UNIVERSAL HOMOTOPY THEORIES

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ABSTRACT. Begin with a small category $\mathcal{C}$. The goal of this short note is to point out that there is such a thing as a 'universal model category built from $\mathcal{C}$'. We describe applications of this to the study of homotopy colimits, the Dwyer-Kan theory of framings, to sheaf theory, and to the homotopy theory of schemes.

CONTENTS

1. INTRODUCTION

Model categories were introduced by Quillen [Q] to provide a framework through which one could do homotopy theory in various settings. They have been astonishingly successful in this regard, and in recent years one of the first things one does when studying any homotopical situation is to try to set up a model structure. The aim of this paper is to introduce a new, but very basic tool into the study of model categories.

Our main observation is that given any small category $\mathcal{C}$ it is possible to expand $\mathcal{C}$ into a model category in a very generic way, essentially by formally adding homotopy colimits. In this way one obtains a 'universal model category built from $\mathcal{C}$'. There is an accompanying procedure which imposes relations into a model category, also in a certain universal sense. These two fundamental techniques are the subject of this paper. Although they are very formal—as any universal constructions would be—we hope to indicate that these ideas can be useful, and have some relevance to quite disparate areas of model category theory.

There are two general themes to single out regarding this material:

(1) Universal model categories give a method for creating a homotopy theory from scratch, based on a category of 'generators' and a set of 'relations'. On the one hand this is a procedure for building model categories in order to study

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some known phenomenon: in fact our original motivation was to 'explain' (if that can be considered the right word) Morel and Voevodsky's construction of a homotopy theory for schemes [MV]. On the other hand, however, this can be a technique for understanding a model category one already has, by asking what kinds of objects and relations are needed to reconstruct that homotopy theory from the ground up (see section 5.5).

(2) If one is trying to prove a theorem which should hold in all model categories—a generic result about the behavior of certain homotopy colimits, for example— then very often it suffices to prove the result just in some 'universal' case. Universal model categories enjoy several nice properties—they are simplicial, proper, cofibrantly-generated, etc.—and so when working in the universal case one has a wealth of tools at one's disposal which are not available in general model categories. This gives a technique for proving theorems analagous to a standard trick in algebra, whereby one proves a result for all rings by first reducing to a universal example like a polynomial ring.

In this paper our goal is to document the basic results about universal model categories, and to generally discuss theme #1. The second theme makes a brief appearance in section (3.5), but a thorough treatment will be postponed for a future paper.

With somewhat more detail, here is an outline of the paper:

In Section 2 we construct a model category $U\mathcal{C}$ and explain in what sense it is the universal model category built from $\mathcal{C}$. This generalizes a construction from category theory in which one 'completes' a category by formally adjoining colimits: the model category $U\mathcal{C}$ is in some sense obtained by formally adding *homotopy* colimits (2.6). The analogs here are very precise: the category-theoretic construction involves looking at categories of diagrams with values in $\mathcal{S}et$, whereas our homotopy-theoretic analog uses diagrams of *simplicial* sets.

Section 3 deals with connections between our universal model categories and the Dwyer-Kan theory of cosimplicial resolutions. These resolutions are a tool for studying 'higher-homotopies' in model categories, and are used for example to obtain explicit formulas for homotopy colimits. Our message in this section is this: a resolution is simply a map from a universal model category. We explain in (3.5) how homotopy colimits can be studied by 'lifting' them to the universal examples $U\mathcal{C}$. These universal examples are actually *simplicial* model categories, and so the theory of homotopy colimits is in this way reduced to a case which is very well understood.

The model categories $U\mathcal{C}$ are a kind of free object, like the free group generated by a set. Just as in the algebraic setting, there turns out to be a way of 'imposing relations' in model categories: This is the well-known process of *localization*, which is reviewed in section 5. It is natural to then ask what kinds of homotopy theories can be described by generators and relations—that is, by starting with a universal model category $U\mathcal{C}$ and then localizing. This question is the subject of section 6.

There is a very broad and useful class of model categories called the *combinatorial* ones, which have been introduced by Jeff Smith. We announce in (6.3) the result that every combinatorial model category is equivalent to a localization of some $U\mathcal{C}$; the proof is too involved to include here, but is instead given in the companion paper [D3]. One immediate consequence is that every combinatorial model category is equivalent to a model category which is simplicial and left proper

(the 'simplicial' part was proven under slightly more restrictive hypotheses in [D1], using very different methods.)

Finally, in sections 7 and 8 we deal with some elementary applications. The first of these is an interpretation of Jardine's model category of simplicial presheaves [J2]: we point out that giving a Grothendieck topology on a category $\mathcal{C}$ amounts to specifying certain 'homotopy-colimit' type relations, and studying the sheaf theory of $\mathcal{C}$ is precisely studying the model category one obtains from $U\mathcal{C}$ by imposing those relations.

The second application is to the Morel-Voevodsky homotopy theory of schemes. We show that their constructions are equivalent to starting with some basic category of schemes $\mathcal{C}$, forming the universal homotopy theory built from $\mathcal{C}$, and then imposing certain geometrically-natural relations. All of this is very formal, and our point is precisely that it *is* formal.

