emptyset^{\check{\mu}_1} \times X^n)_{disj}$$

is the extension by * (and also by !) from $(\mathfrak{W}_n)_{\text{good at } \check{\mu}_1}$. Hence, it it sufficient to establish an isomorphism of twisted D-modules over the open sub-stack appearing in (4.3).

The assertion of the proposition follows now from the fact that the composition

$$\left(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2}\right)_{disj} \underset{(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2})}{\times} \left(\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check{\mu}_1} \times \mathcal{Z}_n^{\check{\mu}_2}\right) \to \left(X_\emptyset^{\check{\mu}_1} \times X_n^{\check{\mu}_2}\right)_{disj} \underset{X_n^{\check{\mu}}}{\times} \mathcal{Z}_n^{\check{\mu}} \to X_\emptyset^{\check{\mu}_1} \times \mathfrak{W}_n$$

factors as

$$\left(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2}\right)_{disj}\underset{(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2})}{\times}\left(\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check\mu_1}\times\mathcal{Z}_n^{\check\mu_2}\right)\to$$

$$\to\left(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2}\right)_{disj}\underset{(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2})}{\times}\left(\mathcal{N}^{\mathrm{mer}}_{\check\mu_1}/\mathcal{N}^{\mathrm{reg}}_{\check\mu_1}\times\mathcal{Z}_n^{\check\mu_2}\right)\simeq$$

$$\simeq\left(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2}\right)_{disj}\underset{(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2})}{\times}\left(\mathcal{N}^{\mathrm{mer}}_{\check\mu_1}\overset{\mathcal{N}^{\mathrm{reg}}_{\check\mu_1}}{\times}\left({}_{\check\mu_1}\mathfrak{W}_n\underset{\mathfrak{W}_n}{\times}\mathcal{Z}_n^{\check\mu_2}\right)\right)\to$$

$$\to\mathcal{N}^{\mathrm{mer}}_{\check\mu_1}\overset{\mathcal{N}^{\mathrm{reg}}_{\check\mu_1}}{\times}{}_{\check\mu_1}\mathfrak{W}_n\overset{\mathrm{act}_{\check\mu_1}}{\to}{}_{\check\mu_1}\mathfrak{W}_n\to(\mathfrak{W}_n)_{\text{good at }\check\mu_1}\hookrightarrow X_\emptyset^{\check\mu_1}\times\mathfrak{W}_n,$$

where the second arrow is the isomorphism, following from the trivialization of the $\mathcal{N}^{\mathrm{reg}}_{\check\mu_1}$-torsor

$$({}_{\check\mu_1}\mathfrak{W}_n\underset{\mathfrak{W}_n}{\times}\mathcal{Z}_n^{\check\mu_2})\underset{X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2}}{\times}\left(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2}\right)_{disj},$$

see Proposition 4.5.

□

4.14. As a corollary of the above proposition, we obtain:

Corollary 4.15. *For $\mathcal{F}\in\mathrm{Whit}_n^c$ and $\check\mu=\check\mu_1+\check\mu_2$, we obtain:*

$$\mathrm{add}^*_{\check\mu_1,\check\mu_2,disj}\left(\pi_*^{\check\mu}('\mathfrak{p}^{-,\check\mu,\cdot}(\mathcal{F}))\right)\simeq\pi_*^{\check\mu_1}('\mathfrak{p}^{-,\check\mu_1,\cdot}(\mathcal{F}_\emptyset))\boxtimes\pi_*^{\check\mu_2}('\mathfrak{p}^{-,\check\mu_2,\cdot}(\mathcal{F}))$$

as objects of $\mathfrak{D}\text{-mod}^c\left(X_\emptyset^{\check\mu_1}\times X_n^{\check\mu_2}\right)_{disj}$. These isomorphisms are compatible with refinements of partitions.

The next step is to analyze the object

$$'\mathcal{L}_\emptyset^{\check\mu}:=\pi_*^{\check\mu}('\mathfrak{p}^{-,\check\mu,\cdot}(\mathcal{F}_\emptyset))\in\mathfrak{D}\text{-mod}^c(X_\emptyset^{\check\mu}).\tag{4.4}$$

We shall prove:

Theorem 4.16.

(1) If $(\check\alpha_\imath,\check\alpha_\imath)_c\notin\mathbb{Z}$ for any $\imath\in\mathcal{I}$, then we a canonical isomorphism $'\mathcal{L}_\emptyset^{\check\mu}\simeq\mathcal{L}_\emptyset^{\check\mu}$ over $\overset{\circ}{X}{}_\emptyset^{\check\mu}$.

(2) If c is irrational, the above isomorphism holds over $X_\emptyset^{\check\mu}$.

Both isomorphisms are compatible with the factorization isomorphisms.

5. Proofs–A

5.1. **Proof of Theorem 4.16(1).** Let us first assume that $\check\mu$ equals the negative of a simple co-root $\check\alpha_\imath$. The scheme $\mathcal{Z}_\emptyset^{\check\alpha_\imath}$ identifies canonically with $X\times\mathbb{G}_a$, and $\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check\alpha_\imath}$ is the complement to the zero section, corresponding to $0\in\mathbb{G}_a$.

Recall that the line bundle $\mathcal{P}_{X_\emptyset^{\check\alpha_\imath}}$ is canonically trivial. However, over $\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check\alpha_\imath}$ we have two trivializations of the the corresponding line bundle: one inherited from that on $X_\emptyset^{\check\alpha_\imath}$, and the other from that on $\mathfrak{W}_{\emptyset,0}$. The discrepancy is given by the map

$$\overset{\circ}{\mathcal{Z}}{}_\emptyset^{\check\alpha_\imath}\simeq X\times\mathbb{G}_m\twoheadrightarrow\mathbb{G}_m\overset{x\mapsto x^{2\cdot d_\imath}}{\longrightarrow}\mathbb{G}_m,$$

where $d_\imath$ is as in Sect. 3.12.

Thus, $\pi_*^{\check{\mu}}('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_\emptyset))$ is equal to the constant D-module on X times the vector space

$$H\left(\mathbb{G}_m, \Psi(2\cdot d_\imath \cdot c)\otimes \exp\right),$$

where $\Psi(\cdot)$ is the Kummer D-module as in (3.8).

Now, the "Gauss sum" formula, i.e., the canonical isomorphism

$$H(\mathbb{G}_m, \Psi(2\cdot d_\imath \cdot c)\otimes \exp)\simeq \mathbb{C}$$

is well-known from the theory of Fourier-Deligne transform.

Let us now assume that $\check{\mu}$ is arbitrary. Corollary 4.15 together with the above computation, imply that the required isomorphism holds after the pull-back to $\left(\prod\limits_\imath X^{m_\imath}\right)$, away from the diagonal divisor, where $\check{\mu} = -\sum\limits_\imath m_\imath \cdot \alpha_\imath$.

We only have to show that the action of the symmetric group $\prod\limits_\imath \Sigma_{m_\imath}$ on the above constant sheaf is given by *sign* character. But this follows from the Künneth formula, as the cohomology $H(\mathbb{G}_m, \Psi(2\cdot d_\imath\cdot c)\otimes\exp)$ is concentrated in degree 0 and $\dim(\mathbb{G}_m)=1$.

□

5.2. **Proof of Proposition 4.10.** Let us first show that for $\mathcal{F}\in \mathrm{Whit}_n$, the object

$$\pi_*^{\check{\mu}}\circ {'\mathfrak{p}}^{-,\check{\mu},\cdot}(\mathcal{F})$$

is concentrated in non-positive degrees.

For that we can assume that $\check{\mu}$ satisfies $\langle \alpha,\check{\mu}\rangle < -(2g-2)$. Indeed, if not, we can replace $\check{\mu}$ by $\check{\mu}_1 = \check{\mu} - k\cdot\check{\rho}$ with k large enough, and then apply Corollary 4.15 combined with Theorem 4.16(1) to factor the extra points away.

For $\check{\mu}$ as above the map $\mathfrak{p}^- : \mathrm{Bun}_{B^-}^{\check{\mu}} \to \mathrm{Bun}_G$ is smooth, and hence, so is the map $'\mathfrak{p}^- : \mathcal{Z}_n^{\check{\mu}} \to \mathfrak{W}_n$. Hence, $'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})$ lives in cohomological degree 0. Hence, our assertion follows from the fact that the morphism $\pi^{\check{\mu}}$ is affine (see [BFGM], Sect. 5.1 for the proof of the latter fact).

