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DISCUSSION ON BENFORD'S LAW AND ITS APPLICATION

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Abstract. The probability that a number in many naturally occurring tables of numerical data has first significant digit (i.e., first non-zero digit) d is predicted by Benford's Law $\text{Prob}\,(d) = \log_{10}\left(1+\frac{1}{d}\right), d = 1, 2\ldots, 9$. Illustrations of Benford's Law from both theoretical and real-life sources on both science and social science areas are shown in detail with some novel ideas and generalizations developed solely by the authors of this paper. Three tests, Chi-Square test, total variation distance, and maximum deviations are adopted to examine the fitness of the datasets to Benford's distribution. Finally, applications of Benford's Law are summarized and explored to reveal the power of this mathematical principle.

1. Introduction

The significant-digit law of statistical folklore is the empirical observation that in many naturally occurring tables of numerical data, the leading significant digits are not uniformly distributed as might be expected, but instead follow a particular logarithmic distribution. Back to 1881, the astronomer and mathematician Simon Newcomb published a 2-page article in the American Journal of Mathematics describing his observation that books of logarithms in the library were dirtier in the beginning and progressively cleaner throughout [1]. He inferred that researchers, be them mathematicians, biologists, sociologists as well as physicists, were looking up numbers starting with 1 much more often than numbers beginning with 2, and numbers with first digit 2 more often than 3, and so on. This ingenious discovery led him to conclude that the probability that a number has first significant digit (i.e., first non-zero digit) d is $\text{Prob}\,(d) = \log_{10}\left(1+\frac{1}{d}\right), \quad d = 1, 2\ldots 9$. In particular, his conjecture stated that the first digit is 1 about 30.1% of the time, and is 9 only about 4.6% of the time. That digits are not equally likely comes as somewhat of a surprise, but to claim an exact law describing their distribution is indeed striking. Passed unnoticed, the proposed law was discovered again and supported by empirical evidence by the physicist Benford who analyzed the frequencies of significant digits from twenty different tabled including such diverse data as surface areas of 335 rivers, specific heat of thousands of chemical compounds, and square-root tables [2]. And this First Digit Law is known as Benford's Law today. But in recognition

of Newcomb's discovery, we can call it Newcomb-Benford's Law. This law applies to stock prices [3], number of hours billed to clients [4] or income tax [5] as well as mathematical series [6]. And the tremendous practical values of Benford's Law were neglected until recently many mathematicians began to focus on the applications of this amazing phenomenon such as the design of computers and analysis of roundoff errors [7, 14, 15], as well as a goodness-of-fit against Benford to detect fraud [8].

In this paper, we will be dealing with heuristic argument and distributional property of the Significant-digit law in section 2; checking how data from various sources fit Benford's Law in section 3; discussing the application of Benford's law in section 4.

2. Mathematical Formulation

2.1 Heuristic Argument

Pietronero and his colleagues gave a general explanation for the origin of the Benford's law in terms of multiplicative processes in 2001 [3]. The explanation was amended slightly in such a way that it can be used to explain the Benford's distribution not specifically to base 10. It stated that many systems such as the stock market prices which is discussed later do not follow the dynamical description by a Brownian process:

$$N(t+1) = \xi + N(t),$$

but rather a multiplicative process:

$$N(t+1) = \xi N(t)$$

where ξ is a stochastic variable. By a simple transformation, we get

$$\ln N(t+1) = \ln \xi + \ln N(t).$$

If you consider $\ln \xi$ as the new stochastic variable, you recover a Brownian dynamics in a logarithmic space; here we mean that a random multiplicative process corresponds to a random additive process in logarithmic space. This implies that as $t \to \infty$, the distribution $\text{Prob}\,(\ln N)$ approaches a uniform distribution. By transforming back to the linear space we have

$$\int \text{Prob}\,(\ln N) d(\ln N) = \int C d(\ln N) = C \int \frac{1}{N} dN,$$

where C is a constant.

