Further Remarks on Multiple p-adic q-L-Function of Two Variables*

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Abstract : The object of this paper is to give several properties and applications of the multiple p-adic q-L-function of two variables $L_{p,q}^{(r)}(s,z,\chi)$. The explicit formulas relating higher order q-Bernoulli polynomials, which involve sums of products of higher order q-zeta function and higher order Dirichlet q-L-function are given. The value of higher order Dirichlet p-adic q-L-function for positive integers is also calculated. Furthermore, the Kummer-type congruences for multiple generalized q-Bernoulli polynomials are derived by making use of the difference theorem of higher order Dirichlet p-adic q-L-function.

Keywords : q-Bernoulli numbers and polynomials, multiple q-Bernoulli numbers and polynomials, p-adic L-function, p-adic q-L-function, multiple p-adic q-L-function, Kummer congruences.

MSC 2000 : 11B68, 11S40, 11M99, 11A07.

1 Introduction

In [28], Kim and Cho defined the following multiple q-L-function:

$$L_q^{(r)}(s,\chi) = \frac{1}{\prod_{j=1}^{r}(s-j)}\frac{1}{[F]_q^r}\sum_{a_1,\ldots,a_r=1}^{F}\chi(a_1+\cdots+a_r)[a_1+\cdots+a_r]_q^{-s+r} \times\sum_{m=0}^{\infty}\binom{r-s}{m}q^{(a_1+\cdots+a_r)m}\left(\frac{[F]_q}{[a_1+\cdots+a_r]_q}\right)^m\beta_{m,q^F}^{(r)}. \tag{1}$$

They also suggested the following question: "*Is it possible to give p-adic analogue of (1) which can be viewed as interpolating, in the same way that $L_{p,q}(s,\chi)$ interpolates $L_q(s,\chi)$ in [25], [27]?*". This question was answered positively by authors in [7] by constructing the following two variable p-adic meromorphic function:

$$L_{p,q}^{(r)}(s,z,\chi) = \frac{1}{[F]_q^r}\frac{1}{\prod_{j=1}^{r}(s-j)}\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{F}\chi(a_1+\cdots+a_r)\langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r} \times\sum_{m=0}^{\infty}\binom{r-s}{m}q^{(a_1+\cdots+a_r+p^*z)m}\left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q}\right)^m\beta_{m,q^F}^{(r)}. \tag{2}$$

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The purpose of this paper is to give further properties of the function $L_{p,q}^{(r)}(s,z,\chi)$ as well as applications related to Kummer-type congruences for multiple q-Bernoulli polynomials.

Kubota and Leopoldt [31] proved the existence of meromorphic function $L_p(s,\chi)$, defined over p-adic number field. $L_p(s,\chi)$ is defined by [10]

$$L_p(s,\chi) = \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{\chi(n)}{n^s} = \left(1 - \chi(p)\, p^{-s}\right) L(s,\chi),$$

where $L(s,\chi)$ is the Dirichlet L-function. $L_p(s,\chi)$ interpolates the values

$$L_p(1-n,\chi) = -\frac{1}{n}\left(1 - \chi_n(p)\, p^{n-1}\right) B_{n,\chi_n}$$

for $n \in \mathbb{Z}$, $n \geqslant 1$, where $B_{n,\chi}$ denotes the generalized Bernoulli numbers associated with the primitive Dirichlet character χ, and $\chi_n = \chi\omega^{-n}$ with ω being the Teichmüller character (cf. [9], [10], [11], [12], [15], [18], [26], [29], [30], [36], [41], [42]).

In [11, 12], Fox derived a meromorphic function $L_p(s,z,\chi)$, which is the two-variable extension of the function $L_p(s,\chi)$. Kim [25] constructed $L_{p,q}(s,z,\chi)$, which serves as a q-extension of $L_p(s,z,\chi)$. In [7], the authors defined the multiple p-adic q-L-function of two variables $L_{p,q}^{(r)}(s,z,\chi)$, which stands for the higher order generalization of Kim's $L_{p,q}(s,z,\chi)$.

Ferrero and Greenberg [10] evaluated the value $(\partial/\partial s)\, L_p(0,\chi)$. In [43], Young gave an extension of this value by using p-adic L-function of two variables $L_p(s,z,\chi)$ under some restrictions on the character χ. Fox [12] derived a formula for $(\partial/\partial s)\, L_p(0,z,\chi)$ without any restrictions. Kim [25] evaluated the value $(\partial/\partial s)\, L_{p,q}(0,z,\chi)$, which is the q-extension and two-variable extension of the result found by Diamond [9] and Ferrero and Greenberg [10]. The authors [7] obtained a formula for $(\partial/\partial s)\, L_{p,q}^{(r)}(0,z,\chi)$, which generalizes the results of Kim [25, 27], Fox [12], Diamond [9] and Ferrero and Greenberg [10]. Further extensions for the value $(\partial/\partial s)\, L_p(0,\chi)$ can be found in [18], [27], [37].

In recent years, many mathematicians and physicists have investigated zeta functions, multiple zeta functions, L-functions and multiple q-Bernoulli numbers and polynomials because mainly of their interest and importance. These functions and polynomials are used not only in Complex Analysis and Mathematical Physics, but also in p-adic Analysis and other areas. In particular, multiple zeta functions and multiple L-functions occur within the context of Knot Theory, Quantum Field Theory, Applied Analysis and Number Theory (see [19], [21], [22], [27], [32], [33], [40]).

The object of the present sequel to earlier work [7] is to derive several properties and applications of the multiple p-adic q-L-function of two variables $L_{p,q}^{(r)}(s,z,\chi)$. We first give a brief summary for zeta and Dirichlet L-functions and related concepts in preliminary section. In Section 3, we review the definition and construction of the multiple q-L-function of two variables and its p-adic analogue. We also find explicit formulas relating higher order q-Bernoulli polynomials, which involve sums of products of $\zeta_q^{(r)}(-n, z_1 + \cdots + z_r)$ and $L_q^{(r)}(-n, z_1 + \cdots + z_r, \chi)$ for integer $n \geqslant 0$. In Section 4, we evaluate the value $L_{p,q}^{(r)}(r,z,\chi)$ for a positive integer r explicitly, from which the value $L_{p,q}(1,z,\chi)$ given by Kim [25] is an immediate consequence. In final section, we purpose to derive Kummer-type congruences for multiple generalized q-Bernoulli polynomials making use of the difference theorem of $L_{p,q}^{(r)}(s,z,\chi)$ and its consequence, which are already proven in [7]. These congruences are generalizations of the congruences given by [3], [6], [11], [13, 14], [16], [20], [35].

2 Preliminaries

In complex number field $\mathbb{C}$, the Bernoulli numbers B_n are defined by means of the generating function

$$F(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad (|t| < 2\pi).$$

$\{B_n\}$ is the sequence of rational numbers first considered by Jacob Bernoulli in the study of finite sums of a given power of consecutive integers. It follows from the generating function definition that

$$B_0 = 1,\, B_1 = -\frac{1}{2},\, B_2 = \frac{1}{6},\, B_3 = 0, B_4 = -\frac{1}{30}, \ldots,$$

and $B_{2k+1} = 0$ for $k \in \mathbb{Z}$, $k \geqslant 1$. For an indeterminate z, Bernoulli polynomials $B_n(z)$ are defined by

$$F(z,t) = \frac{t}{e^t - 1} e^{zt} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!}, \qquad (|t| < 2\pi).$$

One of the curious facts about Bernoulli numbers and polynomials is the relation between the Riemann zeta and the Hurwitz (or generalized) zeta functions.

Theorem 1 *([1]) For every integer $n \geqslant 1$,*

$$\zeta(1-n) = -\frac{B_n}{n} \text{ and } \zeta(1-n,z) = -\frac{B_n(z)}{n},$$

where $\zeta(s)$ and $\zeta(s,z)$ are the Riemann and the Hurwitz (or generalized) zeta functions, defined respectively by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \text{ and } \zeta(s,z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^s},$$

with $s \in \mathbb{C}$, $Re(s) > 1$ and $z \in \mathbb{C}$ with $Re(z) > 0$.