To prove that $\pi_*^{\check{\mu}}\circ {'\mathfrak{p}}^{-,\check{\mu},\cdot}(\mathcal{F})$ is concentrated in non-negative degrees we need to analyze the fibers of the map $\pi^{\check{\mu}}$.

5.3. **Analysis of the fibers.** Let us denote by $\mathcal{Z}^{\check{\mu}}_{loc,x}$ the fiber of $\mathcal{Z}^{\check{\mu}}_1$ over the point $\check{\mu}\cdot x \in X_1^{\check{\mu}}$. For $\check{\lambda}\in\check{\Lambda}$, let us denote by $\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x}$ the pre-image in $\mathcal{Z}^{\check{\mu}}_{loc,x}$ of the sub-stack $\mathfrak{W}_{x,\check{\lambda}}$. Let $\mathcal{P}_{\mathcal{Z}^{\check{\mu}}_{loc,x}}$ and $\mathcal{P}_{\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x}}$ denote the corresponding line bundles, obtained by restriction from $\mathcal{P}_{\mathcal{Z}_1^{\check{\mu}}}$.

As is shown in [BFGM], Sect. 2.6, $\mathcal{Z}^{\check{\mu}}_{loc,x}$ identifies with a closed sub-indscheme of the affine Grassmannian $\mathrm{Gr}_{G,x} = G(\mathcal{K}_x)/G(\mathcal{O}_x)$.

Let $S^{\check{\lambda}}$ denote the $N(\mathcal{K}_x)$-orbit of the point $t^{\check{\lambda}}$, [9] and let $S^{-,\check{\mu}}$ denote the $N^-(\mathcal{K}_x)$-orbit of the point $t^{\check{\mu}}$. Then

$$\mathcal{Z}^{\check{\mu}}_{loc,x} \simeq S^{-,\check{\mu}} \overset{T(\mathcal{O}_x)}{\times} \omega^{\check{\rho}}|_{\mathcal{D}_x} \text{ and } \mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x} \simeq \left(S^{\check{\lambda}}\cap S^{-,\check{\mu}}\right) \overset{T(\mathcal{O}_x)}{\times} \omega^{\check{\rho}}|_{\mathcal{D}_x},$$

where by a slight abuse of notation we denote by $\omega^{\check{\rho}}|_{\mathcal{D}_x}$ the corresponding $T(\mathcal{O}_x)$-torsor.

The line bundles $\mathcal{P}_{\mathcal{Z}^{\check{\mu}}_{loc,x}}$ and $\mathcal{P}_{\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x}}$ are induced from the canonical line bundle on $\mathrm{Gr}_{G,x}$ via the above embeddings.

[9] We denote by t a local parameter on the formal disc $\mathcal{D}_x$ around x; $t^{\check{\lambda}}\in\mathrm{Gr}_G$ is the projection of the point in $G(\mathcal{K}_x)$ corresponding to the map $\mathcal{D}_x^\times \xrightarrow{t} \mathbb{G}_m(\mathcal{K}_x) \xrightarrow{\check{\lambda}} T(\mathcal{K}_x)\hookrightarrow G(\mathcal{K}_x)$.

By Proposition 4.7, for a point $D \in X^{\check{\mu}}_n$ given by $\Sigma\, \check{\mu}'_k \cdot y_k + \Sigma\, \check{\mu}''_j \cdot x_j$ with all the y_k's and x_j's pairwise distinct, its preimage in $\mathcal{Z}^{\check{\mu}}_n$ is isomorphic to the product

$$\prod_k \mathcal{Z}^{\check{\mu}'_k}_{loc,y_k} \times \prod_j \mathcal{Z}^{\check{\mu}''_j}_{loc,x_j}.$$

By Proposition 4.13, an object $\mathcal{F} \in \mathrm{Whit}^c_n$ defines a (complex of) twisted D-modules $\mathcal{F}^{\check{\mu}}_{loc,x}$ for every $\check{\mu}$ and $x \in X$, so that the !-restriction of $'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})$ to the fiber over the point $D \in X^{\check{\mu}}_n$ is isomorphic to the product

$$\left(\underset{k}{\boxtimes} (\mathcal{F}_\emptyset)^{\check{\mu}'_k}_{loc,y_k}\right) \boxtimes \left(\underset{j}{\boxtimes} \mathcal{F}^{\check{\mu}''_j}_{loc,x_j}\right).$$

5.4. **Analysis of the line bundle.** By Lemma 4.9, the line bundle $\mathcal{P}_{\mathcal{Z}^{\check{\mu}}_{loc,x}}$ over $\mathcal{Z}^{\check{\mu}}_{loc,x}$ is canonically *constant* with fiber

$$\omega_x^{-(\check{\mu},\check{\mu}+2\check{\rho})\frac{Kil}{2}}. \tag{5.1}$$

Note that the line bundle $\mathcal{P}_{\mathfrak{W}_{x,\check{\lambda}}}$ over $\mathfrak{W}_{x,\check{\lambda}}$ is also constant with fiber

$$\omega_x^{-(\check{\lambda},\check{\lambda}+2\check{\rho})\frac{Kil}{2}}. \tag{5.2}$$

Hence, the line bundle $\mathcal{P}_{\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x}}$ is also isomorphic to the constant line bundle with the above fiber.

We obtain that $\mathcal{P}_{\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x}}$ admits two trivializations, defined up to a scalar. The discrepancy between them is a function $\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x} \to \mathbb{G}_m$, defined up to a multiplication by a scalar, that we shall denote by $\gamma^{\check{\lambda},\check{\mu}}$. The following assertion will be used in the sequel:

Lemma 5.5. *The function $\gamma^{\check{\lambda},\check{\mu}}$ intertwines the natural $T(\mathcal{O}_x)$ action on $S^{\check{\lambda}} \cap S^{-,\check{\mu}} \subset \mathrm{Gr}_{G,x}$ and the action on $\mathbb{G}_m$ given by the character*

$$T(\mathcal{O}_x) \to T \xrightarrow{(\check{\lambda}-\check{\mu},\cdot)_{Kil}} \mathbb{G}_m.$$

5.6. Let us now return to the proof of Proposition 4.10. Consider the stratification of $X^{\check{\mu}}_n$ by means of the strata formed by divisors $\Sigma\, \check{\mu}'_k \cdot y_k + \Sigma\, \check{\mu}''_j \cdot x_j$ with all the y_k's and x_j's pairwise distinct.

By Sect. 5.3, to prove the desired cohomological estimate, we have to show that for $\mathcal{F} \in \mathrm{Whit}^c_x$ the following holds:

$$\begin{cases} H^i\left(\mathcal{Z}^{\check{\mu}}_{loc,x}, \mathcal{F}^{\check{\mu}}_{loc,x}\right) = 0 \text{ for } i < 0 \text{ and any } \mathcal{F} \\ H^i\left(\mathcal{Z}^{\check{\mu}}_{loc,x}, (\mathcal{F}_\emptyset)^{\check{\mu}}_{loc,x}\right) = 0 \text{ for } i \leq 0 \text{ and } \mathcal{F} = \mathcal{F}_\emptyset \end{cases} \tag{5.3}$$

where by a slight abuse of notation, we view $\mathcal{F}^{\check{\mu}}_{loc,x}$ and $(\mathcal{F}_\emptyset)^{\check{\mu}}_{loc,x}$ as non-twisted D-modules using any trivialization of the line (5.1).

To prove the first assertion in (5.3), we can assume that $\mathcal{F}$ is of the form $\mathcal{F}_{x,\check{\lambda},*}$ for some $\check{\lambda} \in \check{\Lambda}^+$. In this case, the $\mathcal{F}^{\check{\mu}}_{loc,x} \in D(\mathfrak{D}\text{-mod}^c(\mathcal{Z}^{\check{\mu}}_{loc,x}))$ is the *-extension of a complex of twisted D-modules on $\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x}$.

