THERE ARE INFINITELY MANY COUSIN PRIMES

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ABSTRACT. We proved that there are infinitely many cousin primes.

1. Introduction

If p_c and $p_c + 4$ are both primes, then this prime pair is called cousin primes. Hardy and Littlewood conjectured that cousin primes have the same asymptotic density as twin primes[1, 4]. Is there infinitely cousin primes? It is an old unsolved conjecture[2, 3]. We will proved it as theorem 1.1.

Theorem 1.1. *There are infinitely many cousin primes.*

Let P=$\{p_1, p_2, ..., p_v\} = \{2, 3, ..., p_v\}$ be the primes not exceeding $\sqrt{n}$, then the number of primes not exceeding n [2] is,

$$\pi(n) = \begin{cases} (\pi(\sqrt{n}) - 1) + n - \left(\left\lfloor \frac{n}{p_1} \right\rfloor + \left\lfloor \frac{n}{p_2} \right\rfloor + \cdots + \left\lfloor \frac{n}{p_v} \right\rfloor\right) \\ + \left(\left\lfloor \frac{n}{p_1 p_2} \right\rfloor + \left\lfloor \frac{n}{p_1 p_3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p_{v-1} p_v} \right\rfloor\right) + \cdots . \end{cases} \tag{1.1}$$

For simplicity, we can write it as,

$$\begin{aligned} \pi(n) &= (\pi(\sqrt{n}) - 1) + n \left[1 - \frac{1}{p_1}\right] \left[1 - \frac{1}{p_2}\right] \cdots \left[1 - \frac{1}{p_v}\right] \\ &= (\pi(\sqrt{n}) - 1) + n \prod_{i=1}^{v} \left[1 - \frac{1}{p_i}\right], \end{aligned} \tag{1.2}$$

where $n\left[1 - \frac{1}{p_i}\right] = n - \left\lfloor \frac{n}{p_i} \right\rfloor$, $n\left[1 - \frac{1}{p_i}\right]\left[1 - \frac{1}{p_j}\right] = n - \left\lfloor \frac{n}{p_i} \right\rfloor - \left\lfloor \frac{n}{p_j} \right\rfloor + \left\lfloor \frac{n}{p_i p_j} \right\rfloor$. The operator $\left[1 - \frac{1}{p_i}\right]$ will leave the items which are not multiples of p_i.

In this paper, $\lfloor x \rfloor \le x$ is the floor function, $\lceil x \rceil \ge x$ the ceiling function of x. The integral operator '[ ]', which is not the floor function in this paper, has meanings only operating on (real) number: $m[\ ] = [m] = \lfloor m \rfloor$.

2. The number of cousin primes less than p_{v+1}^2

Let $Z = \{1, 2, \cdots, n\} (n < p_{v+1}^2)$ be a natural arithmetic progression, $Z' = Z + 4 = \{5, 6, \cdots, n + 4\}$ be its accompanying arithmetic progression, so that $Z'_k = Z_k + 4, k = 1, 2, \cdots, n$. There are n such pairs.

$$\begin{cases} Z &= \{ \quad 1, \quad 2, \quad \cdots, \quad n \quad \} \\ Z' &= \{ \quad 5, \quad 6, \quad \cdots, \quad n+4 \quad \}. \end{cases} \tag{2.1}$$

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If we delete all the pairs in which one or both items Z_k and Z'_k are composite integers of all primes $p_i \leq \sqrt{n}$, then the pairs left are all cousin primes. Because after deleted the composite integers in sets Z and Z', the items in the middle set $Z'' = Z + 2 = \{3, 4, \cdots, n+2\}$ are all composite integers: if Z''_k is prime, then $Z''_k\mathbf{mod}3 = 1$ or 2, we have either $Z'_k\mathbf{mod}3 = (Z''_k + 2) = 0$ or $Z_k\mathbf{mod}3 = (Z''_k - 2) = 0$, so this item has already been deleted.

For a given p_i, first we delete the multiples of p_i in set Z, or the items of $Z_k\mathbf{mod}p_i = 0$,

$$y(p_i) = \left\lfloor \frac{n}{p_i} \right\rfloor \equiv n\left[\frac{1}{p_i}\right]. \tag{2.2}$$

Secondly we delete the multiples of p_i in Z', i.e. the items of $Z'_k\mathbf{mod}p_i = 0$, or $Z_k\mathbf{mod}p_i = p_i - 4$ (or $p_i - 1 = 2$ for $p_i = 3$, hereafter),

$$y'(p_i) = \left\lfloor \frac{n+4}{p_i} \right\rfloor \equiv n\left[\frac{\widetilde{1}}{p_i}\right]. \tag{2.3}$$

Because $Z_k\mathbf{mod}p_i \neq Z'_k\mathbf{mod}p_i$ when $p_i \neq 2$, the items deleted by $y(p_i)$ and $y'(p_i)$ are not the same. For $p_i = 2$, $Z_k\mathbf{mod}p_i = Z'_k\mathbf{mod}p_i$, we need only delete the items in set Z:

$$y'(p_i) = 0, \quad \text{for} \quad p_i = 2. \tag{2.4}$$

Eq. (2.2) will delete all pairs with $Z_k\mathbf{mod}p_i = 0$ in set Z, and Eq. (2.3) will delete all pairs with $Z_k\mathbf{mod}p_i = p_i - 4$ in set Z.

$$y'(p_i) = n\left[\frac{\widetilde{1}}{p_i}\right] = n\left\lfloor \frac{1}{p_i} \right\rfloor + \delta, \tag{2.5}$$

where $0 \leq \delta \leq 1$ with,

$$\delta = \begin{cases} 0: & 0 \leq n\mathbf{mod}p_i + 4 < p_i \quad \text{for } p_i \geq 5 \\ 2: & n\mathbf{mod}p_i + 1 = p_i \quad \text{for } p_i = 3 \\ 1: & \text{else.} \end{cases} \tag{2.6}$$

The pairs left, after deleted all the multiples of p_i in both Z and Z', have,

$$M(p_i) = n - y(p_i) - y'(p_i) \equiv n\left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right]. \tag{2.7}$$

The operator $\left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right]$, when operating on n, will leaves the items having not those of $Z_k\mathbf{mod}p_i = 0, p_i - 4$ (or 2 for $p_i = 3$).

When the multiples of all primes $p_i \leq p_v < \sqrt{n}$ in both Z and Z' have been deleted, the pairs left will be prime pairs (cousin primes) and have,

$$\begin{aligned} D_0(n) &= n\left[1 - \tfrac{1}{p_1}\right]\left[1 - \tfrac{1}{p_2} - \tfrac{\widetilde{1}}{p_2}\right]\cdots\left[1 - \tfrac{1}{p_v} - \tfrac{\widetilde{1}}{p_v}\right] \\ &= n\left[1 - \tfrac{1}{p_1}\right]\prod_{i=2}^{v}\left[1 - \tfrac{1}{p_i} - \tfrac{\widetilde{1}}{p_i}\right], \end{aligned} \tag{2.8}$$

the meaning is as follows,

$$\left\{\begin{array}{l} n\left[\frac{1}{p_i}\right] = \left\lfloor\frac{n}{p_i}\right\rfloor \\ n\left[\frac{\widetilde{1}}{p_i}\right] = \left\lfloor\frac{n+4}{p_i}\right\rfloor \\ n\left[\frac{1}{p_i}\right]\left[\frac{1}{p_j}\right] = n\left[\frac{1}{p_ip_j}\right] = \left\lfloor\frac{n}{p_ip_j}\right\rfloor \\ n\left[\frac{\widetilde{1}}{p_i}\right]\left[\frac{\widetilde{1}}{p_j}\right] = n\left[\frac{\widetilde{1}}{p_ip_j}\right] = \left\lfloor\frac{n+4}{p_ip_j}\right\rfloor \\ n\left[\frac{1}{p_i}\right]\left[\frac{\widetilde{1}}{p_j}\right] = n\left[\frac{1}{p_i}\frac{\widetilde{1}}{p_j}\right] = \left\lfloor\frac{n+\theta_{i,j}}{p_ip_j}\right\rfloor = \left\lfloor\frac{n+p_ip_j-\lambda_{i,j}}{p_ip_j}\right\rfloor, \end{array}\right. \tag{2.9}$$

where $1 \le \lambda_{i,j} \le p_ip_j$ is the position of the first item with $p_i|Z_{\lambda_{i,j}}$ and $p_j|Z'_{\lambda_{i,j}}$,

$$\left\{\begin{array}{l} Z_{\lambda_{i,j}}\mathbf{mod}p_i = 0 \\ Z_{\lambda_{i,j}}\mathbf{mod}p_j = p_j - 4(\text{or } 2 \text{ for } p_j = 3). \end{array}\right. \tag{2.10}$$

Let $X = \{\lambda p_i, \lambda = 1, 2, \cdots, \left\lfloor\frac{n}{p_i}\right\rfloor\}$, λ_j be the position of the first item with $\lambda_j p_i\mathbf{mod}p_j = p_j - 4$ (or 2 for $p_j = 3$), then $\lambda_{i,j} = \lambda_j p_i$,

$$n\left[\frac{1}{p_i}\right]\left[\frac{\widetilde{1}}{p_j}\right] = \left\lfloor\frac{n+p_ip_j-\lambda_{i,j}}{p_ip_j}\right\rfloor = \left\lfloor\frac{\left\lfloor\frac{n}{p_i}\right\rfloor+p_j-\lambda_j}{p_j}\right\rfloor. \tag{2.11}$$

If there is no such $\lambda_{i,j}$ in Z, i.e., $\lambda_{i,j} \ge (n+1)$, then the last item in Eq. (2.9) will equal zero. Because $1 \le \lambda_j \le (p_j - 1)$, so $p_i \le \theta_{i,j} = p_ip_j - \lambda_{i,j} \le p_i(p_j - 1)$.

