SCHUBERT POLYNOMIALS, THE BRUHAT ORDER, AND THE GEOMETRY OF FLAG MANIFOLDS

NANTEL BERGERON AND FRANK SOTTILE

To the memory of Marcel Paul Schützenberger

Contents

Date: 30 June 1998.

1991 *Mathematics Subject Classification.* 05E15, 14M15, 05E05.

Key words and phrases. Schubert polynomials, flag manifold, Grassmannian, Bruhat order, Schubert variety, Littlewood-Richardson coefficients, Young's lattice.

First author supported in part by an NSERC grant.

Second author supported in part by NSERC grant OGP0170279 and NSF grant DMS-9022140.

To appear in Duke Mathematical Journal.

Introduction

Extending work of Demazure [14] and of Bernstein, Gelfand, and Gelfand [7], Lascoux and Schützenberger [28] defined remarkable polynomial representatives for Schubert classes in the cohomology of a flag manifold, called Schubert polynomials. For each permutation w in $\mathcal{S}_\infty$, there is a Schubert polynomial $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$. Schubert polynomials form an additive basis for this ring. Thus the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \quad = \quad \sum_w c^w_{u\,v} \mathfrak{S}_w$$

defines integral *structure constants* $c^w_{u\,v}$ for the ring of polynomials with respect to its Schubert basis. The $c^w_{u\,v}$ are non-negative: They enumerate flags in a suitable triple intersection of Schubert varieties. Evaluating a Schubert polynomial at certain Chern classes gives a Schubert class in the cohomology of the flag manifold. This exhibits the cohomology of the flag manifold [10] as:

$$\mathbb{Z}[x_1, x_2, \ldots]/\langle \mathfrak{S}_w \,|\, w \notin \mathcal{S}_n \rangle.$$

It remains an open problem to give a bijective formula for these constants. We expect such a formula will have the form

$$c^w_{u\,v} \;=\; \# \left\{ \begin{array}{c} \text{(saturated) chains in the Bruhat order on } \mathcal{S}_\infty \text{ from} \\ u \text{ to } w \text{ satisfying } \textit{some} \text{ condition imposed by } v \end{array} \right\}. \qquad (1)$$

Since every Schur symmetric polynomial $S_\lambda(x_1, \ldots, x_k)$ is a Schubert polynomial, this would generalize the Littlewood-Richardson rule [34] (*cf.* §6.1), as Young tableaux are chains in Young's lattice, a suborder of the Bruhat order. A new proof of Pieri's formula for Grassmannians [51] suggests a geometric rationale for such 'chain-theoretic' formulas. Lastly, known formulas for the $c^w_{u\,v}$ are all of this form. This includes Monk's formula [37], Pieri formulas ([28, 12, 25, 39, 50, 55]), and other formulas of [50].

Here, we illuminate this relation between the Bruhat order and the $c^w_{u\,v}$, refining (1) and proving many new identities among the $c^w_{u\,v}$. This enables us to give a description of the form (1) for some $c^w_{u\,v}$, to compute many more, and to obtain new results about the enumeration of chains in the Bruhat order. Many of these identities have a companion result about the Bruhat order which should imply the identity, were such a formula as (1) known. In fact, they and the Pieri-type formula imply the identities [6]. Our combinatorial analysis leads to a new partial order on $\mathcal{S}_\infty$ which contains Young's lattice. We also compute the effect of many specializations of the variables in Schubert polynomials.

Algebraic structures in the cohomology of a flag manifold yield identities among the $c^w_{u\,v}$ such as $c^w_{u\,v} = c^w_{v\,u}$ (commutativity) or $c^w_{u\,v} = c^{\omega_0 u}_{\omega_0 w\, v} = c^{\overline{w}}_{\overline{u}\,\overline{v}}$, where $\overline{w} := \omega_0 w \omega_0$, (Poincaré duality). Similar identities for the Littlewood-Richardson coefficients have been studied combinatorially [1, 2, 21, 22, 56]. We expect the identities established here will lead to some beautiful combinatorics, once a combinatorial interpretation for the $c^w_{u\,v}$ is known. These identities impose stringent conditions on the form of any combinatorial interpretation and should be useful in finding such an interpretation.

This paper is organized as follows: Section 1 describes our results. Section 2 contains necessary background. Section 3 contains most of our combinatorial analysis. In Section 4, we study the effect on cohomology of certain maps between flag manifolds and compute specializations of the variables in a Schubert polynomial. In Section 5, we prove the identities when $\mathfrak{S}_v$ is a Schur polynomial. In Section 6, we use these identities to compute many of the $c^w_{u\,v}$.

1. Summary

1.1. Suborders of the Bruhat order and the $c^w_{u\,v}$.

The identity

$$\mathfrak{S}_u \cdot S_\lambda(x_1,\ldots,x_k) \quad = \quad \sum_w c^w_{u\,v(\lambda,k)}\,\mathfrak{S}_w \tag{1.1.1}$$

defines integer constants $c^w_{u\,v(\lambda,k)}$ which share many properties with the Littlewood-Richardson coefficients. They are related to chains in the *k-Bruhat order*, $\leq_k$, a suborder of the Bruhat order. Its covers coincide with the index of summation in Monk's formula [37]:

$$\mathfrak{S}_u \cdot \mathfrak{S}_{(k,\,k+1)} \quad = \quad \mathfrak{S}_u \cdot (x_1+\cdots+x_k) \quad = \quad \sum \mathfrak{S}_{u(a,b)},$$

where the sum is over those $a \leq k < b$ with $\ell(u(a,b)) = \ell(u)+1$. Young's lattice of partitions with at most k parts is isomorphic to those permutations comparable to the identity in the k-Bruhat order. These are the Grassmannian permutations with descent k, whose Schubert polynomials are Schur polynomials in $x_1,\ldots,x_k$. If f^λ counts the standard Young tableaux of shape λ, then [36, I.5, Example 2],

$$(x_1+\cdots+x_k)^m \quad = \quad \sum_{\lambda\vdash m} f^\lambda S_\lambda(x_1,\ldots,x_k).$$

Considering the coefficient of $\mathfrak{S}_w$ in the product $\mathfrak{S}_u\cdot(x_1+\cdots+x_k)^m$ and the definition (1.1.1) of $c^w_{u\,v(\lambda,k)}$, we obtain:

Proposition 1.1. *The number of chains in the k-Bruhat order from u to w is*

$$\sum_\lambda f^\lambda c^w_{u\,v(\lambda,k)}.$$

In particular, $c^w_{u\,v(\lambda,k)} = 0$ unless $u \leq_k w$. A chain-theoretic description of the constants $c^w_{u\,v(\lambda,k)}$ should provide a bijective proof of Proposition 1.1. By this we mean a function τ from the set of chains in $[u,w]_k$ to the set of standard Young tableaux T whose shape is a partition λ of $\ell(w)-\ell(u)$ such that $\#\tau^{-1}(T) = c^w_{u\,v(\lambda,k)}$. Schensted insertion [47] furnishes a proof [53] for the Littlewood-Richardson coefficients (*cf.* §6.1), as does Schützenberger's *jeu de taquin* [49]. We show (Theorem 6.3.1) that if τ is a function where $\#\tau^{-1}(T)$ depends only upon the shape of T and satisfies a condition of compatibility with the Pieri formula, then $\#\tau^{-1}(T) = c^w_{u\,v(\lambda,k)}$.

In §3.1 we give a non-recursive description of the k-Bruhat order:

Theorem A. *Let $u, w \in \mathcal{S}_\infty$. Then $u \leq_k w$ if and only if*

I. *$a \leq k < b$ implies $u(a) \leq w(a)$ and $u(b) \geq w(b)$.*
II. *If $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a \leq k < b$.*

We generalize Proposition 1.1 and refine (1). Let P be a *parabolic subgroup* of $\mathcal{S}_\infty$, so P is generated by some adjacent transpositions, $(i, i+1)$. Define the *P-Bruhat order* by its covers. A cover $u \lessdot_P w$ in the P-Bruhat order is a cover in the Bruhat order where $u^{-1}w \notin P$. When P is generated by all adjacent transpositions except $(k, k+1)$, this is the k-Bruhat order.

Let $I \subset \{1, 2, \ldots, n-1\}$ index the adjacent transpositions *not* in P. A *coloured chain* in the P-Bruhat order is a chain together with an element of $I \bigcap \{a, a+1, \ldots, b-1\}$ for each cover $u \lessdot_P u(a,b)$ in the chain [30]. Iterating Monk's rule, we obtain:

$$\left(\sum_{i\in I} \mathfrak{S}_{(i,i+1)}\right)^m \quad = \quad \sum_{v\,:\,\ell(v)=m} f^v_e(P)\; \mathfrak{S}_v, \tag{1.1.2}$$

where $f^v_e(P)$ counts the coloured chains in the P-Bruhat order from e to v. Necessarily, $f^v_e(P) \neq 0$, only if v is minimal in vP. More generally, let $f^w_u(P)$ count the coloured chains in the P-Bruhat order from u to w. Multiplying (1.1.2) by $\mathfrak{S}_u$ and equating coefficients of $\mathfrak{S}_w$, gives a generalization of Proposition 1.1:

Theorem B. *Let $u, w \in \mathcal{S}_\infty$ and $P \subset \mathcal{S}_\infty$ a parabolic subgroup. Then*

$$f^w_u(P) \quad = \quad \sum_v c^w_{u\,v}\, f^v_e(P).$$

Hence if v is minimal in vP, then $c^w_{u\,v} = 0$ unless $u \leq_P w$. This suggests a refinement of (1): Let $u, v, w \in \mathcal{S}_\infty$, and let P be any parabolic subgroup such that v is minimal in vP. Then, for every coloured chain γ in the P-Bruhat order from e to v, we expect that

$$c^w_{u\,v} \;=\; \#\left\{\begin{matrix}\text{coloured chains in the } P\text{-Bruhat order on } \mathcal{S}_\infty \text{ from}\\ u \text{ to } w \text{ which satisfy } \textit{some} \text{ condition imposed by } \gamma\end{matrix}\right\}. \tag{1.1.3}$$

Moreover, this rule should give a bijective proof of Theorem B.

This P-Bruhat order is defined for parabolic subgroups of any Coxeter group. Likewise, the problem of determining the structure constants for a Schubert basis also generalizes. For Weyl groups, this is the Schubert basis of cohomology for a generalized flag manifold G/B or the analogues of Schubert polynomials [8, 17, 20, 45].

For finite Coxeter groups, this is the basis Δ_w in the coinvariant algebra [23]. Likewise, Theorem B and the expectation (1.1.3) have analogues. Of the known formulas [11, 24, 40, 42, 43, 44, 52] (see also the survey [41]), few [11, 24, 40, 52] have been expressed in a chain-theoretic manner.

1.2. **Substitutions and the Schubert basis.** In §§4.3 and 4.4, we study the $c^w_{u\,v}$ when $w(p) = u(p)$ for some p. For $w \in \mathcal{S}_{n+1}$ and $1 \leq p \leq n+1$, let $w/_p \in \mathcal{S}_n$ be defined by deleting the pth row and $w(p)$th column from the permutation matrix of w. If $y \in \mathcal{S}_n$ and $1 \leq q \leq n+1$, then $\varepsilon_{p,q}(y) \in \mathcal{S}_{n+1}$ is the permutation such that $\varepsilon_{p,q}(y)/_p = y$ and $\varepsilon_{p,q}(y)(p) = q$. The index of summation in a particular case of the Pieri formula [4, 28, 50],

$$\mathfrak{S}_v \cdot (x_1 \cdots x_{p-1}) \quad = \quad \sum_{v \xrightarrow{c_p} w} \mathfrak{S}_w,$$

defines the relation $v \xrightarrow{c_p} w$. Let $\Psi_p : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[x_1, x_2, \ldots]$ be

$$\Psi_p(x_j) \;=\; \begin{cases} x_j & \text{if } j < p \\ 0 & \text{if } j = p \\ x_{j-1} & \text{if } j > p \end{cases}.$$

Theorem C. *Let $u, w \in \mathcal{S}_\infty$ and $p \in \mathbb{N}$.*

(*i*) *Suppose $w(p) = u(p)$ and $\ell(w) - \ell(u) = \ell(w/_p) - \ell(u/_p)$. Then*

(*a*) $\varepsilon_{p,u(p)} : [u/_p, w/_p] \xrightarrow{\sim} [u, w]$.

(*b*) *For every $v \in \mathcal{S}_\infty$,*

$$c^w_{u\,v} \quad = \quad \sum_{\substack{y \in \mathcal{S}_\infty \\ v \xrightarrow{c_p} \varepsilon_{p,1}(y)}} c^{w/_p}_{u/_p\, y}.$$

(*ii*) *For $v \in \mathcal{S}_\infty$,*

$$\Psi_p(\mathfrak{S}_v) = \sum_{\substack{y \in \mathcal{S}_\infty \\ v \xrightarrow{c_p} \varepsilon_{p,1}(y)}} \mathfrak{S}_y.$$

We prove the first assertion (Lemma 4.1.1 (*ii*)) using combinatorial arguments. The second (in §4.4) and third (in §4.3) are proven by computing certain maps on cohomology. Since $c^w_{u\,v} = c^w_{v\,u} = c^{\omega_0 u}_{\omega_0 w\, v}$, Theorem C (*i*)(*b*) gives a recursion for $c^w_{u\,v}$ when one of wu^{-1}, wv^{-1}, or $\omega_0 uv^{-1}$ has a fixed point and the condition on lengths is satisfied.

We compute other substitutions of the variables: Let $P \subset \mathbb{N}$ and list the elements of P and $\mathbb{N} - P$ in order:

$$P \;:\; p_1 < p_2 < \cdots \qquad\qquad \mathbb{N} - P \;:\; p^c_1 < p^c_2 < \cdots$$

Define $\Psi_P : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$ by:

$$\Psi_P(x_{p_j}) \;=\; y_j \qquad \text{and} \qquad \Psi_P(x_{p^c_j}) \;=\; z_j.$$

In Remark 4.6.1, we define an infinite set I_P of permutations with the following property:

Theorem D. *For every $w \in \mathcal{S}_\infty$, there exists an integer N such that if $\pi \in I_P$ and $\pi \notin \mathcal{S}_N$, then*

$$\Psi_P(\mathfrak{S}_w) \;=\; \sum_{u,v} c^{(u\times v)\cdot\pi}_{\pi\, w}\; \mathfrak{S}_u(y)\; \mathfrak{S}_v(z).$$

We prove this in §4.6. Theorem D gives infinitely many identities of the form $c^{(u\times v)\cdot\pi}_{\pi\, w} = c^{(u\times v)\cdot\sigma}_{\sigma\, w}$ for $\pi, \sigma \in I_P$. Moreover, for these u, v, π with $c^{(u\times v)\pi}_{\pi\, w} \neq 0$, we have $[\pi,\ (u \times v)\cdot\pi] \simeq [e, u] \times [e, v]$, which suggests a chain-theoretic basis for these identities.

A combinatorial proof of Theorem D may provide insight into the problem of determining the $c^w_{u\,v}$. In particular, it would be interesting to find a proof using one of the combinatorial constructions of Schubert polynomials [3, 4, 9, 18, 27, 54]. Theorem D extends 1.5 of [29], which shows that $\Psi_{[n]}\mathfrak{S}_w$ is a non-negative sum of $\mathfrak{S}_u(y)\mathfrak{S}_v(z)$. The special case of Theorem D when $P = [n]$, together with the formula $c^{w\times z}_{u\times x\ \ v\times y} = c^w_{u\,v}\cdot c^z_{x\,y}$ for $u, v, w \in \mathcal{S}_n$ and $x, y, z \in \mathcal{S}_\infty$ was established by Patras [38] using methods similar to ours. Also, Lascoux and Schützenberger [29] give the special case when $P = \{1\}$.

We consider more general substitutions: Let $P_\bullet := (P_0, P_1, \ldots)$ be any partition of $\mathbb{N}$. For $i > 0$, let $\underline{x}^{(i)} := x^{(i)}_1, x^{(i)}_2, \ldots$ be variables in bijection with P_i. Define $\Psi_{P_\bullet} : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots]$ by

$$\Psi_{P_\bullet}(x_j) \;=\; \begin{cases} 0 & \text{if } j \in P_0 \\ x^{(i)}_l & \text{if } j \text{ is the } l\text{th element of } P_i \end{cases}.$$

Corollary 1.2. *For every partition $P_\bullet$ of $\mathbb{N}$ and $w \in \mathcal{S}_\infty$,*

$$\Psi_{P_\bullet}(\mathfrak{S}_w(x)) \;=\; \sum_{u_1,u_2,\ldots} d^{u_1,u_2,\cdots}_w(P_\bullet)\, \mathfrak{S}_{u_1}(\underline{x}^{(1)})\mathfrak{S}_{u_1}(\underline{x}^{(2)})\cdots,$$

where each $d^{u_1,u_2,\cdots}_w(P_\bullet)$ is an (explicit) sum of products of the $c^z_{v\,y}$.

A *ballot sequence* $A = (a_1, a_2, \ldots)$ is a sequence of non-negative integers where, for each $i, j \geq 1$,

$$\#\{k \leq j \mid a_k = i\} \quad\geq\quad \#\{k \leq j \mid a_k = i + 1\}.$$

(Consider $a_i = 0$ as a vote for 'none of the above'.) Given a ballot sequence A, define $\Psi_A : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[x_1, x_2, \ldots]$ by

$$\Psi_A(x_i) \quad=\quad \begin{cases} 0 & a_i = 0 \\ x_{a_i} & a_i \neq 0 \end{cases}.$$

Corollary 1.3. *For every ballot sequence A and $w \in \mathcal{S}_n$, there exist non-negative integers $d^u_w(A)$ for $u, w \in \mathcal{S}_\infty$ such that*

$$\Psi_A(\mathfrak{S}_w(x)) \quad = \quad \sum_u d^u_w(A)\, \mathfrak{S}_u(x).$$

Moreover, each $d^u_w(A)$ is an (explicit) sum of products of the $c^z_{v\,y}$.

Proof. If $P_0 := \{i \mid a_i = 0\}$ and for $j > 0$

$$P_j \;:=\; \{i \mid a_i \text{ is the } j\text{th occurrence of some integer in } A\},$$

then $\Psi_A = \Delta \circ \Psi_{(P_0, P_1, \ldots)}$, where $\Delta(x^{(i)}_j) = x_j$. ∎

1.3. **Identities when $\mathfrak{S}_v$ is a Schur polynomial.** If λ, μ, and ν are partitions with at most k parts, then the Littlewood-Richardson coefficients $c^\nu_{\mu\,\lambda}$ are defined by the identity

$$S_\mu(x_1, \ldots, x_k) \cdot S_\lambda(x_1, \ldots, x_k) \quad = \quad \sum_\nu c^\nu_{\mu\,\lambda} S_\nu(x_1, \ldots, x_k).$$

They depend only on λ and the skew partition ν/μ. That is, if κ and ρ are partitions with $\kappa/\rho = \nu/\mu$, then for all λ,

$$c^\nu_{\mu\,\lambda} \quad = \quad c^\kappa_{\rho\,\lambda},$$

and the coefficient of $S_\kappa(x_1, \ldots, x_l)$ in $S_\rho(x_1, \ldots, x_l) \cdot S_\lambda(x_1, \ldots, x_l)$ is $c^\kappa_{\rho\,\lambda}$. The order type of the interval in Young's lattice from μ to ν is determined by ν/μ. These facts hold also for the $c^w_{u\,v(\lambda,k)}$.

If $u \leq_k w$, let $[u, w]_k$ be the interval between u and w in the k-Bruhat order. Permutations ζ and η are *shape equivalent* if there exist sets of integers $P = \{p_1 < \cdots < p_n\}$ and $Q = \{q_1 < \cdots < q_n\}$, where ζ (respectively η) acts as the identity on $\mathbb{N} - P$ (respectively $\mathbb{N} - Q$), and

$$\zeta(p_i) \;=\; p_j \quad \Longleftrightarrow \quad \eta(q_i) \;=\; q_j.$$

Theorem E. *Suppose $u \leq_k w$ and $x \leq_l z$ where wu^{-1} is shape equivalent to zx^{-1}. Then the following statements hold*

(*i*) *We have $[u, w]_k \simeq [x, z]_l$. When $wu^{-1} = zx^{-1}$, this isomorphism is given by $v \mapsto vu^{-1}x$.*
(*ii*) *For all partitions λ, $c^w_{u\,v(\lambda,k)} = c^z_{x\,v(\lambda,l)}$.*

Part (*i*) follows from Theorems 3.1.3 and 3.2.3, which are proven using combinatorial arguments. Part (*ii*) is proven in §5.1 using geometric arguments. By Theorem E, we may define the *skew coefficient* c^ζ_λ for $\zeta \in \mathcal{S}_\infty$ and λ a partition by $c^\zeta_\lambda := c^{\zeta u}_{u\,v(\lambda,k)}$ and also define $|\zeta| := \ell(\zeta u) - \ell(u)$ for any $u \in \mathcal{S}_\infty$ with $u \leq_k \zeta u$. This leads (in §3.2) to a partial order $\preceq$ on $\mathcal{S}_\infty$ graded by $|\zeta|$ with the defining property: Let $[e, \zeta]_\preceq$

be the interval in the $\preceq$-order from the identity to ζ. If $u \leq_k \zeta u$, then the map $[e,\zeta]_\preceq \to [u,\zeta u]_k$ defined by

$$\eta \quad \longmapsto \quad \eta\, u$$

is an order isomorphism. Then Proposition 1.1 states that $\sum_\lambda f^\lambda c^\zeta_\lambda$ counts the chains in $[e,\zeta]_\preceq$. This order is studied further in [5].

We express some of the c^ζ_λ in terms of chains in the Bruhat order. If $u \lessdot_k u(a,b)$ is a cover in the k-Bruhat order, label that edge of the Hasse diagram with the integer $u(b)$. The *word* of a chain in the k-Bruhat order is its sequence of edge labels.

Theorem F. *Suppose $u \leq_k w$ and wu^{-1} is shape equivalent to $v(\mu,l)\cdot v(\nu,l)^{-1}$, for some l and partitions μ,ν. Then, for all partitions λ and standard Young tableaux T of shape λ,*

$$c^w_{u\, v(\lambda,k)} \;=\; \#\left\{\begin{array}{c}\textit{Chains in k-Bruhat order from u to w whose word}\\ \textit{has recording tableau T for Schensted insertion}\end{array}\right\}.$$

Theorem F gives a combinatorial proof of Proposition 1.1 for many u, w. It is proven in §6.1.

If a skew partition $\theta = \rho \coprod \sigma$ is the union of incomparable skew partitions ρ and σ, then

$$\rho \coprod \sigma \;\simeq\; \rho\times\sigma$$

as graded posets. The skew Schur function S_θ is defined [36, I.5] to be $\sum_\lambda c^\theta_\lambda S_\lambda$ and $S_{\rho\coprod\sigma} = S_\rho\cdot S_\sigma$ [36, I.5.7]. Thus

$$c^{\rho\coprod\sigma}_\lambda \quad=\quad \sum_{\mu,\,\nu} c^\lambda_{\mu\,\nu}\, c^\rho_\mu\, c^\sigma_\nu. \tag{1.3.1}$$

Permutations ζ and η are *disjoint* if ζ and η have disjoint supports and $|\zeta\eta| = |\zeta| + |\eta|$.