We close this introduction by giving a precise, but brief description of these universal model categories (this is done with more detail in section 2.) To describe the results we need a preliminary definition. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are model categories equipped with functors $r\colon \mathcal{C} \to \mathcal{M}$ and $\gamma\colon \mathcal{C} \to \mathcal{N}$, as depicted below:

$$\mathcal{C} \xrightarrow{\ r\ } \mathcal{M}$$
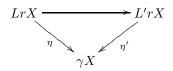$$\gamma \searrow$$
$$\mathcal{N}.$$

We define a **factorization of γ through $\mathcal{M}$** to be the following data:

(i) A Quillen pair $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$, together with
(ii) a natural weak equivalence $\eta\colon L \circ r \xrightarrow{\ \sim\ } \gamma$.

The factorization will be denoted by the triple (L, R, η). In this paper it will be useful to regard a Quillen pair $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$ as a map of model categories $\mathcal{M} \to \mathcal{N}$, which perhaps makes the term 'factorization' seem more appropriate.

We also define the **category of factorizations $\mathbf{Fact}_{\mathcal{M}}(\boldsymbol{\gamma})$**: its objects are triples (L, R, η) as above, and a map $(L, R, \eta) \to (L', R', \eta')$ is a natural transformation $L \to L'$ making the following diagram commute:

$$LrX \longrightarrow L'rX$$
$$\eta \searrow \qquad \swarrow \eta'$$
$$\gamma X$$

(Note that giving a natural transformation $L \to L'$ is equivalent—via adjointness— to giving a transformation $R' \to R$, or to giving two maps $L \to L'$ and $R' \to R$ which are compatible with the adjunctions. So we could have adopted a more symmetric definition of $\mathrm{Fact}_{\mathcal{M}}(\gamma)$, but it would be equivalent to the above).

Here is the basic result:

Proposition 1.1. *There exists a closed model category $U\mathcal{C}$ together with a functor $r\colon \mathcal{C} \to U\mathcal{C}$, such that the following is true: any map $\gamma\colon \mathcal{C} \to \mathcal{M}$ from $\mathcal{C}$ to a model category factors through $U\mathcal{C}$, and moreover the category of such factorizations is contractible.*

When dealing with universal constructions in ordinary category theory one typically finds that the category of choices is a contractible groupoid—this is what is usually

meant by saying that something is 'unique up to unique isomorphism'. When working in the homotopical setting, where the maps of interest are weak equivalences rather than isomorphisms, a category of choices will rarely be a groupoid. The key property of 'homotopically universal' constructions is that the category of choices is contractible. We may interpret the above proposition as saying that $U\mathcal{C}$ is the *universal model category* built from $\mathcal{C}$. (Of course referring to *the* universal model category is somewhat inappropriate, but we will continue this abuse of language throughout the paper). An explicit construction of $U\mathcal{C}$ as a diagram category is given in section 2.

1.2. **Organization of the paper.** We have already given a rough outline, but there are a couple of other points to make. The reader should be warned that section 3, dealing with framings, is somewhat technical and not strictly necessary for the rest of the paper. It follows section 2 because they are closely related, but many readers will want to read section 2 and then skip ahead to section 5 their first time through.

We also need to give a warning about the proofs, which in most cases we have kept very short, only giving a general indication of what one should do. This is on the one hand because of the formal nature of the results: once one decides on what the correct definitions and theorems are, then the results almost prove themselves. On the other hand there is always a certain amount of unpleasant machinery to be dealt with, but inflicting this on the reader would distract from the essentially simple character of the results. Proofs which are decidedly non-trivial are generally postponed until the very last section of the paper, which the reader can refer back to when necessary.

1.3. **Notation and terminology.** Our conventions regarding model categories, framings, and other elements of abstract homotopy theory generally follow those of Hirschhorn [H]. Hovey's book [Ho] is also a good reference. In particular, model categories are assumed to contain small limits and colimits, and to have functorial factorizations.

Following [Ho] we will define a **map** of model categories $L \colon \mathcal{M} \longrightarrow \mathcal{N}$ to be a Quillen pair $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$. That is, a Quillen pair will be regarded as a map of model categories in the direction of the left adjoint. The results in this paper will make it clear why this seems justified. We use $L^{cof}(X)$ to denote an object obtained by taking a cofibrant replacement for X and then applying L to it, and $R^{fib}(Y)$ denotes the result of replacing Y by a fibrant object and then applying R.

If $\mathcal{C}$ is a category and X, Y are objects, then $\mathcal{C}(X, Y)$ denotes the set of maps from X to Y. If $\mathcal{M}$ is a model category then $\underline{\mathcal{M}}(X, Y)$ denotes a homotopy function complex from X to Y.

Finally, we must say something about our conventions regarding homotopy colimits. To define these in general model categories, the approach taken in both [DHK] and [H] is to chose a framing on the model category and then to define homotopy colimits via an explicit formula. The subtlety is that this yields a construction which is only homotopy invariant for diagrams of cofibrant objects. For us, however, when we write 'hocolim' we will *always* mean something which is homotopy invariant: so our hocolim functors are defined by first taking a functorial cofibrant-replacement of every object in our diagram, and only then using the explicit formulas given in [DHK] or [H].