For $n \in \mathbb{Z}$, $n \geqslant 1$, a Dirichlet character to the modulus n is a multiplicative map $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi(a+n) = \chi(a)$ for all $a \in \mathbb{Z}$ and $\chi(a) = 0$ if $(a,n) \neq 1$. Since $a^{\phi(n)} \equiv 1 \,(\text{mod} n)$ for all a such that $(a,n) = 1$, $\chi(a)$ must be the root of unity for such a. If χ is a Dirichlet character to the modulus n, then for any positive multiple m of n, we can induce a Dirichlet character ψ to the modulus m according to $\psi(a) = \chi(a)$ if $(a,m) = 1$ and $\psi(a) = 0$ if $(a,m) \neq 1$. The minimum modulus n for which a character χ cannot be induced from some character to the modulus m, $m < n$, is called the conductor of χ, denoted by $f = f_\chi$. Throughout, it will be assumed that each χ is defined to modulo its conductor. Such a character is said to be primitive. For primitive Dirichlet characters χ and ψ having conductors f_χ and f_ψ, respectively, the product $\chi\psi$ is defined by $\chi\psi(a) = \chi(a)\psi(a)$ for all $a \in \mathbb{Z}$ such that $(a, f_\chi f_\psi) = 1$. The character $\chi = 1$, having conductor $f_1 = 1$ is called the principle character.

Among various generalizations of Bernoulli numbers and polynomials, generalization with a primitive Dirichlet character χ has a special case of attention.

Definition 2 *([15], [42]) For a primitive Dirichlet character χ having conductor $f \in \mathbb{Z}$, $f \geqslant 1$, the generalized Bernoulli numbers $B_{n,\chi}$ and polynomials $B_{n,\chi}(z)$ associated with χ are defined by*

$$\begin{aligned} F_\chi(t) &= \sum_{a=1}^{f} \frac{\chi(a)\, t e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \qquad \left(|t| < \frac{2\pi}{f}\right) \\ F_\chi(z,t) &= \sum_{a=1}^{f} \frac{\chi(a)\, t e^{(a+z)t}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(z) \frac{t^n}{n!}, \quad \left(|t| < \frac{2\pi}{f}\right) \end{aligned}$$

respectively.

Note that the classical Bernoulli numbers are obtained when $\chi = 1$, in that $B_{n,1} = B_n$ if $n \neq 1$ and $B_{1,1} = -B_1$. The generalized Bernoulli numbers and polynomials can be expressed in terms of Bernoulli polynomials as

$$\begin{aligned} B_{n,\chi} &= f^{n-1} \sum_{a=1}^{f} \chi(a) B_n\left(\frac{a}{f}\right), \\ B_{n,\chi}(z) &= f^{n-1} \sum_{a=1}^{f} \chi(a) B_n\left(\frac{a+z}{f}\right). \end{aligned}$$

Given a primitive Dirichlet character χ, having conductor f, the Dirichlet L-function associated with χ is defined by ([1], [42])

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

for $s \in \mathbb{C}$, $Re(s) > 1$. It is well known [42] that $L(s,\chi)$ may be analytically continued to the whole complex plane, except for a simple pole at $s = 1$ when $\chi = 1$, in which case the Riemann zeta function, $\zeta(s) = L(s,1)$ is obtained. The generalized Bernoulli numbers share a particular relationship with the Dirichlet L-function, in that

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}$$

for $n \in \mathbb{Z}$, $n \geqslant 1$.

For $r \in \mathbb{Z}$, $r \geqslant 1$, the Bernoulli numbers $B_n^{(r)}$ and polynomials $B_n^{(r)}(z)$ of order r (also called the multiple Bernoulli numbers and polynomials, respectively) may be defined by means of [34, Chapter 6]

$$\begin{aligned} F^{(r)}(t) &= \left(\frac{t}{e^t - 1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!}, \qquad (|t| < 2\pi) \\ F^{(r)}(z,t) &= \left(\frac{t}{e^t - 1}\right)^r e^{zt} = \sum_{n=0}^{\infty} B_n^{(r)}(z) \frac{t^n}{n!}, \qquad (|t| < 2\pi) \end{aligned}$$

respectively. For $r = 1$, the classical Bernoulli numbers and polynomials are obtained.

Let χ be a Dirichlet character of conductor f. In [16], the multiple generalized Bernoulli numbers $B_{n,\chi}^{(r)}$ attached to χ are defined by

$$F_\chi^{(r)}(t) = \sum_{a_1,\ldots,a_r=1}^{f} \frac{\chi(a_1 + \cdots + a_r) t^r e^{(a_1+\cdots+a_r)t}}{(e^{ft} - 1)^r} = \sum_{n=0}^{\infty} B_{n,\chi}^{(r)} \frac{t^n}{n!}, \qquad \left(|t| < \frac{2\pi}{f}\right)$$

for $r \in \mathbb{Z}$, $r \geqslant 1$. For an indeterminate z, the multiple generalized Bernoulli polynomials $B_{n,\chi}^{(r)}(z)$ attached to χ are naturally given by

$$F_\chi^{(r)}(z,t) = \sum_{a_1,\ldots,a_r=1}^{f} \frac{\chi(a_1 + \cdots + a_r) t^r e^{(z+a_1+\cdots+a_r)t}}{(e^{ft} - 1)^r} = \sum_{n=0}^{\infty} B_{n,\chi}^{(r)}(z) \frac{t^n}{n!}, \qquad \left(|t| < \frac{2\pi}{f}\right).$$

It can be readily seen that

$$B_{n,\chi}^{(r)} = f^{n-r} \sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1 + \cdots + a_r) B_n^{(r)}\left(\frac{a_1 + \cdots + a_r}{f}\right)$$

and

$$B_{n,\chi}^{(r)}(z)=f^{n-r}\sum_{a_1,\ldots,a_r=1}^{f}\chi(a_1+\cdots+a_r)\,B_n^{(r)}\left(\frac{z+a_1+\cdots+a_r}{f}\right).$$

Throughout this paper, p will denote a prime number, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$ will be used to represent, respectively, the p-adic integers, the p-adic numbers, the algebraic closure of $\mathbb{Q}_p$ and the completion of $\overline{\mathbb{Q}}_p$ with respect to p-adic absolute value $|\cdot|_p$, which is normalized so that $|p|_p=p^{-1}$. On $\mathbb{C}_p$, the absolute value is non-Archimedean, and so for any $a,\,b\in\mathbb{C}_p$, $|a+b|_p\leqslant\max\left\{|a|_p\,,|b|_p\right\}$. We denote a particular subring of $\mathbb{C}_p$ as

$$R=\left\{a\in\mathbb{C}_p:|a|_p\leqslant 1\right\}.$$

If $z\in\mathbb{C}_p$ such that $|z|_p\leqslant|p|_p^m$, where $m\in\mathbb{Q}$, then $z\in p^mR$, and this can be also written as $z\equiv 0\,(modp^mR)$.

Let $p^*=4$ if $p=2$ and $p^*=p$ otherwise. Note that there exists $\phi(p^*)$ distinct solutions, modulo p^*, to the equation $x^{\phi(p^*)}-1=0$, and each solution must be congruent to one of the values $a\in\mathbb{Z}$, where $1\leqslant a\leqslant p^*$, $(a,p)=1$. Thus, given $a\in\mathbb{Z}$ with $(a,p)=1$, there exists a unique $\omega(a)\in\mathbb{Z}_p$, where $\omega(a)^{\phi(p^*)}=1$, such that $\omega(a)\equiv a\,(\text{mod}p^*\mathbb{Z}_p)$. Letting $\omega(a)=0$ for $a\in\mathbb{Z}$ such that $(a,p)\neq 1$, it can be seen that ω is actually a Dirichlet character having conductor $f_\omega=p^*$, called the Teichmüller character. Let $\langle a\rangle=\omega^{-1}(a)\,a$. Then $\langle a\rangle\equiv 1\,(\text{mod}p^*\mathbb{Z}_p)$ (see also [15], [25], [42], [43]).

For the context in the sequel, an extension of the definition of the Teichmüller character is needed. If $z\in\mathbb{C}_p$ such that $|z|_p\leqslant 1$, then for any $a\in\mathbb{Z}$, $a+p^*z\equiv a\,(modp^*R)$. Thus, for $z\in\mathbb{C}_p$, $|z|_p\leqslant 1$, $\omega(a+p^*z)=\omega(a)$. Also, for these values of z, let $\langle a+p^*z\rangle=\omega^{-1}(a)\,(a+p^*z)$ (see [11], [12]).

q-extensions of Bernoulli numbers are first studied by Carlitz [2]. The corresponding numbers are called as q-Bernoulli numbers, denoted by $\beta_{n,q}$, and defined by means of the symbolic formula

$$\beta_{0,q}=\frac{q-1}{\log q},\qquad(q\beta_q+1)^n-\beta_{n,q}=\delta_{n,1},$$

with the usual convention about replacing β_q^j by $\beta_{j,q}$ and $\delta_{n,1}$ is the Kronecker symbol (cf. [4], [5], [18], [19], [23], [39]). Note that, $\lim_{q\to1}\beta_{n,q}=B_n$, the classical Bernoulli numbers. q-Bernoulli polynomials $\beta_{n,q}(z)$ are defined by (cf. [19], [25])

$$\beta_{n,q}(z)=\left(q^z\beta_q+[z]_q\right)^n=\sum_{k=0}^{n}\binom{n}{k}q^{kz}\beta_{k,q}\,[z]_q^{n-k},$$

where

$$[z]_q=\frac{1-q^z}{1-q}.$$

When talking about q-extensions, q can be variously considered as an indeterminate, a complex number $q\in\mathbb{C}$ or a p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}$, we assume that $|q|<1$. If $q\in\mathbb{C}_p$, then $|1-q|_p<|p|_p^{1/(p-1)}=p^{-1/(p-1)}$. Thus, for $|x|_p\leqslant 1$, we have $q^x=\exp\left(x\log_p q\right)$, where $\log_p$ is the Iwasawa p-adic logarithm function (see [15]).