Theorem G. *Let ζ and η be disjoint permutations. Then*

(*i*) *The map $(\zeta',\eta') \mapsto \zeta'\eta'$ induces an isomorphism*

$$[e,\zeta]_\preceq \times [e,\eta]_\preceq \xrightarrow{\ \sim\ } [e,\zeta\eta]_\preceq.$$

(*ii*) *For every partition λ,* $c^{\zeta\eta}_\lambda \;=\; \sum_{\mu,\,\nu} c^\lambda_{\mu\,\nu}\, c^\zeta_\mu\, c^\eta_\nu.$

The first statement is proven in §3.3 using combinatorics and the second in §5.2 using geometry.

Our last identity has no analogy with the Littlewood-Richardson coefficients. The n-cycle $(12\ldots n)$ cyclicly permutes $[n]$.

Theorem H. *Suppose $\zeta \in S_n$ and $\eta = \zeta^{(1\,2\,\ldots\,n)}$. Then, for every partition λ, $c^\zeta_\lambda = c^\eta_\lambda$.*

This is proven in §5.3 using geometry. Combined with Proposition 1.1, we obtain:

Corollary 1.4. *If $u \leq_k w$ and $x \leq_k z$ with $wu^{-1}, zx^{-1} \in \mathcal{S}_n$ and $(wu^{-1})^{(1\,2\,\ldots\,n)} = zx^{-1}$, then each of the two intervals $[u,w]_k$ and $[x,z]_k$ have the same number of chains.*

These intervals $[u,w]_k$ and $[x,z]_k$ are typically non-isomorphic: For example, in $\mathcal{S}_4$ let $u = 1234$, $x = 2134$, and $v = 1324$. If $\zeta = (1243)$, $\eta = (1423) = \zeta^{(1234)}$, and $\xi = (1342) = \eta^{(1234)}$, then

$$u \leq_2 \zeta u, \quad x \leq_2 \eta x, \quad \text{and} \quad v \leq_2 \xi v.$$

Figure 1 shows the intervals $[u,\zeta u]_2$, $[x,\eta x]_2$, and $[v,\xi v]_2$.

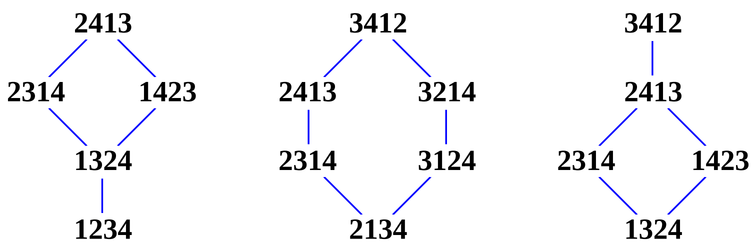


Figure 1. Effect of cyclic shift on intervals

The theorems of this section, together with the 'algebraic' identities $c^w_{u\,v} = c^{\omega_0 w}_{\omega_0 u\,v} = c^{\overline{w}}_{\overline{u}\,\overline{v}}$, greatly reduce the number of distinct coefficients $c^w_{u\,v(\lambda,k)}$ from which all others may be determined. We indicate this for some small symmetric groups in the table in Figure 1. The first row counts the number of $c^w_{u\,v(\lambda,k)}$ with $u \leq_k w$ and $|\lambda| =$

	$\mathcal{S}_4$	$\mathcal{S}_5$	$\mathcal{S}_6$	$\mathcal{S}_7$	$\mathcal{S}_8$
$\#c^w_{u\,v(\lambda,k)}$	208	3600	81669	2285414	79860923
$\#c^\zeta_\lambda$	5	12	62	332	3267

Table 1. Distinct coefficients in different groups

$\ell(w) - \ell(u)$ in $\mathcal{S}_n$, and the second counts those c^ζ_λ from which all the $c^w_{u\,v(\lambda,k)}$ may be determined using the results of this paper. For a discussion of this table, see `http://www.math.yorku.ca/bergeron/coefficients.html`

2. Preliminaries

2.1. **Permutations.** Let $\mathcal{S}_n$ be the group of permutations of $[n] := \{1,2,\ldots,n\}$. Let (a,b) be the transposition interchanging $a < b$. The *length* $\ell(w)$ of $w \in \mathcal{S}_n$ counts the *inversions*, $\{i<j \mid w(i)>w(j)\}$, of w. The Bruhat order $\leq$ on $\mathcal{S}_n$ is the partial order whose cover relation is $w \lessdot w(a,b)$ if $w(a)<w(b)$ and $\ell(w)+1=\ell(w(a,b))$. If $u\leq w$, let $[u,w] := \{v \mid u\leq v\leq w\}$ be the interval between u and w in $\mathcal{S}_n$, a poset graded by $\ell(v)-\ell(u)$. The longest element $\omega_0\in\mathcal{S}_n$ is defined by $\omega_0(j)=n+1-j$. When it is necessary to consider the longest elements in several symmetric groups, we write ω_n for $\omega_0\in\mathcal{S}_n$.

A permutation $w\in\mathcal{S}_n$ acts on $[n{+}1]$, fixing $n+1$. Thus $\mathcal{S}_n\subset\mathcal{S}_{n+1}$. Define $\mathcal{S}_\infty := \bigcup_n \mathcal{S}_n$. For $P=\{p_1<p_2<\cdots\}\subset\mathbb{N}$, define $\phi_P:\mathcal{S}_{\#P}\to\mathcal{S}_\infty$ by requiring that ϕ_P act as the identity on $\mathbb{N}-P$ and $\phi_P(\zeta)(p_i)=p_{\zeta(i)}$. This injection does not preserve length unless $P=\{n+1,n+2,\ldots\}$. For this P, set $1^n\times w := \phi_P(w)$. If there exist permutations ξ,ζ,η and sets of positive integers P,Q such that $\phi_P(\xi)=\zeta$ and $\phi_Q(\xi)=\eta$, then ζ and η are *shape equivalent*.

2.2. **Schubert polynomials.** Lascoux and Schützenberger invented and then developed the elementary theory of Schubert polynomials in a series of papers [28, 29, 30, 31, 32, 33]. For a self-contained exposition of some of this elegant theory see [35].

$\mathcal{S}_n$ acts on polynomials in $x_1,\ldots,x_n$ by permuting the variables. For a polynomial f, $f-(i,i{+}1)f$ is antisymmetric in x_i and x_{i+1}, hence divisible by x_i-x_{i+1}. Define the *divided difference* operator

$$\partial_i \;:=\; (x_i-x_{i+1})^{-1}(e-(i,i{+}1)).$$

If $a_1,a_2,\ldots,a_{\ell(w)}$ is a reduced word for w, then $\partial_{a_1}\circ\cdots\circ\partial_{a_{\ell(w)}}$ depends only upon w, defining the operator ∂_w. For $w\in\mathcal{S}_n$, Lascoux and Schützenberger [28] defined the *Schubert polynomial* $\mathfrak{S}_w$ by

$$\mathfrak{S}_w \;:=\; \partial_{w^{-1}\omega_0}\left(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}\right).$$

The polynomial $\mathfrak{S}_w$ is homogeneous of degree $\ell(w)$ and it is independent of the choice of n. The set of all Schubert polynomials $\{\mathfrak{S}_w \mid w\in\mathcal{S}_\infty\}$ is an integral basis for $\mathbb{Z}[x_1,x_2,\ldots]$.

A *partition* λ is a decreasing sequence $\lambda_1\geq\cdots\geq\lambda_k\geq 0$ of integers. *Young's lattice* is the set of partitions ordered by $\subset$, where $\mu\subset\lambda$ if $\mu_i\leq\lambda_i$ for all i. Write m^l for the partition with l parts, each of size m. If $\lambda_{k+1}=0$, the Schur polynomial $S_\lambda(x_1,\ldots,x_k)$ is

$$S_\lambda(x_1,\ldots,x_k) \;:=\; \frac{\det\left|x_j^{k-i+\lambda_i}\right|_{i,j=1}^{k}}{\det\left|x_j^{k-i}\right|_{i,j=1}^{k}},$$

which is symmetric and homogeneous of degree $|\lambda| := \lambda_1+\cdots+\lambda_k$.

A permutation $w\in\mathcal{S}_\infty$ is *Grassmannian of descent k* if $j\neq k \Rightarrow w(j)<w(j+1)$. Then w defines, and is defined by a partition λ with $\lambda_{k+1}=0$:

$$\lambda_{k+1-j} \;=\; w(j)-j \qquad j=1,\ldots,k.$$

(The condition $w(k+1) < w(k+2) < \cdots$ determines the remaining values of w.) In this case, write $w = v(\lambda, k)$. The *raison d'etre* for this definition is that $\mathfrak{S}_{v(\lambda,k)} = S_\lambda(x_1, \ldots, x_k)$. Thus the Schubert polynomials form a basis for $\mathbb{Z}[x_1, x_2, \ldots]$ which contains all Schur symmetric polynomials $S_\lambda(x_1, \ldots, x_k)$ for all λ and k.

2.3. **The flag manifold.** Let $V \simeq \mathbb{C}^n$. A *flag* $F_\bullet$ in V is a sequence

$$\{0\} \ = \ F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n \ = \ V$$

of subspaces with $\dim_{\mathbb{C}} F_i = i$. Flags $F_\bullet$ and $F_\bullet'$ are *opposite* if $F_{n-j} \cap F_j' = \{0\}$ for all j. The set of all flags is an $\binom{n}{2}$-dimensional complex manifold, $\mathbb{F}\ell V$ (or $\mathbb{F}\ell_n$), called the *flag manifold*. There is a tautological flag $\mathcal{F}_\bullet$ of bundles over $\mathbb{F}\ell V$ whose fibre at $F_\bullet$ is $F_\bullet$. Let $-x_i$ be the first Chern class of the line bundle $\mathcal{F}_i/\mathcal{F}_{i-1}$. Borel [10] showed the cohomology ring of $\mathbb{F}\ell V$ is

$$\mathbb{Z}[x_1, \ldots, x_n]/\langle e_i(x_1, \ldots, x_n) \,|\, i = 1, \ldots, n\rangle,$$

where $e_i(x_1, \ldots, x_n)$ is the ith elementary symmetric polynomial in $x_1, \ldots, x_n$.

Let $\langle S\rangle$ be the linear span of $S \subset V$ and $U - W$ be the set-theoretic difference of subspaces $W \subset U$. An ordered basis $f_1, f_2, \ldots, f_n$ for V determines a flag $E_\bullet := \langle\langle f_1, \ldots, f_n\rangle\rangle$, where $E_i = \langle f_1, \ldots, f_i\rangle$ for $1 \leq i \leq n$. A fixed flag $F_\bullet$ gives a decomposition due to Ehresmann [16] of $\mathbb{F}\ell V$ into affine cells indexed by permutations w of S_n. The cell determined by w is:

$$X_w^\circ F_\bullet \ := \ \{E_\bullet = \langle\langle f_1, \ldots, f_n\rangle\rangle \,|\, f_i \in F_{n+1-w(i)} - F_{n-w(i)},\ 1 \leq i \leq n\}.$$

Its closure is the Schubert subvariety $X_w F_\bullet$, which has codimension $\ell(w)$. Also, $u \leq w \Leftrightarrow X_u F_\bullet \supset X_w F_\bullet$. The *Schubert class* $\mathfrak{S}_w$ is the cohomology class Poincaré dual to the fundamental cycle of $X_w F_\bullet$. These classes form a basis for cohomology. Schubert polynomials were defined so that $\mathfrak{S}_w(x_1, \ldots, x_n) = \mathfrak{S}_w$.

If $F_\bullet$ and $F_\bullet'$ are opposite flags, then $X_u F_\bullet \bigcap X_v F_\bullet'$ is an irreducible, generically transverse intersection, a consequence of [15] (*cf.* [50, §5]). Thus its codimension is $\ell(u) + \ell(v)$, and the fundamental cycle of $X_u F_\bullet \bigcap X_v F_\bullet'$ is Poincaré dual to $\mathfrak{S}_u \cdot \mathfrak{S}_v$. Since

$$\mathbb{Z}[x_1, \ldots, x_n] \ \longrightarrow \ \mathbb{Z}[x_1, \ldots, x_{n+m}]/\langle e_i(x_1, \ldots, x_{n+m})\rangle,$$

is an isomorphism on $\mathbb{Z}\langle x_1^{a_1} \cdots x_n^{a_n} \,|\, a_i < m\rangle$, identities of Schubert polynomials follow from product formulas for Schubert classes. The Schubert basis is self-dual: If $\ell(w) + \ell(v) = \binom{n}{2}$, then

$$\mathfrak{S}_w \cdot \mathfrak{S}_v \quad = \quad \begin{cases} \mathfrak{S}_{\omega_0} & \text{if } v = \omega_0 w \\ 0 & \text{otherwise} \end{cases}. \tag{2.3.1}$$

Let $Grass_k V$ be the Grassmannian of k-dimensional subspaces of V, a $k(n{-}k)$-dimensional manifold. A flag $F_\bullet$ induces a cellular decomposition indexed by partitions $\lambda \subset (n{-}k)^k$. The closure of the cell indexed by λ is the Schubert variety $\Omega_\lambda F_\bullet$:

$$\Omega_\lambda F_\bullet \ := \ \{H \in Grass_k V \,|\, \dim H \bigcap F_{n+j-k-\lambda_j} \geq j,\ j = 1, \ldots, k\}.$$

The cohomology class Poincaré dual to the fundamental cycle of $\Omega_\lambda F_\bullet$ is $S_\lambda(x_1,\ldots,x_k)$, where $x_1,\ldots,x_k$ are negative Chern roots of the tautological k-plane bundle on $Grass_k V$. Write S_λ for $S_\lambda(x_1,\ldots,x_k)$, if k is understood. These *Schubert classes* form a basis for cohomology, $\mu\subset\lambda \Leftrightarrow \Omega_\mu F_\bullet \supset \Omega_\lambda F_\bullet$, and if $F_\bullet, F_\bullet'$ are opposite flags, then

$$[\Omega_\mu F_\bullet \bigcap \Omega_\nu F_\bullet'] \;=\; [\Omega_\mu F_\bullet]\;\cdot\;[\Omega_\nu F_\bullet'] \;=\; \sum_{\lambda\subset(n-k)^k} c^\lambda_{\mu\,\nu}\, S_\lambda,$$

where the $c^\lambda_{\mu\,\nu}$ are the Littlewood-Richardson coefficients [21].

This Schubert basis is self-dual: If $\lambda\subset(n-k)^k$, then let λ^c, the *complement* of λ, be the partition $(n-k-\lambda_k,\ldots,n-k-\lambda_1)$. Suppose $|\lambda|+|\mu| = k(n-k)$, then

$$S_\lambda(x_1,\ldots,x_k)\cdot S_\mu(x_1,\ldots,x_k) \quad=\quad \begin{cases} S_{(n-k)^k} & \text{if } \mu=\lambda^c \\ 0 & \text{otherwise}\end{cases}.$$

We suppress the dependence of λ^c on n and k.

A map $f: X\to Y$ between manifolds induces a group homomorphism $f_*: H^*X \to H^*Y$ via Poincaré duality and the functorial map on homology. This map satisfies the projection formula (*cf.* [19, 8.1.7]): Let $\alpha\in H^*X$ and $\beta\in H^*Y$, then

$$f_*(f^*\alpha\cap\beta) \quad=\quad \alpha\cap f_*\beta. \tag{2.3.2}$$

For a(n oriented) manifold X of dimension d, $H^dX = \mathbb{Z}\cdot[\mathrm{pt}]$ is generated by the class of a point. Let $\deg : H^*X\to\mathbb{Z}$ be the map which selects the coefficient of [pt]. Then $\deg(f_*\beta) = \deg(\beta)$.

Let $\pi_k : \mathbb{F}\ell V \twoheadrightarrow Grass_k V$ be defined by $\pi_k(E_\bullet) = E_k$. Then $\pi_k^{-1}\Omega_\lambda F_\bullet = X_{v(\lambda,k)}F_\bullet$ and $\pi_k : X_{\omega_0 v(\lambda^c,k)}F_\bullet \twoheadrightarrow \Omega_\lambda F_\bullet$ is generically one-to-one. Thus,

$$\begin{aligned}\pi_k^* S_\lambda &= \mathfrak{S}_{v(\lambda,k)}\\ (\pi_k)_*\mathfrak{S}_w &= \begin{cases} S_\lambda & \text{if } w=\omega_0 v(\lambda^c,k)\\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

The cohomology of $\mathbb{F}\ell V\times\mathbb{F}\ell W$ ($\dim W = m$) has an integral basis of classes $\mathfrak{S}_u\otimes\mathfrak{S}_x$ for $u\in\mathcal{S}_n$ and $x\in\mathcal{S}_m$. Likewise the cohomology of $Grass_k V\times Grass_l W$ has a basis $S_\lambda\otimes S_\mu$ for $\lambda\subset(n-k)^k$ and $\mu\subset(m-l)^l$.

While we use the cohomology rings of complex varieties, our results and methods are valid for the Chow rings [19] and l-adic (étale) cohomology [13] of these same varieties over any field.

3. Orders on $\mathcal{S}_\infty$

3.1. **The k-Bruhat order.** The k-Bruhat order, $\leq_k$, is a suborder of the Bruhat order on $\mathcal{S}_\infty$ related to the coefficients $c^w_{u\,v(\lambda,k)}$. It was called the k-coloured Ehresmanoëdre in [30]. Its covers are given by the index of summation in Monk's formula [37]:

$$\mathfrak{S}_u \cdot (x_1 + \cdots + x_k) \quad = \quad \sum_{u \lessdot_k w} \mathfrak{S}_w.$$

Thus w covers u in the k-Bruhat order ($u \lessdot_k w$) if $\ell(w) = \ell(u) + 1$ and $w = u(a,b)$ where $a \leq k < b$. The k-Bruhat order has a non-recursive characterization.

Theorem A. *Let $u, w \in \mathcal{S}_\infty$. Then $u \leq_k w$ if and only if*

I. *$a \leq k < b$ implies $u(a) \leq w(a)$ and $u(b) \geq w(b)$.*
II. *If $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a \leq k < b$.*

Proof. We show the k-Bruhat order is the transitive relation $u \trianglelefteq_k w$ defined by I and II. If $u \lessdot_k u(a,b)$ is a cover, then $u \trianglelefteq_k u(a,b)$. Thus $u \leq_k w$ implies $u \trianglelefteq_k w$. Algorithm 3.1.1 completes the proof. ∎

Algorithm 3.1.1 (Produces a chain in the k-Bruhat order).
`input:` *Permutations $u, w \in \mathcal{S}_\infty$ with $u \trianglelefteq_k w$.*
`output:` *A chain in the k-Bruhat order from w to u.*
Output w. While $u \neq w$, do

1 *Choose $a \leq k$ with $u(a)$ minimal subject to $u(a) < w(a)$.*
2 *Choose $k < b$ with $u(b)$ maximal subject to $w(b) < w(a) \leq u(b)$.*
3 *$w := w(a,b)$, output w.*

At every iteration of 1, $u \trianglelefteq_k w$. Moreover, this algorithm terminates in $\ell(w) - \ell(u)$ iterations and the sequence of permutations produced is a chain in the k-Bruhat order from w to u.

Proof. It suffices to consider a single iteration. We show it is possible to choose a and b, then $u \trianglelefteq_k w(a,b)$, and lastly $w(a,b) \lessdot_k w$.

In 1, $u \neq w$, so one may always choose a. Suppose $u \trianglelefteq_k w \in \mathcal{S}_n$ and it is not possible to choose b. In that case, if $j > k$ and $w(j) < w(a)$, then also $u(j) < w(a)$. Similarly, if $j \leq k$ and $w(j) < w(a)$, then $u(j) \leq w(j) < w(a)$. Thus $\alpha < w(a) \Leftrightarrow uw^{-1}(\alpha) < w(a)$, which contradicts $uw^{-1}(w(a)) = u(a) < w(a)$.

Let $w' := w(a,b)$. Since $w(b) \geq u(a)$ implies I for (u, w'), suppose $w(b) < u(a)$. Set $b_1 := u^{-1}w(b)$. Then $w(b_1) \neq u(b_1)$ and the minimality of $u(a)$ shows that $b_1 > k$ and $w(b_1) < u(b_1)$. Similarly, if $b_2 := u^{-1}w(b_1)$, then $b_2 > k$ and $w(b_2) < u(b_2)$. Continuing, we obtain a sequence $b_1, b_2, \ldots$ with $u(a) > u(b_1) > u(b_2) > \cdots$, a contradiction.

(u, w') satisfies II: Suppose $i < j$ and $u(i) < u(j)$. If $j \leq k$, then $w(i) < w(j)$. To show $w'(i) < w'(j)$, it suffices to consider the case $j = a$. But then $u(i) < u(a)$, and thus $u(i) = w(i) = w'(i)$, by the minimality of $u(a)$. Then $w'(i) < u(a) \leq w(b) = w'(a)$. Similarly, if $k < i$, then $w'(i) < w'(j)$.

Finally, suppose w does not cover w' in the k-Bruhat order. Since $w(a) > w(b)$, there exists a c with $a < c < b$ and $w(a) > w(c) > w(b)$. If $k < c$, then II implies $u(c) > u(b)$ and the maximality of $u(b)$ implies $w(a) < w(c)$, a contradiction. The case $c \leq k$ similarly leads to a contradiction. ∎

Remark 3.1.2. Algorithm 3.1.1 depends only upon $\zeta = wu^{-1}$:
`input:` *A permutation* $\zeta \in \mathcal{S}_\infty$.
`output:` *Permutations* $\zeta, \zeta_1, \ldots, \zeta_m = e$ *such that if* $u \leq_k \zeta u$, *then*

$$u \lessdot_k \zeta_{m-1}u \lessdot_k \cdots \lessdot_k \zeta_1 u \lessdot_k \zeta u \ (= w)$$

is a saturated chain in the k-Bruhat order.

Output ζ. While $\zeta \neq e$, do

1 *Choose α minimal subject to $\alpha < \zeta(\alpha)$.*
2 *Choose β maximal subject to $\zeta(\beta) < \zeta(\alpha) \leq \beta$.*
3 *$\zeta := \zeta(\alpha, \beta)$, output ζ.*

To see this is equivalent to Algorithm 3.1.1, set $\alpha = u(a)$ and $\beta = u(b)$ so that $w(a) = \zeta(\alpha)$ and $w(b) = \zeta(\beta)$. Thus $w(a, b) = \zeta(\alpha, \beta)u$.

More is true, the full interval $[u, w]_k$ depends only upon wu^{-1}:

Theorem 3.1.3. *If $u \leq_k w$ and $x \leq_k y$ with $wu^{-1} = zx^{-1}$, then the map $v \mapsto vu^{-1}x$ induces an isomorphism $[u, w]_k \xrightarrow{\sim} [x, z]_k$.*

This is a consequence of the following lemma.