2. Universal model categories

In this section we introduce the construction of universal model categories, and indicate their basic properties. This generalizes a standard construction in category theory, by which one formally adds colimits to a category $\mathcal{C}$ by passing to the category of diagrams $\mathcal{S}et^{\mathcal{C}^{op}}$. Our universal model category $U\mathcal{C}$ is simply the diagram category $s\mathcal{S}et^{\mathcal{C}^{op}}$ equipped with an appropriate model structure—it may loosely be thought of as the result of formally adding homotopy colimits to $\mathcal{C}$ (see (2.6)). Proposition 2.3 explains the universal property this construction satisfies.

It will be helpful if we review a basic result from category theory. Recall that a category $\mathcal{C}$ is called *co-complete* if every diagram in $\mathcal{C}$ indexed by a small category has a colimit. Given a small category $\mathcal{C}$, there is a universal way of expanding it into a co-complete category, by considering the category of presheaves.

Recall that a **presheaf** on $\mathcal{C}$ is simply a functor $F: \mathcal{C}^{op} \to \mathcal{S}et$, and a map of presheaves is just a natural transformation. We will use $Pre(\mathcal{C})$ to denote the category of presheaves on $\mathcal{C}$—this is just another name for the category of diagrams $\mathcal{S}et^{\mathcal{C}^{op}}$. There is a canonical functor $r: \mathcal{C} \to Pre(\mathcal{C})$ called the Yoneda embedding, which sends an object $X \in \mathcal{C}$ to the presheaf $rX: Z \mapsto \mathcal{C}(Z, X)$. The object rX is called the 'presheaf represented by X'.

One has the following standard result (cf. [AR, Proposition 1.45(i)]):

Proposition 2.1.
(a) Any functor $\gamma: \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ into a co-complete category $\mathcal{D}$ may be factored through a colimit-preserving map $Re: Pre(\mathcal{C}) \to \mathcal{D}$:

$$\mathcal{C} \xrightarrow{\ r\ } Pre(\mathcal{C})$$
$$\gamma \searrow \quad \vdots\, Re$$
$$\mathcal{D}.$$

Moreover, the factorization is unique up to unique isomorphism.
(b) The map Re comes equipped with a right adjoint $Sing: \mathcal{D} \to Pre(\mathcal{C})$.

The proof will be left to the reader, but the basic fact which makes it work is the observation that every presheaf F may be canonically expressed as a colimit of representables. One looks at the overcategory $\mathcal{C} \downarrow F$ determined by the Yoneda embedding $\mathcal{C} \hookrightarrow Pre(\mathcal{C})$, and there is a canonical diagram $(\mathcal{C} \downarrow F) \to Pre(\mathcal{C})$ which sends $[rX \to F]$ to rX. The colimit of this diagram is precisely F, and we'll usually write this as

$$(2.1.1) \qquad\qquad F \cong \operatorname*{colim}_{rX \to F} rX.$$

F may be thought of as the 'formal colimit' of this diagram of representables. The functor Re in the above proposition is a 'realization' functor, which takes a formal colimit in $Pre(\mathcal{C})$ and then builds it in the category $\mathcal{D}$. It's adjoint is the 'singular' functor, defined so that $Sing\,X$ is the presheaf $c \mapsto \mathcal{D}(\gamma c, X)$.

Example 2.2.
(a) Consider the simplicial indexing category Δ. The category $Pre(\Delta)$ is just the category of simplicial sets, and the above result tells us that simplicial sets are just 'formal colimits' built from the basic simplices.

There is an obvious functor $\Delta \to \mathcal{T}op$ which sends $[n]$ to the topological simplex Δ^n. Since $\mathcal{T}op$ is co-complete, the above result gives an adjoint pair $Re : Pre(\Delta) \rightleftarrows \mathcal{T}op : Sing$. Of course these are just the usual realization and singular functors.

(b) There is also an obvious map $\Delta \to \mathcal{C}at$ into the category of small categories: it sends $[n]$ to the category $\{0 \to 1 \to \cdots \to n\}$. Since $\mathcal{C}at$ is co-complete we immediately get functors $Re : Pre(\Delta) \rightleftarrows \mathcal{C}at : Sing$. The functor $Sing$ may be identified with the usual nerve construction, and the functor Re is the usual way of obtaining a category from a simplicial set. (I learned this nice example from Tibor Beke).

Now let $U\mathcal{C}$ denote the category $sPre(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$. There is an obvious embedding $Pre(\mathcal{C}) \to sPre(\mathcal{C})$ which sends any presheaf F to the *discrete* simplicial presheaf containing F in every dimension (with identity maps as faces and degeneracies). Throughout this paper we will implicitly identify $Pre(\mathcal{C})$ with its image in $U\mathcal{C}$. Composing this embedding with the Yoneda map $\mathcal{C} \to Pre(\mathcal{C})$ gives an embedding $\mathcal{C} \hookrightarrow U\mathcal{C}$ which we will also call r, by abuse of notation.

$U\mathcal{C}$ is just the diagram category $s\mathcal{S}et^{\mathcal{C}^{op}}$, and so we can give it a model structure by saying that a map $F \to G$ is a

(a) weak equivalence if every $F(X) \to G(X)$ is a weak equivalence in $s\mathcal{S}et$;

(b) fibration if every $F(X) \to G(X)$ is a fibration in $s\mathcal{S}et$;

(c) cofibration if it has the left-lifting-property with respect to the trivial fibrations.