It should be noted that $\text{Prob}\,(N) = \dfrac{1}{N}$ is not a proper probability distribution, as it diverges or put in another way $\int_0^\infty \dfrac{1}{N}\, dN$ is undefined. However the physical laws

and human conventions usually impose maximums and minimums. The probability that the first significant digit of N is n in base b is given by the following expression:

$$\text{Prob}\,(n) = \frac{\int_n^{n+1} \frac{1}{N} dN}{\int_1^b \frac{1}{N} dN} = \frac{\ln \frac{n+1}{n}}{\ln b} = \log_b \left(1 + \frac{1}{n}\right),$$

for any n which is less than b. We can review $\text{Prob}\,(n) = \log_b \left(1 + \frac{1}{n}\right)$ as a generalized expression of Benford's law to arbitrary base b.

2.2 The Significant-Digit Law and Some Consequences

2.2.1 Significant-Digit Law

The Significant-Digit Law is

$$\text{Prob}\,(D_1 = d_1, \ldots, D_k = d_k) = \log_{10} \left[1 + \left(\sum_{i=1}^{k} d_k \times 10^{k-1}\right)^{-1}\right]$$

where $D_1, D_2, \ldots D_k$ are the first, second $\ldots$ k'th digits respectively.

For example, $\text{Prob}\,(D_1 = 1, D_2 = 2, D_3 = 9) = \log_{10}(1 + (129)^{-1}) \cong 0.00335$, which means there is a probability of 0.00335 that the first three significant digits are 129 in a sample of Benford's distribution [11].

Hill has proved in his papers: "Scale-Invariance implies Base-Invariance", not vice versa [9]; "Base-Invariance implies Benford's Law [10]"; "The logarithmic distribution is the unique continuous base-invariant distribution [9]". He has explained the Central-limit-like Theorem for Significant Digit by saying: "Roughly speaking, this law says that if probability distributions are selected at random and random samples are then taken from each of these distributions in any way so that the overall process is scale (or base) neutral, then the significant-digit frequencies of the combined sample will converge to the logarithmic distribution [11]".

2.2.2 Distributional Properties of D_k's

Based on the Significant-Digit Law, we analyze the statistics and distributional properties of a Benford's Distribution.

2.2.2.1 Mean and Variance for D_k

In this subsection, we compute the numerical values of the means and variances of D_k's using these expressions:

$$E(D_k) = \sum_{1}^{9} n\,\text{Prob}\,(D_k = n)$$

$$\mathrm{Var}\,(D_k) = \sum_{n=1}^{9} n^2\,\mathrm{Prob}\,(D_k = n) - E(D_k)^2,$$

and tabulate them below.

Table 1. Mean and variance of D_k for $k = 1$ to 7

k	$E(D_k)$	Var (D_k)
1	3.44023696712 (5)	6.0565126313757 (6.67)
2	4.18738970693 (4.5)	8.2537786232732 (8.25)
3	4.46776565097 (4.5)	8.2500943647286 (8.25)
4	4.49677537552 (4.5)	8.2500009523513 (8.25)
5	4.49967753636 (4.5)	8.2500000095245 (8.25)
6	4.49996775363 (4.5)	8.2500000000953 (8.25)
7	4.49999677536 (4.5)	8.2500000000016 (8.25)

This shows that by the Significant-digit law the mean of D_k is approaching 4.5 which is the mean if the distribution were uniform and the variance of D_k is approaching 8.25 which is the variance if the distribution were uniform.