Lemma 3.1.4. *Let $u \leq_k w$ and $x \leq_k z$ with $wu^{-1} = zx^{-1}$. Then $u \lessdot_k (\alpha, \beta)u \leq_k w \Longleftrightarrow x \lessdot_k (\alpha, \beta)x \leq_k z$.*

Proof. Let $\zeta = wu^{-1} = zx^{-1}$. The position of γ in u is $u^{-1}(\gamma)$.

Suppose $(\alpha, \beta)x$ does not cover x in the k-Bruhat order, so there is a γ with $\alpha < \gamma < \beta$ and $x^{-1}(\alpha) < x^{-1}(\gamma) < x^{-1}(\beta)$. Then we have

$$\begin{array}{rl} x = & \ldots\ \alpha\ \ldots\ \gamma\ \ldots\ \beta\ \ldots \quad \text{and} \\ z = & \ldots\zeta(\alpha)\ldots\zeta(\gamma)\ldots\zeta(\beta)\ldots \quad . \end{array}$$

Since $u \lessdot_k (\alpha, \beta)u$, either $k < u^{-1}(\beta) < u^{-1}(\gamma)$ or else $u^{-1}(\gamma) < u^{-1}(\alpha) \leq k$. We illustrate u, $(\alpha, \beta)u$, and w for each possibility:

$$\begin{array}{rcc} & k < u^{-1}(\beta) < u^{-1}(\gamma) & u^{-1}(\gamma) < u^{-1}(\alpha) \leq k \\ u: & \ldots\ \alpha\ \ldots\ \beta\ \ldots\ \gamma\ \ldots & \ldots\ \gamma\ \ldots\ \alpha\ \ldots\ \beta\ \ldots \\ (\alpha,\beta)u: & \ldots\ \beta\ \ldots\ \alpha\ \ldots\ \gamma\ \ldots & \ldots\ \gamma\ \ldots\ \beta\ \ldots\ \alpha\ \ldots \\ w: & \ldots\zeta(\alpha)\ldots\zeta(\beta)\ldots\zeta(\gamma)\ldots & \ldots\zeta(\gamma)\ldots\zeta(\alpha)\ldots\zeta(\beta)\ldots \end{array}$$

Assume $k < u^{-1}(\beta) < u^{-1}(\gamma)$. Then Theorem A and $(\alpha, \beta)u \leq_k w$ imply $\gamma \geq \zeta(\gamma)$ and $\zeta(\beta) < \zeta(\gamma)$, since $\alpha < \gamma$ and both have positions greater than k in $(\alpha, \beta)u$. Let $c := x^{-1}(\gamma)$. If $c \leq k$, then $x \leq_k z$ implies $\gamma \leq \zeta(\gamma)$ so $\gamma = \zeta(\gamma)$. Also, $\alpha < \gamma$ implies $\zeta(\alpha) < \zeta(\gamma)$ and thus $\zeta(\gamma) = \gamma < \beta \leq \zeta(\alpha)$, a contradiction. Similarly, $c > k$ or $u^{-1}(\gamma) < u^{-1}(\alpha)$, leads to a contradiction. Thus $x \lessdot_k (\alpha, \beta)x$.

To show $y := (\alpha, \beta)x \leq_k z$, first note that (y, z) satisfies I of Theorem A, because $(\alpha, \beta)u \leq_k w$. For II, we need only show:

a) If $\alpha < \gamma < \beta$ and $x^{-1}(\gamma) < x^{-1}(\alpha)$, so that $\gamma = yx^{-1}(\gamma) < yx^{-1}(\alpha) = \beta$, then $zx^{-1}(\gamma) = \zeta(\gamma) < \zeta(\beta) = zx^{-1}(\alpha)$, and

b) If $\alpha < \gamma < \beta$ and $x^{-1}(\beta) < x^{-1}(\gamma)$, so that $\alpha = yx^{-1}(\beta) < yx^{-1}(\gamma) = \gamma$, then $\zeta(\alpha) < \zeta(\gamma)$.

If $\alpha < \gamma < \beta$, then one of these does occur, as $x \lessdot_k (\alpha, \beta)x = y$. Suppose $x^{-1}(\gamma) < x^{-1}(\alpha)$, as the other case is similar.

Since $x^{-1}(\gamma) < k$ and $x \leq_k z$, we have $\gamma \leq \zeta(\gamma)$, by condition I. If $u^{-1}(\gamma) < u^{-1}(\alpha)$, then $(\alpha\, \beta)u \leq_k w \Rightarrow \zeta(\gamma) < \zeta(\alpha)$. If $u^{-1}(\beta) < u^{-1}(\gamma)$, then $\gamma = \zeta(\gamma)$, and so $\zeta(\gamma) = \gamma < \beta \leq \zeta(\alpha)$. Since $u \leq_k (\alpha, \beta)u$, we cannot have $u^{-1}(\alpha) < u^{-1}(\gamma) < u^{-1}(\beta)$. ∎

Define $\mathrm{up}_\zeta := \{\alpha \mid \alpha < \zeta(\alpha)\}$ and $\mathrm{down}_\zeta := \{\beta \mid \beta > \zeta(\beta)\}$.

Theorem 3.1.5. *Let* $\zeta \in \mathcal{S}_\infty$.

(i) *For* $u \in \mathcal{S}_\infty$, $u \leq_k \zeta u$ *if and only if the following conditions are satisfied.*

  (a) $u^{-1}\mathrm{up}_\zeta \subset \{1, \ldots, k\}$,

  (b) $u^{-1}\mathrm{down}_\zeta \subset \{k+1, k+2, \ldots\}$, *and*

  (c) *For all* $\alpha, \beta \in \mathrm{up}_\zeta$ *(respectively* $\alpha, \beta \in \mathrm{down}_\zeta$*),* $\alpha < \beta$ *and* $u^{-1}(\alpha) < u^{-1}(\beta)$ *together imply* $\zeta(\alpha) < \zeta(\beta)$.

(ii) *If* $\#\mathrm{up}_\zeta \leq k$*, then there is a permutation* u *such that* $u \leq_k \zeta u$.

Proof. Statement (i) follows from Theorem A. For (ii), let $\{a_1, \ldots, a_k\}$ contain up_ζ and possibly some fixed points of ζ, and let $\{a_{k+1}, a_{k+2}, \ldots\}$ be its complement in $\mathbb{N}$. Index these sets so that $\zeta(a_i) < \zeta(a_{i+1})$ for $i \neq k$. Define $u \in \mathcal{S}_\infty$ by $u(i) = a_i$. Then ζu is Grassmannian with descent k, and Theorem A implies $u \leq_k \zeta u$. ∎

3.2. **A new partial order on $\mathcal{S}_\infty$.** For $\zeta \in \mathcal{S}_\infty$, define $|\zeta|$ to be the difference of $\#\{(\alpha, \beta) \in \zeta(\mathrm{up}_\zeta) \times \zeta(\mathrm{down}_\zeta) \mid \alpha > \beta\}$ and

$$\#\{a, b \in \mathrm{up}_\zeta \text{ or } a, b \in \mathrm{down}_\zeta \mid a > b \text{ and } \zeta(a) < \zeta(b)\} \\ + \#\{(a, b) \in \mathrm{up}_\zeta \times \mathrm{down}_\zeta \mid a > b\}.$$

Lemma 3.2.1. *If* $u \leq_k \zeta u$*, then* $\ell(u) + |\zeta| = \ell(\zeta u)$.

Proof. By Theorem 3.1.3, $\ell(\zeta u) - \ell(u)$ depends only upon ζ. Using the permutation u in the proof of Theorem 3.1.5, shows it equals $|\zeta|$: If $c = \zeta(c)$, then the number of inversions involving c is the same for both u and ζu. The first term above counts the remaining inversions in ζu and the last two terms the remaining inversions in u. ∎

By Theorem 3.1.3, $[u, \zeta u]_k$ depends only upon ζ if $u \leq_k \zeta u$. In fact, it is independent of k as well. That is, if $x \leq_l \zeta x$, then the map $v \mapsto xu^{-1}v$ defines an isomorphism $[u, \zeta u]_k \xrightarrow{\sim} [x, \zeta x]_l$.

Definition 3.2.2. For $\zeta, \eta \in \mathcal{S}_\infty$, let $\eta \preceq \zeta$ if there exists $u \in \mathcal{S}_\infty$ and a positive integer k such that $u \leq_k \eta u \leq_k \zeta u$. If u is chosen as in the proof of Theorem 3.1.5, then we see that $\eta \preceq \zeta$ if

1. If $\alpha < \eta(\alpha)$, then $\eta(\alpha) \leq \zeta(\alpha)$.
2. If $\alpha > \eta(\alpha)$, then $\eta(\alpha) \geq \zeta(\alpha)$.
3. If $\alpha, \beta \in \mathrm{up}_\zeta$ (respectively, $\alpha, \beta \in \mathrm{down}_\zeta$) with $\alpha < \beta$ and $\zeta(\alpha) < \zeta(\beta)$, then $\eta(\alpha) < \eta(\beta)$.

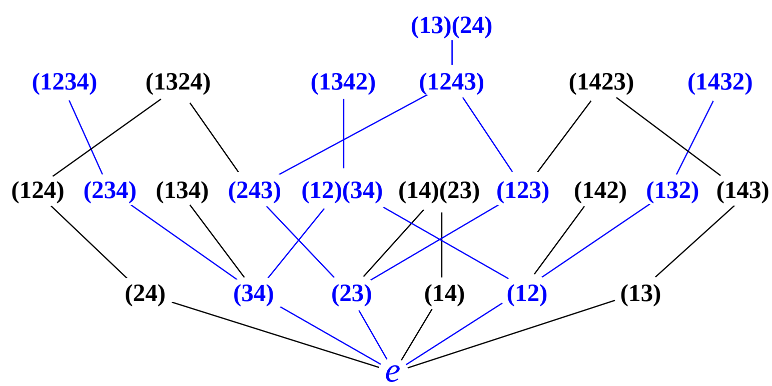


FIGURE 2. $\preceq$ on $\mathcal{S}_4$

Figure 2 illustrates $\preceq$ on $\mathcal{S}_4$. For $\zeta \in \mathcal{S}_n$, define $\overline{\zeta} := \omega_0 \zeta \omega_0$.

Theorem 3.2.3. *Suppose $u, \zeta, \eta, \xi \in \mathcal{S}_\infty$.*

(*i*) *$(\mathcal{S}_\infty, \preceq)$ is a graded poset with rank function $|\zeta|$.*

(*ii*) *The map $\lambda \mapsto v(\lambda, k)$ exhibits Young's lattice of partitions with at most k parts as an induced suborder of $(\mathcal{S}_\infty, \preceq)$.*

(*iii*) *If $u \leq_k \zeta u$, then $\eta \mapsto \eta u$ induces an isomorphism $[e, \zeta]_\preceq \xrightarrow{\sim} [u, \zeta u]_k$.*

(*iv*) *If $\eta \preceq \zeta$, then $\xi \mapsto \xi \eta^{-1}$ induces an isomorphism $[\eta, \zeta]_\preceq \xrightarrow{\sim} [e, \zeta\eta^{-1}]_\preceq$.*

(*v*) *For every infinite set $P \subset \mathbb{N}$, $\phi_P : \mathcal{S}_\infty \to \mathcal{S}_\infty$ is an injection of graded posets. Thus, if $\zeta, \eta \in \mathcal{S}_\infty$ are shape equivalent, then $[e, \zeta]_\preceq \simeq [e, \eta]_\preceq$.*

(*vi*) *The map $\eta \mapsto \eta\zeta^{-1}$ induces an order reversing isomorphism between $[e, \zeta]_\preceq$ and $[e, \zeta^{-1}]_\preceq$.*

(*vii*) *The homomorphism* $\zeta \mapsto \overline{\zeta}$ *on* $\mathcal{S}_n$ *is an automorphism of* $(\mathcal{S}_n, \preceq)$.

Theorem E (i) follows from the definition of $\preceq$ and (v).

Proof. Statements (i)–(v) follow from the definitions. Suppose $u \leq_k \eta u \leq_k \zeta u$ with $u, \eta u, \zeta u \in \mathcal{S}_n$. If $w := \zeta u$, then $w\omega_0 \leq_{n-k} \eta\zeta^{-1}w\omega_0 \leq_{n-k} \zeta^{-1}w\omega_0$, which proves (vi). Similarly, $u \leq_k w \Leftrightarrow \overline{u} \leq_{n-k} \overline{w}$ implies (vii). ∎

Example 3.2.4. Let $\zeta = (24)(153)$ and $\eta = (35)(174) = \phi_{\{1,3,4,5,7\}}(\zeta)$. Then $21345 \leq_2 \zeta \cdot 21345$ and $3215764 \leq_3 \eta \cdot 3215764$. Figure 3 shows $[21342,\, \zeta \cdot 21345]_2$, $[3215764,\, \eta \cdot 3215764]_3$, and $[e, \zeta]_{\preceq}$.

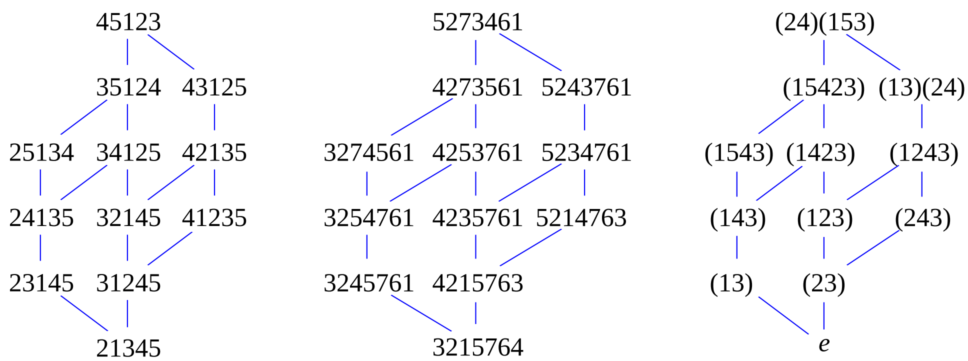


FIGURE 3. Isomorphic intervals in $\leq_2, \leq_3$, and $\preceq$

3.3. **Disjoint permutations.** Let $\zeta \in \mathcal{S}_n$ and $1, \ldots, n$ be the vertices of a convex planar n-gon numbered consecutively. Define the directed geometric graph Γ_ζ to be the union of directed chords $\langle \alpha, \zeta(\alpha) \rangle$ for α in the support, supp_ζ, of ζ.

Permutations ζ and η are *disjoint* if the edge sets of Γ_ζ and Γ_η (drawn on the same n-gon) are disjoint as *subsets of the plane*. This implies (but is not equivalent to) $\mathrm{supp}_\zeta \bigcap \mathrm{supp}_\eta = \emptyset$.

In Figure 4, the pair of cycles on the left is disjoint and the other pair is not. We relate this definition to that given in §1.3.

Lemma 3.3.1. *Let* $\zeta, \eta \in \mathcal{S}_\infty$. *Then the edges of* Γ_ζ *are disjoint from the edges of* Γ_η *if and only if* $\mathrm{supp}_\zeta \bigcap \mathrm{supp}_\eta = \emptyset$ *and* $|\zeta| + |\eta| = |\zeta\eta|$.

Proof. Suppose ζ and η have disjoint support, and let $\langle a, \zeta(a) \rangle$ be an edge of Γ_ζ and $\langle b, \eta(b) \rangle$ be an edge of Γ_η. The contribution of the endpoints of these edges to $|\zeta\eta| - |\zeta| - |\eta|$ is zero if the edges do not cross, which proves the forward implication.

For the reverse, suppose they cross. The contribution is 1 if $a < \zeta(a)$ and $b > \eta(b)$ (or vice-versa), and 0 otherwise. Since each edge is part of a directed cycle, there

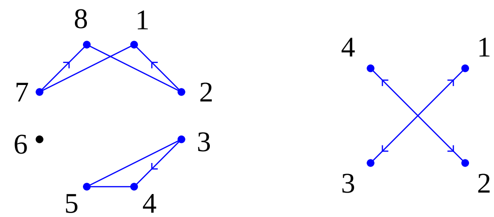


Figure 4. Graphs of the permutations (1782)(345) and (13)(24)

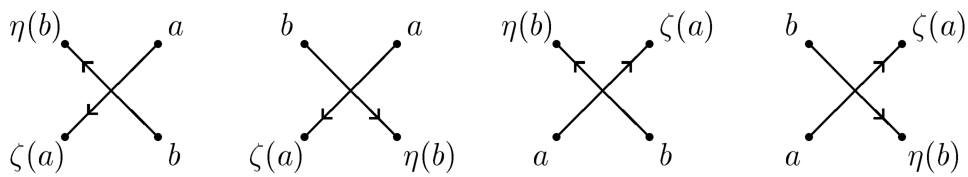


Figure 5. Crossings

are at least four crossings, one of each type shown in Figure 5. There, the numbers increase in a clockwise direction, with the least number in the northeast ($\nearrow$). Thus $|\zeta\eta| > |\zeta| + |\eta|$. ∎

Lemma 3.3.2. *Let $\alpha < \beta$, $\zeta \in \mathcal{S}_\infty$, and suppose $\zeta \lessdot (\alpha, \beta)\zeta$. Then*

(*i*) *α and β are connected in $\Gamma_{(\alpha,\beta)\zeta}$.*
(*ii*) *If $\langle c, d\rangle$ is any chord meeting Γ_ζ, then $\langle c, d\rangle$ meets $\Gamma_{(\alpha,\beta)\zeta}$.*
(*iii*) *If p and q are connnected in Γ_ζ, then they are connected in $\Gamma_{(\alpha,\beta)\zeta}$.*
(*iv*) *If ζ and η are disjoint and $\zeta' \preceq \zeta$, then ζ' and η are disjoint.*

Proof. Suppose $u \in \mathcal{S}_\infty$ with $u \leq_k \zeta u \lessdot_k (\alpha, \beta)\zeta u$. Define i and j by $\zeta u(i) = \alpha$ and $\zeta u(j) = \beta$, and set $a = u(i)$ and $b = u(j)$. Since $\zeta u \lessdot_k (\alpha, \beta)\zeta u$ is a cover, $i \leq k < j$, and thus $a \leq \alpha < \beta \leq b$, as $u \leq_k \zeta u$. Thus the edges $\langle a, \beta\rangle$ and $\langle b, \alpha\rangle$ of $\Gamma_{(\alpha,\beta)\zeta}$ meet, proving (i).

For (ii), note that $\Gamma_{(\alpha,\beta)\zeta}$ differs from Γ_ζ only by the (possible) deletion of edges $\langle a, \alpha\rangle$ and $\langle b, \beta\rangle$ and the addition of the edges $\langle a, \beta\rangle$ and $\langle b, \alpha\rangle$. Checking all possibilities for $\langle c, d\rangle$, $\langle a, \alpha\rangle$, and $\langle b, \beta\rangle$ shows (ii).

Statement (iii) follows from (ii) by considering $\Gamma_\zeta - \langle a, \alpha\rangle - \langle b, \beta\rangle$. The contrapositive of (iv) is also a consequence of (ii); If ζ' and η are not disjoint and $\zeta' \preceq \zeta$, then ζ and η are not disjoint. ∎

Lemma 3.3.3. *Suppose $u, \zeta, \eta \in \mathcal{S}_\infty$ with ζ, η disjoint. Then*

$$u \ \leq_k \ \zeta\eta u \quad \Longleftrightarrow \quad u \ \leq_k \ \zeta u \quad \text{and} \quad u \ \leq_k \ \eta u.$$

Proof. Suppose $u \leq_k \zeta\eta u$. Let $i \leq k$ so that $u(i) \leq \zeta\eta u(i)$. Since $\mathrm{supp}_\zeta \bigcap \mathrm{supp}_\eta = \emptyset$, $u(i) \leq \zeta u(i)$. Similarly, if $k < j$, then $u(j) \geq \zeta u(j)$, showing I of Theorem A holds for the pair $(u, \zeta u)$.

For II, suppose $i < j$, $u(i) < u(j)$, and $\zeta u(i) > \zeta u(j)$. If $j \leq k$, then $u(i) \in \mathrm{supp}_\zeta$. Since $u \leq_k \zeta\eta u$, and ζ, η have disjoint supports, $\zeta u(i) = \eta\zeta u(i) < \eta\zeta u(j)$, thus $u(j) \in \mathrm{supp}_\eta$,and so

$$u(i) \ < \ u(j) \ < \ \zeta u(i) \ < \ \eta u(j).$$

But then the edge $\langle u(i), \zeta u(i)\rangle$ of Γ_ζ meets the edge $\langle u(j), \eta u(j)\rangle$ of Γ_η, a contradiction. The assumption that $k < i$ leads similarly to a contradiction. Thus $u \leq_k \zeta u$ and similarly, $u \leq_k \eta u$.

Suppose now that $u \leq_k \zeta u$ and $u \leq_k \eta u$. Condition I of Theorem A holds for $(u, \zeta\eta u)$ as ζ and η have disjoint support. For II, let $i < j$ with $u(i) < u(j)$ and suppose $j \leq k$. If the set $\{u(i), u(j)\}$ meets at most one of supp_ζ or supp_η, say supp_ζ, then $u \leq_k \zeta u$ implies $\zeta\eta u(i) < \zeta\eta u(j)$. Suppose now that $u(i) \in \mathrm{supp}_\zeta$ and $u(j) \in \mathrm{supp}_\eta$. Since $u \leq_k \zeta u$, we have $\zeta u(i) < \zeta u(j) = u(j)$. But $u \leq_k \eta u$ implies $u(j) \leq \eta u(j)$. Thus $\eta\zeta u(i) = \zeta u(i) < u(j) \leq \eta u(j) = \eta\zeta u(j)$. Similar arguments suffice when $k < i$. ∎

Proof of Theorem G (*i*). Suppose ζ and η are disjoint. By Lemmas 3.3.2 and 3.3.3, the map $[e, \zeta]_\preceq \times [e, \eta]_\preceq \to [e, \zeta\eta]_\preceq$ defined by $(\zeta', \eta') \mapsto \zeta'\eta'$ is an injection. For surjectivity, let $\xi \preceq \zeta\eta$. By Lemma 3.3.2 (*iii*) and downward induction from $\zeta\eta$ to ξ, Γ_ξ has no edges connecting supp_ζ to supp_η. Set $\xi' := \xi|_{\mathrm{supp}_\zeta}$, and $\xi'' := \xi|_{\mathrm{supp}_\eta}$. Then $\xi = \xi'\xi''$, and ξ' and ξ'' are disjoint. Surjectivity will follow by showing $\xi' \preceq \zeta$ and $\xi'' \preceq \eta$.

It suffices to consider the case $\xi \lessdot (\alpha, \beta)\xi = \zeta\eta$ of a cover. By Lemma 3.3.2 (*i*), α and β are connected in $\Gamma_{\zeta\eta}$, so we may assume that $\alpha, \beta \in \mathrm{supp}_\zeta$. Then $\xi'' = \eta$ and $(\alpha, \beta)\xi' = \zeta$. We show that $\xi' \preceq (\alpha, \beta)\xi' = \zeta$ is a cover, which will complete the proof.