For $q\in\mathbb{C}$, $|q|<1$, Kim [25] gave generating functions of q-Bernoulli numbers and polynomials respectively by

$$\begin{aligned}F_q(t)&=\frac{q-1}{\log q}e^{t/(1-q)}-t\sum_{n=0}^{\infty}q^n e^{[n]_q t}=\sum_{n=0}^{\infty}\beta_{n,q}\frac{t^n}{n!},\\ F_q(z,t)&=\frac{q-1}{\log q}e^{t/(1-q)}-t\sum_{n=0}^{\infty}q^{n+z}e^{[n+z]_q t}=\sum_{n=0}^{\infty}\beta_{n,q}(z)\frac{t^n}{n!}\end{aligned}$$

for $|t| < 1$. For a Dirichlet character χ with conductor f, the generalized q-Bernoulli numbers and polynomials associated with χ are defined by the rules

$$\begin{aligned} F_{q,\chi}(t) &= -t\sum_{n=1}^{\infty}\chi(n)\, q^n e^{[n]_q t} = \sum_{n=0}^{\infty}\beta_{n,q,\chi}\frac{t^n}{n!}, \\ F_{q,\chi}(z,t) &= -t\sum_{n=1}^{\infty}\chi(n)\, q^{n+z} e^{[n+z]_q t} = \sum_{n=0}^{\infty}\beta_{n,q,\chi}(z)\frac{t^n}{n!} \end{aligned}$$

for $|t| < 1$, respectively (cf. [25]). From these formulas, it can be easily obtained that

$$\beta_{n,q,\chi}(z) = \sum_{k=0}^{n}\binom{n}{k} q^{kz}\beta_{k,q,\chi}[z]_q^{n-k}.$$

The generalized q-Bernoulli numbers and polynomials can be expressed in terms of q-Bernoulli polynomials as

$$\begin{aligned} \beta_{n,q,\chi} &= [f]_q^{n-1}\sum_{a=1}^{f}\chi(a)\,\beta_{n,q^f}\left(\frac{a}{f}\right), \\ \beta_{n,q,\chi}(z) &= [f]_q^{n-1}\sum_{a=1}^{f}\chi(a)\,\beta_{n,q^f}\left(\frac{a+z}{f}\right). \end{aligned}$$

3 Multiple q-L-Function of Two Variables and its p-adic Analogue

This section is devoted to recall the multiple q-L-function of two variables $L_q^{(r)}(s,z,\chi)$, and its p-adic analogue $L_{p,q}^{(r)}(s,z,\chi)$ which were introduced in [7]. We also summarize the analytic continuation, special values and explicit formulas for these functions.

For $r \in \mathbb{Z}$, $r \geqslant 1$, the multiple q-Bernoulli numbers $\beta_{n,q}^{(r)}$ and the multiple q-Bernoulli polynomials $\beta_{n,q}^{(r)}(z)$, are defined respectively by means of the generating functions (cf. [24])

$$F_q^{(r)}(t) = (-t)^r \sum_{n_1,\ldots,n_r=0}^{\infty} q^{n_1+\cdots+n_r} e^{[n_1+\cdots+n_r]_q t} = \sum_{n=0}^{\infty}\beta_{n,q}^{(r)}\frac{t^n}{n!},$$

and

$$F_q^{(r)}(z,t) = (-t)^r \sum_{n_1,\ldots,n_r=0}^{\infty} q^{z+n_1+\cdots+n_r} e^{[z+n_1+\cdots+n_r]_q t} = \sum_{n=0}^{\infty}\beta_{n,q}^{(r)}(z)\frac{t^n}{n!} \tag{3}$$

for $|t| < 1$. The complex analytic multiple q-zeta function was defined in [24] as follows:

Definition 3 *For $s \in \mathbb{C}$, $Re(s) > r$ and $z \in \mathbb{C}$, $Re(z) > 0$, the multiple q-zeta function $\zeta_q^{(r)}(s,z)$ is defined by means of*

$$\zeta_q^{(r)}(s,z) = \sum_{m_1,\ldots,m_r=0}^{\infty}\frac{q^{z+m_1+\cdots+m_r}}{[z+m_1+\cdots+m_r]_q^s}.$$

For $s \in \mathbb{C}$, the following integral representation was obtained by Kim [24]:

$$\begin{aligned}\zeta_q^{(r)}(s,z) &= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1-r} F_q^{(r)}(z,-t)\,dt \\ &= \sum_{m_1,\ldots,m_r=0}^{\infty} q^{z+m_1+\cdots+m_r}\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1} e^{-[z+m_1+\cdots+m_r]_q t}dt.\end{aligned}$$

This integral representation yields the following assertion:

Theorem 4 *([24]) For $n \in \mathbb{Z}$, $n \geqslant 0$ and $r \in \mathbb{Z}$, $r \geqslant 1$,*

$$\begin{aligned}\zeta_q^{(r)}(-n,z) &= (-1)^r \frac{n!}{(n+r)!}\beta_{n+r,q}^{(r)}(z) \\ &= \frac{(-1)^r}{(n+1)_r}\beta_{n+r,q}^{(r)}(z),\end{aligned}$$

where $(n)_r$ is the Pochhammer symbol defined by

$$(n)_0 = 1 \text{ and } (n)_r = n(n+1)\cdots(n+r-1).$$

For a Dirichlet character χ, the multiple generalized q-Bernoulli numbers $\beta_{n,q,\chi}^{(r)}$ are defined by (cf. [24], [28])

$$\begin{aligned}F_{q,\chi}^{(r)}(t) &= (-t)^r \sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1+\cdots+a_r) \sum_{k_1,\ldots,k_r=0}^{\infty} q^{a_1+k_1f+\cdots+a_r+k_rf} e^{[a_1+k_1f+\cdots+a_r+k_rf]_q t} \\ &= (-t)^r \sum_{n_1,\ldots,n_r=1}^{\infty} \chi(n_1+\cdots+n_r)\, q^{n_1+\cdots+n_r} e^{[n_1+\cdots+n_r]_q t} \\ &= \sum_{n=0}^{\infty} \beta_{n,q,\chi}^{(r)} \frac{t^n}{n!}\end{aligned}$$

For an indeterminate z, this definition can be expanded in order to define the multiple generalized q-Bernoulli polynomials $\beta_{n,q,\chi}^{(r)}(z)$ as follows (cf. [7]):

$$\begin{aligned}F_{q,\chi}^{(r)}(z,t) = (-t)^r &\sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1+\cdots+a_r) \\ &\times \sum_{k_1,\ldots,k_r=0}^{\infty} q^{z+a_1+k_1f+\cdots+a_r+k_rf} e^{[z+a_1+k_1f+\cdots+a_r+k_rf]_q t} \\ &= (-t)^r \sum_{n_1,\ldots,n_r=1}^{\infty} \chi(n_1+\cdots+n_r)\, q^{z+n_1+\cdots+n_r} e^{[z+n_1+\cdots+n_r]_q t} \\ &= \sum_{n=0}^{\infty} \beta_{n,q,\chi}^{(r)}(z) \frac{t^n}{n!}. \qquad (4)\end{aligned}$$

From (3) and (4), it readily seen that

$$\beta_{n,q,\chi}^{(r)}(z) = [f]_q^{n-r} \sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1+\cdots+a_r)\, \beta_{n,q^f}^{(r)}\left(\frac{z+a_1+\cdots+a_r}{f}\right). \qquad (5)$$

The multiple Dirichlet q-L-function of two variables is given by the following definition:

Definition 5 *([7]) For $s \in \mathbb{C}$, $Re(s) > r$, $z \in \mathbb{C}$, $Re(z) > 0$ and a Dirichlet character χ, the multiple Dirichlet q-L-function of two variables $L_q^{(r)}(s,z,\chi)$ is defined by means of*

$$L_q^{(r)}(s,z,\chi) = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\chi(m_1+\cdots+m_r)\, q^{z+m_1+\cdots+m_r}}{[z+m_1+\cdots+m_r]_q^s}.$$

Special values of $L_q^{(r)}(s,z,\chi)$ are given by the following theorem:

Theorem 6 *For $n \in \mathbb{Z}$, $n \geqslant 0$ and $r \in \mathbb{Z}$, $r \geqslant 1$, we have*

$$L_q^{(r)}(-n,z,\chi) = \frac{(-1)^r}{(n+1)_r} \beta_{n+r,q,\chi}^{(r)}(z).$$

Proof. The proof of this theorem was given in [7] by using complex integration. Here, we give another proof.