2.2.2.2 Histogram of D_k for k

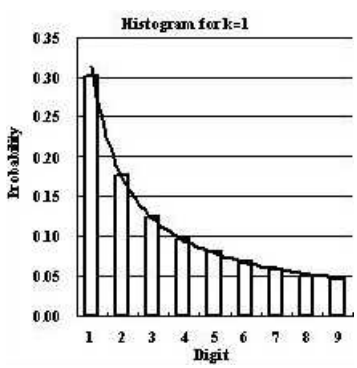

Fig 1. Histograms of D_k for $k = 1$

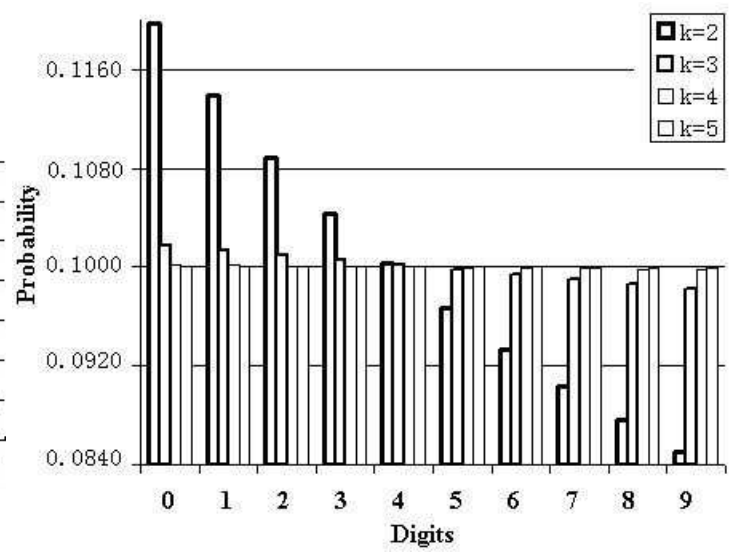

Fig 2. Histograms of D_k for $k = 2, 3, 4, 5$

With the histogram, we illustrate that the distribution of the nth significant digit approaches the uniform distribution rapidly as $n \to \infty$.

2.2.2.3 Total variation distance of D_k with respect to a uniform distribution

By calculating the total variation distance $d(D_1, U)$ of D_k from a uniform distribution on $\{0, 1, 2 \ldots 9\}$ using these expressions:

$$d(D_1, U) = \frac{1}{2} \sum_{n=1}^{9} \left| \text{Prob}\,(D_1 = n) - \frac{1}{9} \right| \text{ for } k = 1$$

$$d(D_k, U) = \frac{1}{2} \sum_{n=0}^{9} \left| \text{Prob}\,(D_k = n) - \frac{1}{10} \right| \text{ for } k \neq 1.$$

As shown in the table, we observe that the D_k is converging to the uniform distribution geometrically.

Table 2. Total Variation Distance of D_k from a uniform distribution on $\{0, 1, 2, \ldots, 9\}$.

k	$d(D_k, U)$
1	0.26872666
2	0.04702863
3	0.00488356
4	0.00048858
5	0.00004886
6	0.00000489
7	0.00000049
$k \to \infty$	$d(D_k, U) \to 0$

2.2.2.4 Correlation coefficient of D_i and D_j

Here, by calculating the correlation coefficient of D_i and D_j,

$$\rho_{D_i D_j} = \frac{\text{Cov}\,(D_i, D_j)}{\sqrt{\text{Var}\,(D_i)\,\text{Var}\,(D_j)}}$$

for $0 < i < j$. we intend to investigate the dependence of one digit on another digit. We conclude that the dependence among significant digits decreases as the distance $(j - i)$ increases.

Table 3.Correlation coefficient of D_i and D_j

$i \backslash j$	2	3	4	5
1	0.0560563	0.0059126	0.0005916	0.0000591
2		0.0020566	0.0002059	0.0000205
3			0.0000228	0.0000022
4				0.0000002

3. Illustrations

The Chi-Square test formula to check the sample's goodness of fit to Benford's distribution is given as below,

$$\chi^2(8^\circ) = \sum_{n=1}^{9} \frac{\left(\log\left(1 + \frac{1}{n}\right) - \text{Prob}\,(D_1 = n)\right)^2}{\log\left(1 + \frac{1}{n}\right)} \times S,$$

where S denotes the sample size. The critical values for 8 degree of freedom at 0.05 and 0.10 level of significance are respectively 15.51 and 20.09.