Choose $u \in \mathcal{S}_\infty$ with $u \leq_k \xi u \leq_k \zeta\eta u$. Let $a := (\xi' u)^{-1}(\alpha)$ and $b := (\xi' u)^{-1}(\beta)$. Since ξ' and η are disjoint, $\alpha, \beta \notin \mathrm{supp}_\eta$ and so $a, b \notin \mathrm{supp}_\eta$. Thus $(\alpha, \beta)\xi'\eta u = \xi'\eta u(a, b)$, showing $a \leq k < b$, as $\xi'\eta u \lessdot_k (\alpha, \beta)\xi'\eta u$.

Since ξ' and η are disjoint and $\xi = \xi'\eta$, Lemma 3.3.3 implies $u \leq_k \xi' u$. Thus $|\xi'| + \ell(u) = \ell(\xi' u)$. But since ξ' and η are disjoint and $\xi'\eta \lessdot \zeta\eta$ is a cover, we have

$$|\zeta| + |\eta| \ = \ |\zeta\eta| \ = \ 1 + |\xi'\eta| \ = \ 1 + |\xi'| + |\eta|,$$

so $\ell(\xi' u) + 1 = \ell(\xi' u(a, b))$. Since $a \leq k < b$ and $\zeta u = \xi' u(a, b)$, this implies $\xi' u \lessdot_k \zeta u$. ∎

Example 3.3.4. Let $\zeta = (2354)$ and $\eta = (176)$, which are disjoint. Let $u = 2316745$. Then

$$u \leq_3 \zeta\eta u = 3571624, \quad u \leq_3 \zeta u = 3516724, \quad \text{and} \quad u \leq_3 \eta u = 2371645.$$

The intervals $[u, \zeta u]_3$, $[u, \eta u]_3$, and $[u,\ \zeta\eta u]_3$ are illustrated in Figure 6.

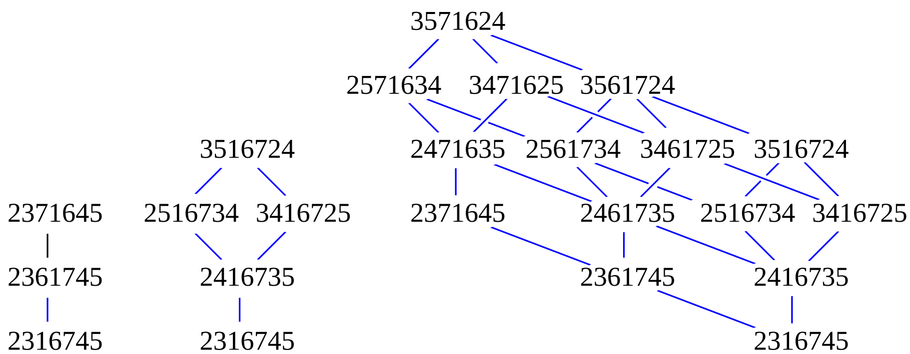


Figure 6. Intervals of disjoint permutations

4. Cohomological formulas and identities for the $c^w_{u\,v}$

4.1. **Maps on $\mathcal{S}_\infty$.** For $p, q \in \mathbb{N}$ and $w \in \mathcal{S}_\infty$, define $\varepsilon_{p,q}(w) \in \mathcal{S}_\infty$:

$$\varepsilon_{p,q}(w)(j) \quad = \quad \begin{cases} w(j) & j < p \text{ and } w(j) < q \\ w(j)+1 & j < p \text{ and } w(j) \geq q \\ q & j = p \\ w(j-1) & j > p \text{ and } w(j) < q \\ w(j-1)+1 & j > p \text{ and } w(j) \geq q \end{cases} .$$

Note that $\varepsilon_{p,p} = \phi_{\mathbb{N}-\{p\}}$. If $p \neq q$, then $\varepsilon_{p,q} : \mathcal{S}_\infty \hookrightarrow \mathcal{S}_\infty$ is not a group homomorphism. The map $\varepsilon_{p,q}$ has a left inverse $/_p$, defined by

$$u/_p(j) \quad = \quad \begin{cases} u(j) & j < p \text{ and } u(j) < u(p) \\ u(j)-1 & j < p \text{ and } u(j) > u(p) \\ u(j+1) & j \geq p \text{ and } u(j) < u(p) \\ u(j+1)-1 & j \geq p \text{ and } u(j) > u(p) \end{cases} .$$

Representing permutations as matrices, $u/_p$ erases the pth row and $u(p)$th column of u and $\varepsilon_{p,q}$ adds a new pth row and qth column consisting mostly of zeroes, but with a 1 in the (p, q)th position. For example,

$$\varepsilon_{3,3}(23154) \;=\; 243165 \qquad \text{and} \qquad 264351/_3 \;=\; 25341.$$

Lemma 4.1.1. *Suppose $u \leq w$ and p, q are positive integers. Then we have the following.*

(i) $\varepsilon_{p,q}(u) \leq \varepsilon_{p,q}(w)$.

(ii) *If* $\ell(w) - \ell(u) = \ell(\varepsilon_{p,q}(w)) - \ell(\varepsilon_{p,q}(u))$, *then*

$$\varepsilon_{p,q} \ : \ [u, w] \ \xrightarrow{\sim} [\varepsilon_{p,q}(u), \varepsilon_{p,q}(w)].$$

(iii) *If $u, w \in \mathcal{S}_n$ and either of p or q is equal to either 1 or $n+1$, then* $\ell(w) - \ell(u) = \ell(\varepsilon_{p,q}(w)) - \ell(\varepsilon_{p,q}(u))$.

(iv) *If $u \leq_k w$ and $u(p) = w(p)$, then $u/_p \leq_{k'} w/_p$ and $[u, w]_k \simeq [u/_p, w/_p]_{k'}$, where k' is equal to k if $k < p$ and $k-1$ otherwise. Furthermore,* $wu^{-1} = \varepsilon_{u(p),u(p)}(w/_p(u/_p)^{-1})$.

Proof. Suppose $u \lessdot u(a, b)$ is a cover. Then $\varepsilon_{p,q}(u) < \varepsilon_{p,q}(u(a, b))$ is a cover if either $p \leq a$ or $b < p$, or else $a < p \leq b$ and either $q \leq u(a)$ or $u(b) < q$. If however, $a < p \leq b$ and $u(a) < q \leq u(b)$, then there is a chain of length 3 from $\varepsilon_{p,q}(u)$ to $\varepsilon_{p,q}(u(a, b)) = \varepsilon_{p,q}(u)(a, b{+}1)$:

$$\varepsilon_{p,q}(u) \ \lessdot \ \varepsilon_{p,q}(u)(a, p) \ \lessdot \ \varepsilon_{p,q}(u)(a, b{+}1, p) \ \lessdot \ \varepsilon_{p,q}(u)(a, b{+}1).$$

The lemma follows from this. For example, under the hypothesis of *(ii)*, $\varepsilon_{p,q}(w)$ and $\varepsilon_{p,q}(u)$ each have the same number of inversions involving q. Thus, if $\varepsilon_{p,q}(u) \leq v \leq \varepsilon_{p,q}(w)$, then $v(p) = q$. ∎

4.2. **An embedding of flag manifolds.** Let $W \subset V$ with $W \simeq \mathbb{C}^n$ and $V \simeq \mathbb{C}^{n+1}$. Suppose $f \in V - W$ so that $V = \langle W, f\rangle$. For $p \in [n{+}1]$ define the injection $\psi_p : \mathbb{F}\ell W \hookrightarrow \mathbb{F}\ell V$ by

$$(\psi_p E_\bullet)_j \quad = \quad \begin{cases} E_j & \text{if } j < p \\ \langle E_{j-1}, f\rangle & \text{if } j \geq p \end{cases}$$

Proposition 4.2.1 ([50], Lemma 12)**.** *Let $E_\bullet \in \mathbb{F}\ell W$ and $w \in \mathcal{S}_n$. Then, for every $p, q \in [n{+}1]$,*

$$\psi_p X_w E_\bullet \quad \subset \quad X_{\varepsilon_{p,q}(w)} \psi_{n+2-q} E_\bullet .$$

Recall that e is the identity permutation.

Corollary 4.2.2. *Let $w \in \mathcal{S}_n$ and $E_\bullet, E'_\bullet \in \mathbb{F}\ell W$ be opposite flags. Then $\psi_1 E_\bullet$ and $\psi_{n+1} E'_\bullet$ are opposite flags in $\mathbb{F}\ell V$ and*

$$\psi_p X_w E_\bullet \ = \ X_{\varepsilon_{p,1}(w)} \psi_{n+1} E_\bullet \bigcap X_{\varepsilon_{p,n+1}(e)} \psi_1 E'_\bullet \ = \ X_{\varepsilon_{p,1}(e)} \psi_{n+1} E'_\bullet \bigcap X_{\varepsilon_{p,n+1}(w)} \psi_1 E_\bullet .$$

Proof. Since $X_e E'_\bullet = \mathbb{F}\ell W$, Proposition 4.2.1 with $q = 1$ or $n+1$ implies $\psi_p X_w E_\bullet$ is a subset of either intersection:

$$X_{\varepsilon_{p,1}(w)} \psi_{n+1} E_\bullet \bigcap X_{\varepsilon_{p,n+1}(e)} \psi_1 E'_\bullet \quad \text{or} \quad X_{\varepsilon_{p,1}(e)} \psi_{n+1} E'_\bullet \bigcap X_{\varepsilon_{p,n+1}(w)} \psi_1 E_\bullet .$$

Since $E_\bullet$ and $E'_\bullet$ are opposite flags, $\psi_{n+1}E_\bullet$ and $\psi_1 E'_\bullet$ are opposite flags, so both intersections are generically transverse and irreducible. Since

$$\ell(\varepsilon_{p,1}(w)) \;=\; \ell(w)+p-1 \quad \text{and} \quad \ell(\varepsilon_{p,n+1}(w)) \;=\; \ell(w)+n+1-p,$$

both intersections have the same dimension as $\psi_p X_w E_\bullet$, which proves equality. ∎

Since $\varepsilon_{p,n+1}(e) = v(n+1-p,\, p)$, where $n+1-p$ is the partition of $n+1-p$ into a single part, we see that $\mathfrak{S}_{\varepsilon_{p,n+1}(e)} = h_{n+1-p}(x_1,\ldots,x_p)$, the complete symmetric polynomial of degree $n+1-p$ in $x_1,\ldots,x_p$. Similarly, $\mathfrak{S}_{\varepsilon_{p,1}(e)} = e_{p-1}(x_1,\ldots,x_{p-1}) = x_1\cdots x_{p-1}$, as $\varepsilon_{p,1} = v(1^{p-1}, p-1)$, where 1^{p-1} is the partition of $p-1$ into $p-1$ equal parts, each of size 1.

Corollary 4.2.3. *Let $w \in \mathcal{S}_n$. In $H^*\mathbb{F}\ell V$,*

$$\mathfrak{S}_{\varepsilon_{p,1}(w)} \cdot h_{n+1-p}(x_1,\ldots,x_p) \quad = \quad \mathfrak{S}_{\varepsilon_{p,n+1}(w)} \cdot x_1\cdots x_{p-1}$$

and these products are equal to $(\psi_p)_*\mathfrak{S}_w$.

We compute ψ_p^*. The Pieri formulas of [50] show that if $u \in \mathcal{S}_n$ and $k,m \leq n$ positive integers, then

$$\mathfrak{S}_u \cdot \mathfrak{S}_{\omega_0 w} \cdot e_m(x_1\cdots x_k) \;=\; \begin{cases} 1 & u \xrightarrow{c_{k,m}} w \\ 0 & \text{otherwise} \end{cases} \tag{4.2.1}$$

$$\mathfrak{S}_u \cdot \mathfrak{S}_{\omega_0 w} \cdot h_{n+1-m}(x_1,\ldots,x_k) \;=\; \begin{cases} 1 & u \xrightarrow{r_{k,m}} w \\ 0 & \text{otherwise} \end{cases}, \tag{4.2.2}$$

where $u \xrightarrow{c_{k,m}} w$ if there is a chain in the k-Bruhat order:

$$u \lessdot_k (\alpha_1,\beta_1)u \lessdot_k \cdots \lessdot_k (\alpha_m,\beta_m)\cdots(\alpha_1,\beta_1)u \;=\; w$$

such that $\beta_1 > \cdots > \beta_m$. When $k = m$, it follows that $\{\alpha_1,\ldots,\alpha_k\} = \{u(1),\ldots,u(k)\}$. When $k = m = p-1$, write $\xrightarrow{c_p}$ for this relation. Similarly, $u \xrightarrow{r_{k,m}} w$ if there is a chain in the k-Bruhat order:

$$u \lessdot_k (\alpha_1,\beta_1)u \lessdot_k \cdots \lessdot_k (\alpha_{n+1-m},\beta_{n+1-m})\cdots(\alpha_1,\beta_1)u \;=\; w$$

such that $\beta_1 < \beta_2 < \cdots < \beta_{n+1-m}$.

Recall that $\omega_n \in \mathcal{S}_n$ is the longest element.

Theorem 4.2.4. *Let $v \in \mathcal{S}_{n+1}$. In $H^*\mathbb{F}\ell_n$,*

(i) $$\psi_p^*\mathfrak{S}_v \;=\; \sum_{\substack{y \in \mathcal{S}_n \\ v \xrightarrow{c_p} \varepsilon_{p,1}(y)}} \mathfrak{S}_y \;=\; \sum_{\substack{y \in \mathcal{S}_n \\ v \xrightarrow{r_{p,n+1-p}} \varepsilon_{p,n+1}(y)}} \mathfrak{S}_y.$$

(ii) $$\psi_p^*(x_i) \;=\; \begin{cases} x_i & i < p \\ 0 & i = p \\ x_{i-1} & i > p \end{cases}.$$

Proof. In $H^*\mathbb{F}\ell_n$,

$$\psi_p^*\mathfrak{S}_v \quad = \quad \sum_{y\in\mathcal{S}_n} \deg(\mathfrak{S}_{\omega_n y}\cdot\psi_p^*\mathfrak{S}_v)\,\mathfrak{S}_y.$$

By the projection formula (2.3.2) and Corollary 4.2.3, we have

$$\deg(\mathfrak{S}_{\omega_n y}\cdot\psi_p^*\mathfrak{S}_v) \quad = \quad \deg(\mathfrak{S}_v\cdot(\psi_p)_*\mathfrak{S}_{\omega_n y}) \quad = \quad \deg(\mathfrak{S}_v\cdot\mathfrak{S}_{\varepsilon_{p,n+1}(\omega_n y)}\cdot x_1\cdots x_{p-1}).$$

Note that $\varepsilon_{p,n+1}(\omega_n y) = \omega_{n+1}\varepsilon_{p,1}(y)$. By (4.2.1), the triple product

$$\mathfrak{S}_v\cdot\mathfrak{S}_{\varepsilon_{p,n+1}(\omega_n y)}\cdot x_1\cdots x_{p-1}$$

is zero unless $v\xrightarrow{c_p}\varepsilon_{p,1}(y)$, and in this case it equals $\mathfrak{S}_{\omega_{n+1}}$. This establishes the first equality of (i). For the second, use the other formula for $(\psi_p)_*\mathfrak{S}_y$ from Corollary 4.2.3 and (4.2.2).

For (ii), let $\mathcal{F}_\bullet$ be the tautological flag on $\mathbb{F}\ell_{n+1}$, $\mathcal{E}_\bullet$ the tautological flag on $\mathbb{F}\ell_n$, and 1 the trivial line bundle. Then

$$\psi_p^*(\mathcal{F}_i/\mathcal{F}_{i-1}) \;=\; \begin{cases} \mathcal{E}_i/\mathcal{E}_{i-1} & \text{if } i<p \\ 1 & \text{if } i=p \\ \mathcal{E}_{i-1}/\mathcal{E}_{i-2} & \text{if } i>p \end{cases},$$

But $-x_i$ is the Chern class of both $\mathcal{F}_i/\mathcal{F}_{i-1}$ and $\mathcal{E}_i/\mathcal{E}_{i-1}$. ∎

4.3. **The endomorphism $x_p\mapsto 0$.** For $p\in\mathbb{N}$ and $v\in\mathcal{S}_\infty$, define

$$A_p(v) \;:=\; \{y\in\mathcal{S}_\infty \mid v\xrightarrow{c_p}\varepsilon_{p,1}(y)\}.$$

Lemma 4.3.1. *If $v\in\mathcal{S}_n$ and $p\le n$, then $A_p(v) \;=\; \{y\in\mathcal{S}_n \mid v\xrightarrow{r_{p,n+1-p}}\varepsilon_{p,n+1}(y)\}$.*

Proof. If $v\in\mathcal{S}_n$, $p\le n$, and $v\xrightarrow{c_p}w$, then $w\in\mathcal{S}_{n+1}$, so $A_p(v)\subset\mathcal{S}_n$. But then $A_p(v)$ and $\{y\in\mathcal{S}_n\mid v\xrightarrow{r_{p,n+1-p}}\varepsilon_{p,n+1}(y)\}$ index the two equal sums in Theorem 4.2.4(i). ∎

Let $\Psi_p:\mathbb{Z}[x_1,x_2,\ldots]\to\mathbb{Z}[x_1,x_2,\ldots]$ be defined by

$$\Psi_p(x_i) \;=\; \begin{cases} x_i & \text{if } i<p \\ 0 & \text{if } i=p \\ x_{i-1} & \text{if } i>p \end{cases}.$$

Theorem C (ii). *For $v\in\mathcal{S}_\infty$ and $p\in\mathbb{N}$, $\Psi_p\mathfrak{S}_v = \sum_{y\in A_p(v)}\mathfrak{S}_y$.*

Proof. For $p\le n+1$, the homomorphism Ψ_p induces the map $\psi_p^*: H^*\mathbb{F}\ell_{n+1}\to H^*\mathbb{F}\ell_n$, by Theorem 4.2.4 (ii). Choosing n large enough completes the proof. ∎

Corollary 4.3.2. *For $w, x, y \in \mathcal{S}_\infty$ and $p \in \mathbb{N}$,*

$$\sum_{u\in A_p(x)} \sum_{v\in A_p(y)} c^w_{u\,v} \quad = \quad \sum_{\substack{z\\ w\in A_p(z)}} c^z_{x\,y}.$$

Proof. Apply Ψ_p to the identity $\mathfrak{S}_x \cdot \mathfrak{S}_y = \sum_z c^z_{x\,y}\mathfrak{S}_z$ to obtain:

$$\sum_{u\in A_p(x)} \sum_{v\in A_p(y)} \mathfrak{S}_u \cdot \mathfrak{S}_v \quad = \quad \sum_z c^z_{x\,y} \sum_{w\in A_p(z)} c^z_{x\,y}\, \mathfrak{S}_w.$$

Expanding the product $\mathfrak{S}_u \cdot \mathfrak{S}_v$ and equating the coefficients of $\mathfrak{S}_w$ proves the identity. ∎

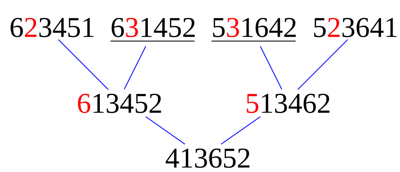
Example 4.3.3. We compute $\Psi_3(\mathfrak{S}_{413652})$. The polynomial $\mathfrak{S}_{413652}$ is

$$\begin{aligned}
&x_1^4x_2x_4x_5 + x_1^3x_2^2x_4x_5 + x_1^3x_2x_4^2x_5 + \\
&x_1^4x_2x_3x_4 + x_1^4x_2x_3x_5 + x_1^4x_3x_4x_5 + x_1^3x_2^2x_3x_4 + x_1^3x_2^2x_3x_5 + x_1^3x_2x_3^2x_4 + \\
&x_1^3x_2x_3^2x_5 + x_1^3x_2x_3x_4^2 + x_1^3x_2x_3^2x_4x_5 + x_1^3x_3x_4^2x_5 + 2\cdot x_1^3x_2x_3x_4x_5.
\end{aligned}$$

Thus $\Psi_3(\mathfrak{S}_{413652}) \;=\; x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4$. However,

$$\begin{aligned}
\mathfrak{S}_{52341} &= x_1^4x_2x_3x_4 && \text{and}\\
\mathfrak{S}_{42531} &= x_1^3x_2^2x_3x_4 + x_1^3x_2x_3^2x_4,
\end{aligned}$$

which shows $\Psi_3(\mathfrak{S}_{413652}) \;=\; \mathfrak{S}_{52341} + \mathfrak{S}_{42531}$. To see this agrees with Theorem C, we compute the permutations w such that $x \xrightarrow{c_3} w$:



Of these, only the two underlined permutations are of the form $\varepsilon_{3,1}(u)$:

$$631452 \;=\; \varepsilon_{3,1}(52341) \quad \text{and} \quad 531642 \;=\; \varepsilon_{3,1}(42531).$$

Lemma 4.3.4. *Let λ be a partition and p, k positive integers. Then $A_p(v(\lambda,k)) = \{(v(\lambda,k')\}$, where $k' = k-1$ if $p \le k$ and k otherwise.*

Proof. By the combinatorial definition of Schur functions [46, §4.4], $\Psi_p(\mathfrak{S}_{v(\lambda,k)}) = \mathfrak{S}_{v(\lambda,k')}$. ∎

Lemma 4.3.4 implies that $v(\lambda,k')$ is the only solution x to $v(\lambda,k)\xrightarrow{c_p}\varepsilon_{p,1}(x)$, a statement about chains in the Bruhat order.