Let $\overset{\circ}{\mathcal{Z}}{}^{\check{\mu}}_{x,\check{\lambda}}$ denote the the pre-image of the locally closed substack $\mathfrak{W}_{x,\check{\lambda}} \subset \mathfrak{W}_x$. This is a smooth scheme, and the map

$$\pi^{\check{\mu}} : \overset{\circ}{\mathcal{Z}}{}^{\check{\mu}}_{x,\check{\lambda}} \to X^{\check{\mu}}_{x,\leq\check{\lambda}}$$

is flat. The complex $\mathcal{F}^{\check{\mu}}_{loc,x}$ is obtained from a lisse twisted D-module on $\overset{\circ}{\mathcal{Z}}{}^{\check{\mu}}_{x,\check{\lambda}}$ by !-restriction to the fiber over the point $\check{\lambda}\cdot x \in X^{\check{\mu}}_{x,\leq\check{\lambda}}$. Hence, it is concentrated in the cohomological degrees $\geq \dim(X^{\check{\mu}}_{x,\leq\check{\lambda}}) = \langle\rho,\check{\lambda}-\check{\mu}\rangle$. However, since $\dim(\mathcal{Z}^{\check{\lambda},\check{\mu}}_{loc,x}) = \dim(S^{\check{\lambda}}\cap S^{-,\check{\mu}}) = \langle\rho,\check{\lambda}-\check{\mu}\rangle$, our assertion follows.

To prove the assertion concerning $\mathcal{F}_\emptyset$, we have to show, that the restriction of $(\mathcal{F}_\emptyset)^{\check{\mu}}_{loc,x}$ to the smooth part of $\mathcal{Z}^{0,\check{\mu}}_{loc,x}$, which is a lisse D-module placed in the cohomological degree $\langle\rho,-\check{\mu}\rangle$, is non-constant on each connected component, where we are using the trivialization of the twisting obtained from trivializing the line (5.1).

Let us describe this lisse D-module explicitly. It is the tensor product of $(\gamma^{0,\check{\mu}})^*(\Psi(c))$ and $\chi^{0\,*}_x(exp)$, where χ^0_x is the map

$$\mathcal{Z}^{0,\check{\mu}}_{loc,x} \simeq S^0\cap S^{-,\check{\mu}} \to N(\mathcal{K}_x)/N(\mathcal{O}_x) \overset{\chi_x}{\to} \mathbb{G}_a.$$

However, by [FGV], Prop. 7.1.7, coupled with [BFGM], Prop. 6.4, it is known that the map χ^0_x is *non-constant* on every irreducible component of $S^0\cap S^{-,\check{\mu}}$. This implies that the above tensor product is non-constant on every component, since the first factor is tame, and the second is not.

6. Proofs–B

6.1. From now till the end of the paper we will assume that c is irrational. The goal of this section is to prove Theorem 4.16(2), as well as the following statement:

Theorem 6.2. *For c irrational there exists an isomorphism*

$$\pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!*}) \simeq \mathcal{L}^{\check{\mu}}_{\overline{x},\overline{\check{\lambda}},!*}.$$

The proofs of the two theorems are largely parallel. We begin with the former.

6.3. By Proposition 4.10, the LHS (i.e., ${}'\mathcal{L}^{\check{\mu}}_\emptyset := \pi^{\check{\mu}}_*\circ{}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_\emptyset)$) is a D-module, which coincides with the RHS (i.e., $\mathcal{L}^{\check{\mu}}_\emptyset$) over the open sub-scheme $\overset{\circ}{X}{}^{\check{\mu}}_\emptyset$.

As a first step, we are going to show that the !-restriction of the LHS to any stratum in $X^{\check{\mu}}_\emptyset - \overset{\circ}{X}{}^{\check{\mu}}_\emptyset$ is concentrated in cohomological degrees ≥ 1. Applying factorization, this is equivalent to the fact that

$$H^i\Big((S^0\cap S^{-,\check{\mu}}), \gamma^{0,\check{\mu}*}(\Psi(c))\otimes\chi^{0\,*}_x(exp)\Big), \tag{6.1}$$

vanishes for $i=-|\check{\mu}|+1$, whenever $|\check{\mu}|>1$, where $|\check{\mu}|$ denotes the length of $\check{\mu}$, i.e., $|\langle\rho,\check{\mu}\rangle|$.

Since the dimension of every irreducible component of $S^0\cap S^{-,\check{\mu}}$ is $|\check{\mu}|$, it is enough to show that for every such component (or its dense open subset) Y,

$$H^i\Big(Y, \gamma^{0,\check{\mu}*}(\Psi(c))\otimes\chi^{0\,*}_x(exp)\Big) \tag{6.2}$$

vanishes for $i=-|\check{\mu}|+1$. We shall now rewrite the expression for the above cohomology.

6.4. Consider the vector space $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] \simeq \mathbb{A}^r$, and let $\chi_{x,univ}$ be the canonical map $N(\mathcal{K}_x) \to \mathbb{A}^r$, where r is the semi-simple rank of G. Our character χ_x can be taken to be the composition of $\chi_{x,univ}$ and *any* functional $\ell : \mathbb{A}^r \to \mathbb{A}$, which is non-zero on all simple roots.

By the projection formula,

$$H^i\Big(Y, \gamma^{0,\check{\mu}*}(\Psi(c)) \otimes \chi_x^{0*}(exp)\Big) \simeq H^i\Bigg(\mathbb{A}^r, (\chi^0_{x,univ}|_Y)_! \left(\gamma^{0,\check{\mu}*}(\Psi(c))\right) \otimes \ell^*(exp)\Bigg).$$

The scheme $S^0 \cap S^{-,\check{\mu}}$, and hence Y, is acted on by $T(\mathcal{O}_x)$, and the map $\chi^0_{x,univ}$ is equivariant, where $T(\mathcal{O}_x)$ acts on $\mathbb{A}^r$ via the projection $T(\mathcal{O}_x) \to T$ and the natural action of the latter on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$.

By Lemma 5.5, the map $\gamma^{0,\check{\mu}} : S^0 \cap S^{-,\check{\mu}} \to \mathbb{G}_m$ is $T(\mathcal{O}_x)$-equivariant against the character $\mu := (\check{\mu}, ?)_{Kil}$. Hence, the complex

$$\mathsf{M}^Y := (\chi^0_{x,univ}|_Y)_! \left(\gamma^{0,\check{\mu}*}(\Psi(c))\right) \tag{6.3}$$

on $\mathbb{A}^r$ is T-equivariant against the character sheaf $\Psi(c \cdot \mu)$. [10] In particular, M^Y is lisse away from the diagonal hyperplanes.

Thus, the cohomology (6.2), *shifted by* $[r]$, is the fiber at $\ell \in (\mathbb{A}^r)^*$ of the Fourier-Deligne transform $\mathrm{Four}(\mathsf{M}^Y)$. By the above equivariance property of M^Y, the complex $\mathrm{Four}(\mathsf{M}^Y)$ is also twisted T-equivariant, and hence is lisse away from the coordinate hyperplanes, and in particular on a neighbourhood of ℓ.

6.5. We are ready now to return to the proof that the cohomology (6.2) vanishes in degree $1 - |\check{\mu}|$.

Let $\mathcal{I}' \subset \mathcal{I}$ be the minimal Dynkin sub-diagram, such that $\check{\mu} \in \mathrm{Span}(\check{\alpha}_{\imath'}, \imath' \in \mathcal{I}')$, and let r' be its rank. We will distinguish two cases: (1) $r' = 1$ and (2) $r' \geq 1$.

In case (1) $\check{\mu} = (-m) \cdot \check{\alpha}_\imath$, where $\alpha_\imath$ is the corresponding simple root. We can describe the intersection $S^0 \cap S^{-,\check{\mu}}$ explicitly. It is irreducible and isomorphic to $\mathbb{A}^{m-1} \times \mathbb{G}_m$, with the map $\chi^0_{x,univ}$ being

$$\mathbb{A}^{m-1} \times \mathbb{G}_m \twoheadrightarrow \mathbb{A}^{m-1} \to \mathbb{A}^1 \overset{\alpha_\imath}{\hookrightarrow} \mathbb{A}^r.$$

(Recall that by assumption $|\check{\mu}| \geq 2$, hence, $m \geq 2$). Since there are no non-constant maps $\mathbb{A}^{m-1} \to \mathbb{G}_m$, the map $\gamma^{0,\check{\mu}}$ factors through the $\mathbb{G}_m$-factor. This implies that the cohomology (6.1) vanishes in all degrees.