Writing $m_i = a_i + k_i f$ with $a_i = 1, 2, \ldots, f$, $k_i = 0, 1, 2, \ldots$, where $i = 1, \ldots, r$,

$$\begin{aligned} L_q^{(r)}(-n,z,\chi) &= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\chi(m_1+\cdots+m_r)\, q^{z+m_1+\cdots+m_r}}{[z+m_1+\cdots+m_r]_q^{-n}} \\ &= \sum_{a_1,\ldots,a_r=1}^{f} \sum_{k_1,\ldots,k_r=0}^{\infty} \chi(a_1+k_1 f+\cdots+a_r+k_r f)\, q^{z+a_1+k_1 f+\cdots+a_r+k_r f} \\ &\quad \times [z+a_1+k_1 f+\cdots+a_r+k_r f]_q^n \\ &= \sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1+\cdots+a_r) \sum_{k_1,\ldots,k_r=0}^{\infty} \left(q^f\right)^{\frac{z+a_1+\cdots+a_r}{f}+k_1+\cdots+k_r} \\ &\quad \times \left[\frac{z+a_1+\cdots+a_r}{f}+k_1+\cdots+k_r\right]_{q^f}^n [f]_q^n. \end{aligned}$$

By Definition 3, last expression equals

$$L_q^{(r)}(-n,z,\chi) = [f]_q^n \sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1+\cdots+a_r)\, \zeta_{q^f}^{(r)}\left(-n, \frac{z+a_1+\cdots+a_r}{f}\right).$$

Making use of Theorem 4, we get

$$L_q^{(r)}(-n,z,\chi) = [f]_q^n \frac{(-1)^r}{(n+1)_r} \sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1+\cdots+a_r)\, \beta_{n+r,q^f}^{(r)}\left(\frac{z+a_1+\cdots+a_r}{f}\right).$$

Now, utilizing equation (5) (with $n \longmapsto n+r$) we obtain

$$L_q^{(r)}(-n,z,\chi) = \frac{(-1)^r}{(n+1)_r} \beta_{n+r,q,\chi}^{(r)}(z),$$

the desired result. ■

In [17, 21], Kim gave closed expressions for the sums of products of q-Bernoulli numbers and generalized q-Bernoulli numbers. Using these sums, we have nested sums over combinations of multiple q-zeta function and multiple q-Dirichlet L-function as follows:

Theorem 7 *We have*

$$\zeta_q^{(r)}(-n, z_1+\cdots+z_r) = \frac{(-1)^r}{(n+1)_r} \sum_{\substack{i_1,\ldots,i_r \geqslant 0 \\ i_1+\cdots+i_r=n+r}}$$

$$\times \sum_{k_1=0}^{n+r-i_1} \sum_{k_2=0}^{n+r-i_1-i_2} \cdots \sum_{k_{r-1}=0}^{n+r-i_1-i_2-\cdots-i_{r-1}} \binom{n+r}{i_1,\ldots,i_r} \binom{n+r-i_1}{k_1} \cdots \binom{n+r-i_1-i_2-\cdots-i_{r-1}}{k_{r-1}}$$

$$\times \beta_{k_1+i_1,q}(z_1)\,\beta_{k_2+i_2,q}(z_2) \cdots \beta_{k_{r-1}+i_{r-1},q}(z_{r-1})\,\beta_{i_r,q}(z_r)\,(q-1)^{k_1+\cdots+k_{r-1}},$$

and

$$L_q^{(r)}(-n, z_1+\cdots+z_r, \chi) = \frac{(-1)^r}{(n+1)_r} [f]_q^n \sum_{a_1,\ldots,a_r=1}^{f} \chi(a_1+\cdots+a_r) \sum_{\substack{i_1,\ldots,i_r \geqslant 0 \\ i_1+\cdots+i_r=n+r}}$$

$$\times \sum_{k_1=0}^{n+r-i_1} \sum_{k_2=0}^{n+r-i_1-i_2} \cdots \sum_{k_{r-1}=0}^{n+r-i_1-i_2-\cdots-i_{r-1}} \binom{n+r}{i_1,\ldots,i_r} \binom{n+r-i_1}{k_1} \cdots \binom{n+r-i_1-i_2-\cdots-i_{r-1}}{k_{r-1}}$$

$$\times \beta_{k_1+i_1,q^f}\left(\frac{a_1+z_1}{f}\right) \beta_{k_2+i_2,q^f}\left(\frac{a_2+z_2}{f}\right) \cdots \beta_{k_{r-1}+i_{r-1},q^f}\left(\frac{a_{r-1}+z_{r-1}}{f}\right)$$

$$\times \beta_{i_r,q^f}\left(\frac{a_r+z_r}{f}\right) (q^f-1)^{k_1+\cdots+k_{r-1}}.$$

Proof. By using the formula of Kim [17]

$$\begin{aligned} \beta_{n,q}^{(r)}(z_1+\cdots+z_r) = & \sum_{\substack{i_1,\ldots,i_r \geqslant 0 \\ i_1+\cdots+i_r=n+r}} \sum_{k_1=0}^{n+r-i_1} \sum_{k_2=0}^{n+r-i_1-i_2} \cdots \sum_{k_{r-1}=0}^{n+r-i_1-i_2-\cdots-i_{r-1}} \\ & \times \binom{n+r}{i_1,\ldots,i_r} \binom{n+r-i_1}{k_1} \cdots \binom{n+r-i_1-i_2-\cdots-i_{r-1}}{k_{r-1}} \\ & \times \beta_{k_1+i_1,q}(z_1)\,\beta_{k_2+i_2,q}(z_2) \cdots \beta_{k_{r-1}+i_{r-1},q}(z_{r-1})\,\beta_{i_r,q}(z_r)\,(q-1)^{k_1+\cdots+k_{r-1}}, \end{aligned} \tag{6}$$

and Theorem 4, we obtain the first formula. The second formula follows from (6), (5) and Theorem 6. ■

Let $F \in \mathbb{Z}$, $F \geqslant 1$, and

$$H_q^{(r)}(s, z : a_1, \ldots, a_r | F) = \sum_{\substack{m_1,\ldots,m_r > 0 \\ m_j \equiv a_j (\mathrm{mod} F)}}^{\infty} \frac{q^{z+m_1+\cdots+m_r}}{[z+m_1+\cdots+m_r]_q^s}$$

be the multiple partial q-zeta function (cf. [7]). For $m_j = a_j + n_j F$, $j = 1, \ldots, r$, it can be written that

$$\begin{aligned} H_q^{(r)}(s, z : a_1, \ldots, a_r | F) &= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{q^{z+a_1+n_1F+\cdots+a_r+n_rF}}{[z+a_1+n_1F+\cdots+a_r+n_rF]_q^s} \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(q^F)^{\frac{z+a_1+\cdots+a_r}{F}+n_1+\cdots+n_r}}{[F]_q^s \left[\frac{z+a_1+\cdots+a_r}{F}+n_1+\cdots+n_r\right]_{q^F}^s} \\ &= [F]_q^{-s}\, \zeta_{q^F}^{(r)}\left(s, \frac{z+a_1+\cdots+a_r}{F}\right). \end{aligned}$$

Thus, the function $H_q^{(r)}(s,z:a_1,\ldots,a_r|F)$ is a meromorphic function for $s\in\mathbb{C}$ with poles at $s=1,2,\ldots,r$ and for $s=-n$, $n\in\mathbb{Z}$, $n\geqslant 0$, it is obvious that

$$\begin{aligned} H_q^{(r)}(-n,z:a_1,\ldots,a_r|F) &= [F]_q^n\,\zeta_{q^F}^{(r)}\left(-n,\frac{z+a_1+\cdots+a_r}{F}\right) \\ &= [F]_q^n\,\frac{(-1)^r}{(n+1)_r}\beta_{n+r,q^F}^{(r)}\left(\frac{z+a_1+\cdots+a_r}{F}\right). \end{aligned} \tag{7}$$