4.4. **Identities for $c^w_{u\,v}$ when $u(p) = w(p)$.**

Lemma 4.4.1. *Let $u, w \in \mathcal{S}_{n+1}$ with $u(p) = w(p)$ for some $p \in [m+1]$ and suppose $\ell(w) - \ell(u) = \ell(w/_p) - \ell(u/_p)$. Then in $H^*\mathbb{F}\ell_{n+1}$,*

$$(\psi_p)_* \left(\mathfrak{S}_{u/_p} \cdot \mathfrak{S}_{\omega_n(w/_p)}\right) \quad = \quad \mathfrak{S}_u \cdot \mathfrak{S}_{\omega_{n+1}w}.$$

Proof. Let $E_\bullet, E'_\bullet$ be opposite flags in W. By Proposition 4.2.1,

$$\psi_p\left(X_{\omega_n(z/_p)}E_\bullet \bigcap X_{x/_p}E'_\bullet\right) \; = \; X_{\omega_{n+1}w}\psi_{w(p)}E_\bullet \bigcap X_u\psi_{n+2-u(p)}E'_\bullet \tag{4.4.1}$$

Note that $\omega_n(w/_p) = (\omega_{n+1}w)/_p$. Since $u(p) = w(p)$, $\psi_{w(p)}E_\bullet$ and $\psi_{n+2-u(p)}E'_\bullet$ are opposite in V. Also, as $\ell(w) - \ell(u) = \ell(w/_p) - \ell(u/_p)$, both sides of (4.4.1) have the same dimension, so they are equal. ∎

Proof of Theorem C (*i*)(*b*). It suffices to compute this in $H^*\mathbb{F}\ell_{n+1}$, for n such that $p \leq n$, $v \in \mathcal{S}_{n+1}$ and $A_p(v) \subset \mathcal{S}_n$. By Lemma 4.4.1,

$$\mathfrak{S}_u \cdot \mathfrak{S}_{\omega_{n+1}w} \; = \; (\psi_p)_* \left(\mathfrak{S}_{u/_p} \cdot \mathfrak{S}_{\omega_n(w/_p)}\right) \; = \; (\psi_p)_* \left(\sum_{y \in \mathcal{S}_n} c^{\omega_n y}_{u/_p \;\; \omega_n(w/_p)} \, \mathfrak{S}_{\omega_n y}\right).$$

Since $c^{\omega_n y}_{x\,\omega_n z} = c^z_{x\,y}$ for $x, y, z \in \mathcal{S}_n$ and $\varepsilon_{p,1}(\omega_n y) = \omega_{n+1}\varepsilon_{p,1}(y)$,

$$\begin{aligned}
\mathfrak{S}_u \cdot \mathfrak{S}_{\omega_{n+1}w} \;\; &= \;\; \sum_{y \in \mathcal{S}_n} c^{w/_p}_{u/_p\,y} (\psi_p)_* \left(\mathfrak{S}_{\omega_n y}\right) \\
&= \;\; \sum_{y \in \mathcal{S}_n} c^{w/_p}_{u/_p\,y} \mathfrak{S}_{\omega_{n+1}\varepsilon_{p,1}(y)} \cdot x_1 \cdots x_{p-1},
\end{aligned}$$

by Corollary 4.2.3. Thus

$$\begin{aligned}
c^w_{u\,v} \;\; &= \;\; \deg\left(\mathfrak{S}_u \cdot \mathfrak{S}_{\omega_{n+1}w} \cdot \mathfrak{S}_v\right) \\
&= \;\; \sum_{y \in \mathcal{S}_n} c^{w/_p}_{u/_p\,y} \cdot \deg\left(\mathfrak{S}_{\omega_{n+1}\varepsilon_{p,1}(y)} \cdot (x_1 \cdots x_{p-1}) \cdot \mathfrak{S}_v\right) \\
&= \;\; \sum_{y \in A_p(v)} c^{w/_p}_{u/_p\,y}. \qquad ∎
\end{aligned}$$

When $p = 1$, this has the following consequence:

Corollary 4.4.2. *If $u(1) = w(1)$, then $c^w_{x\,v} = 0$ unless $v = 1 \times y$. In that case, $c^w_{u\;1\times y} = c^{w/_1}_{u/_1\,y}$.*

4.5. **Products of flag manifolds.** Let $P, Q \in \binom{[n+m]}{n}$, that is, $P, Q \subset [n+m]$ and each has order n. List P, Q, and their complements P^c, Q^c in order.

$$\begin{array}{rclcrcl} P &=& p_1 < \cdots < p_n & \qquad & P^c &:=& [n+m] - P \;=\; p^c_1 < \cdots < p^c_m \\ Q &=& q_1 < \cdots < q_n & & Q^c &:=& [n+m] - Q \;=\; q^c_1 < \cdots < q^c_m \end{array}$$

Define a function $\varepsilon_{P,Q} : \mathcal{S}_n \times \mathcal{S}_m \hookrightarrow \mathcal{S}_{m+n}$ by:

$$\begin{array}{rcll} \varepsilon_{P,Q}(v,w)(p_i) &=& q_{v(i)} & \quad i = 1, \ldots, n \\ \varepsilon_{P,Q}(v,w)(p^c_j) &=& q^c_{w(j)} & \quad i = 1, \ldots, m. \end{array}$$

$\varepsilon_{P,Q}(v,w)$ is the permutation matrix obtained by placing the entries of v in the blocks $P \times Q$ and those of w in the blocks $P^c \times Q^c$. If $P = [n+1] - \{p\}$ and $Q = [n+1] - \{q\}$, then $\varepsilon_{P,Q}(v,e) = \varepsilon_{p,q}(v)$.

Suppose $V \simeq \mathbb{C}^n$, $W \simeq \mathbb{C}^m$, and $P \in \binom{[n+m]}{n}$. Define a map

$$\psi_P \;:\; \mathbb{F}\ell V \times \mathbb{F}\ell W \quad \hookrightarrow \quad \mathbb{F}\ell(V \oplus W)$$

by $\psi_P(E_\bullet, F_\bullet)_j = \langle E_i, F_{i'} \,|\, p_i, p^c_{i'} \leq j \rangle$. Equivalently, if $e_1, \ldots, e_n$ is a basis for V and $f_1, \ldots, f_m$ a basis for W, then

$$\psi_P(\langle\!\langle e_1, \ldots, e_n \rangle\!\rangle, \langle\!\langle f_1, \ldots, f_m \rangle\!\rangle) = \langle\!\langle g_1, \ldots, g_{n+m} \rangle\!\rangle,$$

where $g_{p_i} = e_i$ and $g_{p^c_i} = f_i$. From this, it follows that if $E_\bullet, E'_\bullet \in \mathbb{F}\ell V$ and $F_\bullet, F'_\bullet \in \mathbb{F}\ell W$ are pairs of opposite flags, then $\psi_P(E_\bullet, F_\bullet)$ and $\psi_{\omega_{n+m}P}(E'_\bullet, F'_\bullet)$ are opposite flags in $V \oplus W$.

Lemma 4.5.1. *Let $P, Q \in \binom{[n+m]}{n}$, $v \in \mathcal{S}_n$, and $w \in \mathcal{S}_m$. Then, for $E_\bullet \in \mathbb{F}\ell V$ and $F_\bullet \in \mathbb{F}\ell W$,*

$$\begin{array}{rcl} \psi_P\left(X_{\omega_n v}E_\bullet \times X_{\omega_m w}F_\bullet\right) &\subset& X_{\omega_{n+m}\varepsilon_{P,Q}(v,w)}\psi_Q(E_\bullet, F_\bullet) \\ \psi_P\left(X_v E_\bullet \times X_w F_\bullet\right) &\subset& X_{\varepsilon_{P,Q}(v,w)}\psi_{\omega_{n+m}Q}(E_\bullet, F_\bullet). \end{array}$$

Proof. For a flag $G_\bullet$, define $G^\circ_j := G_j - G_{j-1}$. By the definition of ψ_Q, we have $E^\circ_i \subset \psi_Q(E_\bullet, F_\bullet)^\circ_{q_i}$ and $F^\circ_i \subset \psi_Q(E_\bullet, F_\bullet)^\circ_{q^c_i}$. Since

$$\begin{array}{rcl} \omega_{n+m}Q &=& n+m+1-q_n \;<\; \cdots \;<\; n+m+1-q_1, \\ E_{n+1-j} &\subset& \psi_{\omega_{n+m}Q}(E_\bullet, F_\bullet)_{n+m+1-q_j}, \qquad \text{and} \\ F_{n+1-j} &\subset& \psi_{\omega_{n+m}Q}(E_\bullet, F_\bullet)_{n+m+1-q^c_j}, \end{array}$$

the lemma follows from the definitions of Schubert varieties and ψ_P. ∎

Corollary 4.5.2. *Let $E_\bullet, E'_\bullet \in \mathbb{F}\ell V$ and $F_\bullet, F'_\bullet \in \mathbb{F}\ell W$ be pairs of opposite flags and let $P \in \binom{[n+m]}{n}$. Set $Q = \{m+1, \ldots, m+n\}$. Then, for every $v \in \mathcal{S}_n$ and $w \in \mathcal{S}_m$,*

$$\begin{aligned}\psi_P\left(X_v E_\bullet \times X_w F_\bullet\right) &= X_{\varepsilon_{P,[n]}(v,w)}\psi_Q(E_\bullet, F_\bullet) \bigcap X_{\varepsilon_{P,Q}(e,e)}\psi_{[n]}(E'_\bullet, F'_\bullet) \\ &= X_{\varepsilon_{P,[n]}(v,e)}\psi_Q(E_\bullet, F'_\bullet) \bigcap X_{\varepsilon_{P,Q}(e,w)}\psi_{[n]}(E'_\bullet, F_\bullet) \\ &= X_{\varepsilon_{P,[n]}(e,w)}\psi_Q(E'_\bullet, F_\bullet) \bigcap X_{\varepsilon_{P,Q}(v,e)}\psi_{[n]}(E_\bullet, F'_\bullet) \\ &= X_{\varepsilon_{P,[n]}(e,e)}\psi_Q(E'_\bullet, F'_\bullet) \bigcap X_{\varepsilon_{P,Q}(v,w)}\psi_{[n]}(E_\bullet, F_\bullet).\end{aligned}$$

Proof. Since $\omega_{n+m}[n] = Q$, $X_e E_\bullet = \mathbb{F}\ell V$, and $X_e F_\bullet = \mathbb{F}\ell W$, Lemma 4.5.1 shows that $\psi_P\left(X_v E_\bullet \times X_w F_\bullet\right)$ is a subset of any of the four intersections. Equality follows as they have the same dimension. Indeed, for $x, z \in \mathcal{S}_n$ and $y, u \in \mathcal{S}_m$,

$$\begin{aligned}\ell(\varepsilon_{P,[n]}(x,y)) &= \ell(x) + \ell(y) + \#\{i \in [n], j \in [m] \mid p_i > p_j^c\} \\ \ell(\varepsilon_{P,Q}(z,u)) &= \ell(z) + \ell(u) + \#\{i \in [n], j \in [m] \mid p_j^c > p_i\}.\end{aligned}$$

Thus $\ell(\varepsilon_{P,[n]}(x,y)) + \ell(\varepsilon_{P,Q}(z,u)) = \ell(x) + \ell(y) + \ell(z) + \ell(u) + n \cdot m$ and so

$$\binom{n+m}{2} - \ell(\varepsilon_{P,[n]}(x,y)) - \ell(\varepsilon_{P,Q}(z,u)) = \binom{n}{2} + \binom{m}{2} - \ell(x) - \ell(y) - \ell(z) - \ell(u).$$

If (x,y,z,u) is one of $(v,w,e,e), (v,e,e,w), (e,w,v,e), (e,e,v,w)$, then the left hand side is the dimension of the corresponding intersection, and the right hand side is the dimension of $X_v E_\bullet \times X_w F_\bullet$. ∎

Corollary 4.5.3. *Let $Q = \{m+1, \ldots, m+n\} = \omega_{n+m}[n]$. For every $v \in \mathcal{S}_n$, $w \in \mathcal{S}_m$, and $P \in \binom{[n+m]}{n}$, the identities hold in $H^*\mathbb{F}\ell_{n+m}$:*

$$\begin{aligned}\mathfrak{S}_{\varepsilon_{P,[n]}(v,w)} \cdot \mathfrak{S}_{\varepsilon_{P,Q}(e,e)} &= \mathfrak{S}_{\varepsilon_{P,[n]}(v,e)} \cdot \mathfrak{S}_{\varepsilon_{P,Q}(e,w)} = \\ \mathfrak{S}_{\varepsilon_{P,[n]}(e,w)} \cdot \mathfrak{S}_{\varepsilon_{P,Q}(v,e)} &= \mathfrak{S}_{\varepsilon_{P,[n]}(e,e)} \cdot \mathfrak{S}_{\varepsilon_{P,Q}(v,w)},\end{aligned}$$

and this common cohomology class is $(\psi_P)_*(\mathfrak{S}_v \otimes \mathfrak{S}_w)$.

Theorem 4.5.4. *Let $x \in \mathcal{S}_{n+m}$ and $P \in \binom{[n+m]}{n}$. Then*

(i)
$$\begin{aligned}\psi_P^*\mathfrak{S}_x &= \sum_{v \in \mathcal{S}_n,\ w \in \mathcal{S}_m} c^{\varepsilon_{P,[n]}(v,w)}_{\varepsilon_{P,[n]}(e,e)\ x}\ \mathfrak{S}_v \otimes \mathfrak{S}_w \\ &= \sum_{v \in \mathcal{S}_n,\ w \in \mathcal{S}_m} c^{\varepsilon_{P,[n]}(v,\omega_m)}_{\varepsilon_{P,[n]}(e,\omega_m w)\ x}\ \mathfrak{S}_v \otimes \mathfrak{S}_w \\ &= \sum_{v \in \mathcal{S}_n,\ w \in \mathcal{S}_m} c^{\varepsilon_{P,[n]}(\omega_n,w)}_{\varepsilon_{P,[n]}(\omega_n v,e)\ x}\ \mathfrak{S}_v \otimes \mathfrak{S}_w \\ &= \sum_{v \in \mathcal{S}_n,\ w \in \mathcal{S}_m} c^{\varepsilon_{P,[n]}(\omega_n,\omega_m)}_{\varepsilon_{P,[n]}(\omega_n v,\omega_m w)\ x}\ \mathfrak{S}_v \otimes \mathfrak{S}_w\end{aligned}$$

(*ii*) *Let* $Q = \{m+1, \ldots, m+n\}$. *For every* $v \in \mathcal{S}_n$ *and* $w \in \mathcal{S}_m$, *we have*

$$\begin{array}{ccccccc}
c^{\varepsilon_{P,[n]}(v,w)}_{\varepsilon_{P,[n]}(e,e)\ x} & = & c^{\varepsilon_{P,[n]}(v,\omega_m)}_{\varepsilon_{P,[n]}(e,\omega_m w)\ x} & = & c^{\varepsilon_{P,[n]}(\omega_n,w)}_{\varepsilon_{P,[n]}(\omega_n v,e)\ x} & = & c^{\varepsilon_{P,[n]}(\omega_n,\omega_m)}_{\varepsilon_{P,[n]}(\omega_n v,\omega_m w)\ x} \\
\| & & \| & & \| & & \| \\
c^{\varepsilon_{P,Q}(v,w)}_{\varepsilon_{P,Q}(e,e)\ x} & = & c^{\varepsilon_{P,Q}(v,\omega_m)}_{\varepsilon_{P,Q}(e,\omega_m w)\ x} & = & c^{\varepsilon_{P,Q}(\omega_n,w)}_{\varepsilon_{P,Q}(\omega_n v,e)\ x} & = & c^{\varepsilon_{P,Q}(\omega_n,\omega_m)}_{\varepsilon_{P,Q}(\omega_n v,\omega_m w)\ x}
\end{array}$$

Remark 4.5.5. Each structure constant in (*ii*) is of the form $c^{\zeta y}_{y\,x}$, where ζ is one of $v\times w, v\times\overline{w}^{-1}, \overline{v}^{-1}\times w$, or $\overline{v}^{-1}\times\overline{w}^{-1}$. Each interval $[y, \zeta y]$ is isomorphic to $[e,v]\times[e,w]$. This is consistent with the expectation that the $c^{z}_{y\,x}$ should only depend upon $[y,z]$ and x.

Proof. In (*ii*), the second row is a consequence of the first as $c^{z}_{y\,x} = c^{\omega_{n+m}y}_{\omega_{n+m}z\,x}$ for $x,y,z \in \mathcal{S}_{n+m}$, and the first row is a consequence of the identities in (*i*). For (*i*), there exist constants $d^{v\,w}_{x}$ defined by

$$\psi_P^*\mathfrak{S}_x \quad = \quad \sum d^{v\,w}_{x}\ \mathfrak{S}_v \otimes \mathfrak{S}_w.$$

Since the Schubert basis is self-dual (2.3.1), we have

$$\begin{aligned}
d^{v\,w}_{x} &= \deg\left(\psi_P^*\mathfrak{S}_x \cdot (\mathfrak{S}_{\omega_n v} \otimes \mathfrak{S}_{\omega_m w})\right) \\
&= \deg\left(\mathfrak{S}_x \cdot (\psi_P)_*(\mathfrak{S}_{\omega_n v} \otimes \mathfrak{S}_{\omega_m w})\right).
\end{aligned}$$

Each expression for $(\psi_P)_*(\mathfrak{S}_{\omega_n v} \otimes \mathfrak{S}_{\omega_m w})$ of Corollary 4.5.3 yields one of the sums in (*i*). For example, the last expression yields

$$\begin{aligned}
d^{v\,w}_{x} &= \deg\left(\mathfrak{S}_x \cdot \mathfrak{S}_{\varepsilon_{P,[n]}(e,e)} \cdot \mathfrak{S}_{\omega_{n+m}\varepsilon_{P,[n]}(v,w)}\right) \\
&= c^{\varepsilon_{P,[n]}(v,w)}_{\varepsilon_{P,[n]}(e,e)\ x},
\end{aligned}$$

since $\omega_{n+m}\varepsilon_{P,[n]}(v,w) = \varepsilon_{P,\omega_{n+m}[n]}(\omega_n v, \omega_m w)$. ∎

4.6. **Maps** $\mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$**.** Let $P \subset \mathbb{N}$, define $P^c := \mathbb{N} - P$, and suppose P^c is infinite. List P and P^c:

$$\begin{array}{ccccccl}
P & : & p_1 & < & p_2 & < & \begin{cases} \cdots < p_s & \text{if } \#P = s \\ \cdots & \text{otherwise} \end{cases} \\
P^c & : & p^c_1 & < & p^c_2 & < & \cdots
\end{array}$$

Define $\Psi_P : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$ by

$$x_{p_i} \longmapsto y_i \qquad x_{p^c_i} \longmapsto z_i.$$

Then there exist $d^{u\,v}_{w}(P) \in \mathbb{Z}$ for $u, v, w \in \mathcal{S}_\infty$ defined by:

$$\Psi_P(\mathfrak{S}_w(x)) \;=\; \sum_{u,v} d^{u\,v}_{w}(P)\ \mathfrak{S}_u(y)\ \mathfrak{S}_v(z).$$

For $l, d \in \mathbb{N}$ and $R \subset \{d{+}1, \ldots, d{+}2l\}$ with $\#R = l$, define $\overline{P}(l,d,R) := (P \bigcap [d]) \bigcup R$.

Theorem D′. *Let $P \subset \mathbb{N}$ and $w \in \mathcal{S}_\infty$. For any integers $l > \ell(w)$ and d exceeding the last descent of w and any subset R of $\{d+1, \ldots, d+2l\}$ of cardinality l, set $n := \#\overline{P}(l,d,R)$, $m := d + 2l - n$, and $\pi := \varepsilon_{\overline{P}(l,d,R),\,[n]}(e,e)$. Then $d^{u\,v}_w(P) = 0$ unless $u \in \mathcal{S}_n$ and $v \in \mathcal{S}_m$, and in that case,*

$$d^{u\,v}_w(P) \;=\; c^{(u\times v)\pi}_{\pi\; w}.$$

Moreover, $d^{u\,v}_w(P) \neq 0$ implies that $a := \#P \bigcap [d]$ exceeds the last descent of u and $d - a$ exceeds the last descent of v.

Remark 4.6.1. Theorem D′ generalizes [29, 1.5] (see also [35, 4.19]) where it is shown that $d^{u\,v}_w([a]) \geq 0$. Define I_P to be

$$\{\varepsilon_{\overline{P}(l,l,R),[n]}(e,e) \mid l \in \mathbb{N},\ n = l + \#(P \bigcap [l]),\ \text{and } R \subset \{l+1, \ldots, 3l\}, \#R = l\}.$$

For $w \in \mathcal{S}_n$, choose $N \in \mathbb{N}$ so that $N/3$ exceeds both the $\ell(w)$ and the last descent of w. If $\pi \in I_P$ with $\pi \notin \mathcal{S}_N$, then $\pi = \varepsilon_{\overline{P}(l,d,R),[n]}(e,e)$ for l, d, R satisfying the conditions of Theorem D′ and so $d^{u\,v}_w(P) \;=\; c^{(u\times v)\pi}_{\pi\; w}$ for $\pi \in I_P - \mathcal{S}_N$, which establishes Theorem D.

Apply the ring homomorphism Ψ_P to both sides of the product:

$$\mathfrak{S}_w(x)\ \mathfrak{S}_\gamma(x) \quad = \quad \sum_\zeta c^{\zeta}_{w\,\gamma}\, \mathfrak{S}_\zeta(x).$$

Expand this in terms of $\mathfrak{S}_\eta(y)\mathfrak{S}_\xi(z)$ and equate coefficients to obtain:

Corollary 4.6.2. *Let $w, \gamma, \eta, \xi \in \mathcal{S}_\infty$, and $P \subset \mathbb{N}$. Then there exists an integer $N \in \mathbb{N}$ such that if $\pi \in I_P - \mathcal{S}_N$, then*

$$\sum_\zeta c^{(\eta\times\xi)\pi}_{\pi\;\zeta}\ c^{\zeta}_{w\,\gamma} \quad = \quad \sum_{u,v,\alpha,\beta} c^{(u\times v)\pi}_{\pi\; w}\ c^{(\alpha\times\beta)\pi}_{\pi\;\gamma}\ c^{\eta}_{u\,\alpha}\ c^{\xi}_{v\,\beta}.$$

Proof of Theorem D′. First, $\mathfrak{S}_\pi(x) \in \mathbb{Z}[x_1, \ldots, x_s]$ whenever s exceeds the last descent of π [28] (see also [35, 4.13]). Thus, $\mathfrak{S}_w(x) \in \mathbb{Z}[x_1, \ldots, x_d]$, and if $d^{u\,v}_w(P) \neq 0$, then $\mathfrak{S}_u(y) \in \mathbb{Z}[y_1, \ldots, y_a]$ and $\mathfrak{S}_v(z) \in \mathbb{Z}[z_1, \ldots, z_b]$, hence a, respectively, b, exceeds the last descent of u, respectively v. Since $\deg \mathfrak{S}_w(x) \leq l$, both $\deg \mathfrak{S}_u(y)$ and $\deg \mathfrak{S}_v(z)$ are at most l. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}[x_1, \ldots, x_d] & \xhookrightarrow{\ \iota\ } & \mathbb{Z}[x_1, \ldots, x_{n+m}] & \xrightarrow{\ \overline{\Psi_P}\ } & \mathbb{Z}[y_1, \ldots, y_n, z_1, \ldots, z_m] \\
 & & \downarrow & & \downarrow \\
 & & H^*\mathbb{F}\ell_{n+m} & \xrightarrow{\ \psi^*_{\overline{P}}\ } & H^*\mathbb{F}\ell_n \otimes H^*\mathbb{F}\ell_m
\end{array}$$

Here, $\overline{\Psi_P}$ is the restriction of $\Psi_{\overline{P}}$ to $\mathbb{Z}[x_1,\ldots,x_{n+m}]$. The vertical arrows are injective on the module $\mathbb{Z}\langle x_1^{\alpha_1}\cdots x_d^{\alpha_d} \mid \alpha_i \leq l\rangle$ and its image

$$\mathbb{Z}\langle y_1^{\beta_1}\cdots y_a^{\beta_a} z_1^{\gamma_1}\cdots z_b^{\gamma_b} \mid \beta_i,\gamma_j \leq l\rangle \quad \subset \quad \mathbb{Z}[y_1,\ldots,y_n,z_1,\ldots,z_m].$$

Moreover, as $P\bigcap[d] = \overline{P}\bigcap[d]$, the composition, $\overline{\Psi_P}\circ\iota$, of the top row coincides with $\Psi_P\circ\iota$. Since $\mathfrak{S}_w(x) \in \mathbb{Z}\langle x_1^{\alpha_1}\cdots x_d^{\alpha_d} \mid \alpha_i \leq l\rangle$, the formula for $\psi^*_{\overline{P}}(\mathfrak{S}_w)$ in Theorem 4.5.4 computes $\Psi_P(\mathfrak{S}_w(x))$. ▛

4.7. **Products of Grassmannians.** Let $k \leq n$ and $l \leq m$ be integers, $V \simeq \mathbb{C}^n$, and $W \simeq \mathbb{C}^m$. Define $\varphi_{k,l} : Grass_kV \times Grass_lW \hookrightarrow Grass_{k+l}(V\oplus W)$ by

$$\varphi_{k,l} \;:\; (H,K) \quad \longmapsto \quad H\oplus K.$$

Theorem 4.7.1.