Let us now consider case (2). By Sect. 6.4, it is enough to show that M^Y itself lives in perverse cohomological degrees $\geq -\langle \rho, \check{\mu} \rangle + 2$.

The equivariance property of M^Y against the character sheaf $\Psi(c \cdot \mu)$ on T, and since c is irrational, implies that the complex M^Y is the extension by zero from the complement to the union of coordinate hyperplanes in $\mathbb{A}^{r'}$. However, over this open subset, the fibers of the map $\chi^0_{x,univ} : Y \to \mathbb{A}^{r'}$ have dimension $\leq \langle \rho, \check{\mu} \rangle - r'$, and we are done since $r' \geq 2$.

6.6. Let us proceed with the proof of Theorem 4.16(2). We obtain, that the map

$$\pi^{\check{\mu}}_! \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_\emptyset) \to j^{Diag}_*(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_\emptyset),$$

resulting from the isomorphism of Theorem 4.16(1), is injective.

[10] The latter is the pull-back of $\Psi(c)$ by means of the map $\mu : T \to \mathbb{G}_m$.

Since we are dealing with holonomic D-modules, the direct images with compact supports are well-defined, and by Verdier duality, we obtain that the map

$$j_!^{Diag}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\emptyset}) \to \pi^{\check{\mu}}_! \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{\emptyset})$$

is surjective.

Consider the composition

$$j_!^{Diag}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\emptyset}) \twoheadrightarrow \pi^{\check{\mu}}_! \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{\emptyset}) \to \pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{\emptyset}) \hookrightarrow j_*^{Diag}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\emptyset}),$$

where the middle arrow is the canonical map as in Theorem 4.11. This map restricts to the tautological isomorphism over $\overset{\circ}{X}{}^{\check{\mu}}_{\emptyset}$; hence, it is the canonical map over the entire $X^{\check{\mu}}_{\emptyset}$.

Applying Theorem 4.11 (which will be proven in the next section), we deduce that

$$\pi^{\check{\mu}}_! \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{\emptyset}) \simeq j_{!*}^{Diag}(\overset{\circ}{\mathcal{L}}{}^{\check{\mu}}_{\emptyset}) \simeq \pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{\emptyset}),$$

as required.

6.7. **Proof of Theorem 6.2.** To simplify the notation, we will assume that $n = 1$, i.e., $\overline{x}$ consists of one point x (and $\overline{\check{\lambda}}$ is just one co-weight $\check{\lambda}$). The proof in the general case is the same.

Both D-modules:

$$\pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!*}) \text{ and } \mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*}$$

are supported on the sub-scheme $X^{\check{\mu}}_{x,\leq\check{\lambda}} \subset X^{\check{\mu}}_x$. By Proposition 4.13, the desired isomorphism holds over the open sub-scheme $\overset{\circ}{X}{}^{\check{\mu}}_{x,\leq\check{\lambda}}$.

Moreover, by Theorem 4.16, the isomorphism in question holds over a larger open sub-scheme: one consisting of divisors $D = \sum\limits_k \check{\mu}_k \cdot y_k + \check{\lambda} \cdot x$ with $y_k \neq x$. Thus, we have to show that the isomorphisms holds also over this divisor.

We will argue by induction on $\check{\lambda} - \check{\mu}$. The case $\check{\lambda} = \check{\mu}$ is evident. We assume that the assertion is true for all $\check{\mu}'$ with $|\check{\lambda} - \check{\mu}'| < |\check{\lambda} - \check{\mu}|$. Then, by factorization, the two D-modules appearing in the theorem are isomorphic away from the point $\check{\mu} \cdot x \overset{i^{\check{\mu}}_x}{\hookrightarrow} X^{\check{\mu}}_{x,\leq\check{\lambda}}$. Let us denote the corresponding open embedding $X^{\check{\mu}}_{x,\leq\check{\lambda}} - \{\check{\mu} \cdot x\} \hookrightarrow X^{\check{\mu}}_{x,\leq\check{\lambda}}$ by j^{pole}.

6.8. First, we claim that the !-fiber of $\pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!*})$ at the above point is concentrated in cohomological degrees ≥ 1.

Indeed, the above fiber is given (up to a cohomological shift by $|\check{\lambda} - \check{\mu}|$) by

$$H^\bullet\left(\mathcal{Z}^{\check{\mu}}_{loc,x}, (\mathcal{F}_{x,\check{\lambda},!*})^{\check{\mu}}_{loc,x}\right). \tag{6.4}$$

We need to show that the above cohomology vanishes in degree $-|\check{\lambda} - \check{\mu}|$. This is equivalent to the fact that the D-module $(\mathcal{F}_{x,\check{\lambda},!*})^{\check{\mu}}_{loc,x}$ is non-constant on an open part of each irreducible component of $\mathcal{Z}^{\check{\mu}}_{loc,x}$.

Replacing $\mathcal{Z}^{\check{\mu}}_{loc,x}$ by a dense open subset in the support of the D-modules in question, we are reduced to the study of $S^{\check{\lambda}} \cap S^{-,\check{\mu}}$, and the D-module $(\gamma^{\check{\lambda},\check{\mu}})^*(\Psi(c)) \otimes \chi^{\check{\lambda}*}_x(exp)$ on it, where $\gamma^{\check{\lambda},\check{\mu}}$ is as in Sect. 5.4, and $\chi^{\check{\lambda}}_x$ is the function $S^{\check{\lambda}} \to \mathbb{G}_a$, induced by the character $\chi_x : N(\mathcal{K}_x) \to \mathbb{G}_a$, which is defined up to a shift.

Let us distinguish two cases: If the the function $\chi_x^{\check{\lambda}}$ is non-constant on the given irreducible component, the assertion follows as in the proof of Proposition 4.10 (see the end of Sect. 5.6).

If $\chi_x^{\check{\lambda}}$ is constant, the sheaf in question is $(\gamma^{\check{\lambda},\check{\mu}})^*(\Psi(c))$, and it is non-constant, since c is irrational, and hence $c\cdot(\check{\lambda}-\check{\mu},\cdot)_{Kil}$ is not an integral character of T.

6.9. The rest of the proof is similar to that of Theorem 4.16(2). Indeed, we obtain that the map

$$\pi_*^{\check{\mu}}\circ{}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!*})\to j_*^{pole}\left(j^{pole*}(\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*})\right)$$

is injective. Dually, we obtain a surjective map

$$j_!^{pole}\left(j^{pole*}(\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*})\right)\to\pi_!^{\check{\mu}}\circ{}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!*}).$$

The composition

$$j_!^{pole}\left(j^{pole*}(\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*})\right)\to\pi_!^{\check{\mu}}\circ{}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!})\to\pi_*^{\check{\mu}}\circ{}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!})\to j_*^{pole}\left(j^{pole*}(\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*})\right)$$

is the canonical map

$$j_!^{pole}\left(j^{pole*}(\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*})\right)\to j_*^{pole}\left(j^{pole*}(\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*})\right),$$

because this is so over $X^{\check{\mu}}_{x,\leq\check{\lambda}}-\{\check{\mu}\cdot x\}$.

Now, Theorem 4.11 implies the desired isomorphism

$$\pi_!^{\check{\mu}}\circ{}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!*})\simeq\mathcal{L}^{\check{\mu}}_{x,\check{\lambda},!*}\simeq\pi_*^{\check{\mu}}\circ{}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!*}).$$

7. Cleanness

In this section we will prove Theorem 4.11.

7.1. We introduce the stack $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$ (see [BG], Sect. 1.2), whose definition is the same as that of $\mathfrak{W}_\emptyset$ (with B replaced by B^- and κ^λ replaced by $\kappa^{\lambda,-}$) and the difference being that we allow an arbitrary T-bundle of degree $(2g-2)\check{\rho}-\check{\mu}$.