Let F be a positive integer multiple of the conductor f. The multiple Dirichlet q-L-function of two variables $L_q^{(r)}(s,z,\chi)$ is given in terms of $H_q^{(r)}(s,z:a_1,\ldots,a_r|F)$ as (cf. [7])

$$L_q^{(r)}(s,z,\chi)=\sum_{a_1,\ldots,a_r=1}^{F}\chi(a_1+\cdots+a_r)\,H_q^{(r)}(s,z:a_1,\ldots,a_r|F).$$

For $n\in\mathbb{Z}$, $n\geqslant 0$,

$$L_q^{(r)}(-n,z,\chi)=\sum_{a_1,\ldots,a_r=1}^{F}\chi(a_1+\cdots+a_r)\,H_q^{(r)}(-n,z:a_1,\ldots,a_r|F),$$

thus from (7), it can be written that

$$L_q^{(r)}(-n,z,\chi)=\frac{(-1)^r}{(n+1)_r}[F]_q^n\sum_{a_1,\ldots,a_r=1}^{F}\chi(a_1+\cdots+a_r)\,\beta_{n+r,q^F}^{(r)}\left(\frac{z+a_1+\cdots+a_r}{F}\right). \tag{8}$$

By substituting the expansion

$$\beta_{n+r,q^F}^{(r)}\left(\frac{z+a_1+\cdots+a_r}{F}\right)=\sum_{k=0}^{n+r}\binom{n+r}{k}\beta_{k,q^F}^{(r)}q^{k(z+a_1+\cdots+a_r)}\left[\frac{z+a_1+\cdots+a_r}{F}\right]_{q^F}^{n+r-k},$$

of multiple q-Bernoulli polynomials on the right of (8), the equation

$$\begin{aligned} L_q^{(r)}(-n,z,\chi) &= \frac{(-1)^r}{(n+1)_r}[F]_q^{-r}\sum_{a_1,\ldots,a_r=1}^{F}\chi(a_1+\cdots+a_r)\,[z+a_1+\cdots+a_r]_q^{n+r} \\ &\quad\times\sum_{k=0}^{n+r}\binom{n+r}{k}\beta_{k,q^F}^{(r)}q^{k(z+a_1+\cdots+a_r)}\left(\frac{[F]_q}{[z+a_1+\cdots+a_r]_q}\right)^k \end{aligned}$$

can be obtained. Last equation yields

$$\begin{aligned} L_q^{(r)}(s,z,\chi) &= \frac{1}{[F]_q^r}\frac{1}{\prod_{j=1}^{r}(s-j)}\sum_{a_1,\ldots,a_r=1}^{F}\chi(a_1+\cdots+a_r)\,[z+a_1+\cdots+a_r]_q^{-s+r} \\ &\quad\times\sum_{k=0}^{\infty}\binom{-s+r}{k}\beta_{k,q^F}^{(r)}q^{k(z+a_1+\cdots+a_r)}\left(\frac{[F]_q}{[z+a_1+\cdots+a_r]_q}\right)^k, \end{aligned}$$

which can be used to construct the p-adic analogue of the function $L_q^{(r)}(s,z,\chi)$ (cf. [7]).

Let $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-1/(p-1)}$. For an arbitrary character χ and $n\in\mathbb{Z}$, let $\chi_n=\chi\omega^{-n}$ in the sense of product of characters, where ω being the Teichmüller character. Also, let $\langle a\rangle_q=$

$\omega^{-1}(a)[a]_q$. Then $\langle a\rangle_q \equiv 1\,(mod p^* R)$. If $z \in \mathbb{C}_p$, $|z|_p \leqslant 1$, then for any $a \in \mathbb{Z}$, it can be written that $\langle a+p^*z\rangle_q = \omega^{-1}(a)[a+p^*z]_q$ so that $\langle a+p^*z\rangle_q \equiv 1\,(mod p^* R)$ for $z \in \mathbb{C}_p$, $|z|_p \leqslant 1$ (cf. [5], [7], [25], [26], [37]).

Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n,$$

$a_{n,j} \in \mathbb{C}_p$, $j = 0, 1, \ldots$, be a sequence of formal power series, each of which converges in a fixed subset

$$D = \left\{ s \in \mathbb{C}_p : |s|_p \leqslant |p^*|_p^{-1} p^{-1/(p-1)} \right\}$$

of $\mathbb{C}_p$ such that $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for all n, and for each $s \in D$ and $\epsilon > 0$, there exists $n_0 = n_0(s, \epsilon)$ such that

$$\left| \sum_{n \geqslant n_0}^{\infty} a_{n,j} s^n \right|_p < \epsilon$$

for all j. Then, $\lim_{j\to\infty} A_j(s) = A_0(s)$ for any $s \in D$. This fact was used by Washington [41] to construct the p-adic equivalent of Dirichlet L-function $L(s,\chi)$.

By making use of this method, the authors [7] constructed the multiple p-adic q-L-function of two variables as follows:

$$\begin{aligned} L_{p,q}^{(r)}(s,z,\chi) &= \frac{1}{[F]_q^r} \frac{1}{\prod_{j=1}^{r}(s-j)} \sum_{\substack{a_1,\ldots,a_r=1 \\ (a_1+\cdots+a_r,p)=1}}^{F} \chi(a_1+\cdots+a_r) \langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r} \\ &\quad \times \sum_{k=0}^{\infty} \binom{-s+r}{k} \beta_{k,q^F}^{(r)} q^{k(a_1+\cdots+a_r+p^*z)} \left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q} \right)^k, \end{aligned} \tag{9}$$

where F is a positive integer multiple of p^* and the conductor f of χ. The analytical properties and analytic continuation of $L_{p,q}^{(r)}(s,z,\chi)$ are given by the following theorem ([7]):

Theorem 8 *Let F be a positive integer multiple of f and p^*, and $L_{p,q}^{(r)}(s,z,\chi)$ be defined as (9). Then, $L_{p,q}^{(r)}(s,z,\chi)$ is analytic for $z \in \mathbb{C}_p$, $|z|_p \leqslant 1$, provided $s \in D$, except $s = 1, 2, \ldots, r$ when $\chi = 1$, and meromorphic with simple poles at $s = 1, 2, \ldots, r$ when $\chi = 1$. Furthermore, for each $n \in \mathbb{Z}$, $n \geqslant 0$,*

$$L_{p,q}^{(r)}(-n,z,\chi) = \frac{(-1)^r}{(n+1)_r} \left(\beta_{n+r,q,\chi_{n+r}}^{(r)}(p^*z) - \chi_{n+r}(p)[p]_q^n \beta_{n+r,q^p,\chi_{n+r}}^{(r)}(p^{-1}p^*z) \right).$$

In next sections, we will consider some properties and applications of the function $L_{p,q}^{(r)}(s,z,\chi)$.

4 The Value of $L_{p,q}^{(r)}(s,z,\chi)$ at $s=r$

In this section, we evaluate the value $L_{p,q}^{(r)}(r,z,\chi)$ for a positive integer r.

Theorem 9 *Let χ be a Dirichlet character of conductor f and let F be a positive integer multiple of f and p^*. Then for $r \in \mathbb{Z}$, $r \geqslant 1$,*

$$\begin{aligned} L_{p,q}^{(r)}(r,z,\chi) &= \frac{1}{[F]_q^r}\frac{1}{(r-1)!}\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{F}\chi(a_1+\cdots+a_r)\left\{-\beta_{0,q^F}^{(r)}\log_p\langle a_1+\cdots+a_r+p^*z\rangle_q\right.\\ &\left.+\sum_{k=r}^{\infty}\frac{(-1)^k}{k}\beta_{k,q^F}^{(r)}q^{k(a_1+\cdots+a_r+p^*z)}\left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q}\right)^k\right\}. \end{aligned}$$

Proof. It follows from (9) that

$$\begin{aligned} L_{p,q}^{(r)}(s,z,\chi) = \frac{1}{[F]_q^r}&\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{F}\chi(a_1+\cdots+a_r)\frac{\langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r}}{\prod_{j=1}^{r}(s-j)}\\ &\times\sum_{k=0}^{r-1}\binom{-s+r}{k}\beta_{k,q^F}^{(r)}q^{k(a_1+\cdots+a_r+p^*z)}\left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q}\right)^k\\ &+\frac{1}{[F]_q^r}\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{F}\chi(a_1+\cdots+a_r)\langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r}\\ &\times\sum_{k=r}^{\infty}\frac{\binom{-s+r}{k}}{\prod_{j=1}^{r}(s-j)}\beta_{k,q^F}^{(r)}q^{k(a_1+\cdots+a_r+p^*z)}\left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q}\right)^k. \end{aligned}$$