(*i*) *For every Schubert class $S_\lambda \in H^*Grass_{k+l}V\oplus W$,*

$$\varphi^*_{k,l}(S_\lambda) \quad = \quad \sum_{\mu,\nu} c^\lambda_{\mu\,\nu} S_\mu \otimes S_\nu.$$

(*ii*) *If $S_{\mu^c}\otimes S_{\nu^c} \in H^*Grass_kV \otimes H^*Grass_lW$, then*

$$(\varphi_{k,l})_*(S_{\mu^c}\otimes S_{\nu^c}) \quad = \quad \sum_\lambda c^\lambda_{\mu\,\nu} S_{\lambda^c},$$

where λ^c,μ^c, and ν^c are defined by $\mu^c_i = n-k-\mu_{k+1-i}$, $\nu^c_i = m-l-\nu_{l+1-i}$, and $\lambda^c_i = m+n-k-l-\lambda_{k+l+1-i}$.

Remark 4.7.2. Suppose $-x_1,\ldots,-x_k$ are Chern roots of the tautological bundle over $Grass_kV$, $-y_1,\ldots,-y_l$ those of the tautological bundle over $Grass_lW$, and $f \in H^*Grass_{k+l}V\oplus W$ (which is a symmetric polynomial in the negative Chern roots of the tautological bundle over $Grass_{k+l}V\oplus W$). Then

$$\varphi^*_{k,l}f \;=\; f(x_1,\ldots,x_k,y_1,\ldots,y_l).$$

Let $\Lambda = \Lambda(z)$ be the ring of symmetric functions, which is the inverse limit (in the category of graded rings) of the rings of symmetric polynomials in the variables $z_1,\ldots,z_n$. Fixing λ and choosing k,l,n, and m large enough gives a new proof of [36, I.5.9]:

Proposition 4.7.3 ([36, I.5.9])**.** *Let λ be a partition and x, y be infinite sets of variables. Then*

$$S_\lambda(x,y) \;=\; \sum_{\mu,\nu} c^\lambda_{\mu\,\nu}\, S_\mu(x)\cdot S_\nu(y),$$

where S_μ are Schur functions in the ring Λ of symmetric functions.

If we define a linear map $\Delta : \Lambda(z) \to \Lambda(x) \otimes_{\mathbb{Z}} \Lambda(y)$ by $\Delta(f(z)) = f(x,y)$, then Δ is induced by the maps $\varphi^*_{k,l}$. Moreover, the obvious commutative diagrams of spaces give a new proof of [36, I.5.25], that Λ is a cocommutative Hopf algebra with comultiplication Δ.

Proof of Theorem 4.7.1. The first statement is a consequence of the second: Schubert classes form a basis for the cohomology ring, so there exist integral constants $d^{\mu\,\nu}_{\lambda}$ such that

$$\varphi^*_{k,l}(S_\lambda) \quad = \quad \sum_{\mu,\nu} d^{\mu\,\nu}_{\lambda} S_\mu \otimes S_\nu.$$

Since the Schubert basis diagonalizes the intersection pairing,

$$d^{\mu\,\nu}_{\lambda} \quad = \quad \deg(\varphi^*_{k,l}(S_\lambda) \cdot (S_{\mu^c} \otimes S_{\nu^c})).$$

Apply $(\varphi_{k,l})_*$ and use the second assertion to obtain

$$\begin{aligned} d^{\mu\,\nu}_{\lambda} &= \deg(S_\lambda \cdot (\varphi_{k,l})_*(S_{\mu^c} \otimes S_{\nu^c})) \\ &= S_\lambda \cdot \sum_{\kappa} c^{\kappa}_{\mu\,\nu} S_{\kappa^c} \\ &= c^{\lambda}_{\mu\,\nu}. \end{aligned}$$

The second assertion is a consequence of the following lemma.

Lemma 4.7.4. *Suppose μ, ν are partitions with $\mu \subset (n-k)^k$ and $\nu \subset (m-l)^l$. Let $E_\bullet \in \mathbb{F}\ell V$ and $F_\bullet \in \mathbb{F}\ell W$ and let $G'_\bullet$ be any flag opposite to $\psi_{[n]}(E_\bullet, F_\bullet)$ with $G'_m = W$. Then*

$$\varphi_{k,l}\left(\Omega_{\mu^c} E_\bullet \times \Omega_{\nu^c} F_\bullet\right) \quad = \quad \Omega_{\rho^c}\psi_{[n]}(E_\bullet, F_\bullet) \bigcap \Omega_{(n-k)^l} G'_\bullet, \tag{4.7.1}$$

where ρ is the partition

$$\nu_1 + (n-k) \ \geq \ \cdots \ \geq \ \nu_l + (n-k) \ \geq \ \mu_1 \ \geq \cdots \ \geq \mu_k.$$

We finish the proof of Theorem 4.7.1. Lemma 4.7.4 implies

$$\begin{aligned} \left(\varphi_{k,l}\right)_* \left(S_{\mu^c} \otimes S_{\nu^c}\right) &= \left[\Omega_{\rho^c}\psi_{[n]}(E_\bullet, F_\bullet) \bigcap \Omega_{(n-k)^l} G'_\bullet\right] \\ &= \sum_{\lambda} c^{\lambda^c}_{\rho^c\,(n-k)^l}\, S_{\lambda^c}. \end{aligned}$$

Since $\deg(S_\alpha \cdot S_\beta \cdot S_\gamma) = c^{\alpha^c}_{\beta\,\gamma}$, we see that

$$c^{\lambda^c}_{\rho^c\,(n-k)^l} \ = \ c^{\rho}_{(n-k)^l\,\lambda} \ = \ c^{\rho/(n-k)^l}_{\lambda} \ = \ c^{\mu \coprod \nu}_{\lambda} \ = \ c^{\lambda}_{\mu\,\nu}.$$

Here, $\mu \coprod \nu$ is a skew partition with two components μ and ν and the last equality is a special case of (1.3.1) in §1.3. ∎

Proof of Lemma 4.7.4. Since

$$\Omega_{(n-k)^l} G'_\bullet \;=\; \{M \in Grass_{k+l} V \oplus W \mid \dim M \bigcap G'_m \geq l\}$$

and $G'_m = W$, we see that $\varphi_{k,l}(Grass_k V \times Grass_l W) \subset \Omega_{(n-k)^l} G'_\bullet$. The inclusion in (4.7.1) follows, as the definitions imply

$$\varphi_{k,l}(\Omega_{\mu^c} E_\bullet \times \Omega_{\nu^c} F_\bullet) \;\subset\; \Omega_{\rho^c} \psi_{[n]}(E_\bullet, F_\bullet).$$

Equality follows, as the cycles have the same dimension: The intersection has dimension $|\rho| - |(n-k)^l| \;=\; |\mu| + |\nu| \;=\; \dim \Omega_{\mu^c} E_\bullet \times \Omega_{\nu^c} F_\bullet$. ∎

5. Identities among the $c^w_{u\,v(\lambda,k)}$

5.1. **Proof of Theorem E (*ii*).** Combining Lemma 4.3.4 with Theorem C $(i)(b)$, we deduce:

Lemma 5.1.1. *Suppose $x \leq_k z$ and $x(p) = z(p)$. Let $k' = k-1$ if $p < k$ and $k' = k$ otherwise. Then for all partitions λ, we have*

$$c^z_{x\;v(\lambda,k)} \quad = \quad c^{z/p}_{x/p\;v(\lambda,k')}.$$

By Lemma 4.1.1 (iv), zx^{-1} and $z/_p(x/_p)^{-1}$ are shape-equivalent.

Lemma 5.1.2. *Let $x, z, u, w \in \mathcal{S}_n$. Suppose $x \leq_k z$, $u \leq_k w$, and $zx^{-1} = wu^{-1}$. Further suppose that w is Grassmannian with descent k, the permutation wu^{-1} has no fixed points, and, for $k < i \leq n$, $u(i) = x(i)$. Then, for all partitions λ with at most k parts,*

$$c^w_{u\,v(\lambda,k)} \quad = \quad c^z_{x\,v(\lambda,k)}.$$

Proof of Theorem E (*ii*) using Lemma 5.1.2. We reduce Theorem E (ii) to Lemma 5.1.2. First, by Lemma 5.1.1, it suffices to prove Theorem E (ii) when $x, z, u, w \in \mathcal{S}_n$, $k = l$, with $wu^{-1} = zx^{-1}$ and the permutation wu^{-1} has no fixed points.

Define $s \in \mathcal{S}_n$ by

$$s(i) \quad := \quad \begin{cases} u(i) & 1 \leq i \leq k \\ x(i) & k < i \leq n \end{cases}$$

and set $t := wu^{-1}s$. Then $s \leq_k t$ and

$$t(i) \quad = \quad \begin{cases} w(i) & 1 \leq i \leq k \\ z(i) & k < i \leq n \end{cases} .$$

It suffices to show that $c^w_{u\,v(\lambda,k)}$ and $c^t_{s\,v(\lambda,k)}$ each equal $c^z_{t\,v(\lambda,k)}$. Thus we may further assume $u(i) = x(i)$ for $1 \leq i \leq k$ or $u(i) = x(i)$ for $k < i \leq n$.

Suppose that $u(i) = x(i)$ for $1 \leq i \leq k$. If for $v \in \mathcal{S}_n$, $\overline{v} := \omega_0 v \omega_0$,

$$c^z_{x\;v(\lambda,k)} \quad = \quad c^w_{u\;v(\lambda,k)} \quad \Longleftrightarrow \quad c^{\overline{z}}_{\overline{x}\;\overline{v(\lambda,k)}} \quad = \quad c^{\overline{w}}_{\overline{u}\;\overline{v(\lambda,k)}}.$$

Set $l = n - k$ and λ^t the partition conjugate to λ. Then $\overline{x} \leq_{k'} \overline{z}$, $\overline{u} \leq_{k'} \overline{w}$, $\overline{z}(\overline{x}^{-1}) = \overline{wu}^{-1}$, $\overline{v(\lambda, k)} = v(\lambda^t, l)$, and $\overline{x}(i) = \overline{u}(i)$ for $l < i \leq n$. Thus we may assume $x(i) = u(i)$ for $1 \leq i \leq k$.

Finally, there is a permutation $s \in \mathcal{S}_n$ such that $wu^{-1}s$ is Grassmannian of descent k. Thus it suffices to further assume that w is Grassmannian with descent k, the situation of Lemma 5.1.2. ∎

We prove Lemma 5.1.2 by studying two intersections of Schubert varieties and their image under the projection $\mathbb{F}\ell V \twoheadrightarrow Grass_k V$. Let $e_1, \ldots, e_n$ be a basis for V and set $F_\bullet = \langle\langle e_1, \ldots, e_n \rangle\rangle$. Let $M(w) \subset M_{n\times n}\mathbb{C}$ be the set of matrices satisfying the conditions:

(a) $M(w)_{i,w(i)} = 1$
(b) $M(w)_{i,j} = 0$ if either $w(i) < j$ or else $w^{-1}(j) < i$.

Then $M(w) \simeq \mathbb{C}^{\ell(w)}$ as the only unconstrained entries of $M(w)$ are $M(w)_{i,j}$ when $j < w(i)$ and $i < w^{-1}(j)$, and there are $\ell(w)$ such entries.

Example 5.1.3. $M(25134)$ is the set of matrices

$$\left\{ \begin{bmatrix} a & 1 & 0 & 0 & 0 \\ b & 0 & c & d & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \middle| (a,b,c,d) \in \mathbb{C}^4 \right\}$$

Fix a basis $e_1, \ldots, e_n$ for V. For $\alpha \in M(w)$, and $1 \leq i \leq n$, define the vector $f_i(\alpha) := \sum_j \alpha_{i,j} e_j$. Then $f_1(\alpha), \ldots, f_n(\alpha)$ are the 'row vectors' of the matrix α and they form a basis for V as α has determinant $(-1)^{\ell(w)}$. Set $E_\bullet(\alpha) = \langle\langle f_1(\alpha), \ldots, f_n(\alpha) \rangle\rangle$. Since $f_i(\alpha) \in F_{w(i)} - F_{w(i)-1}$, we see that $E_\bullet(\alpha) \in X_{\omega_0 w} F_\bullet$. In fact, $M(w)$ parameterizes the Schubert cell $X^\circ_{\omega_0 w} F_\bullet$. When w is Grassmannian with descent k, matrices in $M(w)$ have a simple form: if $k < i$, then $f_i(\alpha) = e_{w(i)}$.

For opposite flags $F_\bullet, F'_\bullet$, $\mathfrak{S}_{\omega_0 w} \cdot \mathfrak{S}_u$ is the class Poincaré dual to the fundamental cycle of $X_{\omega_0 w} F_\bullet \bigcap X_u F'_\bullet$. We use the projection formula (2.3.2) to compute the coefficient $c^w_{u\, v(\lambda,k)}$:

$$\begin{aligned} c^w_{u\, v(\lambda,k)} &= \deg(S_\lambda(x_1, \ldots, x_k) \cdot \mathfrak{S}_{\omega_0 w} \cdot \mathfrak{S}_u) \\ &= \deg(\pi_k)_*(S_\lambda(x_1, \ldots, x_k) \cdot \mathfrak{S}_{\omega_0 w} \cdot \mathfrak{S}_u) \\ &= \deg(S_\lambda \cdot (\pi_k)_*(\mathfrak{S}_{\omega_0 w} \cdot \mathfrak{S}_u)) \end{aligned}$$

Thus Lemma 5.1.2 is a consequence of Lemma 5.1.4, which shows:

$$(\pi_k)_*(\mathfrak{S}_{\omega_0 w} \cdot \mathfrak{S}_u) = (\pi_k)_*(\mathfrak{S}_{\omega_0 z} \cdot \mathfrak{S}_x).$$

Lemma 5.1.4. *Let u, w, x, z satisfy the hypotheses of Lemma 5.1.2. Then, if $F_\bullet$ and $F_\bullet'$ are opposite flags in V,*

$$\pi_k\left(X_{\omega_0 w}F_\bullet \bigcap X_u F_\bullet'\right) \quad = \quad \pi_k\left(X_{\omega_0 z}F_\bullet \bigcap X_x F_\bullet'\right)$$

and the projections π_k onto the image cycle have the same degree.

Proof. Let $e_1, \ldots, e_n$ be a basis for V such that $F_\bullet = \langle\langle e_1, \ldots, e_n\rangle\rangle$ and $F_\bullet' = \langle\langle e_n, \ldots, e_1\rangle\rangle$, and define $M(w)$ as before. Let $A \subset M(w)$ consist of those matrices α such that $E_\bullet(\alpha) \in X_u^\circ F_\bullet'$. If $j > k$, set $g_j(\alpha) = f_j(\alpha) = e_{w(j)}$. For $j \leq k$ construct $g_j(\alpha)$ inductively, setting $g_j(\alpha)$ to be the intersection of $F'_{n+1-u(j)}$ and the affine space $f_j(\alpha)+ \langle g_i(\alpha) \,|\, i < j \text{ and } u(i) < u(j)\rangle$. Since $E_\bullet(\alpha) \in X_u F_\bullet$ and $\dim E_j(\alpha) \bigcap F'_{n+1-u(j)} = \#\{i \leq j \,|\, u(i) > u(j)\}$, this intersection consists of a single, non-zero vector, $g_j(\alpha)$.

The algebraic map $A \ni \alpha \mapsto (g_1(\alpha), \ldots, g_n(\alpha)) \in V^n$ parameterizes a basis of V. Moreover, for $\alpha \in A$, $E_\bullet(\alpha) = \langle\langle g_1(\alpha), \ldots, g_n(\alpha)\rangle\rangle$, and if $1 \leq j \leq k$, then $g_j(\alpha) \in F'_{n+1-u(j)} \bigcap F_{w(j)}$. For $\alpha \in A$,

$$G_\bullet(\alpha) \;:=\; \langle\langle g_{u^{-1}x(1)}(\alpha), \ldots, g_{u^{-1}x(n)}(\alpha)\rangle\rangle \;\in\; X_{\omega_0 z}F_\bullet \bigcap X_x F_\bullet', \tag{5.1.1}$$

thus A parameterizes a subset of $X_{\omega_0 z}F_\bullet \bigcap X_x F_\bullet'$. Indeed, for $1 \leq j \leq k$, $g_{u^{-1}x(j)} \in F'_{n+1-x(j)} \bigcap F_{y(j)}$. Also for $j > k$, we have $u^{-1}x(j) = j = w^{-1}y(j)$, thus $g_j(\alpha) = f_j(\alpha) = e_{y(j)}$ and $G_j(\alpha) = E_j(\alpha)$. Then the definition of Schubert varieties in §2.3 implies (5.1.1).

Both cycles $X_{\omega_0 w}F_\bullet \bigcap X_u F_\bullet'$ and $X_{\omega_0 z}F_\bullet \bigcap X_x F_\bullet'$ are irreducible and have the same dimension, $\ell(w) - \ell(u) = |wu^{-1}|$. Since $G_\bullet(\alpha) = G_\bullet(\beta)$ if and only if $\alpha = \beta$, the loci of flags $\{G_\bullet(\alpha) \,|\, \alpha \in A\}$ is dense in $X_x F_\bullet' \bigcap X_{\omega_0 y}F_\bullet$. Since the association $E_\bullet(\alpha) \mapsto G_\bullet(\alpha)$ induces a rational map

$$X_{\omega_0 w}F_\bullet \bigcap X_u F_\bullet' \;\dashrightarrow\; X_{\omega_0 z}F_\bullet \bigcap X_x F_\bullet'$$

covering the projections π_k, these projections have the same degree, which completes the proof. ∎

5.2. **Proof of Theorem G (*ii*).** We show that if ζ and η are disjoint permutations and λ any partition, then

$$c^{\zeta\eta}_\lambda \quad = \quad \sum_{\mu,\nu} c^\lambda_{\mu\,\nu}\; c^\zeta_\mu\; c^\eta_\nu.$$

Lemma 5.2.1. *Let $\zeta, \eta \in \mathcal{S}_{n+m}$ be disjoint permutations. Suppose $k \geq \#\mathrm{up}_\zeta$, $l \geq \#\mathrm{up}_\eta$, $n \geq \#\mathrm{supp}_\zeta$, and $m \geq \#\mathrm{supp}_\eta$. Let $u \in \mathcal{S}_{n+m}$ be a permutation such that $u \leq_{k+l} \zeta\eta u$. Let Q be any element of $\binom{[n+m]-\mathrm{supp}_\eta}{n}$ which contains supp_ζ for which $k = \#u^{-1}(Q)\bigcap[k+l]$. Set $Q^c := [n+m] - Q$.*

Define $\zeta' \in \mathcal{S}_n$ *and* $\eta' \in \mathcal{S}_m$ *by* $\phi_Q(\zeta') = \zeta$ *and* $\phi_{Q^c}(\eta') = \eta$. *Set* $P = u^{-1}(Q)$, $P^c = u^{-1}(Q^c)$, *and define* $v \in \mathcal{S}_n$ *and* $w \in \mathcal{S}_m$ *by* $u(p_i) = q_{v(i)}$ *and* $u(p_i^c) = q_{w(i)}^c$, *where*

$$\begin{array}{lllllllllll} P &=& p_1 &<& p_2 &<& \cdots &<& p_n \\ Q &=& q_1 &<& q_2 &<& \cdots &<& q_n \end{array} \qquad \begin{array}{lllllllllll} P^c &=& p_1^c &<& p_2^c &<& \cdots &<& p_m^c \\ Q^c &=& q_1^c &<& q_2^c &<& \cdots &<& q_m^c \end{array}$$

Then

(*i*) $v \leq_k \zeta' v$ *and* $w \leq_l \eta' w$,
(*ii*) $u = \varepsilon_{P,Q}(v, w)$ *and* $\zeta\eta u = \varepsilon_{P,Q}(\zeta' v, \eta' w)$, *and*
(*iii*) *For all pairs of opposite flags* $E_\bullet, E'_\bullet \in \mathbb{F}\ell_n$ *and* $F_\bullet, F'_\bullet \in \mathbb{F}\ell_m$,

$$\psi_P\left[\left(X_{\omega_n\zeta' v}E_\bullet \bigcap X_v E'_\bullet\right) \times \left(X_{\omega_m\eta' w}F_\bullet \bigcap X_w F'_\bullet\right)\right] = X_{\omega_{n+m}\zeta\eta u}\psi_Q(E_\bullet, F_\bullet) \bigcap X_u \psi_{\omega_0^{(m+n)}Q}(E'_\bullet, F'_\bullet).$$

Proof. Since $u \leq_{k+l} \zeta\eta u$, (*i*) follows from Theorem A. Statement (*ii*) is also immediate. For (*iii*), Lemma 4.5.1 shows the inclusion $\subset$. Since ζ' is shape equivalent to ζ, η' to η, and ζ and η are disjoint, $|\zeta\eta| = |\zeta'| + |\eta'|$, showing both cycles have the same dimension, and hence are equal, as $\psi_Q(E_\bullet, F_\bullet)$ and $\psi_{\omega_0^{(m+n)}Q}(E'_\bullet, F'_\bullet)$ are opposite.