The total number of cousin primes in n is,

$$\begin{aligned} D(n) &= D_0(n) + D(\sqrt{n}) - D_1 \\ &= n\left[1 - \frac{1}{p_1}\right]\prod_{i=2}^{v}\left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] + D(\sqrt{n}) - D_1, \end{aligned} \tag{2.12}$$

where $D(\sqrt{n}) \ge 0$ is the number of cousin primes $[p_i, p_{i+1}]$ when $p_i \le p_v < \sqrt{n}$, where $p_{i+1} = p_i + 4$ and,

$$D_1 = \left\{\begin{array}{ll} 0 & \text{if } p_v \ge 5 \\ 1 & \text{if } p_v \le 3. \end{array}\right. \tag{2.13}$$

D_1 is the number of pairs which are not cousin primes and have not been deleted by the process. When $n + 4 > p_{v+1}^2$, some pairs which are not cousin primes may not be deleted.

If $p_v \ge 5$ then $D_1 = 0$. Because if $n = p_{v+1}^2 - 1, p_{v+1}^2 - 3$, then $n\mathbf{mod}2 = 0$, if $n = p_{v+1}^2 - 4$, then $(n+4)\mathbf{mod}p_{v+1} = 0$, these items are already deleted. The only item which may not be deleted is the one $n = p_{v+1}^2 - 2$. But after deleted the items of $Z_k\mathbf{mod}2 = 0, Z_k\mathbf{mod}3 = 0, Z_k\mathbf{mod}3 = 2$ in set Z, the items left in set Z must have the form $Z_k\mathbf{mod}6 = 1$ and set Z' have the form $Z'_k\mathbf{mod}6 = 5$ correspondingly. Besides, when $p_{v+1} \ge 5$, all primes must have the form $p_{v+1} = 6R \pm 1$, where R is an integer, so $p_{v+1}^2\mathbf{mod}6 = 1$. Therefore, $(p_{v+1}^2 - 2)\mathbf{mod}6 = 5$ (not 1), and the pair including this item has also been deleted. Certainly, the pair (1,5) has been deleted by $Z'_k\mathbf{mod}5 = 0$

If $p_v \le 3$ then $D_1 = 1$. Because the pair of (1,5) is not one of cousin primes and has not been deleted. Besides, we should note that if $p_v = 2$, then one number (11) in the cousin primes (7,11) is not less than $p_2^2 = 9$, and if we delete this pair, then $D_1 = 2$ for $p_i = 2$.

Later, we will suppose that $p_v \ge 5$, and so $D_1 = 0$ for 'large' number n.

Example 2.1. Let $n = p_4^2 - 1 = 48$, then $v = 3$, $p_i = [2, 3, 5]$, $Z = [1, 2, \cdots, 48]$.
From Eq. (2.8),

$$\begin{aligned}
D_0(48) &= n\left[1-\tfrac{1}{2}\right]\left[1-\tfrac{1}{3}-\tfrac{\tilde{1}}{3}\right]\left[1-\tfrac{1}{5}-\tfrac{\tilde{1}}{5}\right] \\
&= 48\Big[1-\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{6}-\tfrac{\tilde{1}}{3}+\tfrac{1}{2}\tfrac{\tilde{1}}{3}-\tfrac{1}{5}+\tfrac{1}{10}+\tfrac{1}{15}-\tfrac{1}{30}+\tfrac{1}{5}\tfrac{\tilde{1}}{3}-\tfrac{1}{10}\tfrac{\tilde{1}}{3} \\
&\quad -\tfrac{\tilde{1}}{5}+\tfrac{1}{2}\tfrac{\tilde{1}}{5}+\tfrac{1}{3}\tfrac{\tilde{1}}{5}-\tfrac{1}{6}\tfrac{\tilde{1}}{5}+\tfrac{\tilde{1}}{15}-\tfrac{1}{2}\tfrac{\tilde{1}}{15}\Big] \\
&= 48-\left\lfloor\tfrac{48}{2}\right\rfloor-\left\lfloor\tfrac{48}{3}\right\rfloor+\left\lfloor\tfrac{48}{6}\right\rfloor-\left[\tfrac{\widetilde{48}}{3}\right]+\left[\tfrac{48}{2}\tfrac{\tilde{1}}{3}\right] \\
&\quad -\left\lfloor\tfrac{48}{5}\right\rfloor+\left\lfloor\tfrac{48}{10}\right\rfloor+\left\lfloor\tfrac{48}{15}\right\rfloor-\left\lfloor\tfrac{48}{30}\right\rfloor+\left[\tfrac{48}{5}\tfrac{\tilde{1}}{3}\right]-\left[\tfrac{48}{10}\tfrac{\tilde{1}}{3}\right] \\
&\quad -\left[\tfrac{\widetilde{48}}{5}\right]+\left[\tfrac{48}{2}\tfrac{\tilde{1}}{5}\right]+\left[\tfrac{48}{3}\tfrac{\tilde{1}}{5}\right]-\left[\tfrac{48}{6}\tfrac{\tilde{1}}{5}\right]+\left[\tfrac{\widetilde{48}}{15}\right]-\left[\tfrac{48}{2}\tfrac{\tilde{1}}{15}\right] \\
&= 48-24-16+8-\left[\tfrac{48+4}{3}\right]+\left\lfloor\frac{\lfloor\frac{48}{2}\rfloor+3-1}{3}\right\rfloor \\
&\quad -9+4+3-1+\left\lfloor\frac{\lfloor\frac{48}{5}\rfloor+3-1}{3}\right\rfloor-\left\lfloor\frac{\lfloor\frac{48}{10}\rfloor+3-2}{3}\right\rfloor \\
&\quad -\left[\tfrac{48+4}{5}\right]+\left\lfloor\frac{\lfloor\frac{48}{2}\rfloor+5-3}{5}\right\rfloor+\left\lfloor\frac{\lfloor\frac{48}{3}\rfloor+5-2}{5}\right\rfloor-\left\lfloor\frac{\lfloor\frac{48}{6}\rfloor+5-1}{5}\right\rfloor+\left[\tfrac{48+4}{15}\right]-\left\lfloor\frac{\lfloor\frac{48}{2}\rfloor+15-12}{15}\right\rfloor \\
&= 16-17+8 \\
&\quad -3+3-1 \\
&\quad -10+5+4-2+3-1 \\
&= 5.
\end{aligned}$$

We can check it directly,
$\overrightarrow{Z_k\mathbf{mod}2 \neq 0}$ $[1, 3, \cdots, 47]$
$\overrightarrow{Z_k\mathbf{mod}3 \neq 0}$ $[1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47]$
$\overrightarrow{Z_k\mathbf{mod}5 \neq 0}$ $[1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47]$
$\overrightarrow{Z_k\mathbf{mod}3 \neq 2}$ $[1, 7, 13, 19, 31, 37, 43]$
$\overrightarrow{Z_k\mathbf{mod}5 \neq 1}$ $[7, 13, 19, 37, 43]$.
So $D_0(48) = 5$.

It is the same as before. From $D(\sqrt{48}) = 0$ for $p_i \leq 5$, $D_1(48) = 0$ for $p_3 \geq 5$. So $D(48) = D_0(48) + D(\sqrt{48}) - D_1 = 5 + 0 - 0 = 5$.