Now, by using Taylor expansion at $s=r$, we have

$$\begin{aligned} &\frac{\langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r}}{\prod_{j=1}^{r}(s-j)}\\ &\quad=-\frac{\log_p\langle a_1+\cdots+a_r+p^*z\rangle_q}{\prod_{j=1}^{r-1}(s-j)}\left(1+\{\text{terms involving the powers of }(-s+r)\}\right), \end{aligned}$$

so that

$$\lim_{s\to r}\frac{\langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r}}{\prod_{j=1}^{r}(s-j)}=-\frac{\log_p\langle a_1+\cdots+a_r+p^*z\rangle_q}{(r-1)!},$$

and

$$\frac{\binom{-s+r}{k}}{\prod_{j=1}^{r}(s-j)}=\frac{(-1)^k s(s+1)\cdots(s-r+k-1)}{k!},$$

so that

$$\lim_{s\to r}\frac{\binom{-s+r}{k}}{\prod_{j=1}^{r}(s-j)}=\frac{1}{(r-1)!}\frac{(-1)^k}{k}.$$

We therefore get

$$
\begin{aligned}
L_{p,q}^{(r)}(r,z,\chi) &= \lim_{s\to r} L_{p,q}^{(r)}(s,z,\chi) \\
&= \lim_{s\to r}\frac{1}{[F]_q^r}\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{F}\chi(a_1+\cdots+a_r)\frac{\langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r}}{\prod_{j=1}^{r}(s-j)} \\
&\times \sum_{k=0}^{r-1}\binom{-s+r}{k}\beta_{k,q^F}^{(r)}q^{k(a_1+\cdots+a_r+p^*z)}\left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q}\right)^k
\end{aligned}
$$

$$
\begin{aligned}
&+\lim_{s\to r}\frac{1}{[F]_q^r}\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{F}\chi(a_1+\cdots+a_r)\langle a_1+\cdots+a_r+p^*z\rangle_q^{-s+r} \\
&\quad\times\sum_{k=r}^{\infty}\frac{\binom{-s+r}{k}}{\prod_{j=1}^{r}(s-j)}\beta_{k,q^F}^{(r)}q^{k(a_1+\cdots+a_r+p^*z)}\left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q}\right)^k. \\
&=\frac{1}{[F]_q^r}\frac{1}{(r-1)!}\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{F}\chi(a_1+\cdots+a_r)\Bigg\{-\beta_{0,q^F}^{(r)}\log_p\langle a_1+\cdots+a_r+p^*z\rangle_q \\
&\quad+\sum_{k=r}^{\infty}\frac{(-1)^k}{k}\beta_{k,q^F}^{(r)}q^{k(a_1+\cdots+a_r+p^*z)}\left(\frac{[F]_q}{[a_1+\cdots+a_r+p^*z]_q}\right)^k\Bigg\},
\end{aligned}
$$

the desired result. ■

Note that for $r=1$, Theorem 9 reduces to Theorem 8 of Kim [25].

5 Congruences for Multiple Generalized q-Bernoulli Polynomials

Congruences related to classical and generalized Bernoulli numbers have found an amount of interest. One of the most celebrated examples is the Kummer congruences for classical Bernoulli numbers (cf. [42]):

$$p^{-1}\Delta_c\frac{B_n}{n}\in\mathbb{Z}_p,$$

where $c\in\mathbb{Z}$, $c\geqslant 1$, $c\equiv 0\,(mod\,(p-1))$, and $n\in\mathbb{Z}$ is positive, even and $n\not\equiv 0\,(mod\,(p-1))$. Here, Δ_c is the forward difference operator which operates on a sequence $\{x_n\}$ by

$$\Delta_c x_n = x_{n+c}-x_n.$$

The powers Δ_c^k of Δ_c are defined by Δ_c^0 =identity and $\Delta_c^k=\Delta_c\circ\Delta_c^{k-1}$ for positive integers k, so that

$$\Delta_c^k x_n=\sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}x_{n+mc}.$$

More generally, it can be shown that

$$p^{-k}\Delta_c^k\frac{B_n}{n}\in\mathbb{Z}_p,$$

where $k \in \mathbb{Z}$, $k \geqslant 1$ and c and n are as above, but with $n > k$.

Kummer congruences for generalized Bernoulli numbers $B_{n,\chi}$ was first regarded by Carlitz [3]:

For positive $c \in \mathbb{Z}$, $c \equiv 0\,(mod\,(p-1))$, $n, k \in \mathbb{Z}$, $n > k \geqslant 1$, and χ such that $f = f_\chi \neq p^m$, where $m \in \mathbb{Z}$, $m \geqslant 0$,

$$p^{-k}\Delta_c^k \frac{B_{n,\chi}}{n} \in \mathbb{Z}_p\left[\chi\right].$$

Here, $\mathbb{Z}_p\left[\chi\right]$ denotes the ring of polynomials in χ, whose coefficients are in $\mathbb{Z}_p$.

Shiratani [35] applied the operator Δ_c^k to $-\left(1-\chi_n\left(p\right)p^{n-1}\right)B_{n,\chi_n}/n$ for similar c and χ, and showed that Carlitz's congruence is still true without the restriction $n > k$, requiring only that $n \geqslant 1$. He also established that the divisibility conditions on c can be removed, and proved

$$\left(p^*\right)^{-k}\Delta_c^k\left(1-\chi_n\left(p\right)p^{n-1}\right)\frac{B_{n,\chi_n}}{n} \in \mathbb{Z}_p\left[\chi\right].$$

As an extension of the Kummer congruence, Gunaratne [13, 14] showed that the value

$$p^{-k}\Delta_c^k\left(1-\chi_n\left(p\right)p^{n-1}\right)\frac{B_{n,\chi_n}}{n},$$

modulo $p\mathbb{Z}_p$, is independent of n and

$$p^{-k}\Delta_c^k\left(1-\chi_n\left(p\right)p^{n-1}\right)\frac{B_{n,\chi_n}}{n} \equiv p^{-k'}\Delta_c^{k'}\left(1-\chi_{n'}\left(p\right)p^{n'-1}\right)\frac{B_{n',\chi_{n'}}}{n'}\left(modp\mathbb{Z}_p\right),$$

if $p > 3$, $c, n, k \in \mathbb{Z}$ are positive, $\chi = \omega^h$, where $h \in \mathbb{Z}$, $h \not\equiv 0\,(mod\,(p-1))$, $n', k' \in \mathbb{Z}$, $k \equiv k'\,(mod\,(p-1))$. Furthermore, by means of the binomial coefficient operator

$$\binom{p^{-1}\Delta_c}{k}x_n = \frac{1}{k!}\left(\prod_{j=0}^{k-1}\left(p^{-1}\Delta_c - j\right)\right)x_n,$$

it has been shown that for similar character χ,

$$\binom{p^{-1}\Delta_c}{k}\left(1-\chi_n\left(p\right)p^{n-1}\right)\frac{B_{n,\chi_n}}{n} \in \mathbb{Z}_p,$$

and this value, modulo $p\mathbb{Z}_p$, is independent of n.

Fox [11] derived congruences similar to those above for the generalized Bernoulli polynomials without restrictions on the character χ. In [20], Kim gave a proof of Kummer-type congruence for the q-Bernoulli numbers of higher order. In [6], Cenkci and Kurt extended Fox's and Kim's results to generalized q-Bernoulli polynomials. For other versions of Kummer-type congruences related other numbers and polynomials we refer [4], [38].

In order to derive a collection of congruences, similar to the results above, relating to the multiple generalized q-Bernoulli polynomials, we utilize the following theorem and its immediate consequence, found in [7]:

Theorem 10 *For the character χ of conductor f, let $F_0 = lcm\left(f, p^*\right)$, F be a positive integer multiple of $p\left(p^*\right)^{-1}rF_0$, $r \in \mathbb{Z}$, $r \geqslant 1$, $z \in \mathbb{C}_p$, $\left|z\right|_p \leqslant 1$ and $s \in D$, except $s \neq 1, 2, \ldots, r$ if $\chi = 1$. Then*

$$\begin{aligned} &L_{p,q}^{(r)}\left(s, z+rF, \chi\right) - L_{p,q}^{(r)}\left(s, z, \chi\right) \\ &\quad = -\sum_{\substack{a_1,\ldots,a_r=1 \\ (a_1+\cdots+a_r,p)=1}}^{p^*F}\chi_r\left(a_1+\cdots+a_r\right)q^{a_1+\cdots+a_r+p^*z}\left\langle a_1+\cdots+a_r+p^*z\right\rangle_q^{-s}.\end{aligned}$$

Corollary 11 *Let F, r and s be as in Theorem 10. Then*

$$\begin{aligned} & L_{p,q}^{(r)}(s, rF, \chi) - L_{p,q}^{(r)}(s, \chi) \\ & \quad = - \sum_{\substack{a_1, \ldots, a_r = 1 \\ (a_1 + \cdots + a_r, p) = 1}}^{p^* F} \chi_r(a_1 + \cdots + a_r) q^{a_1 + \cdots + a_r} \langle a_1 + \cdots + a_r \rangle_q^{-s}, \end{aligned}$$

where $L_{p,q}^{(r)}(s, \chi) = L_{p,q}^{(r)}(s, 0, \chi)$.