Another test, which we have used to analyze the sample, is the total variation distance (denoted as d_1 here) of the sample from the Benford's distribution on

$\{1, 2, \ldots, 9\}$:

$$d_1 = \frac{1}{2}\sum_{n=1}^{0}\left|\text{Prob}\,(D_1 = n) - \log(1 + \frac{1}{n})\right|.$$

The maximum of deviations from a uniform distribution for each digit could also be considered as a test to check the goodness of fit:

$$d_{\max} = \max_{1\le n\le 9}\left\{\left|\text{Prob}\,(D_1 = n) = \log\left(1 + \frac{1}{n}\right)\right|\right\}.$$

3.1 Physical Constants

Many literatures on Benford's law cite the table of physical constants as an example to illustrate Benford's law [9, 10, 11]. It was only Burke and his colleagues who actually attempted to check whether the physical constants would match Benford's law [12].

However they only chose the constants from an introductory physics text. The sample size is too small to be significant. Here the 183 constants from the 1998 Committee On Data for Science and Technology recommended complete listing of the fundamental physical constants (http://physics.nist.gov/constants) were analyzed for the first time.

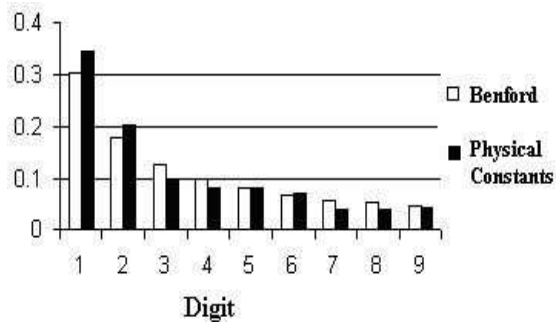

Fig 3. Histogram on fitness of physical constant

Table 4. First Significant Digit Test for the physical constants.

Digits	Counts in Sample Frequency	Predicted by Benford's Law	Frequency Observed in Sample	The Difference
1	63	0.3010	0.3443	0.0432
2	37	0.1761	0.2022	0.0261
3	18	0.1249	0.0984	-0.0266
4	15	0.0969	0.0820	-0.0149
5	15	0.0792	0.0820	0.0028
6	13	0.0669	0.0710	0.0041
7	7	0.0580	0.0383	-0.0197
8	7	0.0512	0.0383	-0.0129
9	8	0.0458	0.0437	-0.0020
Sample Size: 183			Chi-Square Test: 5.206	
Total variation distance: 0.0762			Maximum of deviations: 0.0432	

3.2 Stock Prices & One-day Returns on Stock Index

Interesting results on the frequency of first significant digit of one-day returns on stock index were found by Ley [13]. However, investigations on stock prices were not done. According to Pietronero, due to the multiplicative process, a stock's prices over a long period of time should conform to Benford's Law. Here, we look at all stock's prices on certain days rather than a single stock's prices on a series of days. We collected data on local stock market SGX main board from http://www.sgx.com.

Table 5. Analysis of Stock Prices

	Sample Source	Sample Size	χ^2	d_1	$d_{\max}$
A	One-trading-day (13/10/2003)	548	2.23	0.0277	0.01734
B	20-trading-Day (10/10/2003- 7/11/2003)	11,015	45.5	0.0255	0.01501
C	32-trading-Day (10/10/2003- 26/11/2003)	17,214	79.3	0.0284	0.01834

The observed frequencies roughly agree with the theoretical frequency predicted by Benford's law, indicted by the relatively small values of the total variation distance (0.0277, 0.0255 and 0.0284) of the sample and the Benford's distribution. However, if we performed the usual chi-square test of goodness of fit, we would reject the null hypothesis for datasets B and C, as the chi-square values are much greater than 15.51 for the critical chi-square test value of 8 degrees of freedom at 5% significant level or 20.09 at 1% significant level. It is because of the large number of observations - that is, the classical acceptance region shrinks with the sample size, given a significant level. On the contrary, the Benford's hypothesis would not be rejected for dataset A.