We have a tautological open embedding $\jmath^-:\mathrm{Bun}^{\check{\mu}}_{B^-}\hookrightarrow\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$, and the maps

$$\mathrm{Bun}^{\check{\mu}}_T\xleftarrow{\overline{\mathfrak{q}}^-}\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}\xrightarrow{\overline{\mathfrak{p}}^-}\mathrm{Bun}_G,$$

that extend the corresponding maps for $\mathrm{Bun}^{\check{\mu}}_{B^-}$.

Recall the line bundle $\mathcal{P}_{\mathrm{Bun}_T}$ over Bun_T and the line bundle $\mathcal{P}_{\mathrm{Bun}_G}$ over Bun_G. We let $\mathcal{P}^T_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}}$ and $\mathcal{P}^G_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}}$, respectively, denote their pull-backs to $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$. The two are canonically isomorphic over the open sub-stack $\mathrm{Bun}^{\check{\mu}}_{B^-}$.

7.2. We introduce the compactified Zastava space $\overline{\mathcal{Z}}^{\check{\mu}}_n$ as the open sub-stack of $\mathfrak{W}_n\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$ corresponding to the condition that all the compositions

$$\omega^{\langle\lambda,\check{\rho}\rangle}\xrightarrow{\kappa^\lambda}\mathcal{V}^\lambda_{\mathfrak{F}_G}\xrightarrow{\kappa^{-,\lambda}}\lambda(\mathfrak{F}_T)$$

are non-zero. We let ${}'\overline{\mathfrak{p}}^-:\overline{\mathcal{Z}}^{\check{\mu}}_n\to\mathfrak{W}_n$ and ${}'\mathfrak{p}:\overline{\mathcal{Z}}^{\check{\mu}}_n\to\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$ denote the corresponding base-changed maps.

As in the case of the usual Zastava spaces, we have a natural map

$$\pi^{\check{\mu}}:\overline{\mathcal{Z}}^{\check{\mu}}_n\to X^{\check{\mu}}_n,$$

and an analog of Proposition 4.7 holds (with the same proof):

$$\left(X^{\check{\mu}_1}_\emptyset \times X^{\check{\mu}_2}_n\right)_{disj} \underset{X^{\check{\mu}}_n}{\times} \overline{\mathcal{Z}}^{\check{\mu}}_n \simeq \left(X^{\check{\mu}_1}_\emptyset \times X^{\check{\mu}_2}_n\right)_{disj} \underset{(X^{\check{\mu}_1}_\emptyset \times X^{\check{\mu}_2}_n)}{\times} \left(\overline{\mathcal{Z}}^{\check{\mu}_1}_\emptyset \times \overline{\mathcal{Z}}^{\check{\mu}_2}_n\right). \tag{7.1}$$

There is a tautological isomorphism of line bundles:

$$'\mathfrak{p}^*(\mathcal{P}^T_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}})^{\otimes -1} \simeq (\overline{\pi}^{\check{\mu}})^*(\mathcal{P}_{X^{\check{\mu}}_n}).$$

We shall denote by $\mathfrak{D}\text{-mod}^c(\overline{\mathcal{Z}}^{\check{\mu}}_n)$ the corresponding category of twisted D-modules.

Let us denote by $'\jmath^-$ the open embedding $\mathcal{Z}^{\check{\mu}}_n \hookrightarrow \overline{\mathcal{Z}}^{\check{\mu}}_n$. For $\mathcal{F} \in \mathrm{Whit}^c_n$ we can consider $'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})$ as an object of $\mathfrak{D}\text{-mod}^c(\mathcal{Z}^{\check{\mu}}_n)$; this is due to the identification of the line bundles $\mathcal{P}^T_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}}$ and $\mathcal{P}^G_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}}$ over $\mathrm{Bun}^{\check{\mu}}_{B^-}$. We will deduce Theorem 4.11 from the next assertion:

Theorem 7.3. *Assume that c is irrational. Then for $\mathcal{F} \in \mathrm{Whit}^c_n$, the object $'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})$ is clean with respect to $'\jmath^-$, i.e., the map*

$$'\jmath^-_!('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})) \to {'\jmath^-_*}('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}))$$

is an isomorphism in $\mathfrak{D}\text{-mod}^c(\overline{\mathcal{Z}}^{\check{\mu}}_n)$ (in particular, the LHS is well-defined).

7.4. **Proof of Theorem 4.11.** The basic observation is that the map $\overline{\pi}^{\check{\mu}}$ is proper. Indeed, $\overline{\mathcal{Z}}^{\check{\mu}}_n$ is a closed sub-scheme of the corresponding relative (i.e., Beilinson-Drinfeld) version of the affine Grassmannian over $X^{\check{\mu}}_n$.

Hence, it remains to notice that

$$\pi^{\check{\mu}}_!('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})) \simeq \overline{\pi}^{\check{\mu}}_!\left('\jmath^-_!('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}))\right) \text{ and } \pi^{\check{\mu}}_*('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})) \simeq \overline{\pi}^{\check{\mu}}_*\left('\jmath^-_*('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}))\right).$$

□

7.5. The rest of this section is devoted to the proof of Theorem 7.3. First, we will establish a cleanness-type result purely on $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$.

Let $\mathcal{P}^{ratio}_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}}$ denote the ratio of the two line bundles $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$

$$\mathcal{P}^{ratio}_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}} := {}^G\mathcal{P}_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}} \otimes \left({}^T\mathcal{P}_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}}\right)^{\otimes -1}.$$

We note that the restriction of this line bundle to the open part $\mathrm{Bun}^{\check{\mu}}_{B^-}$ is canonically trivial. (In fact, in [BFG], Theorem 11.6 it was shown that, after passing from $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$ to its normalization, the inverse of the corresponding section of $\mathcal{P}^{ratio}_{\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}}$ is regular and its locus of zeroes is $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-} - \mathrm{Bun}^{\check{\mu}}_{B^-}$.)

We introduce $\mathfrak{D}\text{-mod}^{ratio,c}(\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-})$ as the corresponding category of twisted D-modules on $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$.

For a given c let $\mathsf{Const}^c_{\mathrm{Bun}^{\check{\mu}}_{B^-}}$ denote canonical "constant" object of the category

$$\mathfrak{D}\text{-mod}^{ratio,c}(\mathrm{Bun}^{\check{\mu}}_{B^-}) \simeq \mathfrak{D}\text{-mod}(\mathrm{Bun}^{\check{\mu}}_{B^-}).$$

Our main technical tool is the following:

Theorem 7.6. *For c irrational, the object $\mathsf{Const}^c_{\mathrm{Bun}^{\check\mu}_{B^-}} \in \mathfrak{D}\text{-}mod^{ratio,c}(\mathrm{Bun}^{\check\mu}_{B^-})$ is clean with respect to $\jmath^-$, i.e., the maps*

$$\jmath^-_!(\mathsf{Const}^c_{\mathrm{Bun}^{\check\mu}_{B^-}}) \to \jmath^-_{!*}(\mathsf{Const}^c_{\mathrm{Bun}^{\check\mu}_{B^-}}) \to \jmath^-_*(\mathsf{Const}^c_{\mathrm{Bun}^{\check\mu}_{B^-}})$$

are isomorphisms in $\mathfrak{D}\text{-}mod^{ratio,c}(\overline{\mathrm{Bun}}^{\check\mu}_{B^-})$.

7.7. **Proof of Theorem 7.6.** We will only sketch the proof, as it follows very closely the IC sheaf computation in [FFKM] or [BFGM], Sect. 5.

We will work with all connected components $\overline{\mathrm{Bun}}^{\check\mu}_{B^-}$, so $\check\mu$ will be dropped from the notation. We represent $\overline{\mathrm{Bun}}_{B^-}$ as a union of open sub-stacks ${}^{\leq\check\nu}\overline{\mathrm{Bun}}_{B^-}$, where the latter classifies the data $\{\mathfrak{F}_G, \mathfrak{F}_T, (\kappa^{-,\lambda})\}$, where the length of the quotient $\lambda(\mathfrak{F}_T)/\operatorname{Im}(\kappa^{-,\lambda})$ is $\leq \langle\lambda,\check\nu\rangle$. Let ${}^{=\check\nu}\overline{\mathrm{Bun}}_{B^-}$ be the closed sub-stack of ${}^{\leq\check\nu}\overline{\mathrm{Bun}}_{B^-}$, when the length of the above quotient is exactly $\langle\lambda,\check\nu\rangle$.