This is called the **Bousfield-Kan model structure** (see [BK, p. 314]). It is known to be cofibrantly-generated, proper, simplicial, and to have a wealth of other nice properties: it inherits essentially any nice property of $s\mathcal{S}et$. The weak equivalences are generally called **objectwise weak equivalences**, and likewise for the fibrations.

Proposition 2.3. *Any functor $\gamma \colon \mathcal{C} \to \mathcal{M}$ from $\mathcal{C}$ into a model category $\mathcal{M}$ may be 'factored' through $U\mathcal{C}$ in the sense that there is a Quillen pair $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$ and a natural weak equivalence $\eta \colon Re \circ r \xrightarrow{\sim} \gamma$:*

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ r\ } & U\mathcal{C} \\
 & {}_{\gamma}\searrow \ \Downarrow & \downarrow {}^{Re} \\
 & & \mathcal{M}.
\end{array}
$$

Moreover, the category of such factorizations (as defined in the introduction) is contractible.

Idea of proof. Proposition 2.1 allows us to extend γ to an adjoint pair of categories $Pre(\mathcal{C}) \rightleftarrows \mathcal{M}$. To extend this further to $U\mathcal{C}$ we must add a simplicial direction, and figure out what the realization of objects like '$X \otimes \Delta^n$' should be for $X \in \mathcal{C}$. This is accomplished by the theory of cosimplicial resolutions, discussed in the next section. The proof will be completed at that time. $\square$

Note that the representables rX are always cofibrant in $U\mathcal{C}$, and therefore their images $Re(rX)$ are cofibrant in $\mathcal{M}$. This is why we needed the natural transformation η, because the above triangle won't be able to commute on the nose unless γ actually took its values in the cofibrant objects.

Example 2.4. If we take pt to be the trivial category with one object and an identity map, then $U(pt)$ is just the model category $sSet$. So the homotopy theory of simplicial sets is just the universal homotopy theory on a point. This is really a silly statement, as simplicial sets are in some sense built into the very fabric of what people have decided they mean by a 'homotopy theory'. We will see a more interesting statement along these lines in Example 5.6.

Example 2.5. Let G be a finite group, and let $G\mathcal{T}op$ denote the usual model category of G-spaces. Consider the orbit category $\mathcal{O}_G$, which is the full subcategory of $G\mathcal{T}op$ whose objects are the orbits G/H. The inclusion $\mathcal{O}_G \hookrightarrow G\mathcal{T}op$ gives rise to a Quillen pair

$$U(\mathcal{O}_G) \rightleftarrows G\mathcal{T}op,$$

and it is easy to check that the singular functor associates to any G-space X the diagram of (singular complexes of) its fixed spaces $G/H \mapsto X^H$. It is a classical theorem of equivariant topology that this Quillen pair is an equivalence—the homotopy theory of G-spaces is the universal homotopy theory generated by the orbit category $\mathcal{O}_G$.

2.6. Cofibrant replacement in $U\mathcal{C}$. We close this section with two generalizations of (2.1.1), which will explain in what sense every object of $U\mathcal{C}$ is a formal homotopy colimit of objects in $\mathcal{C}$. This is accomplished by writing down two very convenient cofibrant-replacement functors in $U\mathcal{C}$. Knowing nice versions of cofibrant-replacement is often an important point in dealings with these model categories.

Let F be an object in $Pre(\mathcal{C})$. Define $\tilde{Q}F$ to be the simplicial presheaf whose nth level is

$$(\tilde{Q}F)_n = \coprod_{rX_n \to \cdots \to rX_0 \to F} (rX_n)$$

and whose face and degeneracy maps are the obvious candidates (d_i means omit X_i, etc.) In the language of [BK], this is the **simplicial replacement** of the canonical diagram $(\mathcal{C} \downarrow F) \to Pre(\mathcal{C})$. (Notice that $\tilde{Q}F$ is in some sense the formal homotopy colimit of this diagram.) Also note that there is a natural map $\tilde{Q}F \to \pi_0(\tilde{Q}F)$, and that the codomain is just F by (2.1.1).

Lemma 2.7. *The simplicial presheaf $\tilde{Q}F$ is cofibrant, and the map $\tilde{Q}F \to F$ is a weak equivalence.*

If F_* is an arbitrary simplicial presheaf then applying the functor $\tilde{Q}$ in every dimension gives a bisimplicial presheaf, and we let QF denote the diagonal. Once again there is a natural map $QF \to F$.

Proposition 2.8 (Resolution by representables)**.** *For any simplicial presheaf F one has that QF is cofibrant, and the map $QF \to F$ is a weak equivalence.*

Proofs. Both the lemma and proposition are proven in section 9.1. $\quad\square$

Notice that QF is a simplicial presheaf which in every dimension is a coproduct of representables. We can think of it as the realization—or homotopy colimit—of the diagram of representables

$$\cdots \coprod_{rX_1 \to rX_0 \to F_1} (rX_1) \rightrightarrows \coprod_{rX_0 \to F_0} (rX_0),$$

or as the 'formal' homotopy colimit of the same diagram back in $\mathcal{C}$. The above proposition tells us that every simplicial presheaf is canonically a homotopy colimit of representables, which of course is the direct analog of (2.1.1).