Note that if $u \leq_k \zeta u$, then

$$c_\lambda^\zeta = \deg(S_\lambda \cdot (\pi_k)_*(\mathfrak{S}_{\omega_0\zeta u} \cdot \mathfrak{S}_u)).$$

Thus the skew coefficients c_λ^ζ are defined by the identity in H^*Grass_kV:

$$(\pi_k)_*(\mathfrak{S}_{\omega_0\zeta u} \cdot \mathfrak{S}_u) = \sum_{\lambda\subset(n-k)^k} c_\lambda^\zeta\, S_{\lambda^c}. \tag{5.2.1}$$

Proof of Theorem G (*ii*). We use the notation of Lemma 5.2.1. The following diagram commutes, since $[k+l] = \{p_1, \ldots, p_k, p_1^c, \ldots, p_l^c\}$.

$$\begin{array}{ccc} \mathbb{F}\ell_n \times \mathbb{F}\ell_m & \xrightarrow{\quad\psi_P\quad} & \mathbb{F}\ell_{n+m} \\ {\scriptstyle \pi_k \times \pi_l}\downarrow & & \downarrow{\scriptstyle \pi_{k+l}} \\ Grass_k\mathbb{C}^n \times Grass_l\mathbb{C}^m & \xrightarrow{\quad\varphi_{k,l}\quad} & Grass_{k+l}\mathbb{C}^{n+m} \end{array}$$

From this and Lemma 5.2.1, we see that

$$\pi_{k+l}\left(X_{\omega_{n+m}\zeta\eta u}\psi_Q(E_\bullet, F_\bullet) \bigcap X_u \psi_{\omega_0^{(m+n)}Q}(E'_\bullet, F'_\bullet)\right)$$

is equal to

$$\varphi_{k,l}\left(\pi_k\left(X_{\omega_n\zeta' v}E_\bullet \bigcap X_v E'_\bullet\right) \times \pi_l\left(X_{\omega_m\eta' w}F_\bullet \bigcap X_w F'_\bullet\right)\right).$$

Thus $(\pi_{k+l})_*\left(\mathfrak{S}_{\omega_{n+m}\zeta\eta u}\cdot\mathfrak{S}_u\right)$ is equal to

$$(\varphi_{k,l})_*\left((\pi_k)_*\left(\mathfrak{S}_{\omega_n\zeta' v}\cdot\mathfrak{S}_v\right)\otimes(\pi_l)_*\left(\mathfrak{S}_{\omega_m\eta' w}\cdot\mathfrak{S}_w\right)\right).$$

This, together with (5.2.1) and Theorem 4.7.1 (*ii*), gives

$$\begin{aligned}\sum_\lambda c^{\zeta\eta}_\lambda S_{\lambda^c} &= (\pi_{k+l})_*\left(\mathfrak{S}_{\omega_{n+m}\zeta\eta u}\cdot\mathfrak{S}_u\right)\\ &= (\varphi_{k,l})_*\left(\sum_\mu c^{\zeta'}_\mu S_{\mu^c}\otimes\sum_\nu c^{\eta'}_\nu S_{\nu^c}\right)\\ &= \sum_{\mu,\nu} c^{\zeta'}_\mu c^{\eta'}_\nu(\varphi_{k,l})_*\left(S_{\mu^c}\otimes\ S_{\nu^c}\right)\\ &= \sum_{\mu,\nu} c^{\zeta'}_\mu c^{\eta'}_\nu\sum_\lambda c^\lambda_{\mu\,\nu}S_{\lambda^c}.\end{aligned}$$

We are done, as ζ',ζ and η',η are shape equivalent pairs. ∎

5.3. **Cyclic Shift. Theorem H′ (Cyclic Shift)** *Let $u,w,x,z\in\mathcal{S}_\infty$ with $u\leq_k w$ and $x\leq_l z$. Suppose $wu^{-1}\in\mathcal{S}_n$ and zx^{-1} is shape equivalent to $(wu^{-1})^{(1\,2\,\ldots\,n)^t}$, for some t. Then, for every partition λ,*

$$c^w_{u\,v(\lambda,k)} \quad=\quad c^z_{x\,v(\lambda,l)}.$$

Proof. It suffices to prove a restricted case. Suppose $u,w\in\mathcal{S}_n$, $u\leq_k w$, and w is Grassmannian with descent k. We construct permutations $x,z\in\mathcal{S}_n$ with $x\leq_k z$ and $zx^{-1}=(wu^{-1})^{(1\,2\,\ldots\,n)}$ for which

$$\pi_k\left(X_{\omega_0 w}F_\bullet\bigcap X_uF'_\bullet\right) \quad=\quad \pi_k\left(X_{\omega_0 z}G_\bullet\bigcap X_xG'_\bullet\right), \tag{5.3.1}$$

where $e_1,\ldots,e_n$ be a basis for V and the flags $F_\bullet, F'_\bullet, G_\bullet$, and $G'_\bullet$ are

$$\begin{aligned}F_\bullet &= \langle\langle e_1,\ldots,e_n\rangle\rangle & F'_\bullet &= \langle\langle e_n,\ldots,e_1\rangle\rangle\\ G_\bullet &= \langle\langle e_n,e_1,\ldots,e_{n-1}\rangle\rangle & G'_\bullet &= \langle\langle e_{n-1},\ldots,e_1,e_n\rangle\rangle.\end{aligned}$$

Then (5.3.1) implies $c^w_{u\,v(\lambda,k)}=c^z_{x\,v(\lambda,k)}$, which completes the proof.

If $wu^{-1}(n)=n$, then $zx^{-1}=1\times wu^{-1}$, which is shape equivalent to wu^{-1}, and the result follows by Theorem E (*ii*). Assume $wu^{-1}(n)\neq n$. Then $w(k)=n$ and $u(k)<n$, as w is Grassmannian with descent k. Set $m:=u(k)$, $p:=u^{-1}(n)(>k)$, and $l := w(p)$. Define $x \in \mathcal{S}_n$ by:

$$x(j) \quad = \quad \begin{cases} u(j)+1 & 1 \leq j < k \text{ or } p < j \\ 1 & j = k \\ m+1 & j = k+1 \\ u(j-1)+1 & k+1 < j \leq p \end{cases} .$$

Then $x \leq_k z := (wu^{-1})^{(1\,2\,\ldots\,n)}x$ where

$$z(j) \quad = \quad \begin{cases} w(j)+1 & 1 \leq j < k \text{ or } p < j \\ l+1 & j = k \\ 1 & j = k+1 \\ w(j-1)+1 & k+1 < j \leq p \end{cases} .$$

To show (5.3.1), let $g_1(\alpha), \ldots, g_n(\alpha)$ for $\alpha \in A$ be the parameterized basis for flags $E_{\bullet}(\alpha) \in X_u^\circ F'_{\bullet} \bigcap X_{\omega_0 w}^\circ F_{\bullet}$ constructed in the proof of Lemma 5.1.4. Since $g_k(\alpha) \in F'_{n+1-u(k)} \bigcap F_{w(k)}$, $u(k) = m$, and $w(k) = n$, there exist regular functions $\beta_j(\alpha)$ on A such that

$$g_k(\alpha) = e_n + \sum_{j=m}^{n-1} \beta_j(\alpha) e_j.$$

Since $F'_1 = \langle e_n \rangle \subset E_p(\alpha) - E_{p-1}(\alpha)$ and $g_p(\alpha) = e_l$, there exist regular functions $\delta_j(\alpha)$ on A with $\delta_p(\alpha)$ nowhere vanishing such that

$$\begin{aligned} e_n &= \sum_{j=1}^{p} \delta_j(\alpha) g_j(\alpha) \\ &= g_k(\alpha) + \sum_{j=1}^{k-1} \delta_j(\alpha) g_j(\alpha) + \sum_{j=k+1}^{p} \delta_j(\alpha) e_{w(j)}, \end{aligned}$$

as $g_k(\alpha)$ is the only $g_j(\alpha)$ whose e_n-coefficient is has a non-zero. Thus

$$e_n - \sum_{j=k+1}^{p} \delta_j(\alpha) e_{w_j} \;=\; g_k(\alpha) + \sum_{j=1}^{k-1} \delta(\alpha) g_j(\alpha) \;\in\; E_k(\alpha) - E_{k-1}(\alpha).$$

Define a basis $h_1(\alpha), \ldots, h_n(\alpha)$ for V by

$$h_j(\alpha) \quad = \quad \begin{cases} g_j(\alpha) & 1 \leq j < k \text{ or } p < j \\ e_n - \left(\sum_{j=k+1}^{p} \delta_j(\alpha) e_{w(j)} \right) & j = k \\ e_n & j = k+1 \\ g_{j-1}(\alpha) & k+1 < j \leq p \end{cases} .$$

We claim $E'_\bullet(\alpha) := \langle\langle h_1(\alpha), \ldots, h_n(\alpha)\rangle\rangle$ is a flag in $X_{\omega_0 z}G_\bullet \bigcap X_x G'_\bullet$, which implies (5.3.1): Since $h_k(\alpha) \in E_k(\alpha) - E_{k-1}(\alpha)$ and $h_j(\alpha) = g_j(\alpha)$ for $j < k$, we have

$$E'_k(\alpha) \quad = \quad \langle h_1(\alpha), \ldots, h_k(\alpha)\rangle \quad = \quad E_k(\alpha).$$

Thus if $\alpha \neq \alpha'$, then $E'_\bullet(\alpha) \neq E'_\bullet(\alpha')$ and so $\{E'_\bullet(\alpha) \mid \alpha \in A\}$ is a subset of the intersection $X_{\omega_0 z}G_\bullet \bigcap X_x G'_\bullet$ of dimension equal to $\dim A = \ell(w) - \ell(u) = \ell(z) - \ell(x)$, the dimension of $X_{\omega_0 z}G_\bullet \bigcap X_x G'_\bullet$. Thus $\{E'_\bullet(\alpha) \mid \alpha \in A\}$ is dense, and so $E'_k(\alpha) = E_k(\alpha)$ implies (5.3.1).

For notational convenience, set $G^\circ_j := G_j - G_{j-1}$, and similarly for F°_j. To establish this claim, we first show that $h_j(\alpha) \in G^\circ_{z(j)}$ for $j = 1, \ldots, n$, which shows $h_1(\alpha), \ldots, h_n(\alpha)$ is a parameterized basis for V and $E'_\bullet(\alpha) \in X_{\omega_0 z}G_\bullet$. Then, for a fixed $\alpha \in A$, we construct $h'_1, \ldots, h'_n$ which satisfy $E'_\bullet(\alpha) = \langle\langle h'_1, \ldots, h'_n\rangle\rangle$ and $h'_j \in G'_{n+1-x(j)}$ for $j = 1, \ldots, n$, showing $E'_\bullet(\alpha) \in X_x G'_\bullet$.

Note that if $i < n$, then $G_{i+1} = \langle e_n, F_i\rangle$. Thus $h_j(\alpha) \in F^\circ_{w(j)} \subset G^\circ_{z(j)}$ for $1 \leq j < k$ and $p < j$, and if $k+1 < j \leq p$, then $h_j(\alpha) \in F^\circ_{w(j-1)} \subset G^\circ_{z(j)}$. Then, since $G_1 = \langle e_n\rangle$, we see that $h_{k+1}(\alpha) = e_n \in G'_1 = G'_{n+1-x(k+1)}$. Finally, since w is Grassmannian of descent k, if $k+1 \leq i \leq p$, then $w(i) \leq w(p) = l$, which shows $h_k(\alpha) \in G^\circ_{l+1} = G^\circ_{z(k)}$. Thus $E'_\bullet(\alpha) \in X^\circ_{\omega_0 z}G_\bullet$.

We now show that $E'_\bullet(\alpha) \in X_x G'_\bullet$. Note that if $a \leq b < n$, then $F'_{n+1-a} \bigcap F_b \subset G'_{n-a} \cap G_{b+1}$. Thus if $1 \leq j < k$, $h_j(\alpha) = g_j(\alpha) \in F'_{n+1-u(j)} \bigcap F_{w(j)} \subset G'_{n+1-x(j)}$. Since $x(k) = 1$, we see that $h_k(\alpha) \in G'_{n+1-x(k)} = V$. Fix $\alpha \in A$ and set $h'_j = h_j(\alpha)$ for $1 \leq j \leq k$. Define

$$h'_{k+1} \;:=\; g_k(\alpha) - e_n \;=\; \sum_{j=m}^{n-1} \beta_j(\alpha) e_j \;\in\; G'_{n+1-(m+1)} \;=\; G'_{n+1-x(k+1)}.$$

Since $h'_{k+1} + h_{k+1}(\alpha) = g_k(\alpha)$, we see that $E'_{k+1}(\alpha) = \langle E'_k(\alpha), H'_{k+1}\rangle$.

Finally, since $E_\bullet(\alpha) \in X_u F'_\bullet$, if $k < j$ there exists a vector

$$g'_j \;:=\; \sum_{i \leq j} \gamma_{i,j} g_i(\alpha) \;\in\; F'_{n+1-u(j)}$$

such that $\langle E_{j-1}(\alpha), g'_j\rangle = E_j(\alpha)$. For $k+1 < j \leq p$, set

$$h'_j \;=\; g'_{j-1} - \gamma_{k,j-1} e_n \;\in\; \langle e_{n-1} \ldots, e_{n+1-u(j-1)}\rangle \;=\; G'_{n+1-x(j)},$$

as as $g_k(\alpha)$ is the only vector among $\{g_1(\alpha), \ldots, g_n(\alpha)\}$ which is not in the span of $e_1, \ldots, e_{n-1}$. If $p < j$, set $h'_j = g'_j - \gamma_{k,j} e_n \in G'_{n+1-x(j)}$. Then $\langle\langle h'_1, \ldots, h'_n\rangle\rangle = E'_\bullet(\alpha)$, completing the proof. ∎

6. Formulas for some $c^w_{u\,v(\lambda,k)}$

6.1. **A chain-theoretic interpretation.** We give a chain-theoretic interpretation for some coefficients c^ζ_λ similar to the results of [50]. If either $u \lessdot_k (\alpha,\beta)u$ or $\zeta \prec\!\!\cdot\, (\alpha,\beta)\zeta$ is a cover, label that edge in the Hasse diagram with the integer $\beta = \max\{\alpha,\beta\}$. Given a saturated chain in the k-Bruhat order from u to ζu, equivalently, a saturated $\preceq$-chain from e to ζ, the *word* of that chain is its sequence of edge labels. Given a word $\omega = a_1.a_2\ldots a_m$, Schensted insertion [47] or [46, §3.3] of ω into the empty tableau gives a pair (S,T) of Young tableaux, where S is the *insertion tableau* and T the *recording tableau* of ω.

Let $\mu \subset \lambda$ be partitions. A permutation ζ is *shape-equivalent* to a skew Young diagram λ/μ if there is a k such that ζ is shape-equivalent to $v(\lambda,k)\cdot v(\mu,k)^{-1}$. It follows that ζ is shape equivalent to some skew partition λ/μ if and only if whenever $\alpha,\beta \in \mathrm{up}_\zeta$ or $\alpha,\beta\in\mathrm{down}_\zeta$,

$$\alpha \;<\; \beta \quad\Longleftrightarrow\quad \zeta(\alpha) \;<\; \zeta(\beta).$$

Theorem F′. *Let $\mu\subset\lambda$ be partitions and suppose $\zeta\in\mathcal{S}_\infty$ is shape equivalent to λ/μ. Then, for every partition ν*

(*i*) $c^\zeta_\nu = c^{\lambda/\mu}_\nu$, *and*

(*ii*) *For every standard Young tableau T of shape ν,*

$$c^\zeta_\nu \;=\; \#\left\{\begin{array}{c}\preceq\text{-chains from } e \text{ to } \zeta \text{ whose}\\ \text{word has recording tableau } T\end{array}\right\}.$$

Equivalently, if $u\leq_k w$ and $wu^{-1}=\zeta$, then

$$c^w_{u\,v(\nu,k)} \;=\; \#\left\{\begin{array}{c}\text{Chains in } k\text{-Bruhat order from } u \text{ to}\\ w \text{ whose word has recording tableau } T\end{array}\right\}.$$

Remark 6.1.1. Theorem F′ (*ii*) gives a combinatorial proof of Proposition 1.1, when wu^{-1} is shape equivalent to a skew partition. Theorem F′ (*ii*) is deceptively similar to Theorem 8 of [50]:

Theorem 8 [50]**.** *Suppose $\nu=(p,1^{q-1})$. Then for every $u,w\in\mathcal{S}_\infty$ and $k\in\mathbb{N}$, the constant $c^w_{u\,v(\nu,k)}$ counts either set*

(*i*) $\left\{\begin{array}{l}\text{Chains in } k\text{-Bruhat order from } u \text{ to } w \text{ with}\\ \text{word } \ a_1<\cdots<a_p>a_{p+1}>\cdots>a_{p+q-1}.\end{array}\right\}.$

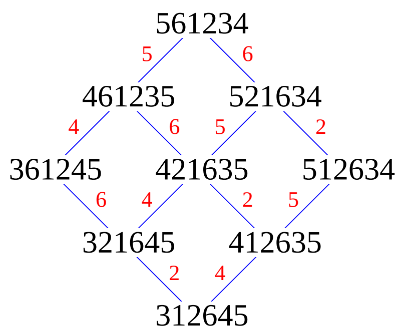
(*ii*) $\left\{\begin{array}{l}\text{Chains in } k\text{-Bruhat order from } u \text{ to } w \text{ with}\\ \text{word } \ a_1>\cdots>a_q<a_{q+1}<\cdots<a_{p+q-1}.\end{array}\right\}.$

The recording tableaux of words in (*i*) have $1,2,\ldots,p$ in the first row and $1,p+1,\ldots,p+q-1$ in the first column, and these are the only words with this recording tableau. Similarly, the recording tableaux of words in (*ii*) have $1,2,\ldots,q$ in the first column and $1, q{+}1, \ldots, p{+}q{-}1$ in the first row. However, Theorem F is *not* a generalization of this result: The permutation $\zeta := (143652)$ is not shape equivalent to any skew partition as $4, 5 \in \mathrm{down}_\zeta$ but $\zeta(4) > \zeta(5)$. Nevertheless, $c^{\zeta}_{(4,1)} = 1$. Interestingly, ζ does satisfy the conclusions of Theorem F′.

While the hypothesis of Theorem F′ is not necessary for the conclusion to hold, some hypotheses are necessary: Let $\zeta = (162)(354)$, a product of two disjoint 3-cycles. Then $\zeta^{(1\ldots 6)} = (132)(465) = v(\text{(3,2)}, 2) \cdot v(\text{(2)}, 2)^{-1}$. Hence, by Theorem H, we have:

$$c^{\zeta}_{(4)} \;=\; c^{\zeta}_{(3,1)} \;=\; c^{\zeta}_{(2,2)} \;=\; 1.$$

(This is also a consequence of Theorem G and the form of the Pieri formula in [28], or of [50], Theorem 5.) If $u = 312645$, then $\zeta u = 561234$ and the labeled Hasse diagram of $[u, \zeta u]_2$ is:



The labels of the six chains are:

$$2456,\ 2465,\ 2645,\ 4526,\ 4256,\ 4265$$

and these have (respective) recording tableaux:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 4\\\cline{1-1} 1&2&3\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 3\\\cline{1-1} 1&2&4\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 3\\\cline{1-1} 1&2&4\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 2\\\cline{1-1} 1&3&4\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 2&4\\\hline 1&3\\\hline\end{array}\ .$$

This list omits $\begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end{array}$, and the third and fourth tableaux are identical.

Proof of Theorem F′. Suppose $\zeta = v(\lambda, k) \cdot v(\mu, k)^{-1}$. Then

$$[e, \zeta]_{\preceq} \quad\simeq\quad [v(\mu, k),\ v(\lambda, k)]_k \quad\simeq\quad [\mu, \lambda]_{\subset}.$$

The first isomorphism preserves the edge labels, and in the second these labels correspond to diagonals in a Young diagram: If $\nu \subset\!\!\cdot\ \nu'$ is a cover in Young's lattice, there is a unique row i such that $\nu_i \neq \nu'_i$. In that case, $\nu_i + 1 = \nu'_i$ and the label of the corresponding edge in the k-Bruhat order is $k - i + \nu'_i$, the diagonal on which the new box of ν' lies.

A chain in Young's lattice from μ to λ is a standard skew tableau R of shape λ/μ. Consider its word, $a_1 \cdots a_m$, as a two-rowed array:

$$w \quad = \quad \left(\begin{array}{cccc} 1 & 2 & \cdots & m \\ a_1 & a_2 & \cdots & a_m \end{array} \right).$$

Then the entry i of R is in the a_ith diagonal.

Let S and T be, respectively, the insertion and recording tableaux of w. Consider the two-rowed array consisting of the columns $\binom{a_i}{i}$ arranged in lexicographic order. Then the insertion and recording tableaux of this new array are T and S, respectively [26, 48].

The second row of this new array, the word inserted to obtain T, is the 'diagonal' word of R; the entries of R ordered lexicographically by diagonal. By Lemma 6.1.2, the diagonal word is Knuth-equivalent to the original word. Thus T is the unique tableau of partition shape Knuth-equivalent to R. This gives a combinatorial bijection

$$\left\{ \begin{array}{c} \preceq\text{-chains from } e \text{ to } \zeta \text{ whose} \\ \text{word has recording tableau } T \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Skew tableaux } R \text{ of shape} \\ \lambda/\mu \text{ Knuth-equivalent to } T \end{array} \right\},$$

proving the theorem in this case, as it is well-known that

$$c_\nu^{\lambda/\mu} \quad = \quad \# \left\{ \begin{array}{c} \text{Skew tableaux } R \text{ of shape} \\ \lambda/\mu \text{ Knuth-equivalent to } T \end{array} \right\}.$$

Now suppose ζ is shape-equivalent to $v(\lambda,k)\cdot v(\mu,k)^{-1}$. By Theorem E (ii), $c_\nu^\zeta = c_\nu^{\lambda/\mu}$, proving (i). Assume λ, μ, and k have been chosen so that $\zeta = \phi_P\left(v(\lambda,k)\cdot v(\mu,k)^{-1}\right)$, for some P. By Theorem 3.2.3 (iii), ϕ_P induces an isomorphism

$$\phi_P \;:\; [e,\, v(\lambda,k)\cdot v(\mu,k)^{-1}]_\preceq \;\xrightarrow{\sim}\; [e,\,\zeta]_\preceq.$$

If $\eta \lessdot (\alpha,\beta)\eta$ is a cover in $[e,\, v(\lambda,k)\cdot v(\mu,k)^{-1}]_\preceq$, then $\phi_P\eta \lessdot \phi_P((\alpha,\,\beta)\eta)$ is a cover in $[e,\zeta]_\preceq$ with label p_β, where $P = p_1 < p_2 < \cdots$. Thus, if γ is a chain in $[e,\, v(\lambda,k)\cdot v(\mu,k)^{-1}]_\preceq$ whose word $a_1,\ldots,a_m$ has recording tableau T, then $\phi_P(\gamma)$ is a chain in $[e,\zeta]_\preceq$ with word $p_{a_1},\ldots,p_{a_m}$, which also has recording tableau T. $\blacksquare$

Order the diagonals of a skew Young tableau R beginning with the diagonal incident to the end of the first column of R. The *diagonal word* of R is the entries of R listed in lexicographic order by diagonal, with magnitude breaking ties. The tableau on the left below has diagonal word 7 58 379 148 26 26 5 8. Schensted insertion of the initial segment 7 58 379 148, (those diagonals incident upon the first column), gives the tableau on the right, whose row word is this initial segment.

7	8	9		
5	7	8		
3	4	6	6	
1	2	2	5	8

7		
5	8	
3	7	9
1	4	8

This observation is the key to the proof of the following lemma.