The set: $Z = \{7, 13, 19, 37, 43\}, Z' = Z + 4 = \{11, 17, 23, 41, 47\}$. The five cousin primes are $(7, 11), (13, 17), (19, 23), (37, 41)$ and $(43, 47)$. $\square$

Definition 2.2. Let $X = \{X_1, X_2, \cdots, X_t\}$ be an (any) integer set, then

$$t\left[1-\tfrac{1'}{p_i}-\tfrac{1''}{p_i}\right] := \sum_{X_k\mathbf{mod}p_i \neq 0, p_i-4} 1 = t-\left[\tfrac{t+\phi_i'}{p_i}\right]-\left[\tfrac{t+\phi_i''}{p_i}\right] \geq 0 \tag{2.14}$$

is the number of items left after deleted the items of $X_k\mathbf{mod}p_i = 0, p_i - 4$ (2 for $p_i = 3$).

Usually $m\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right] \neq \left(m\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]\right)\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right]$, We can express it as

$$m\left[1-\tfrac{1}{p_i}-\tfrac{\tilde{1}}{p_i}\right]\left[1-\tfrac{1}{p_j}-\tfrac{\tilde{1}}{p_j}\right] = \left(m\left[1-\tfrac{1}{p_i}-\tfrac{\tilde{1}}{p_i}\right]\right)\left[1-\tfrac{1'}{p_j}-\tfrac{1''}{p_j}\right], \tag{2.15}$$

is the number of items left when we first delete those $Z_k\mathbf{mod}p_i = 0, p_i - 4$ from set Z, and then delete those $X_{k'}\mathbf{mod}p_j = 0, p_j - 4$ from set $X = \{Z, X_{k'}\mathbf{mod}p_i \neq$

$0, p_i - 4, k' = 1, 2, \cdots, m\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right]\}$, where set X is no longer an arithmetic sequence. In general,

(2.16)
$$m\prod_{i=1}^{i_m}\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right]\left[1 - \frac{1}{p_j} - \frac{\tilde{1}}{p_j}\right] = \left(m\prod_{i=1}^{i_m}\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right]\right)\left[1 - \frac{1'}{p_j} - \frac{1''}{p_j}\right].$$

3. Some property

(3.1)
$$\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right]\left[1 - \frac{1}{p_j} - \frac{\tilde{1}}{p_j}\right] = \left[1 - \frac{1}{p_j} - \frac{\tilde{1}}{p_j}\right]\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right].$$

(3.2)
$$\left[\frac{m}{p_i}\right] = \left[\frac{m_1+m_2}{p_i}\right] = \left[\frac{m_1}{p_i}\right] + \left[\frac{m_2}{p_i}\right] + \left[\frac{m_1 \mathbf{mod} p_i + m_2 \mathbf{mod} p_i}{p_i}\right].$$

(3.3)
$$m\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] = ap_i\left(1 - \frac{2}{p_i}\right) + b\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] \quad \text{for } m = ap_i + b.$$

(3.4)
$$-2 \le (m_1 + m_2)\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] - m_1\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] - m_2\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] \le 1.$$

Proof. Let $\alpha = m_1 \mathbf{mod} p_i, \beta = m_2 \mathbf{mod} p_i$,

$$\begin{aligned}
&\delta_{12} = (m_1 + m_2)\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] - m_1\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] - m_2\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] \\
&= m_1 + m_2 - \left[\frac{m_1+m_2}{p_i}\right] - \left[\frac{m_1+m_2+4}{p_i}\right] - m_1 + \left[\frac{m_1}{p_i}\right] + \left[\frac{m_1+4}{p_i}\right] \\
&\quad -m_2 + \left[\frac{m_2}{p_i}\right] + \left[\frac{m_2+4}{p_i}\right] \\
&= -\left[\frac{\alpha+\beta}{p_i}\right] - \left[\frac{\alpha+\beta+4}{p_i}\right] + \left[\frac{\alpha+4}{p_i}\right] + \left[\frac{\beta+4}{p_i}\right].
\end{aligned}$$

The minimum: $min(\delta_{12}) = -1-1 = -2$, when $\alpha+\beta \ge p_i$, $\alpha+4 < p_i$, $\beta+4 < p_i$, and $\alpha + \beta + 4 \ge p_i$.

The maximum: $max(\delta_{12}) = -0 - 1 + 1 + 1 = 1$, when $\alpha + \beta < p_i$, $\alpha + 4 \ge p_i$, $\beta + 4 \ge p_i$ (it is true for $p_i = 3$). □

We can represent it as

(3.5)
$$\begin{aligned}
&(m_1 + m_2)\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] = m_1 + m_2 - \left[\frac{m_1+m_2}{p_i}\right] - \left[\frac{m_1+m_2+4}{p_i}\right] \\
= \;& m_1 - \left[\frac{m_1}{p_i}\right] - \left[\frac{m_1+4}{p_i}\right] + m_2 - \left[\frac{m_1 \mathbf{mod} p_i + m_2}{p_i}\right] - \left[\frac{m_2+(m_1+4)\mathbf{mod} p_i}{p_i}\right] \\
= \;& m_1\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] + m_2\left[1 - \frac{1'}{p_i} - \frac{1''}{p_i}\right].
\end{aligned}$$

$m_2\left[\frac{1'}{p_i}\right]$ will delete the items of $X_k \mathbf{mod} p_i = 0$ in the sequence of $X = \{m_1 + 1, m_1 + 2, \cdots, m_1 + m_2\}$, and $m_2\left[\frac{1''}{p_i}\right]$ will delete the items of $X'_k \mathbf{mod} p_i = 0$ in the sequence of $X' = \{X + 4\}$ or the items of $X_k \mathbf{mod} p_i = p_i - 4$ in set X.

For any m_2, $0 \le m_2\left[1 - \frac{1'}{p_i} - \frac{1''}{p_i}\right] \le m_2$, so,

(3.6)
$$m'\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] \ge m\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i}\right] \quad \text{for } m' \ge m.$$

4. Some lemma

Lemma 4.1. *For $m \ge 4p_i$,*

$$m\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right] \ge \left\lceil m\left(1-\frac{3}{p_i}\right)\right\rceil. \tag{4.1}$$

Proof.

$$\begin{aligned} left &= \left\lceil m - \left\lfloor \frac{m}{p_i} \right\rfloor - \left\lfloor \frac{m+\theta_i}{p_i} \right\rfloor \right\rceil \ge \left\lceil m - \left\lfloor \frac{m}{p_i} \right\rfloor - \left\lfloor \frac{m}{p_i} \right\rfloor - 2 \right\rceil \\ &\ge \left\lceil m - \frac{2m}{p_i} - \frac{m}{p_i} \right\rceil = right. \end{aligned}$$

□

Lemma 4.2. *For $m \ge p_j^2$, $p_j > p_i \ge 3$,*

$$m\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right] \ge \left\lceil m\left(1-\frac{3}{p_j}\right)\right\rceil \left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]. \tag{4.2}$$

$$m\left[1-\frac{1}{p_i}\right]\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right] \ge \left\lceil m\left(1-\frac{3}{p_j}\right)\right\rceil \left[1-\frac{1}{p_i}\right] \quad \textit{for } p_i = 2. \tag{4.3}$$

Proof of (4.2). Let $m = sp_ip_j + t$, $t = ap_j + b$, $0 \le a \le (p_i - 1)$, $0 \le b \le (p_j - 1)$, Because $m \ge p_j^2$, $p_j > p_i$, so $s \ge 1$. From Eq. (3.2), (3.3),