There is another cofibrant-replacement functor for $U\mathcal{C}$ which looks a little different from the one above, but is useful in some settings. Consider the functor $\mathcal{C} \times \Delta \to U\mathcal{C}$ defined by $A \times [n] \mapsto rA \otimes \Delta^n$. For a simplicial presheaf $F \in U\mathcal{C}$ we may form the overcategory $(\mathcal{C} \times \Delta \downarrow F)$, whose objects correspond to the data $[A \times [n], rA \otimes \Delta^n \to F]$. This category comes equipped with a canonical functor $(\mathcal{C} \times \Delta \downarrow F) \to U\mathcal{C}$ sending $[A \times [n], rA \otimes \Delta^n \to F]$ to $rA \otimes \Delta^n$, and the colimit of this functor is easily seen to be F itself. The *homotopy* colimit is called the **canonical homotopy colimit of F with respect to $\mathcal{C}$**, and will be denoted by **hocolim$(\mathcal{C} \times \Delta \downarrow F)$**. (To form this homotopy colimit recall that $U\mathcal{C}$ is a simplicial model category, and so we can use the formulas from [BK].)

The object hocolim$(\mathcal{C} \times \Delta \downarrow F)$ is a homotopy colimit of a diagram in which all the objects have the form $rA \otimes \Delta^n$, and in particular are cofibrant. Therefore hocolim$(\mathcal{C} \times \Delta \downarrow F)$ is cofibrant as well. The natural map from the homotopy colimit to the colimit gives a map hocolim$(\mathcal{C} \times \Delta \downarrow F) \to F$, and we claim that this is always a weak equivalence:

Proposition 2.9. *Let $\mathcal{Q}$ be the functor defined by $\mathcal{Q}F = \mathrm{hocolim}(\mathcal{C} \times \Delta \downarrow F)$. Then each $\mathcal{Q}F$ is cofibrant, and the natural map $\mathcal{Q}F \to F$ is a weak equivalence.*

Proof. See section 9.   □

The above proposition is of course another generalization of (2.1.1), as it shows how to canonically express any $F \in U\mathcal{C}$ as a homotopy colimit of representables.

Example 2.10. To see the difference between the functors $\mathcal{Q}$ and Q consider the case where $\mathcal{C}$ is the trivial category with one object and an identity map. Then $U\mathcal{C}$ is the category $s\mathcal{S}et$. Given a simplicial set K, QK is just K again. To get $\mathcal{Q}K$, though, we take the category of simplices ΔK of K and consider the diagram $\Delta K \to s\mathcal{S}et$ which sends the n-simplex σ to Δ^n. The homotopy colimit of this diagram is $\mathcal{Q}K$. It is weakly equivalent to K, but is a much bigger object.

Canonical homotopy colimits are extremely important in [D3], where a treatment is given for all model categories.

3. Connections with the theory of framings

This section continues our discussion of the basic theory of universal model categories. What we will see is that working with these universal model categories is exactly the same as working with *cosimplicial resolutions*, in the sense of Dwyer and Kan. Our 'universal' perspective has some advantages, however, in that it efficiently captures the limited amount of adjointness that resolutions exhibit. We explain in (3.5) a technique by which many theorems whose proofs require resolutions can be immediately reduced to the case of *simplicial* model categories, which are usually easier to deal with. For instance, most standard results in the theory of homotopy colimits can be deduced from the simplicial case by this method.

We begin by reviewing what resolutions are. First recall the notion of a cylinder object: If $X \in \mathcal{M}$ then a cylinder object for X is an object of $\mathcal{M}$ which 'looks and

feels' like '$X \times \Delta^1$'. It is an object X_1 together with maps

$$X \amalg X \rightarrowtail X_1 \xrightarrow{\sim} X,$$

where the first map is a cofibration and the second a weak equivalence. These maps can be assembled into the beginning of a cosimplicial object:

$$X \mathrel{\substack{\longleftarrow \\[-0.3em] \longrightarrow \\[-0.3em] \longrightarrow}} X_1.$$

The Dwyer-Kan theory of resolutions [DK] is a massive generalization of this, which gives a way of talking about objects which 'look and feel' like '$X \times \Delta^n$' for any n. This is actually what is called a *cosimplicial* resolution. There are also *simplicial* resolutions, which give a way of dealing with objects which look and feel like X^{Δ^n}, in the same way that path objects are substitutes for X^{Δ^1}. The theories of cosimplicial and simplicial resolutions are completely dual.

In a simplicial model category $\mathcal{M}$ the object $X \otimes \Delta^*$ is a particularly nice element of $c\mathcal{M}$—the category of cosimplicial objects—with the property that the object in each level is weakly equivalent to X (at least if X is cofibrant!) The main part of what one must come to terms with is what should be meant by 'particularly nice'—just as for cylinder objects, this should translate into certain maps being cofibrations. The reader can consult [DK, Section 4.3] for a precise formulation. It turns out that there is a natural model structure on $c\mathcal{M}$ called the *Reedy model structure* whose cofibrant objects are precisely what we want. We will not recall Reedy model categories here, but refer the reader to [Ho, Chapter 5].