Lemma 6.1.2. *The diagonal word of a skew tableau is Knuth-equivalent to its column word.*

Proof. Let $d(R)$ be the diagonal word of a skew tableau R. We show $d(R)$ is Knuth equivalent to the word $c.d(R')$, where c is the first column of R and R' is R with c removed. An induction completes the proof.

Suppose the first column of R has length b and R has r diagonals. For $1 \leq j \leq b$ let $w_j := a^j_1 \ldots a^j_{s_j}$ be the subword of $d(R)$ consisting of the jth diagonal. Then $a^j_1 < \cdots < a^j_{s_j}$, $s_1 \leq s_2 \leq \cdots \leq s_b$, and if $k \leq s_j$, then $a^j_k > a^{j+1}_k > \cdots > a^b_k$, as these are consecutive entries in the kth column of R.

For $1 \leq l \leq b$, let T_l be the insertion tableau of the word $w_1.w_2.\ldots.w_l$. Then the kth column of T_l is $a^j_k > \cdots > a^l_k$, where $s_{j-1} < k \leq s_j$. Hence $c.d(R') = c.row(T').w_{b+1}\ldots w_r$, where $row(T')$ is the row word of T_b with its first column, c, removed. Since the column word of a tableau is Knuth-equivalent to its row word, we have the Knuth-equivalences: $c.row(T') \equiv_K c.col(T') = col(T_b) \equiv_K row(T_b)$, which completes the proof. ∎

6.2. **Skew permutations.** Define the set of *skew permutations* to be the smallest set of permutations containing all skew partitions $v(\lambda, k) \cdot v(\mu, k)^{-1}$ which is closed under:

1. Shape equivalence. If η is shape equivalent to a skew permutation ζ, then η is skew.
2. Cyclic shift. If $\zeta \in \mathcal{S}_n$ is skew, then so is $\zeta^{(1\,2\,\ldots\,n)}$.
3. Products of disjoint permutations. If ζ, η are disjoint and skew, then $\zeta\eta$ is skew.

A *shape* of a skew permutation ζ is a (non-unique!) skew partition θ which is defined inductively. If ζ is shape equivalent to λ/μ, then ζ has shape λ/μ. If $\zeta \in \mathcal{S}_n$ is a skew permutation with shape θ, then $\zeta^{(1\,2\,\ldots\,n)}$ has shape θ. If ζ and η are disjoint skew permutations with respective shapes ρ and σ, then $\zeta\eta$ has skew shape $\rho \coprod \sigma$.

Theorem 6.2.1. *Let ζ be a skew permutation with shape θ, then*

(*i*) *For all partitions ν,*

$$c^\zeta_\nu \quad = \quad c^\theta_\nu.$$

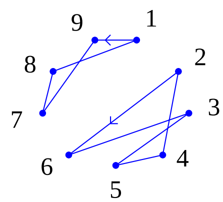
(*ii*) *The number of chains in the interval $[e, \zeta]_\preceq$ is equal to the number of standard Young tableaux of shape θ.*

Proof. The number of standard skew tableaux of shape θ is $\sum_\lambda f^\lambda c^\theta_\lambda$, hence (*ii*) is consequence of (*i*) and Proposition 1.1. To show (*i*), we need only consider the last part (3) of the recursive definition of skew permutations, by Theorems E (*ii*) and H.

Suppose ζ and η are disjoint skew permutations with respective shapes ρ and σ, and for all partitions ν, $c^\zeta_\nu = c^\rho_\nu$ and $c^\eta_\nu = c^\sigma_\nu$. Then by Theorem G (*ii*),

$$\begin{aligned} c^{\zeta\eta}_\nu &= \sum_{\lambda,\mu} c^\nu_{\lambda\,\mu}\, c^\zeta_\lambda\, c^\eta_\mu \\ &= \sum_{\lambda,\mu} c^\nu_{\lambda\,\mu}\, c^\rho_\lambda\, c^\sigma_\mu \\ &= c^{\rho \amalg \sigma}_\nu. \end{aligned}$$ ∎

Example 6.2.2. Consider the graph of (1978)(26354):



Thus the two cycles $\zeta = (1978)$ and $\eta = (26354)$ are disjoint.

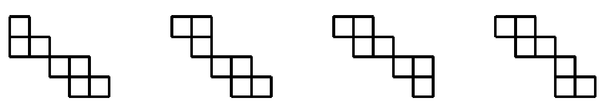
Note that ζ is shape equivalent to (1423) and $(1423)^{(1234)} = (1342)$. Similarly, η is shape equivalent to (15243) and $(15243)^{(12345)} = (13542)$. Both of these cycles, (1423) and (15243), are skew partitions: Let $\lambda = \square$, $\mu = \boxplus$, $\nu = \boxplus\!\square$. Then

$$v(\lambda, 2) \;=\; 13245, \quad v(\mu, 2) \;=\; 34125, \quad v(\nu, 2) \;=\; 35124$$

and

$$\begin{aligned} v(\lambda, 2) &\leq_2 & (1342) \cdot v(\lambda, 2) &= v(\mu, 2), \\ v(\lambda, 2) &\leq_2 & (13542) \cdot v(\lambda, 2) &= v(\nu, 2). \end{aligned}$$

Hence, for every partition κ, $c^\zeta_\kappa = c^{\mu/\lambda}_\kappa$ and $c^\eta_\kappa = c^{\nu/\lambda}_\kappa$. Thus it follows that $c^{\zeta\eta}_\kappa = c^\rho_\kappa$, where ρ is any of the four skew partitions:



6.3. **Further remarks.** For small symmetric groups, it is instructive to examine all permutations and determine to which class they belong. In Table 2, we enumerate each class in $\mathcal{S}_4$, $\mathcal{S}_5$, and $\mathcal{S}_6$. If ζ is one of the 42 permutations in $\mathcal{S}_6$ that is not a skew permutation, and ζ is not one of

$$(125634),\ (145236),\ (143652),\ (163254),\ (153)(246),\ \text{or}\ (135)(264), \qquad (6.3.1)$$

then there is a skew partition θ such that $c^\zeta_\nu = c^\theta_\nu$ for all partitions ν. It would be interesting to understand why this occurs for all but these 6 permutations. Can one characterize those permutations ζ such that there exists a skew partition θ with $c^\zeta_\nu = c^\theta_\nu$ for all partitions νΓ

	skew partitions	shape equivalent to a skew partition	skew permutation
$\mathcal{S}_4$	14	21	24
$\mathcal{S}_5$	42	79	120
$\mathcal{S}_6$	132	311	678

TABLE 2.

For each of these (6.3.1) six 'exceptional' permutation ζ, there is a skew partition θ for which $c^\zeta_\nu = c^\theta_\nu$ for all $\nu \subset a^b$, where $a = \#\mathrm{up}_\zeta$ and $b = \#\mathrm{down}_\zeta$. For these we have, $\theta \not\subset a^b$. For example, let $\zeta = (153)(246)$. If $u = 214365$, then $u \leq_3 \zeta u$ and there are 42 chains in $[u,\, \zeta u]_3$. Also

$$c^\zeta_{(3,3)} \;=\; 1, \qquad c^\zeta_{(3,2,1)} \;=\; 2, \qquad \text{and} \qquad c^\zeta_{(2,2,2)} \;=\; 1,$$

which verifies Proposition 1.1 as $f^{(3,3)} = 5$, $f^{(3,2,1)} = 16$, and $f^{(3,3)} = 5$. In this case, $\theta = (4,3,2,1)/(2,2)$. Since $\mathrm{up}_\zeta = \{1,2,4\}$ and $\mathrm{down}_\zeta = \{6,5,3\}$, we see that $a = b = 3$, however $\theta \not\subset (3,3,3) = a^b$.

A bijective interpretation of the $c^w_{u\,v(\lambda,k)}$ should also give a bijective proof of Proposition 1.1. We show that a function τ from chains to standard tableaux satisfying some extra conditions will provide a bijective interpretation of the $c^w_{u\,v(\lambda,k)}$.

Let $\mathrm{ch}[u,w]_k$ denote the set of (saturated) chains in the interval $[u,w]_k$. For a partition μ and integer m, let $\mu * m$ be the set of partitions λ with $\lambda - \mu$ a horizontal strip of length m. These arise in the classical Pieri's formula:

$$S_\mu(x_1,\ldots,x_k)\cdot h_m(x_1,\ldots,x_k) \;=\; \sum_{\lambda\in\mu*m} S_\lambda(x_1,\ldots,x_k).$$

If T is a standard tableau of shape μ and m any integer, let $T*m$ be the set of tableaux U which contain T as an initial segment such that $U-T$ is a horizontal strip whose entries increase from left to right.

Theorem 6.3.1. *Suppose that for every $u \leq_k w$, there is a map*

$$\begin{array}{rcl} ch[u,w]_k & \longrightarrow & \left\{\begin{array}{c}\textit{Standard Young tableau } T \textit{ whose}\\ \textit{shape is a partition of } \ell(w)-\ell(u)\end{array}\right\} \\ \gamma & \longmapsto & \tau(\gamma) \end{array}$$

such that

1. *$d^w_{u\,v(\lambda,k)} := \#\{\gamma \in ch[u,w]_k \mid \tau(\gamma) = T\}$ depends only upon the shape λ of the standard tableau T.*
2. *If $\gamma = \delta.\varepsilon$ is the concatenation of two chains δ and ε, then $\tau(\delta)$ is a subtableau of $\tau(\gamma)$. (This means that $\tau(\gamma)$ is a recording tableau.)*

3. *Suppose* $\gamma = \delta.\varepsilon$ *with* $\delta \in ch[u,x]_k$, *and hence* $\varepsilon \in ch[x,w]_k$. *Then* $\tau(\delta.\varepsilon) \in \tau(\delta) * m$ *only if* $x \xrightarrow{r_{k,m}} w$, *and* $\varepsilon(\delta) := \varepsilon \in ch[x,w]_k$ *is unique for this to occur.*

Then, for every standard tableau T *of shape* λ *and* $u \leq_k w$,

$$c^w_{u\, v(\lambda,k)} \;=\; d^w_{u\, v(\lambda,k)}.$$

Such a map τ is a generalization of Schensted insertion. In that respect, the existence of such a map would generalize Theorem F′.

Proof. We induct on λ. Assume the theorem holds for all u, w, and partitions π with fewer rows than λ, or if λ and π have the same number of rows, then the last row of λ exceeds the last row of π.

The form of the Pieri formulas expressed in [50, 55] (also §4.2) and condition (3) prove the theorem when λ consists of a single row. Assume that λ has more than one row and set μ to be λ with its last row removed. Let m be the length of the last row of λ and T be any tableau of shape μ. Recall that $U \mapsto \text{shape}(U)$ gives a one-to-one correspondence between $T * m$ and $\mu * m$.

By the definition of $c^y_{u\, v(\mu,k)}$, we have

$$\mathfrak{S}_u \cdot S_\mu(x_1,\ldots,x_k) \;=\; \sum_{u \leq_k y} c^y_{u\, v(\mu,k)}\, \mathfrak{S}_y.$$

By the Pieri formula for Schubert polynomials,

$$\mathfrak{S}_u \cdot S_\mu(x_1,\ldots,x_k)\cdot h_m(x_1,\ldots,x_k) \;=\; \sum_w \sum_{\substack{u \leq_k y \\ y \xrightarrow{r_{k,m}} w}} c^y_{u\, v(\mu,k)} \mathfrak{S}_w.$$

By the classical Pieri's formula, this also equals

$$\mathfrak{S}_u \cdot \sum_{\pi \in \mu * m} S_\pi(x_1,\ldots,x_k) \;=\; \sum_w \sum_{\pi\in\mu*m} c^w_{u\, v(\pi,k)} \mathfrak{S}_w.$$

Hence

$$\sum_{\pi \in \mu * m} c^w_{u\, v(\pi,k)} \;=\; \sum_{\substack{u \leq_k y \\ y \xrightarrow{r_{k,m}} w}} c^y_{u\, v(\mu,k)}.$$

We exhibit a bijection between the two sets

$$M_{T,k,m} \;:=\; \coprod_{\substack{u \leq_k y \\ y \xrightarrow{r_{k,m}} w}} \{\delta \in \text{ch}[u,y]_k \mid \tau(\delta) = T\}$$

and $\coprod_{\pi\in\mu*m} L_\pi$, where

$$L_\pi \;:=\; \{\gamma \in \text{ch}[u,w]_k \mid \tau(\gamma) \in T * m \text{ and } \tau(\gamma) \text{ has shape } \pi\}.$$

This will complete the proof. Indeed, by the induction hypothesis

$$\#M_{T,k,m} \;=\; \sum_{\substack{u \leq_k y \\ y \xrightarrow{r_{k,m}} w}} c^{y}_{u\, v(\mu,k)}$$

and for $\pi \in \mu * m$ with $\pi \neq \lambda$,

$$\#L_\pi \;=\; c^{w}_{u\, v(\pi,k)}.$$

Thus the bijection shows

$$c^{w}_{u\, v(\lambda,k)} \;=\; \sum_{\substack{u \leq_k y \\ y \xrightarrow{r_{k,m}} w}} c^{y}_{u\, v(\mu,k)} - \sum_{\pi \in \mu * m,\ \pi \neq \lambda} c^{w}_{u\, v(\pi,k)} \;=\; \#L_\lambda,$$

which is $\#\tau^{-1}(U)$, for any U of shape λ.

To construct the desired bijection, consider first the map

$$M_{T,k,m} \;\longrightarrow\; \coprod_{\pi \in \mu * m} L_\pi$$

defined by $\delta \in \mathrm{ch}[u,y]_k \mapsto \delta.\varepsilon(\delta)$. By property 3, $\tau(\delta.\varepsilon(\delta)) \in T * m$, so this injective map has the stated range. To see it is surjective, let $\pi \in \mu * m$ and $\gamma \in L_\pi$. Let δ be the first $|\mu|$ steps in the chain γ, so that $\gamma = \delta.\varepsilon$ and suppose $\delta \in \mathrm{ch}[u,y]_k$. Then $\tau(\delta) = T$ so $\tau(\delta.\varepsilon) \in \tau(\delta) * m$. By 3, this implies $y \xrightarrow{r_{k,m}} w$, and hence $\delta \in M_{T,k,m}$. ∎

References

[1] G. Benkart, F. Sottile, and J. Stroomer, *Tableau switching: Algorithms and applications*, J. Comb. Th. Ser. A, 76 (1996), pp. 11–43.

[2] A. Berenstein and A. Zelevinsky, *Triple multiplicities for $\mathfrak{sl}(r+1,\mathbb{C})$ and the spectrum of the exterior algebra of the adjoint representation*, J. Alg. Comb., 1 (1992), pp. 7–22.

[3] N. Bergeron, *A combinatorial construction of the Schubert polynomials*, J. Combin. Theory, Ser. A, 60 (1992), pp. 168–182.

[4] N. Bergeron and S. Billey, *RC-Graphs and Schubert polynomials*, Experimental Math., 2 (1993), pp. 257–269.

[5] N. Bergeron and F. Sottile, *A moniod for the Grassmann-Bruhat order*. Revised version of MSRI preprint 1997-073, 1997.

[6] ——, *Skew Schubert functions and the Pieri formula for flag manifolds*. MSRI preprint 1997-096, 1997.

[7] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Russian Mathematical Surveys, 28 (1973), pp. 1–26.

[8] S. Billey and M. Haiman, *Schubert polynomials for the classical groups*, J. AMS, 8 (1995), pp. 443–482.

[9] S. Billey, W. Jockush, and R. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combinatorics, 2 (1993), pp. 345–374.

[10] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. Math., 57 (1953), pp. 115–207.
[11] C. Chevalley, *Sur les décompositions cellulaires des espaces G/B*, in Algebraic Groups and their Generalizations: Classical Methods, American Mathematical Society, 1994, pp. 1–23. Proceedings and Symposia in Pure Mathematics, vol. 56, Part 1.
[12] I. Ciocan-Fontanine, *On quantum cohomology of partial flag varieties.* Institut Mittag-Leffler preprint. 1997.
[13] P. Deligne, *Cohomologie Etale, $SGA4\frac{1}{2}$*, no. 569 in Springer Lecture Notes, Springer-Verlag, 1977.
[14] M. Demazure, *Désingularization des variétés de Schubert généralisées*, Ann. Sc. E. N. S. (4), 7 (1974), pp. 53–88.
[15] V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Invent. Math., 79 (1985), pp. 499–511.
[16] C. Ehresmann, *Sur la topologie de certains espaces homogènes*, Ann. Math., 35 (1934), pp. 396–443.
[17] S. Fomin and A. N. Kirillov, *Combinatorial B_n-analogs of Schubert polynomials*, Trans. AMS, 348 (1996), pp. 3591–3620.
[18] S. Fomin and R. Stanley, *Schubert polynomials and the nilCoxeter algebra*, Adv. Math., 103 (1994), pp. 196–207.
[19] W. Fulton, *Intersection Theory*, no. 2 in Ergebnisse der Math., Springer-Verlag, 1984.
[20] ———, *Determinantal formulas for orthogonal and symplectic degeneracy loci*, J. Diff. Geom., 43 (1996), pp. 276–290.
[21] ———, *Young Tableaux*, Cambridge University Press, 1996.
[22] P. Hanlon and S. Sundaram, *On a bijection between Littlewood-Richardson fillings of conjugate shape*, J. Comb. Th. Ser. A, 60 (1992), pp. 1–18.
[23] H. Hiller, *Schubert calculus of a Coxeter group*, Enseign. Math., 27 (1981), pp. 57–84.
[24] H. Hiller and B. Boe, *Pieri formula for SO_{2n+1}/U_n and SP_n/U_n*, Adv. in Math., 62 (1986), pp. 49–67.
[25] A. Kirillov and T. Maeno, *Quantum double Schubert polynomials, quantum Schubert polynomials, and the Vafa–Intriligator formula.* math.QA/9610162. 52 pp., 1996.
[26] D. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math., 34 (1970), pp. 709–727.
[27] A. Kohnert, *Weintrauben, Polynome, Tableaux*, Bayreuth Math. Schrift., 38 (1990), pp. 1–97.
[28] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris, 294 (1982), pp. 447–450.
[29] ———, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C. R. Acad. Sci. Paris, 295 (1982), pp. 629–633.
[30] ———, *Symmetry and flag manifolds*, in Invariant Theory, (Montecatini, 1982), vol. 996 of Lecture Notes in Math., Springer-Verlag, 1983, pp. 118–144.
[31] ———, *Interpolation de Newton à plusieurs variables*, in Sém. d'Algèbre M. P. Malliavin 1984-3, vol. 1146 of Lecture Notes in Math., Springer-Verlag, 1985, pp. 161–175.
[32] ———, *Schubert polynomials and the Littlewood-Richardson rule*, Lett. Math. Phys., 10 (1985), pp. 111–124.
[33] ———, *Symmetrization operators on polynomial rings*, Funkt. Anal., 21 (1987), pp. 77–78.
[34] D. E. Littlewood and A. R. Richardson, *Group characters and algebra*, Philos. Trans. Roy. Soc. London., 233 (1934), pp. 99–141.

[35] I. G. Macdonald, *Notes on Schubert Polynomials*, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montréal, Montréal, 1991.
[36] ———, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995. second edition.
[37] D. Monk, *The geometry of flag manifolds*, Proc. London Math. Soc., 9 (1959), pp. 253–286.
[38] F. Patras, *Le calcul de schubert des permutations décomposables*, Séminaire Lotharingien de combinatoire, B35f (1995). 10 pages, `http://cartan.u-strasbg.fr:80/~slc/wpapers/s35patras.html`.
[39] A. Postnikov, *On a quantum version of Pieri's formula*. To appear in *Progress in Geometry*, J.-L. Brylinski and R. Brylinski, eds., 1997.
[40] P. Pragacz, *Algebro-geometric applications of Schur S-and Q-polynomials*, in Topics in Invariant Theory, Séminaire d'Algèbre Dubreil-Malliavin 1989-90, Springer-Verlag, 1991, pp. 130–191.
[41] ———, *Symmetric polynomials and divided differences in formulas of intersection theory*, in Parameter Spaces, vol. 36 of Banach Center Publications, Banach Center workshop, 1994, Institute of Mathematics, Polish Academy of Sciences, 1996, pp. 125–177.
[42] P. Pragacz and J. Ratajski, *Pieri-type formula for $SP(2m)/P$ and $SO(2m+1)/P$*, C. R. Acad. Sci. Paris, 317 (1993), pp. 1035–1040.
[43] ———, *Pieri-type formula for Lagrangian and odd orthogonal Grassmannians*, J. reine agnew. Math., 476 (1996), pp. 143–189.
[44] ———, *A Pieri-type theorem for even orthogonal Grassmannians*. Max-Planck Institut preprint, 1996.
[45] ———, *Formulas for Lagrangian and orthogonal loci: The $\tilde{Q}$-polynomial approach*. Composito Math., to appear, 1997.
[46] B. Sagan, *The Symmetric Group; Representations, Combinatorics, Algorithms & Symmetric Functions*, Wadsworth & Brooks/Cole, 1991.
[47] C. Schensted, *Longest increasing and decreasing subsequence*, Can. J. Math., 13 (1961), pp. 179–191.
[48] M.-P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Math. Scand., 12 (1963), pp. 117–128.
[49] ———, *La correspondence de Robinson*, in Combinatoire et Représentation du Groupe Symétrique, D. Foata, ed., vol. 579 of Lecture Notes in Math., Springer-Verlag, 1977, pp. 59–135.
[50] F. Sottile, *Pieri's formula for flag manifolds and Schubert polynomials*, Annales de l'Institut Fourier, 46 (1996), pp. 89–110.
[51] ———, *Pieri's formula via explicit rational equivalence*. Can. J. Math., 46 (1997), pp. 1281–1298.
[52] J. Stembridge, *Shifted tableaux and the projective representations of the symmetric group*, Adv. Math., 74 (1989), pp. 87–134.
[53] G. Thomas, *On Schensted's construction and the multiplication of Schur functions*, Adv. in Math., 30 (1978), pp. 8–32.
[54] R. Winkel, *Diagram rules for the generation of Schubert polynomials*. manuscript, `http://www.iram.rwth-aachen.de/~winkel/pp.html`, 1995.
[55] ———, *On the multiplication of Schubert polynomials*. to appear in Adv. in Appl. Math., manuscript at `http://www.iram.rwth-aachen.de/~winkel/pp.html`, 1996.
[56] A. Zelevinsky, *A generalization of the Littlewood-Richardson rule and the Robinson-Schensted-Knuth correspondence*, J. Algebra, 69 (1981), pp. 82–94.

Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, CANADA
E-mail address, Nantel Bergeron: bergeron@mathstat.yorku.ca
URL, Nantel Bergeron: http://www.math.yorku.ca/bergeron

Department of Mathematics, University of Wisconsin, Van Vleck Hall, 480 Lincoln Drive, Madison, Wisconsin 53706-1388, USA
E-mail address, Frank Sottile: sottile@math.wisc.edu
URL, Frank Sottile: http://www.math.wisc.edu/~sottile