$$\begin{aligned} \varepsilon &= m\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right] - \left\lceil m\left(1-\frac{3}{p_j}\right)\right\rceil \left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right] \\ &= sp_ip_j\left(1-\frac{2}{p_i}\right)\left(1-\frac{2}{p_j}\right) - sp_ip_j\left(1-\frac{2}{p_i}\right)\left(1-\frac{3}{p_j}\right) \\ &\quad + t - \left\lfloor \frac{t}{p_i} \right\rfloor - \left\lfloor \frac{t+4}{p_i} \right\rfloor - \left\lfloor \frac{t}{p_j} \right\rfloor + \left\lfloor \frac{t}{p_ip_j} \right\rfloor + \left\lfloor \frac{t+\theta_{j,i}}{p_ip_j} \right\rfloor - \left\lfloor \frac{t+4}{p_j} \right\rfloor + \left\lfloor \frac{t+\theta_{i,j}}{p_ip_j} \right\rfloor + \left\lfloor \frac{t+4}{p_ip_j} \right\rfloor \\ &\quad - \left(t - \left\lfloor \frac{3t}{p_j} \right\rfloor\right) + \left\lfloor \frac{t - \left\lfloor \frac{3t}{p_j} \right\rfloor}{p_i} \right\rfloor + \left\lfloor \frac{t+4-\left\lfloor \frac{3t}{p_j} \right\rfloor}{p_i} \right\rfloor \\ &= s(p_i - 2) \\ &\quad + \left\lfloor \frac{3t}{p_j} \right\rfloor - \left\lfloor \frac{t}{p_j} \right\rfloor - \left\lfloor \frac{t+4}{p_j} \right\rfloor + \left\lfloor \frac{t}{p_ip_j} \right\rfloor + \left\lfloor \frac{t+\theta_{j,i}}{p_ip_j} \right\rfloor + \left\lfloor \frac{t+\theta_{i,j}}{p_ip_j} \right\rfloor + \left\lfloor \frac{t+4}{p_ip_j} \right\rfloor \\ &\quad - \left\lfloor \frac{\left\lfloor \frac{3t}{p_j} \right\rfloor}{p_i} \right\rfloor - \varepsilon_1 - \left\lfloor \frac{\left\lfloor \frac{3t}{p_j} \right\rfloor}{p_i} \right\rfloor - \varepsilon_2 \\ &= s(p_i - 2) \\ &\quad + 3a + \left\lfloor \frac{3b}{p_j} \right\rfloor - a - a - \left\lfloor \frac{b+4}{p_j} \right\rfloor + 0 + \left\lfloor \frac{ap_j+b+\theta_{j,i}}{p_ip_j} \right\rfloor + \left\lfloor \frac{ap_j+b+\theta_{i,j}}{p_ip_j} \right\rfloor + \left\lfloor \frac{ap_j+b+4}{p_ip_j} \right\rfloor \\ &\quad - 2\left\lfloor \frac{3a+\left\lfloor \frac{3b}{p_j} \right\rfloor}{p_i} \right\rfloor - \varepsilon_1 - \varepsilon_2 \\ &= s(p_i - 2) + a + \left\lfloor \frac{3b}{p_j} \right\rfloor + \left\lfloor \frac{ap_j+b+\theta_{j,i}}{p_ip_j} \right\rfloor + \left\lfloor \frac{ap_j+b+\theta_{i,j}}{p_ip_j} \right\rfloor + \left\lfloor \frac{ap_j+b+4}{p_ip_j} \right\rfloor \\ &\quad - \varepsilon_1 - \varepsilon_2 - 2\varepsilon_3 - \varepsilon_4, \end{aligned}$$

where

$$\begin{cases} \varepsilon_1 &= \left\lfloor \frac{\left\lfloor \frac{3t}{p_j} \right\rfloor \mathbf{mod} p_i + \left(t - \left\lfloor \frac{3t}{p_j} \right\rfloor\right) \mathbf{mod} p_i}{p_i} \right\rfloor \leq 1 \\ \varepsilon_2 &= \left\lfloor \frac{\left\lfloor \frac{3t}{p_j} \right\rfloor \mathbf{mod} p_i + \left(t + 4 - \left\lfloor \frac{3t}{p_j} \right\rfloor\right) \mathbf{mod} p_i}{p_i} \right\rfloor \leq 1 \\ \varepsilon_3 &= \left\lfloor \frac{3a + \left\lfloor \frac{3b}{p_j} \right\rfloor}{p_i} \right\rfloor \leq 2 \\ \varepsilon_4 &= \left\lfloor \frac{b+4}{p_j} \right\rfloor \leq 1 (\text{except for } p_j = 3 \text{ and b=2}), \end{cases}$$

$$\Delta\varepsilon = \varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 + \varepsilon_4 \leq 7 \quad (p_j \geq 5).$$

(1) If $p_i \geq 11$, then $\varepsilon \geq s(11-2) - \Delta\varepsilon \geq 9 - 7 > 0$.
(2) If $p_i = 7$, then
  (a) If $a \geq 2$, then $\varepsilon \geq s(p_i - 2) + a - \Delta\varepsilon \geq 5 + 2 - 7 = 0$.
  (b) If $a \leq 1$, then $\varepsilon_3 = \left\lfloor \frac{3a + \left\lfloor \frac{3b}{p_j} \right\rfloor}{p_i} \right\rfloor \leq \left\lfloor \frac{3+2}{7} \right\rfloor = 0$, $\Delta\varepsilon \leq 3$, so $\varepsilon \geq s(p_i - 2) - \Delta\varepsilon \geq 5 - 3 > 0$.
(3) If $p_i = 5$, then
  (a) If $a \geq 4$, then $\varepsilon \geq s(p_i - 2) + a - \Delta\varepsilon \geq 3 + 4 - 7 = 0$.
  (b) If $a = 0$, then $\varepsilon_3 = 0$, $\Delta\varepsilon \leq 3$, so $\varepsilon \geq s(p_i - 2) - \Delta\varepsilon \geq 3 - 3 = 0$.
  (c) If $a = 1$: if $\left\lfloor \frac{3b}{p_j} \right\rfloor \geq 1$ then $\varepsilon_3 \leq 1$, $\Delta\varepsilon \leq 5$, so $\varepsilon \geq s(p_i - 2) + a + \left\lfloor \frac{3b}{p_j} \right\rfloor - \Delta\varepsilon \geq 3 + 1 + 1 - 5 = 0$; else for $\left\lfloor \frac{3b}{p_j} \right\rfloor = 0$ then $\varepsilon_3 = 0$, $\Delta\varepsilon \leq 3$, so $\varepsilon \geq s(p_i - 2) + a - \Delta\varepsilon \geq 3 + 1 - 3 > 0$.
  (d) If $a = 2$, then $\varepsilon_3 = 1$, $\Delta\varepsilon \leq 5$, so $\varepsilon \geq s(p_i - 2) + a - \Delta\varepsilon \geq 3 + 2 - 5 = 0$.
  (e) If $a = 3$: if $\left\lfloor \frac{3b}{p_j} \right\rfloor \geq 1$ then $\varepsilon \geq s(p_i - 2) + a + \left\lfloor \frac{3b}{p_j} \right\rfloor - \Delta\varepsilon \geq 3 + 3 + 1 - 7 = 0$; else for $\left\lfloor \frac{3b}{p_j} \right\rfloor = 0$ then $\varepsilon_3 = \left\lfloor \frac{3a + \left\lfloor \frac{3b}{p_j} \right\rfloor}{p_i} \right\rfloor = 1$, $\Delta\varepsilon \leq 5$, so $\varepsilon \geq s(p_i - 2) + a - \Delta\varepsilon \geq 3 + 3 - 5 > 0$.
(4) If $p_i = 3$, then $a \leq 2$,

$$\begin{cases} a - 2\varepsilon_3 &= a - 2\left\lfloor \frac{3a + \left\lfloor \frac{3b}{p_j} \right\rfloor}{3} \right\rfloor = a - 2a = -a \\ \varepsilon_1 &= \left\lfloor \frac{\left(3a + \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3 + \left(ap_j + b - 3a - \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3}{3} \right\rfloor \\ &= \left\lfloor \frac{\left(\left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3 + \left(ap_j + b - \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3}{3} \right\rfloor \\ \varepsilon_2 &= \left\lfloor \frac{\left(3a + \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3 + \left(ap_j + b + 4 - 3a - \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3}{3} \right\rfloor \\ &= \left\lfloor \frac{\left(\left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3 + \left(ap_j + b + 1 - \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 3}{3} \right\rfloor, \end{cases}$$

$$\begin{aligned}\varepsilon \quad &= s - a + \left\lfloor \frac{3b}{p_j} \right\rfloor + \left\lfloor \frac{ap_j+b+\theta_{j,i}}{3p_j} \right\rfloor + \left\lfloor \frac{ap_j+b+\theta_{i,j}}{3p_j} \right\rfloor + \left\lfloor \frac{ap_j+b+4}{3p_j} \right\rfloor \\ &-\varepsilon_1 - \varepsilon_2 - \varepsilon_4.\end{aligned}$$

(a) If $\left\lfloor \frac{3b}{p_j} \right\rfloor = 0$, then $\varepsilon_1 = \varepsilon_2 = 0$,

(i) if $p_j \geq 7$, then $b < p_j/3,\ b+4 < p_j/3+4 = (p_j+12)/3 < p_j$, so $\varepsilon_4 = 0$. If $a \leq 1$ then $\varepsilon \geq s - a \geq 0$. Else for $a = 2(< p_i)$, because $\theta_{j,i} = p_i p_j - \lambda p_j \geq p_j(\ 1 \leq \lambda \leq p_i - 1)$, $\left\lfloor \frac{ap_j+b+\theta_{j,i}}{3p_j} \right\rfloor \geq \left\lfloor \frac{2p_j+p_j}{3p_j} \right\rfloor = 1$. So $\varepsilon \geq s - a + 1 \geq 0$.