We have

$$\overline{\mathrm{Bun}}_{B^-} - \mathrm{Bun}_{B^-} = \bigcup_{\check\nu\in\check\Lambda^{pos}-0} {}^{=\check\nu}\overline{\mathrm{Bun}}_{B^-}.$$

We will argue by induction and assume that the assertion of the theorem is valid over $\overline{\mathrm{Bun}}^{\leq\check\nu'}_{B^-}$ for all $\check\nu'$ with $|\check\nu'| < |\check\nu|$.

Let $\mathcal{Z}^{-,\check\nu}$ be the corresponding Zastava space, i.e., an open subset of $\mathrm{Bun}_N \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}^{\check\nu}_{B^-}$. Let $\mathcal{P}^{ratio}_{\mathcal{Z}^{-,\check\nu}}$ be the pull-back of $\mathcal{P}^{ratio}_{\overline{\mathrm{Bun}}^{\check\nu}_{B^-}}$ under the natural projection. We let $\mathfrak{D}\text{-mod}^{ratio,c}(\mathcal{Z}^{-,\check\nu})$ denote the corresponding category of twisted D-modules.

Let $\overset{\circ}{\mathcal{Z}}{}^{-,\check\nu} \overset{'\jmath^-}{\hookrightarrow} \mathcal{Z}^{-,\check\nu}$ be the open sub-scheme equal to the pre-image of $\mathrm{Bun}_{B^-} \overset{\jmath^-}{\hookrightarrow} \overline{\mathrm{Bun}}_{B^-}$. One shows as in [BFGM], Sect. 3, that the problem of extension of the twisted D-module $\mathsf{Const}^c_{\mathrm{Bun}_{B^-}}$ from Bun_{B^-} to ${}^{\leq\check\nu}\overline{\mathrm{Bun}}_{B^-}$ is equivalent to that of extension of the corresponding twisted D-module $\mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check\mu}}$ from $\overset{\circ}{\mathcal{Z}}{}^{-,\check\nu}$ to $\mathcal{Z}^{-,\check\nu}$.

Consider the projection $\pi^{-,\check\nu} : \mathcal{Z}^{-,\check\nu} \to X^{\check\nu}$. Since the pull-back of $\mathcal{P}_{\mathrm{Bun}_G}$ to Bun_N is canonically trivial, we have an identification of line bundles

$$\mathcal{P}^{ratio}_{\mathcal{Z}^{-,\check\nu}} \simeq (\pi^{-,\check\nu})^*(\mathcal{P}_{X^{\check\nu}}),$$

where $\mathcal{P}_{X^{\check\nu}}$ is the canonical line bundle over $X^{\check\nu}$ defined as in Sect. 3.2. In particular, we have a well-defined direct image functors

$$\pi^{-,\check\nu}_!, \pi^{-,\check\nu}_* : D(\mathfrak{D}\text{-mod}^{ratio,c}(\mathcal{Z}^{-,\check\nu})) \to D(\mathfrak{D}\text{-mod}^{ratio,c}(X^{\check\nu})).$$

As in [BFGM], Sect. 2.2., the map $\pi^{-,\check\nu}$ admits a canonical section, denoted $\mathfrak{s}^{-,\check\nu}$, compatible with the above identification of line bundles. The image of $\mathfrak{s}^{-,\check\nu}$ is the locus ${}^{=\check\nu}\mathcal{Z}^{-,\check\nu}$ equal to the pre-image of ${}^{=\check\nu}\overline{\mathrm{Bun}}_{B^-} \subset \overline{\mathrm{Bun}}_{B^-}$.

By the induction hypothesis, the cleanness assertion for $\mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check\nu}}$ with respect to $'\jmath^-$ holds away from the image of the section $\mathfrak{s}^{-,\check\nu}$. Hence, to prove the theorem, it suffices to show that

$$(\mathfrak{s}^{-,\check\nu})^! \circ ('\jmath^-)_{!*}(\mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check\mu}}) = 0.$$

However, since the morphism $\pi^{-,\check\nu}$ is affine and there exists a $\mathbb{G}_m$-action along its fibers that contracts $\mathcal{Z}^{-,\check\nu}$ onto the image of $\mathfrak{s}^{-,\check\nu}$, with $\mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check\mu}}$ being equivariant, we have:

$$(\mathfrak{s}^{-,\check\nu})^! \circ ('\jmath^-)_{!*}(\mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check\nu}}) \simeq (\pi^{-,\check\nu}_!) \circ ('\jmath^-)_{!*}(\mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check\nu}}) := \mathsf{K}^{-,\check\nu}.$$

Again, by the induction hypothesis and factorization, $\mathsf{K}^{-,\check{\nu}}$ vanishes away from the main diagonal $X \subset X^{\check{\nu}}$. Moreover, by the defining property of the Goresky-MacPherson extension, $\mathsf{K}^{\check{\nu}}$ lives in the cohomological degrees > 0.

Hence, it suffices to see that its *-fiber at every point $x \in X$ lives in the cohomological degrees < 0. By base change and the the induction hypothesis, the fiber in question is given by

$$H_c(S^0 \cap S^{-,\check{\nu}}, \mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check{\nu}}_{loc,x}}).$$

As in the proof of Proposition 4.10, the complex of D-modules $\mathsf{Const}^c_{\overset{\circ}{\mathcal{Z}}{}^{-,\check{\nu}}_{loc,x}}$ is isomorphic to the pull-back of $\Psi(c)$ by means of the function $\gamma^{0,\check{\nu}}$, cohomologically shifted by $[2|\check{\nu}|]$.

Since $\dim(S^0 \cap S^{-,\check{\nu}}) = |\check{\nu}|$, non-zero cohomology can only exist in degrees ≤ 0 (and from the above we already know that it vanishes for degrees strictly less than 0). Hence, it suffices to show that the above cohomology vanishes in degree 0. The latter happens if and only if the lisse D-module, obtained by restricting $\gamma^{0,\check{\nu}*}(\Psi(c))[2|\check{\nu}|]$ to a smooth open part of every irreducible component of $S^0 \cap S^{-,\check{\nu}}$, is non-constant.

However, from Lemma 5.5, we know that the D-module in question is equivariant with respect to the $T(\mathcal{O}_x)$-action on the scheme $S^0 \cap S^{-,\check{\nu}}$ against the character sheaf $\Psi(c \cdot \nu)$, where $\nu \in \Lambda$ is $(\check{\nu}, \cdot)_{Kil}$, and $c \cdot \nu \notin \mathbb{Z}$ unless $\check{\nu} = 0$, since c is irrational.

□

7.8. We shall now show how to deduce Theorem 7.3 from Theorem 7.6. We will use the following assertion, which can be proved by the same argument as [BG], Sect. 5.3:

Proposition 7.9. *For every open sub-stack of finite type $U \subset \mathrm{Bun}_G$ and $\check{\nu}, \check{\mu} \in \check{\Lambda}^{pos}$ there exists $\check{\mu}' \in \check{\Lambda}$ with $\check{\mu} \geq \check{\mu}'$, such that the twisted D-module $\jmath^-_*(\mathsf{Const}^c_{\mathrm{Bun}^{\check{\mu}'}_{B^-}})$, restricted to ${}^{\leq \check{\nu}}\overline{\mathrm{Bun}}^{\check{\mu}'}_{B^-}$ is ULA* [11] *with respect to the map $\overline{\mathfrak{p}}^-$ over U.*

Given an object $\mathcal{F} \in \mathrm{Whit}^c_n$, let $\overline{\check{\lambda}} = (\check{\lambda}_1, ..., \check{\lambda}_n)$ be the bound on the order of the poles of the maps κ^λ contained in the support of $\mathcal{F}$ in $\mathfrak{W}_n$.

