ON D.K. BISS' PAPERS "THE HOMOTOPY TYPE OF THE MATROID GRASSMANNIAN" ANNALS OF MATHEMATICS 158 (2003) 929-952 AND "ORIENTED MATROIDS, COMPLEX MANIFOLDS, AND A COMBINATORIAL MODEL FOR BU" ADVANCES IN MATHEMATICS 179 (2003) 250-290

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My very unfortunate duty is to point out a serious flaw in papers [Bis03a, Bis03b] by Daniel Biss devoted to homotopy type of matroid Grassmannians. Four years passed after the publication, but no errata became available. Meanwhile the problem was already acknowledged and discussed in private correspondence between experts two years ago.

The mistake is the same for both papers, it is very simple, looks almost like a typo, but it is located in the key propositions for the induction towers – Proposition 4.5 [Bis03a] on page 948 and Proposition 7.3 [Bis03b] on page 285. Here the aim of Biss' reasons is to show that his natural combinatorial models for Schubert cells are correctly attached one to another. In the surgery of such an attachment some important complex called $||S^+||$ appears. Biss needs to prove that $||S^+||$ is contractible. Biss covers $||S^+||$ by contractible *open* set $O = (||S^+|| \setminus ||A||)$ and a *closed* set $C = ||B||$ homotopy equivalent to $C \cap O$ and concludes that it follows that $||S^+||$ is contractible. This is not correct. For example one can easily cover a circle by open and closed intervals with the same property and thus prove that the circle is also contractible.

So the statement in the final lines of the proof of [Bis03a, Proposition 4.5, 11-13 lines from the top of the page 948] :

> *"...and thus the inclusion $||S^+|| \setminus ||A|| \hookrightarrow ||S^+||$ induces a homotopy equivalence, and $||S^+||$ is therefore contractible"*

is wrong.

The same argument is used by Biss in the proof of [Bis03b, Proposition 7.3, page 285 lines 11-13 from the top]. There the role of $||S^+||$ is played by $||S_+^{\mathfrak{M}}||$.

Unfortunately this simple mistake destroys the main theorems of both papers. If one tries to continue the cut and paste induction over $||S^+||$ by cutting it into smaller natural pieces then one will quickly face "ball-like" posets formed by *weak maps* of oriented matroids which have a well known bad habit to be homotopy nontrivial (see for example [MRG93]). Surprisingly the weak maps of matroids do not appear up to this hidden point in Biss' scheme at all. This looks strange by the reasons of dimension of Grassmannians.

Personally, I don't currently see a way to save Biss' theorems in those strongest forms (that finite matroid posets can model finite-dimensional Grassmannians). On the other hand in stable infinite dimensional case when Grassmannian became classifying space for stable vector bundles I think that matroid models really may work very well and one can use much more geometric topology to prove it.

References

[Bis03a] Daniel K. Biss. The homotopy type of the matroid Grassmannian. *Ann. of Math. (2)*, 158(3):929–952, 2003.

[Bis03b] Daniel K. Biss. Oriented matroids, complex manifolds, and a combinatorial model for BU. *Adv. Math.*, 179(2):250–290, 2003.

[MRG93] Nicolai E. Mnëv and Jürgen Richter-Gebert. Two constructions of oriented matroids with disconnected extension space. *Discrete Comput. Geom.*, 10(3):271–285, 1993.