(ii) else for $p_j = 5$, from $\left\lfloor \frac{3b}{p_j} \right\rfloor = 0$, we have $b = 0, 1$. If b=0 we follow the process above. For b=1, $\varepsilon_4 = 1$.

$$\varepsilon \quad = s - a + 0 + \left\lfloor \frac{5a+1+\theta_{j,i}}{15} \right\rfloor + \left\lfloor \frac{5a+1+\theta_{i,j}}{15} \right\rfloor + \left\lfloor \frac{a+1}{3} \right\rfloor - 1$$

- for a=0, $\varepsilon \geq s - 1 \geq 0$;
- for a=2, $\theta_{j,i} \geq 5$ (it comes from one of 5,10,15), $\varepsilon \geq s - a + 1 + 0 + 1 - 1 \geq 0$;
- for a=1, if $s \geq 2$ then $\varepsilon \geq s - a - 1 \geq 0$; else for s=1, from $s = \left\lfloor \frac{m}{p_i p_j} \right\rfloor \geq \left\lfloor \frac{p_j^2}{3p_j} \right\rfloor = 1$, and $5^2 = 3 \cdot 5 + 2 \cdot 5$, we have $a = 2$. So this case is impossible.

(b) If $\left\lfloor \frac{3b}{p_j} \right\rfloor \geq 1$ (note $p_j \geq 5$), then $\varepsilon' = s + \left\lfloor \frac{ap_j+b+\theta_{i,j}}{3p_j} \right\rfloor - \varepsilon_4 \geq 1$.

Proof. (i) If $s \geq 2$ then $\varepsilon' \geq 1$.

(ii) If $s = 1$, i.e., $s = \left\lfloor \frac{m}{p_i p_j} \right\rfloor \geq \left\lfloor \frac{p_j^2}{3p_j} \right\rfloor = 1$, so $p_j = 5$. From $5^2 = 3 \cdot 5 + 2 \cdot 5$, we have $a = 2$. Let us consider $\theta_{i,j} = p_i p_j - \lambda_{i,j}$, $\lambda_{i,j} = \lambda p_i = 3\lambda$, with the condition $(3\lambda + 4)\mathbf{mod}5 = 0$ i.e. $\lambda = 2$, $\theta_{i,j} = p_i p_j - 3\lambda = 15 - 6 = 9$. $\left\lfloor \frac{ap_j+b+\theta_{i,j}}{p_i p_j} \right\rfloor \geq \left\lfloor \frac{2 \cdot 5+9}{3 \cdot 5} \right\rfloor = 1$, Therefore,

$$\varepsilon' = s + \left\lfloor \frac{ap_j+b+\theta_{i,j}}{3p_j} \right\rfloor - \varepsilon_4 \geq 1 \quad \text{if} \quad s = 1.$$

□

Besides, $\varepsilon'' \ = \left\lfloor \frac{3b}{p_j} \right\rfloor - \varepsilon_1 - \varepsilon_2 \geq 0.$

Proof. (i) if $\left\lfloor \frac{3b}{p_j} \right\rfloor = 2$, then $\varepsilon'' \geq 0$.

(ii) if $\left\lfloor \frac{3b}{p_j} \right\rfloor = 1$, because,

$$\begin{aligned}&\varepsilon_1 + \varepsilon_2 = \left\lfloor \frac{1+(ap_j+b-1)\mathbf{mod}3}{3} \right\rfloor + \left\lfloor \frac{1+(ap_j+b)\mathbf{mod}3}{3} \right\rfloor \\ &= \begin{cases} 1+0=1 & \text{if} \quad (ap_j+b)\mathbf{mod}3 = 0 \\ 0+0=0 & \text{if} \quad (ap_j+b)\mathbf{mod}3 = 1 \\ 0+1=1 & \text{if} \quad (ap_j+b)\mathbf{mod}3 = 2. \end{cases}\end{aligned}$$

So $\varepsilon_1 + \varepsilon_2 \leq 1$ and $\varepsilon'' \ = \left\lfloor \frac{3b}{p_j} \right\rfloor - \varepsilon_1 - \varepsilon_2 \geq 0$.

□

Therefore,

$$\begin{aligned}\varepsilon &= \varepsilon' - a + \varepsilon'' + \left\lfloor \frac{ap_j+b+\theta_{j,i}}{p_ip_j} \right\rfloor + \left\lfloor \frac{ap_j+b+2}{p_ip_j} \right\rfloor \\ &\geq 1 - a + \left\lfloor \frac{ap_j+b+\theta_{j,i}}{p_ip_j} \right\rfloor .\end{aligned}$$

If $a \leq 1$ then $\varepsilon \geq 1 - a \geq 0$.
Else for $a = 2(< p_i)$, because $\theta_{j,i} = p_ip_j - \lambda p_j \geq p_j$ $(1 \leq \lambda \leq p_i - 1)$, $\left\lfloor \frac{ap_j+b+\theta_{j,i}}{p_ip_j} \right\rfloor \geq \left\lfloor \frac{2p_j+p_j}{3p_j} \right\rfloor = 1$. So $\varepsilon \geq 1 - a + 1 \geq 0$.

In summary, $\varepsilon \geq 0$ for all $p_i < p_j \leq \sqrt{m}$. □

Proof of (4.3). when $p_i = 2$, $\left[\frac{\tilde{1}}{p_i}\right] \equiv 0$, $m = 2sp_j + t$, $t = ap_j + b$, $0 \leq a \leq 1$, $0 \leq b \leq p_j - 1$, $s \geq 1$.

$$\begin{aligned}\varepsilon &= m\left[1 - \tfrac{1}{2}\right]\left[1 - \tfrac{1}{p_j} - \tfrac{\tilde{1}}{p_j}\right] - \left\lceil m\left(1 - \tfrac{3}{p_j}\right)\right\rceil\left[1 - \tfrac{1}{2}\right] \\ &= 2sp_j\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{2}{p_j}\right) - 2sp_j\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{3}{p_j}\right) \\ &\quad + t - \left\lfloor \tfrac{t}{2} \right\rfloor - \left\lfloor \tfrac{t}{p_j} \right\rfloor + \left\lfloor \tfrac{t}{2p_j} \right\rfloor - \left\lfloor \tfrac{t+4}{p_j} \right\rfloor + \left\lfloor \tfrac{t+\theta_{i,j}}{2p_j} \right\rfloor \\ &\quad - \left(t - \left\lfloor \tfrac{3t}{p_j} \right\rfloor\right) + \left\lfloor \frac{t - \left\lfloor \frac{3t}{p_j} \right\rfloor}{2} \right\rfloor \\ &= s + \left\lfloor \tfrac{3t}{p_j} \right\rfloor - \left\lfloor \tfrac{t}{p_j} \right\rfloor + 0 - \left\lfloor \tfrac{t+4}{p_j} \right\rfloor + \left\lfloor \tfrac{t+\theta_{i,j}}{2p_j} \right\rfloor - \left\lfloor \frac{\left\lfloor \frac{3t}{p_j} \right\rfloor}{2} \right\rfloor - \varepsilon_1 \\ &= s + 3a + \left\lfloor \tfrac{3b}{p_j} \right\rfloor - a - a - \left\lfloor \tfrac{b+4}{p_j} \right\rfloor + \left\lfloor \tfrac{ap_j+b+\theta_{i,j}}{2p_j} \right\rfloor - \left\lfloor \frac{3a + \left\lfloor \frac{3b}{p_j} \right\rfloor}{2} \right\rfloor - \varepsilon_1 \\ &= s + \left\lfloor \tfrac{3b}{p_j} \right\rfloor + \left\lfloor \tfrac{ap_j+b+\theta_{i,j}}{2p_j} \right\rfloor - \varepsilon_1 - \varepsilon_3 - \varepsilon_4,\end{aligned}$$

where

$$\begin{cases} \varepsilon_1 &= \left\lfloor \frac{\left\lfloor 3a + \frac{3b}{p_j} \right\rfloor \mathbf{mod} 2 + \left(ap_j + b - 3a - \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 2}{2} \right\rfloor \\ &= \left\lfloor \frac{\left\lfloor a + \frac{3b}{p_j} \right\rfloor \mathbf{mod} 2 + \left(ap_j + b - a - \left\lfloor \frac{3b}{p_j} \right\rfloor\right) \mathbf{mod} 2}{2} \right\rfloor \leq 1 \\ \varepsilon_3 &= \left\lfloor \frac{a + \left\lfloor \frac{3b}{p_j} \right\rfloor}{2} \right\rfloor \leq 1 \\ \varepsilon_4 &= \left\lfloor \frac{b+4}{p_j} \right\rfloor \leq 1 (\leq 2 \ for p_j = 3), \end{cases}$$