Consider now the support of ${}'\jmath^-_*({}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}))$ on $\overline{\mathcal{Z}}{}^{\check{\mu}}_n$; denote it by Y; this is a stack of finite type. The image of Y in $\mathfrak{W}_n$ is contained in an open sub-stack ${}^{\leq \check{\eta}}\mathfrak{W}_{n,\leq\check{\lambda}}$ of $\mathfrak{W}_{n,\leq\check{\lambda}}$, where the total amount of zeroes of the maps κ^λ does not exceed $\check{\eta}$. The image of ${}^{\leq \check{\eta}}\mathfrak{W}_{n,\leq\check{\lambda}}$ in Bun_G under $\mathfrak{p}$ is contained in an open sub-stack of finite type that we will denote by U. Similarly, the image of Y in $\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$ is contained in ${}^{\leq \check{\nu}}\overline{\mathrm{Bun}}^{\check{\mu}}_{B^-}$ for some $\check{\nu}$.

Let us take $\check{\mu}'$ with $\check{\mu} \geq \check{\mu}'$ given by Proposition 7.9 for the above U and $\check{\nu}$. Consider the open sub-scheme ${}^U\overline{\mathcal{Z}}{}^{\check{\mu}'}_n$ of $\overline{\mathcal{Z}}{}^{\check{\mu}'}_n$ equal to the pre-image of U under the forgetful map $\mathcal{Z}^{\check{\mu}'}_n \to \mathrm{Bun}_G$. We claim that it it sufficient to show that the cleanness statement holds over ${}^U\overline{\mathcal{Z}}{}^{\check{\mu}'}_n$.

Indeed, by factorization (i.e., (7.1)) we can complement any given point of $\overline{\mathcal{Z}}{}^{\check{\mu}}_n$ by points in $\overset{\circ}{\mathcal{Z}}{}^{\check{\mu}_k}_{loc,y_k}$ with $\check{\mu} - \check{\mu}' = \sum\limits_k \check{\mu}_k$, and y_k being away from the support of the divisor equal to the image of our point under $\overline{\pi}^{\check{\mu}}$, to get a point of $\overline{\mathcal{Z}}{}^{\check{\mu}'}_n$. The image of this new point in $\mathfrak{W}_n$ will still be contained in ${}^{\leq \check{\eta}}\mathfrak{W}_{n,\leq\check{\lambda}}$; and hence its image in Bun_G will be contained in U.

[11] See [BG], Sect. 5.1, for a review of the ULA property.

Let us note that the line bundle $'\mathfrak{p}^*(\mathcal{P}^T_{\overline{\mathrm{Bun}}^{\check{\mu}'}_{B^-}})^{\otimes -1}$ over $\overline{\mathcal{Z}}^{\check{\mu}'}_n$ is the tensor product

$$('\overline{\mathfrak{p}}^-)^*(\mathcal{P}_{\mathfrak{W}_n}) \otimes {}'\mathfrak{p}^*(\mathcal{P}^{ratio}_{\overline{\mathrm{Bun}}^{\check{\mu}'}_{B^-}}).$$

Hence, for $\mathcal{F}' \in \mathfrak{D}^c\text{-mod}(\mathfrak{W}_n)$ and $\mathcal{F}'' \in \mathfrak{D}\text{-mod}(\overline{\mathrm{Bun}}^{\check{\mu}'}_{B^-})^{ratio,c}$ it makes sense to consider

$$\mathcal{F}' \underset{\mathrm{Bun}_G}{\boxtimes} \mathcal{F}'' \in D(\mathfrak{D}\text{-mod}^c(\overline{\mathcal{Z}}^{\check{\mu}'}_n)).$$

Now, the assertion of Theorem 7.3 follows from the following general statement:

7.10. Let $\mathcal{Y}$ be a smooth scheme (or stack), $f : \mathcal{Y}' \to \mathcal{Y}$ a map, and $\jmath : \overset{\circ}{\mathcal{Y}}{}' \hookrightarrow \mathcal{Y}'$ and open sub-stack. Let $\mathcal{F} \in D(\mathfrak{D}\text{-mod}(\overset{\circ}{\mathcal{Y}}{}'))$ be an object which is clean with respect to $\jmath$, i.e., such that

$$\jmath_!(\mathcal{F}) \to \jmath_*(\mathcal{F})$$

is an isomorphism.

Let now $Z \to \mathcal{Y}$ be a map and let $\mathcal{L}$ be an object of $D(\mathfrak{D}\text{-mod}(Z))$. Set

$$\overset{\circ}{Z}{}' := Z \underset{\mathcal{Y}}{\times} \overset{\circ}{\mathcal{Y}}{}' \text{ and } Z' := Z \underset{\mathcal{Y}}{\times} \mathcal{Y}',$$

and let $\jmath'$ be the corresponding open embedding $\overset{\circ}{Z}{}' \hookrightarrow Z'$.

Lemma 7.11. *Assume that $\jmath_*(\mathcal{F})$ is ULA with respect to f. Then $\mathcal{L} \underset{\mathcal{Y}}{\boxtimes} \mathcal{F} \in D(\mathfrak{D}\text{-mod}(\overset{\circ}{Z}{}'))$ is clean with respect to $\jmath'$, i.e.,*

$$\jmath'_!(\mathcal{L} \underset{\mathcal{Y}}{\boxtimes} \mathcal{F}) \to \jmath'_*(\mathcal{L} \underset{\mathcal{Y}}{\boxtimes} \mathcal{F})$$

is an isomorphism.

8. Equivalence

8.1. In this section we shall prove Theorem 3.11. Thus, we have to construct a functor

$$\mathsf{G}_n : \mathrm{Whit}^c_n \to \widetilde{\mathrm{FS}}^c_n;$$

show that its image belongs to FS^c_n, and prove that the above functor is an equivalence.

The first step has been essentially carried out already: for $\mathcal{F} \in \mathrm{Whit}^c_n$, we set

$$\mathsf{G}_n(\mathcal{F})^{\check{\mu}} := \pi^{\check{\mu}}_*('\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F})) \in \mathfrak{D}\text{-mod}^c(X^{\check{\mu}}_n).$$

These D-modules satisfy the required factorization property by Corollary 4.15 and Theorem 4.16. The functor G is exact by Proposition 4.10.

8.2. Let us fix (distinct) points $\overline{x} := (x_1, ..., x_n)$, and consider the corresponding functor

$$\mathsf{G}_{\overline{x}} : \mathrm{Whit}^c_{\overline{x}} \to \widetilde{\mathrm{FS}}^c_{\overline{x}}.$$

We will first prove:

Theorem 8.3. *The functor $\mathsf{G}_{\overline{x}}$ has its image in $\mathrm{FS}^c_{\overline{x}}$ and induces an equivalence with the latter sub-category.*

Proof. By Proposition 2.8 and Theorem 6.2, we obtain that $\mathsf{G}_{\overline{x}}$ does indeed send $\mathrm{Whit}^c_{\overline{x}}$ to $\mathrm{FS}^c_{\overline{x}}$, is faithful, and defines a bijection on the level of irreducible objects.

Thus, applying Theorem 3.9(b), we obtain that the assertion of Theorem 8.3 reduces to the statement that the category $\mathrm{Whit}^c_{\overline{x}}$ (with c irrational) is semi-simple. By Lemma 2.7, there are no non-trivial self-extensions of $\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!}$. Hence, semi-simplicity of $\mathrm{Whit}^c_{\overline{x}}$ is equivalent to the next statement:

Proposition 8.4. *For c irrational and any $\overline{\check{\lambda}} \in \check{\Lambda}^+$, the maps*

$$\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!} \to \mathcal{F}_{\overline{x},\overline{\check{\lambda}},!*} \to \mathcal{F}_{\overline{x},\overline{\check{\lambda}},*}$$

are isomorphisms.

□

8.5. **Proof of Proposition 8.4.** For simplicity we shall assume that $n = 1$, i.e., $\overline{x} = \{x\}$ and $\overline{\check{\lambda}} = \{\check{\lambda}\}$. The proof in the general case is the same.