One interesting thing, which in three sets of data $d_{\max}$ always occurs at $n = 2$, attracts our attention. By plotting the histograms of three datasets against a Benford's distribution, it is found that the distributions of stock price data are in quite agreement among themselves but a little different from Benford's distribution.

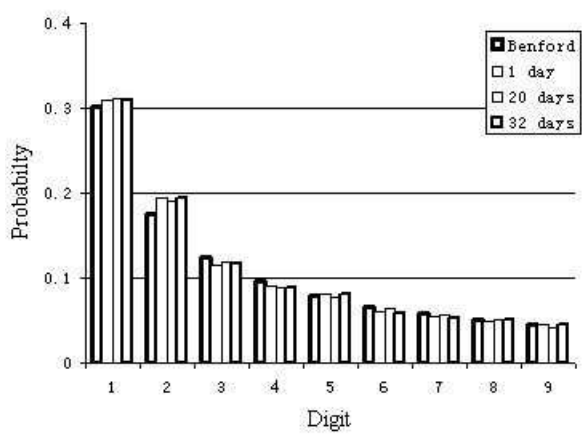

Fig 4. Histogram on fitness of stock prices

We attribute this to the fact that a simple multiplicative process cannot precisely reproduce the complicated stock process thus the distribution of stock prices data may follow a curve that is steeper than the log curve of Benford's Law.

3.3 Some Mathematical Series

3.3.1 Fibonacci Series

Here we provide a simple explanation for the conformance of Fibonacci series to Benford's law. Fibonacci series is characterized by the recursion relation: $a_{n+2} = a_{n+1} + a_n$. This formula does not provide the explicit information of a general term from the series that may suggest the origin of Fibonacci series' conformance to Benford's law. However, it could be easily derived that say for a series with $a_1 = 1$, $a_2 = 2$:

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3\sqrt{5}+5}{10}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{5-3\sqrt{5}}{10}\right).$$

Moreover, the magnitude of the second term is $\left|\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{5-3\sqrt{5}}{10}\right)\right|$ which is always less than 1 and approaches zero when n gets larger and larger. Thus to analyze the first significant digit of a_n, we only need to analyze:

$$a'_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{5+3\sqrt{5}}{10}\right)$$

instead. Now we have $\ln a'_n = (n-1)\ln\left(\frac{1+\sqrt{5}}{2}\right) + \ln\left(\frac{5+3\sqrt{5}}{10}\right)$. When the sample size is large, distribution on n could be considered as uniform, with $\ln\left(\frac{1+\sqrt{5}}{2}\right)$ and $\ln\left(\frac{5+3\sqrt{5}}{10}\right)$ being constants, $\{\ln a'_n\}$ will be uniform. It is then followed that the $\{a'_n\}$ and therefore $\{a_n\}$ will be uniformly distributed in logarithmic space and will conform to Benford's law.

3.3.2 The Prime-number Series

The prime number series is rather uniform below 100000, with the probability of each possible first significant digit being between 12.5% and 10.4%. Moreover, using the upper and lower bounds of function pi from the prime number theorem, it can be shown that the prime number sequence approximates a uniform distribution.

3.3.3 Sequence α^n in base b, where $\alpha \in R$, $n \in N$

The origin of Benford's distribution in α^n is often attributed to its scale invariance which means that any power law relation is scale invariant:

$$\text{Prob}\,(\lambda N) = f(\lambda)\,\text{Prob}\,(N).$$

An even simpler argument can be done. For α^n in base b, where α is a constant, the sufficient condition for which the first significant digit of α^n is d is $d \times b^k \leq \alpha^n < (d+1) \times b^k$ (that is $\log_b d + k \leq n \log_b \alpha < \log b(d+1+k)$ for an integer k. Thus for each k, the probability for the first significant digit of α^n being d is

$$\frac{(\log_b(d+1)+k) - (\log_b d + k)}{(k+1)-k} = \log_n \left(\frac{d+1}{d}\right).$$

3.3.4 Factorial

For the first 160 factorial numerical values, the chi-square test does not reject the Benford's distribution; however the total variation distance is too big for it to be considered as fit.