(1) If $p_j \geq 5$, then

$$\Delta\varepsilon = \varepsilon_1 + \varepsilon_3 + \varepsilon_4 \leq 3.$$

(a) If $\left\lfloor \frac{3b}{p_j} \right\rfloor = 0$, then $\varepsilon_3 = 0$, $\varepsilon_1 = \left\lfloor \frac{a + (ap_j + b - a)\mathbf{mod} 2}{2} \right\rfloor$:
If $a = 0$, then $\varepsilon_1 = 0$, $\Delta\varepsilon = \varepsilon_4 \leq 1$, $\varepsilon \geq s - \Delta\varepsilon \geq 0$;
Else for $a = 1$, $\varepsilon_1 = \left\lfloor \frac{1 + b\mathbf{mod} 2}{2} \right\rfloor$. $\lambda_{i,j} = 2p_j - 4$, $\theta_{i,j} = 2p_j - \lambda_{i,j} = 4$.
$\left\lfloor \frac{ap_j+b+\theta_{i,j}}{2p_j} \right\rfloor - \varepsilon_4 = \left\lfloor \frac{p_j+b+4}{2p_j} \right\rfloor - \left\lfloor \frac{b+4}{p_j} \right\rfloor = \begin{cases} 0 - 0 = 0 & \text{if} \quad b + 4 < p_j \\ 1 - 1 = 0 & \text{if} \quad b + 4 \geq p_j, \end{cases}$

$$\left\lfloor \frac{3b}{p_j} \right\rfloor - \varepsilon_3 = \left\lfloor \frac{3b}{p_j} \right\rfloor - \left\lfloor \frac{a + \left\lfloor \frac{3b}{p_j} \right\rfloor}{2} \right\rfloor \geq \begin{cases} 0 - 0 = 0 & \text{if } \left\lfloor \frac{3b}{p_j} \right\rfloor = 0 \\ 1 - 1 = 0 & \text{if } \left\lfloor \frac{3b}{p_j} \right\rfloor = 1 \\ 2 - 1 = 1 & \text{if } \left\lfloor \frac{3b}{p_j} \right\rfloor = 2, \end{cases}$$

So $\varepsilon = s + \left(\left\lfloor \frac{3b}{p_j} \right\rfloor - \varepsilon_3 \right) + \left(\left\lfloor \frac{ap_j + b + \theta_{i,j}}{2p_j} \right\rfloor - \varepsilon_4 \right) - \varepsilon_1 \geq 1 + 0 + 0 - 1 = 0.$

(b) If $\left\lfloor \frac{3b}{p_j} \right\rfloor = 1$, then $\varepsilon \geq s + 1 + 0 - \varepsilon_1 - \varepsilon_3 - 1 \geq 1 - \varepsilon_1 - \varepsilon_3$:
If $a = 0$, then $\varepsilon_3 = 0$, $\varepsilon \geq 1 - \varepsilon_1 - 0 \geq 0$;
Else for $a = 1$, then $\varepsilon_1 = 0$, $\varepsilon \geq 1 - 0 - \varepsilon_3 \geq 0$.

(c) If $\left\lfloor \frac{3b}{p_j} \right\rfloor = 2$, then $\varepsilon \geq s + 2 + 0 - \varepsilon_1 - \varepsilon_3 - \varepsilon_4 \geq 0$.

(2) If $p_j = 3$, then for $b \leq 1$, we can follow the process of $p_j \geq 5$, else for $b = 2$,

$$\begin{cases} \varepsilon_1 &= \left\lfloor \frac{\left\lfloor 3a + \frac{3b}{3} \right\rfloor \mathbf{mod} 2 + \left(3a + b - 3a - \left\lfloor \frac{3b}{3} \right\rfloor\right) \mathbf{mod} 2}{2} \right\rfloor \\ &= \left\lfloor \frac{a \mathbf{mod} 2 + 0}{2} \right\rfloor = 0 \\ \varepsilon_3 &= \left\lfloor \frac{a + \left\lfloor \frac{3b}{3} \right\rfloor}{2} \right\rfloor = 1 \\ \varepsilon_4 &= \left\lfloor \frac{b+4}{3} \right\rfloor = 2, \end{cases}$$

$$\begin{aligned} \varepsilon &= s + \left\lfloor \tfrac{3b}{3} \right\rfloor + \left\lfloor \frac{3a + b + \theta_{i,j}}{6} \right\rfloor - \varepsilon_1 - \varepsilon_3 - \varepsilon_4 \\ &\geq s + 2 + 0 - 3 \geq 0 \end{aligned}$$

In summary, $\varepsilon \geq 0$ for all $2 = p_i < p_j \leq \sqrt{m}$. □

This lemma means that, from equation (4.1) and (3.5), we can let $m \left[1 - \frac{1}{p_j} - \frac{\tilde{1}}{p_j} \right] = \left\lceil m \left(1 - \frac{3}{p_j} \right) \right\rceil + t, t \geq 0$, and operated by $\left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i} \right]$ to get the items which have no multiples of p_i and p_j in Z andZ'.

$$\begin{aligned} & m \left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i} \right] \left[1 - \frac{1}{p_j} - \frac{\tilde{1}}{p_j} \right] = \left(\left\lceil m \left(1 - \frac{3}{p_j} \right) \right\rceil + t \right) \left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i} \right] \\ & = \left\lceil m \left(1 - \frac{3}{p_j} \right) \right\rceil \left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i} \right] + t \left[1 - \frac{1'}{p_i} - \frac{1''}{p_i} \right] \\ & \geq \left\lceil m \left(1 - \frac{3}{p_j} \right) \right\rceil \left[1 - \frac{1}{p_i} - \frac{\tilde{1}}{p_i} \right]. \end{aligned} \tag{4.4}$$

Lemma 4.3.

$$m \left[1 - \tfrac{1}{2} \right] \left[1 - \tfrac{1}{3} - \tfrac{\tilde{1}}{3} \right] \geq \left\lfloor \tfrac{m}{6} \right\rfloor. \tag{4.5}$$

Proof. For $p_i = 2, p_j = 3$, let $m = 6s + t$,

$$\begin{aligned} & m \left[1 - \tfrac{1}{2} \right] \left[1 - \tfrac{1}{3} - \tfrac{\tilde{1}}{3} \right] = 6s \left[1 - \tfrac{1}{2} \right] \left[1 - \tfrac{1}{3} - \tfrac{\tilde{1}}{3} \right] + t \left[1 - \tfrac{1}{2} \right] \left[1 - \tfrac{1}{3} - \tfrac{\tilde{1}}{3} \right] \\ & \geq 6s \left(1 - \tfrac{1}{2} \right) \left(1 - \tfrac{2}{3} \right) = s = \left\lfloor \tfrac{m}{6} \right\rfloor. \end{aligned}$$

For the residual class of modulo 6, $X = \{6s+1, 6s+2, \cdots, 6s+6\}$, $X_k \mathbf{mod} 6 = \{1, 2, 3, 4, 5, 6\}$, there are 4 elements $(2, 3, 4, 6)$ of multiples of 2 or 3. For the other elements $X_k \mathbf{mod} 6 = \{1, 5\}$, there is at least one with $X_k \mathbf{mod} 6 = 1$, i.e., $X_k \mathbf{mod} 2 \neq 0, X_k \mathbf{mod} 3 \neq 0$, and $X_k \mathbf{mod} 3 \neq 2$. The cousin primes have at least $\left\lfloor \frac{m}{6} \right\rfloor$ items, or $m \left[1 - \frac{1}{2} \right] \left[1 - \frac{1}{3} - \frac{\tilde{1}}{3} \right] \geq \left\lfloor \frac{m}{6} \right\rfloor$. □

Lemma 4.4. *For* $i = 1, 2, \cdots, i_m, j$,

$$m \prod_{i=1}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] \geq \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=1}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right], \tag{4.6}$$

where $\frac{\widetilde{1}}{p_i} \equiv 0$ *for* $p_i = 2$.

Proof. Suppose that for $1 < r \leq i_m$,

$$m \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] = \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] + t, \tag{4.7}$$

where $t \geq 0$. It means that the effect of the operator $\left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right]$ when operating on m is that dividing $Z = \{1, 2, \cdots, m\}$ into two effective sets $X = \{1, 2, \cdots, m' = \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil\}$ and $X' = \{X'_1, X'_2, \cdots, X'_t, t = m - m' \geq 0\}$. From equation (2.15), (2.16), (4.1), (4.2), and (4.7), we have

$$\begin{aligned}
& m \prod_{i=r-1}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] \\
&= m \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] \left[1 - \frac{1}{p_{r-1}} - \frac{\widetilde{1}}{p_{r-1}}\right] \\
&= \left(\left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] + t\right) \left[1 - \frac{1'}{p_{r-1}} - \frac{1''}{p_{r-1}}\right] \\
&\geq \left(\left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right]\right) \left[1 - \frac{1'}{p_{r-1}} - \frac{1''}{p_{r-1}}\right] \\
&= \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r-1}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right].
\end{aligned}$$

Or

$$\begin{aligned}
& m \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] \left[1 - \frac{1}{p_{r-1}} - \frac{\widetilde{1}}{p_{r-1}}\right] \\
&= m \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] - m \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] \left[\frac{1}{p_{r-1}}\right] \\
&\quad - m \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] \left[\frac{\widetilde{1}}{p_{r-1}}\right] \\
&= \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] + t \\
&\quad - \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[\frac{1}{p_{r-1}}\right] - t \left[\frac{1'}{p_{r-1}}\right] \\
&\quad - \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[\frac{\widetilde{1}}{p_{r-1}}\right] - t \left[\frac{1''}{p_{r-1}}\right] \\
&= \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_{r-1}} - \frac{\widetilde{1}}{p_{r-1}}\right] + t \left[1 - \frac{1'}{p_{r-1}} - \frac{1''}{p_{r-1}}\right] \\
&\geq \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=r-1}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right].
\end{aligned}$$

In fact, if

$$m \prod_{i=1}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right] \left[1 - \frac{1}{p_j} - \frac{\widetilde{1}}{p_j}\right] < \left\lceil m \left(1 - \frac{3}{p_j}\right)\right\rceil \prod_{i=1}^{i_m} \left[1 - \frac{1}{p_i} - \frac{\widetilde{1}}{p_i}\right],$$

then for any p_i, p_j, before deleting the multiples of other primes, it must have

$$m\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right]<\left\lceil m\left(1-\frac{3}{p_j}\right)\right\rceil\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right].$$

which contradicts Lemma 4.2. So this lemma is true.