Let $\mathcal{F}'$ be the kernel of the map $\mathcal{F}_{x,\check{\lambda},!} \to \mathcal{F}_{x,\check{\lambda},!*}$. It is sufficient to show that $\mathsf{G}_{\overline{x}}(\mathcal{F}') = 0$. For $\check{\mu} \in \check{\Lambda}$, consider the corresponding short exact sequence:

$$0 \to \pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}') \to \pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!}) \to \pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!*}) \to 0. \tag{8.1}$$

As in Sect. 6.7, we will argue by induction on $|\check{\lambda} - \check{\mu}|$ that $\pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}') = 0$. The base of the induction $\check{\mu} = \check{\lambda}$ trivially holds. The induction hypothesis and factorization imply that $\pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}')$ is supported at the point $\check{\mu} \cdot x \in X^{\check{\mu}}_{x,\leq\check{\lambda}}$.

Let $i^{\check{\mu}}_x$ denote the embedding of this point into $X^{\check{\mu}}_{x,\leq\check{\lambda}}$. It is sufficient to show that the 0-th cohomology of $(i^{\check{\mu}}_x)^! \left(\pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}')\right)$ vanishes.

However, by Theorem 3.9(b), the short exact sequence (8.1) is split. So, it is sufficient to show that the 0-th cohomology of $(i^{\check{\mu}}_x)^! \left(\pi^{\check{\mu}}_* \circ {}'\mathfrak{p}^{-,\check{\mu},\cdot}(\mathcal{F}_{x,\check{\lambda},!})\right)$ vanishes. But the latter calculation has been performed in Sect. 6.8.

□

8.6. We are now ready to show that the functor G_n maps Whit^c_n to FS^c_n.

Recall the categories $\mathrm{FS}^c_{\overset{\circ}{n}} \subset \widetilde{\mathrm{FS}}^c_{\overset{\circ}{n}}$, (see Sect. 3.5). Let $\mathrm{Whit}^c_{\overset{\circ}{n}}$ be the corresponding Whittaker category over $\overset{\circ}{X}{}^n$. We have a functor

$$\mathsf{G}_{\overset{\circ}{n}} : \mathrm{Whit}^c_{\overset{\circ}{n}} \to \widetilde{\mathrm{FS}}^c_{\overset{\circ}{n}} :$$

The following results from Theorem 3.9(b) and and Theorem 8.3:

Lemma 8.7.
(a) Every object of $\mathrm{Whit}^c_{\overset{\circ}{n}}$ is isomorphic to a direct sum $\underset{\overline{\check{\lambda}}}{\oplus} \mathcal{F}_{\overset{\circ}{n},\overline{\check{\lambda}},!*} \otimes \mathcal{M}(\overline{\check{\lambda}})$, where $\mathcal{F}_{\overset{\circ}{n},\overline{\check{\lambda}},!*}$ is a relative (over $\overset{\circ}{X}{}^n$) version of $\mathcal{F}_{\overline{x},\overline{\check{\lambda}},!*}$, and $\mathcal{M}(\overline{\check{\lambda}})$ is a D-module on $\overset{\circ}{X}{}^n$.

(b) Every object of $\mathrm{FS}^c_{\overset{\circ}{n}}$ is isomorphic to a direct sum $\underset{\overline{\check{\lambda}}}{\oplus} \mathcal{L}_{\overset{\circ}{n},\overline{\check{\lambda}},!*} \otimes \mathcal{M}(\overline{\check{\lambda}})$, where $\mathcal{L}_{\overset{\circ}{n},\overline{\check{\lambda}},!*}$ is a relative (over $\overset{\circ}{X}{}^n$) version of $\mathcal{L}_{\overline{x},\overline{\check{\lambda}},!*}$ and $\mathcal{M}(\overline{\check{\lambda}})$ is a D-module on $\overset{\circ}{X}{}^n$.

(c) The functor $\mathsf{G}_{\overset{\circ}{n}}$ induces an equivalence

$$\mathrm{Whit}^c_{\overset{\circ}{n}} \simeq \mathrm{FS}^c_{\overset{\circ}{n}} . \tag{8.2}$$

For any partition $\overline{n}$: $n = n_1 + ... + n_k$ we have the mutually adjoint functors

$$(\Delta_{\overline{n}})^!, (\Delta_{\overline{n}})_* : D^+(\mathrm{Whit}^c_n) \leftrightarrows D^+(\mathrm{Whit}^c_{\overset{\circ}{k}}),$$

which [12] are intertwined by means of G with the corresponding functors

$$(\Delta_{\overline{n}})^!, (\Delta_{\overline{n}})_* : D^+(\widetilde{\mathrm{FS}}^c_n) \leftrightarrows D^+(\widetilde{\mathrm{FS}}^c_{\overset{\circ}{k}}).$$

The commutative diagrams

$$\begin{tikzcd}
\mathrm{Whit}^c_n \arrow[r, "(\Delta_{\overline{n}})^!"] \arrow[d, "\mathsf{G}_n"'] & D^+(\mathrm{Whit}^c_{\overset{\circ}{k}}) \arrow[d, "\mathsf{G}_{\overset{\circ}{k}}"'] \\
\widetilde{\mathrm{FS}}^c_n \arrow[r, "(\Delta_{\overline{n}})^!"] & D^+(\widetilde{\mathrm{FS}}^c_{\overset{\circ}{k}})
\end{tikzcd}$$

and (8.2) imply that G_n indeed sends Whit^c_n to FS^c_n, as required.

8.8. Finally, let us show that G_n is an equivalence.

Let $\mathrm{Whit}^c_{Diag(n)} \subset \mathrm{Whit}^c_n$ and $\mathrm{FS}^c_{Diag(n)} \subset \mathrm{FS}^c_n$ be the subcategories of objects, consisting of twisted D-modules that supported over the diagonal divisor $X^{Diag(n)} \subset X^n$. By induction on n we can assume that G_n induces an equivalence

$$\mathsf{G}_{Diag(n)} : \mathrm{Whit}^c_{Diag(n)} \simeq \mathrm{FS}^c_{Diag(n)}\,. \tag{8.3}$$

Let $i_{Diag(n)}$ denote the morphism $X^{Diag(n)} \to X^n$. It induces the functors

$$(i_{Diag(n)})_* : \mathrm{Whit}^c_{Diag(n)} \to \mathrm{Whit}^c_n \text{ and } \mathrm{FS}^c_{Diag(n)} \to \mathrm{FS}^c_n,$$

which are intertwined by G. The same is true for the functors

$$(j^{poles})_* : \mathrm{Whit}^c_{\overset{\circ}{n}} \to \mathrm{Whit}^c_n \text{ and } \mathrm{FS}^c_{\overset{\circ}{n}} \to \mathrm{FS}^c_n\,.$$

Hence, in order to prove the theorem, it suffices to show the following: for every $\mathcal{F} \in \mathrm{Whit}^c_{\overset{\circ}{n}}$ there exists an inverse family of objects $\mathcal{F}_i \in \mathrm{Whit}^c_{\overset{\circ}{n}}$ with $(j^{poles})^*(\mathcal{F}_i) \simeq \mathcal{F}$, such that for any $\mathcal{F}' \in \mathrm{Whit}^c_{Diag(n)}$ the direct limits

$$\varinjlim_i \mathrm{Ext}^k_{\mathrm{Whit}^c_n}\left(\mathcal{F}_i, (i_{Diag(n)})_*(\mathcal{F}')\right) \text{ and } \varinjlim_i \mathrm{Ext}^k_{\mathrm{FS}^c_n}\left(\mathsf{G}_n\left(\mathcal{F}_i\right), \mathsf{G}_n\left((i_{Diag(n)})_*(\mathcal{F}')\right)\right) \tag{8.4}$$

vanish for $k = 0, 1$. Note that in both cases, the corresponding Hom and Ext^1 can be computed inside the ambient category of D-modules.

Consider the pro-object in the category of D-modules on $\mathfrak{W}_n$, given by $(j^{poles})_!(\mathcal{F})$. It is easy to see that it can be represented as the limit of an inverse family of objects from $\mathrm{Whit}^c_{\overset{\circ}{n}}$. We take $\mathcal{F}_i$ to be this family.

With this choice, the vanishing of the first limit in (8.4) is automatic. The vanishing of the second limit follows from Theorem 4.11 and the $(\pi^{\tilde{\mu}}_!, \pi^{\tilde{\mu}!})$ adjunction.

[12] Here again $D(\cdot)$ is understood as the derived category of the abelian category, but it is easy to see that the above functors induce the usual functors on the level of underlying twisted D-modules.

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