We also incorporate the polynomial structure

$$B_n^r(z, q, \chi) = \frac{(-1)^r}{(n+1)_r} \left(\beta_{n+r,q,\chi_{n+r}}^{(r)}(p^* z) - \chi_{n+r}(p) [p]_q^n \beta_{n+r,q^p,\chi_{n+r}}^{(r)}(p^{-1} p^* z) \right)$$

and the set structure

$$R^* = \left\{ x \in \mathbb{Z}_p : |x|_p < p^{-1/(p-1)} \right\}.$$

Throughout, we assume that $q \in \mathbb{Z}_p$ with $|1-q|_p < p^{-1/(p-1)}$, so that $q \equiv 1 \,(mod R^*)$.

Theorem 12 *Let n, c, k, r be positive integers and $z \in p(p^*)^{-1} rF_0 R^*$. Then the quantity*

$$(p^*)^{-k} \Delta_c^k B_n^r(z, q, \chi) - (p^*)^{-k} \Delta_c^k B_n^r(0, q, \chi) \in R^*[\chi],$$

and, modulo $p^* R^*[\chi]$, is independent of n.

Proof. Since Δ_c is a linear operator, Corollary 11 implies that

$$\begin{aligned} & \Delta_c^k L_{p,q}^{(r)}(-n, rF, \chi) - \Delta_c^k L_{p,q}^{(r)}(-n, \chi) \\ & \quad = - \sum_{\substack{a_1, \ldots, a_r = 1 \\ (a_1 + \cdots + a_r, p) = 1}}^{p^* F} \chi_r(a_1 + \cdots + a_r) q^{a_1 + \cdots + a_r} \Delta_c^k \langle a_1 + \cdots + a_r \rangle_q^n. \end{aligned}$$

Thus

$$\begin{aligned} & \Delta_c^k B_n^r(rF, q, \chi) - \Delta_c^k B_n^r(0, q, \chi) \\ & \quad = - \sum_{\substack{a_1, \ldots, a_r = 1 \\ (a_1 + \cdots + a_r, p) = 1}}^{p^* F} \chi_r(a_1 + \cdots + a_r) q^{a_1 + \cdots + a_r} \Delta_c^k \langle a_1 + \cdots + a_r \rangle_q^n. \end{aligned}$$

Note that

$$\begin{aligned} \Delta_c^k \langle a_1 + \cdots + a_r \rangle_q^n & = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} \langle a_1 + \cdots + a_r \rangle_q^{n+mc} \\ & = \langle a_1 + \cdots + a_r \rangle_q^n \left(\langle a_1 + \cdots + a_r \rangle_q^c - 1 \right)^k. \end{aligned}$$

Now, $\langle a_1 + \cdots + a_r \rangle_q \equiv 1 \,(mod p^* R^*)$ for $(a_1 + \cdots + a_r, p) = 1$, which implies that $\langle a_1 + \cdots + a_r \rangle_q^c \equiv 1 \,(mod p^* R^*)$, and thus $\Delta_c^k \langle a_1 + \cdots + a_r \rangle_q^n \equiv 0 \left(mod \,(p^*)^k R^* \right)$. Therefore

$$\Delta_c^k B_n^r(rF, q, \chi) - \Delta_c^k B_n^r(0, q, \chi) \equiv 0 \left(mod \,(p^*)^k R^*[\chi] \right),$$

and so

$$(p^*)^{-k}\,\Delta_c^k B_n^r(rF,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)\in R^*[\chi].$$

Also, since $\langle a_1+\cdots+a_r\rangle_q^c\equiv 1\,(mod p^*R^*)$, the equation

$$\begin{aligned}&\Delta_c^k B_n^r(rF,q,\chi)-\Delta_c^k B_n^r(0,q,\chi)\\ &\quad=\;-\sum_{\substack{a_1,\ldots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{p^*F}\chi_r(a_1+\cdots+a_r)\,q^{a_1+\cdots+a_r}\langle a_1+\cdots+a_r\rangle_q^n\left(\frac{\langle a_1+\cdots+a_r\rangle_q^c-1}{p^*}\right)^k\end{aligned}$$

implies that the value of $(p^*)^{-k}\,\Delta_c^k B_n^r(rF,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)$, modulo $p^*R^*[\chi]$, is independent of n.

Let $z\in p(p^*)^{-1}rF_0R^*$. Since the set of positive integers in $p(p^*)^{-1}rF_0\mathbb{Z}$ is dense in $p(p^*)^{-1}rF_0R^*$, there exists a sequence $\{z_j\}$ in $p(p^*)^{-1}rF_0\mathbb{Z}$ with $z_j>0$ for each j, such that $z_j\to z$. Now, $B_n^r(z,q,\chi)$ is a polynomial, which implies that $B_n^r(z_j,q,\chi)\to B_n^r(z,q,\chi)$. Therefore,

$$\lim_{j\to\infty}\left(\Delta_c^k B_n^r(z_j,q,\chi)-\Delta_c^k B_n^r(0,q,\chi)\right)=\Delta_c^k B_n^r(z,q,\chi)-\Delta_c^k B_n^r(0,q,\chi).$$

The left side of this equality is 0 modulo $(p^*)^k R^*[\chi]$, which implies that

$$\Delta_c^k B_n^r(z,q,\chi)-\Delta_c^k B_n^r(0,q,\chi)\equiv 0\left(mod\,(p^*)^k R^*[\chi]\right),$$

and so

$$(p^*)^{-k}\,\Delta_c^k B_n^r(z,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)\in R^*[\chi].$$

Furthermore, for a positive integer n'

$$\begin{aligned}&\lim_{j\to\infty}\Big\{\left((p^*)^{-k}\,\Delta_c^k B_n^r(z_j,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)\right)\\&\qquad-\left((p^*)^{-k}\,\Delta_c^k B_{n'}^r(z_j,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_{n'}^r(0,q,\chi)\right)\Big\}\\&=\Big\{\left((p^*)^{-k}\,\Delta_c^k B_n^r(z,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)\right)\\&\qquad-\left((p^*)^{-k}\,\Delta_c^k B_{n'}^r(z,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_{n'}^r(0,q,\chi)\right)\Big\}.\end{aligned}$$

Since $z_j\in p(p^*)^{-1}rF_0\mathbb{Z}$ for all j, the quantity on the left must be 0 modulo $p^*R^*[\chi]$. Therefore, the value $(p^*)^{-k}\,\Delta_c^k B_n^r(z,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)$, modulo $p^*R^*[\chi]$, is independent of n. ■

Theorem 13 *Let n, c, r, k, k' be positive integers, $k\equiv k'\,(mod\,(p-1))$ and let $z\in p(p^*)^{-1}rF_0R^*$. Then*

$$\begin{aligned}&(p^*)^{-k}\,\Delta_c^k B_n^r(z,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)\\&\quad\equiv\;(p^*)^{-k'}\,\Delta_c^{k'} B_n^r(z,q,\chi)-(p^*)^{-k'}\,\Delta_c^{k'} B_n^r(0,q,\chi)\,(modpR^*[\chi]).\end{aligned}$$

Proof. Let k and k' be positive integers such that $k\equiv k'\,(mod\,(p-1))$. Without loss of generality, assume that $k\geqslant k'$. Then

$$\begin{aligned}&\left((p^*)^{-k}\,\Delta_c^k B_n^r(rF,q,\chi)-(p^*)^{-k}\,\Delta_c^k B_n^r(0,q,\chi)\right)\\&\qquad-\;\left((p^*)^{-k'}\,\Delta_c^{k'} B_n^r(rF,q,\chi)-(p^*)^{-k'}\,\Delta_c^{k'} B_n^r(0,q,\chi)\right)\end{aligned}$$

$$= -\sum_{\substack{a_1,\dots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{p^*F} \chi_r(a_1+\cdots+a_r)\, q^{a_1+\cdots+a_r} \langle a_1+\cdots+a_r\rangle_q^n$$
$$\times \left\{ \left(\frac{\langle a_1+\cdots+a_r\rangle_q^c - 1}{p^*}\right)^k - \left(\frac{\langle a_1+\cdots+a_r\rangle_q^c - 1}{p^*}\right)^{k'} \right\}$$
$$= -\sum_{\substack{a_1,\dots,a_r=1\\(a_1+\cdots+a_r,p)=1}}^{p^*F} \chi_r(a_1+\cdots+a_r)\, q^{a_1+\cdots+a_r} \langle a_1+\cdots+a_r\rangle_q^n$$
$$\times \left(\frac{\langle a_1+\cdots+a_r\rangle_q^c - 1}{p^*}\right)^{k'} \left\{ \left(\frac{\langle a_1+\cdots+a_r\rangle_q^c - 1}{p^*}\right)^{k-k'} - 1 \right\}.$$