If $p_{r-1}=2$, then

$$\begin{aligned} &m\left[1-\frac{1}{p_{r-1}}\right]\prod_{i=r}^{i_m}\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right] \\ &\geq\left\lceil m\left(1-\frac{3}{p_j}\right)\right\rceil\prod_j\left[1-\frac{1}{p_{r-1}}\right]\prod_{i=r}^{i_m}\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]. \end{aligned} \tag{4.8}$$

□

With Lemma 4.2, the operator $\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right]$ can be represented by $\left(1-\frac{3}{p_j}\right)$, and other operator $\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]$ can operate on this inequality unchanged.

5. Explantation

Let $m=sp_ip_j+ap_j+b\geq p_j^2$, then for all $p_i<p_j$, $m\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right]$ is effective to a natural sequence $X=\{f(Z)\}$ whose number is not less than $\lceil m(1-3/p_j)\rceil$. The reason is as follows. When a natural sequence is deleted by the items of $Z_k\mathbf{mod}p_j=0,p_j-4$, the sequence is subtracted by $\left[\frac{m}{p_j}\right]+\left[\frac{m+4}{p_j}\right]$. We can arrange the m items in a table of p_j rows (Table 1). $\left[\frac{m}{p_j}\right]$ will delete the p_jth row, and $\left[\frac{m+4}{p_j}\right]$ will delete the $(p_j-4)th$ row. Thus there are (p_j-2) rows left in which each item $Z_k\mathbf{mod}p_j\neq 0,p_j-4$.

Table 1. Set Z

1	p_j+1	$\cdots$	$(p_i-1)p_j+1$	$\cdots$	$(sp_i-1)p_j+1$	sp_ip_j+1	$\cdots$	$sp_ip_j+ap_j+1$
2	p_j+2	$\cdots$	$(p_i-1)p_j+2$	$\cdots$	$(sp_i-1)p_j+2$	sp_ip_j+2	$\cdots$	$sp_ip_j+ap_j+2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
b	p_j+b	$\cdots$	$(p_i-1)p_j+b$	$\cdots$	$(sp_i-1)p_j+b$	sp_ip_j+b	$\cdots$	$sp_ip_j+ap_j+b$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
p_j	$2p_j$	$\cdots$	p_ip_j	$\cdots$	sp_ip_j	$sp_ip_j+p_j$	$\cdots$	

But every p_i items $(0\leq Z_k\mathbf{mod}p_i\leq p_i-1)$ in any row of the first sp_i columns consist in a complete systems of residues modulo p_i, because $C_1=\{1,p_j+1,2p_j+1,\cdots,(p_i-1)p_j+1\}$ and $C_r=\{C_1+r\}$ are both complete system of residues modulo p_i, where r is any (row or column) constant. There are (p_j-2) such rows or $sp_i(p_j-2)$ items left. These items are effective to a nature sequence when deleting the items of $Z_k\mathbf{mod}p_i=0,p_i-4$.

$sp_ip_j\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]\left[1-\frac{1}{p_j}-\frac{\tilde{1}}{p_j}\right]=sp_i(p_j-2)\left[1-\frac{1}{p_i}-\frac{\tilde{1}}{p_i}\right]=s(p_i-2)(p_j-2).$

Let $t=ap_j+b$, $0\leq b\leq p_j-1$, $0\leq a\leq p_i-1$. $t\left[\frac{1}{p_j}+\frac{\tilde{1}}{p_j}\right]$ will delete at most $a+(a+1)\leq t\frac{1}{p_j}+sp_ip_j\frac{1}{p_j}=\frac{m}{p_j}$ items. If we add these items by removing those from the end of sequence then the sequence is again effective to a nature sequence, the sequence left has at least,

$$\begin{aligned} M(j) &\geq sp_i(p_j-2)+t\left[1-\frac{1}{p_j}-\widetilde{\frac{1}{p_j}}\right]-[a+(a+1)] \\ &\geq sp_i(p_j-3)+t+sp_i-4a-2 \\ &= sp_i(p_j-3)-3\left\lfloor\frac{t}{p_j}\right\rfloor+t+(sp_i-a-2) \\ &\geq sp_ip_j(1-\frac{3}{p_j})+\left\lceil t-\frac{3t}{p_j}\right\rceil+(sp_i-a-2) \\ &= \left\lceil(sp_ip_j+t)(1-\frac{3}{p_j})\right\rceil+(sp_i-a-2) \\ &= \left\lceil m(1-\frac{3}{p_j})\right\rceil+(sp_i-a-2). \end{aligned}$$

For $s \geq 2$ or $a \leq p_i - 2$, we have $(sp_i - a - 2) = (s-1)p_i + (p_i - a - 1) - 1 \geq 0$, $M(j) \geq \left\lceil m(1-\frac{3}{p_j})\right\rceil$.

For $s = 1$ and $a = p_i - 1$, the items of t have p_j rows, $p_i - 1$ columns and some b items. In each of the first b rows, there are exact p_i items which consist in a complete system of residues modulo p_i, and these items can be considered as an effective nature sequence when deleting the multiples of p_i ($Z_k\mathbf{mod}p_i = 0, p_i - 4$). The other items have at most p_j rows and $p_i - 1$ columns where the multiples of p_j have at most $2(p_i - 1)$. As before, we can add these items to make the t as an effective nature sequence, therefore,

$$\begin{aligned} M(j) &\geq sp_i(p_j-2)+t\left[1-\frac{1}{p_j}-\widetilde{\frac{1}{p_j}}\right]-2(p_i-1) \\ &\geq \left\lceil m(1-\frac{3}{p_j})\right\rceil+(sp_i-a-1)\geq\left\lceil m(1-\frac{3}{p_j})\right\rceil. \end{aligned}$$

Thus for any $p_i < p_j$, the original sequence of $m \geq p_j^2$, when deleted by the items of $Z_k\mathbf{mod}p_j = 0, p_j - 4$ from Z, is effective to reconstruct a new nature sequence having at least $\left\lceil m(1-\frac{3}{p_j})\right\rceil$ items.

The analysis above is for $p_j \geq 7$. For $p_j = 5$, we can reach at the same conclusion.

Example 5.1. $n = 48$, $P = \{2, 3, 5\}$, $Z = \{1, 2, \cdots, 48\}$.

For $p_j = 5$, after deleted the item of $Z_k\mathbf{mod}p_j = 0, 5-4$, it becomes
$Z \to Z' = \{2, 3, 4, 7, 8, 9, 12, 13, 14, 17, 18, 19, 22, 23, 24, 27, 28, 29, 32, 33, 34, 37, 38,$-$39, 42, 43, 44, 47, 48\}$. we can rearrange these items as $Z' = \{(37), 2, 3, 4, (23), (24),$-$7, 8, 9, (22), (29), 12, 13, 14, (27), (28), 17, 18, 19, (32), ||33, 34, 38, 39, 42, 43, 44, 47, 48\}$. The first $\left\lceil n\left(1-\frac{3}{p_j}\right)\right\rceil = 20$ items can be taken as an effective nature sequence ($\{1, 2, \cdots, 20\}$) from the original one when deleting the items of $Z_k\mathbf{mod}3 = 0, 2$. The other sequence $X = \{33, 34, , 38, 39, 42, 43, 44, 47, 48\}$, having at least zero item when deleting the multiples of all primes, will be neglected in further process.

6. Proof of Theorem 1.1

For a given n, consider the possible cousin primes of $[Z_k, Z_k + 4]$ in set $Z = \{1, 2, \cdots, n\}$, $p_v^2 \leq n < p_{v+1}^2$.