Definition 3.1. *Let $\mathcal{M}$ be a model category and let X be an object.*

*(a) A **cosimplicial resolution** of X is a Reedy cofibrant object $\Gamma \in c\mathcal{M}$ together with a map $\Gamma \to c^*X$ which is a weak equivalence in every dimension. Here c^*X is the constant cosimplicial object with X in every level.*

*(b) A **simplicial resolution** of X is a Reedy fibrant object Φ in $s\mathcal{M}$ together with a map $c_*X \to \Phi$ which is a weak equivalence in every dimension.*

If X is itself cofibrant then it has a cosimplicial resolution whose 0th object is actually equal to X, and the standard practice is to choose such a resolution when possible.

The above definition has the following immediate generalization:

Definition 3.2. *Let $\mathcal{C}$ be a category with a functor $\gamma \colon \mathcal{C} \to \mathcal{M}$. A **cosimplicial resolution** of γ is*
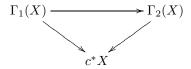
(i) a functor $\Gamma \colon \mathcal{C} \to c\mathcal{M}$ such that each $\Gamma(X)$ is Reedy cofibrant, and

(ii) a natural weak equivalence $\Gamma(X) \xrightarrow{\sim} c^*X$,

*   **Simplicial resolutions** of γ are defined similarly.*

The convention in [DHK] and [H] is to use 'framings' rather than resolutions—the difference is, for instance, that in a cosimplicial framing the objects $\Gamma(X)$ are Reedy cofibrant only if X itself was cofibrant. The advantage of framings is that in a simplicial model category the assignment $X \mapsto X \otimes \Delta^*$ is a cosimplicial framing, whereas to get a cosimplicial resolution we must use $X \mapsto X^{cof} \otimes \Delta^*$ for some cofibrant replacement $X^{cof} \xrightarrow{\sim} X$. The disadvantage of framings is that formulas which use them don't always yield the 'correct' answer—to get the correct answer one must use resolutions.

We will also need to talk about maps between resolutions:

Definition 3.3. *Let* Γ_1 *and* Γ_2 *be two cosimplicial resolutions of a map* $\gamma : \mathcal{C} \to \mathcal{M}$. *A* **map of resolutions** $\Gamma_1 \to \Gamma_2$ *is a natural transformation* $\Gamma_1(X) \to \Gamma_2(X)$ *making the following triangle commute:*

$$\Gamma_1(X) \xrightarrow{\hspace{3cm}} \Gamma_2(X)$$
$$c^*X$$

The category of cosimplicial resolutions on γ *will be denoted* **coRes(γ)**. *A map of simplicial resolutions, as well as the category* $\mathrm{sRes}(\gamma)$, *are defined similarly.*

Proposition 3.4. *Let* $\mathcal{C}$ *be a small category and let* $\gamma \colon \mathcal{C} \to \mathcal{M}$ *be a map. Then giving a factorization of* γ *through* $U\mathcal{C}$ *is precisely the same as giving a cosimplicial resolution on* γ. *Even more, there is a natural equivalence of categories*

$$\mathrm{Fact}_{\mathcal{M}}(\gamma) \simeq \mathrm{coRes}(\gamma).$$

In words, the proposition says that a cosimplicial resolution is just a map from a universal model category.

Proof. This is not hard, but requires some machinery. See section 9.5. $\square$

Proof of Proposition 2.3. We have just seen that factoring a functor $\gamma \colon \mathcal{C} \to \mathcal{M}$ through $U\mathcal{C}$ is equivalent to giving a cosimplicial resolution on γ. But it is a standard result in the theory of resolutions that (1) any diagram $\gamma \colon \mathcal{C} \to \mathcal{M}$ has a cosimplicial resolution, and (2) the category $\mathrm{coRes}(\gamma)$ is contractible (both are proven in [H]). So Proposition 2.3 is just a re-casting of these classical facts. $\square$

3.5. Application to homotopy colimits.

Let $X \colon \mathcal{C} \to \mathcal{M}$ be a diagram whose homotopy colimit we wish to study. When $\mathcal{M}$ is a *simplicial* model category then there is a well known formula for the homotopy colimit due to Bousfield and Kan. We can use universal model categories to reduce to the simplicial case in a very natural way:

The map $X \colon \mathcal{C} \to \mathcal{M}$ will factor through the universal model category $U\mathcal{C}$,

$$\mathcal{C} \xrightarrow{\tilde{X}} U\mathcal{C}$$
$$X \searrow \quad \downarrow Re$$
$$\mathcal{M}.$$

Note that $\tilde{X}$ doesn't really have anything to do with X—it is the same functor we have been calling r, and it's the universal example of a diagram in a model category with indexing category $\mathcal{C}$. For present purposes it's convenient to think of it as a lifting of the diagram X, though.

$U\mathcal{C}$ happens to be a simplicial model category, and so we can use the Bousfield-Kan formula to compute the homotopy colimit of $\tilde{X}$:

$$\mathrm{hocolim}\, \tilde{X} \simeq \mathrm{coeq}\left[\coprod_{\beta \to \gamma} \tilde{X}_\beta \otimes B(\gamma \downarrow \mathcal{C})^{op} \rightrightarrows \coprod_{\alpha} \tilde{X}_\alpha \otimes B(\alpha \downarrow \mathcal{C})^{op} \right].$$

Now the realization Re is a left Quillen functor, and so whatever we mean by the homotopy colimit of X in $\mathcal{M}$ will have to be weakly equivalent to $Re(\mathrm{hocolim}\, \tilde{X})$.