3.3.5 Sequence of Power, i.e. $n^2, n^3, n^4, n^5, \ldots$

For n^k series, as the constant k increases, the distribution becomes closer to the Benford's law. All values of the three tests get smaller as k increases, as shown in the analysis of n^2, n^5, n^{20} and n^{50} for $n = 1$ to 30,000.

3.3.6 Numbers in Pascal Triangle

We suspect that the numbers in Pascal Triangle obeys Benford's law, because it relates to Fibonacci Series closely while in another perspective it could be considered as a mixture of sequences of power. However in this paper, no data on Pascal triangle were analyzed.

Table 6. Analysis of some mathematical series

Sample	Sample Size	χ^2	d_1	$d_{\max}$
Fibonacci Series (7 series)	10, 317	0.0125	3.8×10^{-4}	1.7×10^{-4}
Prime-number below 1,000	168	45.0	0.2271	0.1522
Prime-number below 100,000	9761	3247	0.4905	0.1761
1.007^n, Specifically for $n = 1$ to 30,000	30,000	0.410	1.2×10^{-3}	5.9×10^{-4}
1.007^n, Specifically for $n = 1$ to 65,028	65,028	0.0329	2.5×10^{-4}	$1,2 \times 10^{-4}$
Factorial of 1 to 100	100	6.95	0.0651	0.04885
Factorial of 1 to 130	130	8.97	0.0871	0.03492
Factorial of 1 to 160	160	10.10	0.0834	0.03635
n^2, Specifically for $n = 1$ to 30,000	30,000	3.16×10^3	0.1409	0.09900
n^5, Specifically for $n = 1$ to 30,000	30,000	2.76×10^2	0.0394	0.03640
n^{20}, Specifically for $n = 1$ to 30,000	30,000	20.8	0.0112	0.00681
n^{50}, Specifically for $n = 1$ to 30,000	30,000	3.7	0.0048	0.00256

3.4 Demographic Statistics and other social science data

The demographic data that the author used in this section are obtained from Energy Information Administration United States Department of Energy.

Table 7. Analysis of demographic data

	Sample Size	χ^2	d_1	$d_{\max}$
GDP total for each country in 1992-2001	1606	33.0	0.1009	0.03352
World Consumption of Primary Energy by Selected Country Groups, 1992-2001	480	40.5	0.1113	0.05936
Population for each country in year 2001	212	6.59	0.0792	0.04160
Crude Oil Production, Import and Export, Stock Build Statistics for Each Country in 1990-2001	4196	32.2	0.0383	0.02166

The chi-square test rejects the null hypothesis (the data's obeys the Benford's distribution) for all except the population statistics; while the total variation distance says that the oil statistics fit the Benford's distribution better than the other three.

Other two sets of statistics in social science obtained from *The World Affair Companion* by Gerald Segal (1996) are also analyzed. The null hypothesis is accepted in both cases in the chi-square test.

Table 8. Analysis of social science data

	Sample Size	χ^2	d_1	$d_{\max}$
Annual average number of people reported killed or affected by disasters per region and country in 1968-1992	333	11.05	0.0715	0.03222
Weapon imports of 50 leading recipients in US dollars in 1990-1994	291	11.06	0.0832	0.02872

3.5 Numbers appeared in Magazines and Newspaper

Among the data compiled in Benford's original paper [2], "numbers appeared in Reader's Digest" attracts our attention. Numbers appeared in a magazine or newspaper seems random at first glance; its conformance to Benford's law was deemed as a coincidence rather a rule. However by referring to the Hill's paper, "selected at random and random samples ... then the significant-digit frequencies of the combined sample will converge to the logarithmic distribution" [11], we hypothesize that

numbers appeared in Reader's Digest could be considered as a mixture of data of different distributions and more specialized magazines containing biased mixture of data will deviate from Benford's distribution more significantly. Here numbers appeared in both Innovation (science orientated) and The Economist (current affairs orientated, more on social side) were analyzed. It was found that both do not conform well to the Benford's distribution, however the combination dataset conforms to Benford much better with total variation distance of 0.0474. This supports our hypothesis.