If $a_1+\cdots+a_r$ such that
$$\langle a_1+\cdots+a_r\rangle_q^c - 1 \not\equiv 0 \,(modpp^*R^*)\,,$$
then, since $k-k' \equiv 0 \,(\mathrm{mod}\,(p-1))$, we have
$$\left(\frac{\langle a_1+\cdots+a_r\rangle_q^c - 1}{p^*}\right)^{k-k'} - 1 \equiv 0 \,(modpR^*)\,.$$

Thus
$$\begin{aligned} &(p^*)^{-k} \Delta_c^k B_n^r(rF,q,\chi) - (p^*)^{-k} \Delta_c^k B_n^r(0,q,\chi) \\ &\quad\equiv\quad (p^*)^{-k'} \Delta_c^{k'} B_n^r(rF,q,\chi) - (p^*)^{-k'} \Delta_c^{k'} B_n^r(0,q,\chi) \,(modpR^*[\chi])\,. \end{aligned}$$

Now let $z \in p\,(p^*)^{-1} rF_0R^*$. Then there exists a sequence $\{z_j\}$ in $p\,(p^*)^{-1} rF_0\mathbb{Z}$ with $z_j > 0$ for each j, such that $z_j \to z$. Consider

$$\begin{aligned} &\lim_{j\to\infty} \Big\{ \Big((p^*)^{-k} \Delta_c^k B_n^r(z_j,q,\chi) - (p^*)^{-k} \Delta_c^k B_n^r(0,q,\chi) \Big) \\ &\qquad - \Big((p^*)^{-k'} \Delta_c^{k'} B_n^r(z_j,q,\chi) - (p^*)^{-k'} \Delta_c^{k'} B_n^r(0,q,\chi) \Big) \Big\} \\ &= \Big\{ \Big((p^*)^{-k} \Delta_c^k B_n^r(z,q,\chi) - (p^*)^{-k} \Delta_c^k B_n^r(0,q,\chi) \Big) \\ &\qquad - \Big((p^*)^{-k'} \Delta_c^{k'} B_n^r(z,q,\chi) - (p^*)^{-k'} \Delta_c^{k'} B_n^r(0,q,\chi) \Big) \Big\}. \end{aligned}$$

Since the left side of this equality must be 0 modulo $pR^*[\chi]$, the proof follows. ■

The binomial coefficients operator $\binom{T}{k}$ associated to an operator T is defined by writing the binomial coefficients
$$\binom{X}{k} = \frac{X(X-1)\cdots(X-k+1)}{k!}$$
for $k \geqslant 0$ as a polynomial in X, and replacing X by T.

In the proof of next theorem, we need special numbers, namely the Stirling numbers of the first kind $s(n,k)$, which are defined by means of the generating function

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} s(n,k)\,\frac{t^n}{n!},$$

for $k \in \mathbb{Z}$, $k \geqslant 0$. Since there is no constant term in the expansion of $\log(1+t)$, $s(n,k) = 0$ for $0 \leqslant n < k$. Also, $s(n,n) = 1$ for all $n \geqslant 0$. The numbers $s(n,k)$ are integers and satisfy the following relation related to binomial coefficients:

$$\binom{x}{k} = \frac{1}{n!} \sum_{k=0}^{n} s\left(n,k\right) x^{k}. \tag{10}$$

For further information for Stirling numbers, we refer [8].

Theorem 14 *Let n, c, k be positive integers and $z \in p\left(p^{*}\right)^{-1} F_{0} R^{*}$. Then the quantity*

$$\binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} B_{n}^{r}\left(z,q,\chi\right) - \binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} B_{n}^{r}\left(0,q,\chi\right) \in R^{*}\left[\chi\right],$$

and, modulo $p^{*}R^{*}\left[\chi\right]$, is independent of n.

Proof. Since the binomial coefficients operator is a linear operator, Corollary 11 implies that

$$\begin{aligned} &\binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} L_{p,q}^{(r)}\left(-n,rF,\chi\right) - \binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} L_{p,q}^{(r)}\left(-n,\chi\right) \\ &= -\sum_{\substack{a_{1},\ldots,a_{r}=1 \\ (a_{1}+\cdots+a_{r},p)=1}}^{p^{*}F} \chi_{r}\left(a_{1}+\cdots+a_{r}\right) q^{a_{1}+\cdots+a_{r}} \binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} \left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{n}. \end{aligned}$$

Then

$$\begin{aligned} &\binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} B_{n}^{r}\left(rF,q,\chi\right) - \binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} B_{n}^{r}\left(0,q,\chi\right) \\ &= -\sum_{\substack{a_{1},\ldots,a_{r}=1 \\ (a_{1}+\cdots+a_{r},p)=1}}^{p^{*}F} \chi_{r}\left(a_{1}+\cdots+a_{r}\right) q^{a_{1}+\cdots+a_{r}} \binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} \left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{n}. \end{aligned}$$

By using (10), we can write

$$\begin{aligned} &\binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} \left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{n} \\ &= \frac{1}{k!}\sum_{m=0}^{k} s\left(k,m\right)\left(p^{*}\right)^{-m} \Delta_{c}^{m} \left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{n} \\ &= \frac{1}{k!}\sum_{m=0}^{k} s\left(k,m\right)\left(p^{*}\right)^{-m} \left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{n} \left(\left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{c}-1\right)^{m}. \end{aligned}$$

Thus

$$\begin{aligned} &\binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} B_{n}^{r}\left(rF,q,\chi\right) - \binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} B_{n}^{r}\left(0,q,\chi\right) \\ &= -\sum_{\substack{a_{1},\ldots,a_{r}=1 \\ (a_{1}+\cdots+a_{r},p)=1}}^{p^{*}F} \chi_{r}\left(a_{1}+\cdots+a_{r}\right) q^{a_{1}+\cdots+a_{r}} \\ &\times \binom{\left(p^{*}\right)^{-1}\Delta_{c}}{k} \left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{n} \binom{\left(p^{*}\right)^{-1}\left(\left\langle a_{1}+\cdots+a_{r}\right\rangle_{q}^{c}-1\right)}{k}. \end{aligned}$$

Since $(p^*)^{-1}\left(\langle a_1+\cdots+a_r\rangle_q^c-1\right)\in R^*$ for $(a_1+\cdots+a_r,p)=1$, we see that

$$\langle a_1+\cdots+a_r\rangle_q^n\binom{(p^*)^{-1}\left(\langle a_1+\cdots+a_r\rangle_q^c-1\right)}{k}\in R^*.$$

This then implies

$$\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(rF,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(0,q,\chi)\in R^*[\chi].$$

Furthermore, since $\langle a_1+\cdots+a_r\rangle_q^c\equiv 1\,(mod p^*R^*)$, the value of this quantity, modulo $p^*R^*[\chi]$, is independent of n.

Now let $z\in p\,(p^*)^{-1}rF_0R^*$, and let $\{z_j\}$ be a sequence in $p\,(p^*)^{-1}rF_0\mathbb{Z}$, with $z_j>0$ for each j, such that $z_j\to z$. Then

$$\begin{aligned}\lim_{j\to\infty}&\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(z_j,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(0,q,\chi)\\&=\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(z,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(0,q,\chi)\end{aligned}$$

must be in $R^*[\chi]$. Now let $n'\in\mathbb{Z}$, $n'>0$, and consider

$$\begin{aligned}&\lim_{j\to\infty}\left\{\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(z_j,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(0,q,\chi)\right.\\&\qquad\left.-\binom{(p^*)^{-1}\Delta_c}{k}B_{n'}^r(z_j,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_{n'}^r(0,q,\chi)\right\}\\&=\left\{\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(z,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(0,q,\chi)\right.\\&\qquad\left.-\binom{(p^*)^{-1}\Delta_c}{k}B_{n'}^r(z,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_{n'}^r(0,q,\chi)\right\}.\end{aligned}$$

The quantity on the left must be 0 modulo $p^*R^*[\chi]$, which implies that the value of

$$\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(z,q,\chi)-\binom{(p^*)^{-1}\Delta_c}{k}B_n^r(0,q,\chi),$$

modulo $p^*R^*[\chi]$, is independent of n. ∎

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