So we have uncovered the formula

$$\text{hocolim}\, X \simeq Re(\text{hocolim}\, \tilde{X})$$

$$\simeq \text{coeq}\left[\coprod_{\beta \to \gamma} Re\big(\tilde{X}_\beta \otimes B(\gamma \downarrow \mathcal{C})^{op}\big) \rightrightarrows \coprod_{\alpha} Re\big(\tilde{X}_\alpha \otimes B(\alpha \downarrow \mathcal{C})^{op}\big)\right]$$

The Dwyer-Kan theory gives a formula for the homotopy colimit in terms of resolutions: one chooses a cosimplicial resolution for the map $\mathcal{C} \to \mathcal{M}$, and then writes down an analog of the Bousfield-Kan formula in which 'tensoring' has been replaced by an operation involving resolutions. Our choice for the factorization of X through $U\mathcal{C}$ was precisely a choice of resolution for X, and it is an easy exercise to check that the formula we obtained is precisely the Dwyer-Kan formula.

One consequence of this perspective is that the basic properties of homotopy colimits, once known for simplicial model categories, immediately follow in the general case: one knows the property for the universal case $U\mathcal{C}$, and then simply pushes it to general $\mathcal{M}$ using the left Quillen functor Re. (The full power of this technique requires the ability to impose 'relations' into $U\mathcal{C}$, as discussed in section 5). The idea is analogous to a standard trick in algebra, where one proves a result for all rings by first reducing to some universal example like a polynomial ring. We hope to give a more detailed treatment of this material in a future paper.

4. Universal model categories for homotopy limits

The categories $U\mathcal{C}$ we have been talking about perhaps should have been called 'co-universal' model categories. There is of course a strictly dual notion which we will denote $V\mathcal{C}$—this will be discussed briefly in this section. Just as $U\mathcal{C}$ was very relevant to the study of homotopy colimits, $V\mathcal{C}$ pertains to the theory of homotopy *limits*. The material in this section is not needed in the rest of the paper, but is included for the sake of completeness.

Let $V\mathcal{C}$ denote the category $[s\mathcal{S}et^{\mathcal{C}}]^{op}$. Note that there is an obvious 'Yoneda embedding' $r\colon \mathcal{C} \to V\mathcal{C}$. The diagram category $s\mathcal{S}et^{\mathcal{C}}$ may be given the usual Bousfield-Kan structure, and we give $[s\mathcal{S}et^{\mathcal{C}}]^{op}$ the opposite model structure: a map $D_1 \to D_2$ in $V\mathcal{C}$ is a weak equivalence (resp. cofibration, fibration) precisely when $D_2 \to D_1$ is a weak equivalence (resp. fibration, cofibration) in the model category $s\mathcal{S}et^{\mathcal{C}}$.

Given two functors $\gamma\colon \mathcal{C} \to \mathcal{M}$ and $r\colon \mathcal{C} \to \mathcal{N}$ from $\mathcal{C}$ to model categories $\mathcal{M}$ and $\mathcal{N}$, define a **co-factorization of γ through $\mathcal{N}$** to be:

(i) A Quillen pair $L : \mathcal{M} \rightleftarrows \mathcal{N} : R$, together with
(ii) a natural weak equivalence $\gamma \xrightarrow{\sim} R \circ r$.

We leave it to the reader to define the **category of co-factorizations coFact$_{\mathcal{N}}(\gamma)$**: it is strictly dual to the category of factorizations. (Note that in this section it is more convenient to think of a Quillen pair as a map of model categories in the direction of the *right* adjoint!)

Proposition 4.1.
(a) If $\gamma\colon \mathcal{C} \to \mathcal{M}$ is a map from $\mathcal{C}$ to a model category then there is a co-factorization of γ through $V\mathcal{C}$, and the category of all such co-factorizations is contractible.
(b) The category of co-factorizations is naturally equivalent to the category of simplicial resolutions on $\mathcal{C} \to \mathcal{M}$.

All of the results from sections 2 and 3 can be repeated in this context. In particular, the objects of $V\mathcal{C}$ can be thought of as formal homotopy limits of the objects of $\mathcal{C}$.

5. Imposing relations via localizaton

Now that we have a notion of universal object for model categories, it is natural to ask if there is some procedure for 'imposing relations', and then if every model category can be obtained from a universal one in this way. These questions will be addressed in this section and the next. Our method of imposing relations is the well-known procedure of *localization*: given a model category $\mathcal{M}$ and a set of maps S, one forms a new model structure $\mathcal{M}/S$ in which the elements of S have been added to the weak equivalences. A very thorough account of localization machinery is contained in [H], but in the beginning of this section we summarize the relevant material.

Model categories of the form $U\mathcal{C}/S$ will be our central concern for the rest of the paper, and in (5.5) we give some basic examples: the most notable of these is Segal's Γ-spaces, which we can interpret in terms of universal constructions. In (5.8) we end with some indications that the objects $U\mathcal{C}/S$ are something like 'cofibrant' model categories.

5.1. **Review of localization.** Our basic definition of localizations for model categories is a slight variant of what is called 'left localization' by Hirschhorn [H]:

Definition 5.2. *Let $\mathcal{M}$ be a model category and let S be a set of maps in $\mathcal{M}$. An S-localization of $\mathcal{M}$ is a model category $\mathcal{M}/S$ and a map $F\colon \mathcal{M} \to \mathcal{M}/S$ such that the following holds:*
(a) F^{cof} takes maps in S to weak-equivalences, and
(b) F is initial among maps satisfying (a).
(Recall that F^{cof} —which we call the left derived functor of F —denotes any functor obtained by pre-composing F with a cofibrant-replacement functor.)