Lemma 6.1. *The number of cousin prime pairs in $n \geq p_3^2$,*

$$D(n) \geq \left\lfloor \frac{p_v}{3}\prod_{i=3}^{v-1}\frac{p_{i+1}-3}{p_i}\right\rfloor + D(\sqrt{n}). \tag{6.1}$$

Proof. Because $m(v) = n \geq p_v^2$. Let $m(j-1) = \lceil m(j)(1-3/p_j) \rceil$, then

$$\begin{aligned} m(j-1) &= \lceil m(j)(1-3/p_j) \rceil \geq p_j^2(1-3/p_j) = p_j(p_j-3) \\ &\geq (p_{j-1}+2)(p_{j-1}-1) = p_{j-1}^2 + p_i - 2 \geq p_{j-1}^2. \end{aligned}$$

And for any $i \leq (j-1)$, we have $m(i) \geq p_i^2$.

From equation (2.8), (4.6), (4.8) and (4.5),

$$\begin{aligned} D_0(n) &= n\left[1-\tfrac{1}{2}\right] \prod_{i=2}^{v} \left[1 - \tfrac{1}{p_i} - \tfrac{\tilde{1}}{p_i}\right] \\ &\geq \left\lceil n\left(1-\tfrac{3}{p_v}\right)\right\rceil \left[1-\tfrac{1}{2}\right] \prod_{i=2}^{v-1} \left[1 - \tfrac{1}{p_i} - \tfrac{\tilde{1}}{p_i}\right] \\ &\geq \left\lceil \left\lceil n\left(1-\tfrac{3}{p_v}\right)\right\rceil \left(1-\tfrac{3}{p_{v-1}}\right)\right\rceil \left[1-\tfrac{1}{2}\right] \prod_{i=2}^{v-2} \left[1 - \tfrac{1}{p_i} - \tfrac{\tilde{1}}{p_i}\right] \\ &\geq \cdots \\ &\geq \left\lceil n \prod_{i=3}^{v} \left(1-\tfrac{3}{p_i}\right)\right\rceil \left[1-\tfrac{1}{2}\right] \left[1 - \tfrac{1}{p_2} - \tfrac{\tilde{1}}{p_2}\right] \\ &\geq \left\lfloor \tfrac{n}{6} \prod_{i=3}^{v} \left(1-\tfrac{3}{p_i}\right)\right\rfloor, \end{aligned}$$

but

$$\begin{aligned} \prod_{i=3}^{v} \left(1-\tfrac{3}{p_i}\right) &= \tfrac{p_3-3}{p_3} \tfrac{p_4-3}{p_4} \cdots \tfrac{p_{v-1}-3}{p_{v-1}} \tfrac{p_v-3}{p_v} \\ &= \tfrac{p_3-3}{p_v} \tfrac{p_4-3}{p_3} \tfrac{p_5-3}{p_4} \cdots \tfrac{p_v-3}{p_{v-1}} = \tfrac{2}{p_v} \prod_{i=3}^{v-1} \tfrac{p_{i+1}-3}{p_i}, \end{aligned}$$

and $n \geq p_v^2$, so

$$D_0(n) \geq \left\lfloor \tfrac{n}{3p_v} \textstyle\prod_{i=3}^{v} \left(1-\tfrac{3}{p_i}\right)\right\rfloor \geq \left\lfloor \tfrac{p_v}{3} \textstyle\prod_{i=3}^{v-1} \tfrac{p_{i+1}-3}{p_i}\right\rfloor.$$

From equation (2.12),(2.13), for $p_v \geq 5$,

$$\begin{aligned} D(n) &= D_0(n) + D(\sqrt{n}) - D_1 \\ &\geq \left\lfloor \tfrac{p_v}{3} \textstyle\prod_{i=3}^{v-1} \tfrac{p_{i+1}-3}{p_i}\right\rfloor + D(\sqrt{n}). \end{aligned}$$

□

Proof of Theorem 1.1. Suppose that there is no cousin prime when greater than enough large number n_M.

$$D(n) \leq D(n_M) \quad \text{for enough } n > n_M. \tag{6.2}$$

Consider the pairs of cousin prime in the range $[1, n]$, where $n = n_M^2$,

$$D(n) \geq \left\lfloor \tfrac{p_V}{3} \textstyle\prod_{i=3}^{V-1} \tfrac{p_{i+1}-3}{p_i}\right\rfloor + D(\sqrt{n}). \tag{6.3}$$

where p_V is the maximum prime in n.

Then the cousin primes between n_M and $n = n_M^2$ will have,

$$\Delta D(n) = D(n) - D(n_M) = D(n) - D(\sqrt{n}) \geq \left\lfloor \tfrac{p_V}{3} \textstyle\prod_{i=3}^{V-1} \tfrac{p_{i+1}-3}{p_i}\right\rfloor. \tag{6.4}$$

if

$$W(V) := p_V \prod_{i=3}^{V-1} \frac{p_{i+1}-3}{p_i} \geq 3, \tag{6.5}$$

then $\Delta D(n) \geq 1$, there will be at least one cousin prime between n_M and $n = n_M^2$.

For $V = V_0 = 4, p_{V_0} = 7$,

$$W(4) = 7\frac{4}{5} = 5.6 > 3. \tag{6.6}$$

Suppose that for V, $W(V) = p_V \prod_{i=3}^{V-1} \frac{p_{i+1}-3}{p_i} > 3$, then for $V+1$,

$$\begin{aligned} W(V+1) &= p_{V+1} \prod_{i=3}^{V} \frac{p_{i+1}-3}{p_i} \\ &= p_V \prod_{i=3}^{V-1} \frac{p_{i+1}-3}{p_i} \frac{p_{V+1}}{p_V} \frac{p_{V+1}-3}{p_V} \\ &= W(V)\frac{p_{V+1}^2-3p_{V+1}}{p_V^2}. \end{aligned} \tag{6.7}$$

Because, $p_{V+1} = p_V + 2\Delta, \Delta \geq 1, p_V \geq 5$,

$$\begin{aligned} & p_{V+1}^2 - 3p_{V+1} - p_V^2 = (p_V + 2\Delta)^2 - 3(p_V + 2\Delta) - p_V^2 \\ & = p_V^2 + 4\Delta p_V + 4\Delta^2 - 3p_V - 6\Delta - p_V^2 \\ & = 3(\Delta - 1)p_V + \Delta(p_V + 4\Delta - 6) > 0. \end{aligned}$$

Therefore, $\frac{p_{V+1}^2-3p_{V+1}}{p_V^2} > 1$, and

$$W(V+1) > W(V) > 3. \tag{6.8}$$

So that $\Delta D(n) \geq 1$. It contradicts the supposition of Eq. (6.2). Therefore 'there are infinitely many pairs of cousin prime'. From Eq. (6.4) and (6.5), $\Delta D(n)$ approaches infinity as n grows without bound. The proof is completed. □

Example 6.2 (Actual vs. Simplified Formula)**.** Let

$$D'(n) = \left\lceil \frac{p_v}{3} \prod_{i=3}^{v-1} \frac{p_{i+1}-3}{p_i} \right\rceil. \tag{6.9}$$

Then from Eq. (6.3), it should have $D(n) \geq D'(n)$.

Figure 1 shows the actual pairs of cousin prime $D(n)$ (solid line) in the range of $[1, p_v^2]$, and its simplified formula $D'(n)$(dashed line) from Eq. (6.9). It clearly shows that $D(n) \geq D'(n)$, and $D'(n)$ has no up bound for enough large n.

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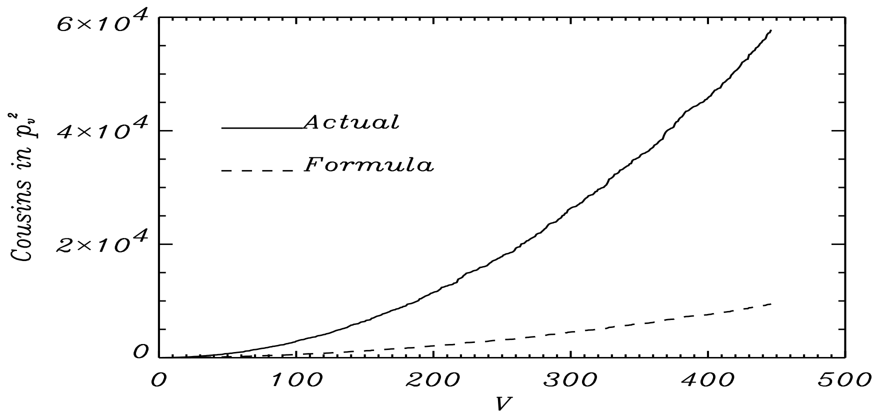

FIGURE 1. The minimum number of actual cousin primes (solid) and its simplified formula (dashed) against v.