Table 9. Analysis of numbers appeared in magazines

Numbers appeared in	Sample Size	x^2	d_1
Innovation (Vol. 2 No. 4. 2002)	152	6.57	0.0747
The Economist (17th May 2003)	449	16.24	0.0788
Combination of the above two	601	9.22	0.0474

3.6 Other Datasets

Other kinds of datasets, which are usually quoted as example of Benford's law include: survival distribution [14], the magnitude of the gradient of an image & Laplacian Pyramid Code [15], radioactive half-lives of unhindered alpha decays [11]. Datasets compiled by Benford in his original paper include: Rivers area, population, newspapers, specific heat, molecular weights, atomic weights, Reader's Digest, X−Ray volts, American league, addresses, death rate ... [2] They provide a wide range of examples to illustrate Benford's Law, however the certainty of their conformance to Benford's law varies largely.

4. Application of Benford's Law

The analysis of first few significant digit frequencies provides us a potential framework to examine the accuracy and authenticity of data values in numerical data set.

4.1 The Design of Computers and Analysis of Roundoff Errors

Peter Schatte has determined that based on an assumption of Benford input, the computer design that minimizes expected storage space (among all computers with binary-power base) is base 8 [16], and other researchers have started exploring the use of logarithmic computers to speed up calculations [17]. It will be crucial for the development of computing science in the information age.

4.2 A Goodness-of-fit against Benford to Detect Fraud

A. Rose and J. Rose developed a VBA code for detecting fraud under Excel working environment [8]. However the program can only do a first digit test but not a

first significant digit test (for example, the 1 in 0.150 will not be counted by the program), and the program will encounter error upon reading text (not numbers). We improved the program to eliminate all the limitations of the original program.

However, we argue that the use of a goodness-of-fit test against Benford to detect fraud is dubious. It has many limitations. Firstly, this method to detect fraud will not be accurate if the data in the account has very close built-in maximums and minimums. Secondly, a one-off embezzlement or very few entries of amendments made in an accounting record will not be detected using Benford's Law. Moreover the public or specifically here the fraudsters learn very quickly, after a few cases of successful detections of fraud using Benford's law being reported, they may take special cares upon changing the entries in the account table in order to conform to Benford's law. Therefore, we conclude that Benford's law can only serve as an initial strategy to detect fraud and it will become less useful later.

4.3. Others

Mark Nigrini mentioned in his paper [4] that Benford's law could be used to test the accuracy of measuring equipment. Hill in his paper suggested Benford's law as a test of reasonableness of forecasts of a proposed model [9].

5. Conclusions

Among the large range of samples we have analyzed, by using Chi-square test and calculating total variation distance and maximum of deviations, Fibonacci series, Sequence α^n, population and number in magazines fit Benford's distribution well. Fitness to Benford's law of other data such as physical constants and stock prices are controversial, and they require further investigations. Two common explanations for the origin of Benford's distribution are multiplicative process and scale invariance. One simple multiplicative process cannot reproduce most of natural occurring phenomena thus by virtue of multiplicative process it cannot explain the origin of Benford's distribution for all datasets that are fit. The understanding of the origin of scale invariance has been one of the fundamental tasks of modern statistical science. How systems with many interacting degrees of freedom can spontaneously organize into scale invariant states will be the focus of further research.

It is generally believed that the more chaotic and diversified the probability distributions are, the better the overall data set fits the Benford's Law. Thus, we put forward a bold conjecture that Benford's Law is an act of nature and can be used to indicate the randomness of our world. Therefore it can be related to the second thermodynamics law of entropy. This will be an interesting area for further investigation.

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