Unfortunately, S-localization need not always exist. The questions of when they exist and what they might look like can be very hairy, but there are certain classes of 'nice' model categories where the situation is well under control. We regard the process of localization as a way of 'imposing relations' in a model category, hence the notation $\mathcal{M}/S$—other authors have used $S^{-1}\mathcal{M}$ or $L_S\mathcal{M}$ for the same concept.

Bousfield [Bo] was the first to give a systematic approach to what S-localizations might look like, and we now recall this.

Definition 5.3.
(a) An S-local object of $\mathcal{M}$ is a fibrant object X such that for every map $A \to B$ in S, the induced map of homotopy function complexes $\underline{\mathcal{M}}(B, X) \to \underline{\mathcal{M}}(A, X)$ is a weak equivalence of simplicial sets.
(b) An S-local equivalence is a map $A \to B$ such that $\underline{\mathcal{M}}(B, X) \to \underline{\mathcal{M}}(A, X)$ is a weak equivalence for every S-local object X.

In words, the S-local objects are the ones which see every map in S as if it were a weak equivalence. The S-local equivalences are those maps which are seen as weak equivalences by every S-local object. The idea is that the S-local equivalences are

the maps which are *forced* into being weak equivalences as soon as we expand our notion of weak equivalence to include the maps in S.

Definition 5.4. *A **Bousfield S-localization** of $\mathcal{M}$ is a model category $\mathcal{M}/S$ with the properties that*

(a) The underlying category of $\mathcal{M}/S$ is $\mathcal{M}$;

(b) The cofibrations in $\mathcal{M}/S$ are the same as those in $\mathcal{M}$;

(c) The weak equivalences in $\mathcal{M}/S$ are the S-local equivalences.

Hirschhorn has proven that Bousfield S-localizations are indeed S-localizations as defined above. Bousfield localizations also need not always exist; if they do exist, they are clearly unique. The fibrant objects in $\mathcal{M}/S$ will be precisely the S-local objects, but the fibrations may be somewhat mysterious. From now on whenever we speak of localizations we will always mean Bousfield localizations.

There are two main classes of model categories where localizations are always known to exist (for any set of maps S). These are the left proper, cellular model categories of Hirschhorn [H], and the left proper, combinatorial model categories of Smith [Sm]. We will not recall the definitions of these classes here, but suffice it to say that the model categories $U\mathcal{C}$ belong to both of them, and so we are free to localize. In general, the model categories $U\mathcal{C}$ are about as nice as one could possibly want.

5.5. Basic examples of model categories $U\mathcal{C}/S$.

Example 5.6. In Example 2.4 we saw that the homotopy theory of topological spaces was the universal homotopy theory on a point, but that this was almost a tautological statement. A more interesting example can be obtained as follows: The way we usually think of simplicial sets is as objects formally built from the basic simplices, so let us look at $U\Delta$, the universal homotopy theory built from Δ. The obvious map $\Delta \to \mathcal{T}op$ gives rise to a Quillen pair $Re : U\Delta \rightleftarrows Top : Sing$, but this is not a Quillen equivalence. The first problem one encounters is that there is nothing in $U\Delta$ saying that the objects Δ^n are contractible. In fact this turns out to be the only problem. If we localize $U\Delta$ at the set of maps $S = \{\Delta^n \to *\}$, then our Quillen functors descend to a pair

$$Re : U\Delta/\{\Delta^n \to *\} \rightleftarrows \mathcal{T}op : Sing.$$

It can be seen that this is now a Quillen equivalence—this can be deduced from [D1, Proposition 5.2], but in fact it was the present observation which inspired that result. So the homotopy theory of simplicial sets is the universal homotopy theory built from Δ in which the Δ^n's are contractible.

Example 5.7 (Gamma-spaces)**.** In this example we need the observation that everything we've done with universal model categories can be duplicated in a *pointed* context. Namely, every small category $\mathcal{C}$ gives rise to a universal pointed model category built from $\mathcal{C}$, denoted $U_*\mathcal{C}$. Instead of using presheaves of simplicial sets one uses presheaves of pointed simplicial sets, and all the same results work with identical proofs.

Now let $\mathcal{S}pectra$ denote your favorite model category of spectra—for convenience we'll choose Bousfield-Friedlander spectra— and let S denote the sphere spectrum. Let $\mathcal{C}$ be the subcategory whose objects are S, $S \times S$, $S \times S \times S$, etc., and whose

morphisms are generated by the 'obvious' maps one can write down: e.g., projections $p_1, p_2 \colon S \times S \to S$, inclusions into a factor $i_1, i_2 \colon S \to S \times S$, diagonal maps $S \to S \times S$, etc.

Now $\mathcal{C}$ is almost the same as the category called Γ in [BF]—the only difference is that Γ contains an extra object corresponding to the trivial spectrum $*$. In any case the inclusion $\mathcal{C} \hookrightarrow \mathcal{S}pectra$ extends to a Quillen pair $Re \colon U_* \mathcal{C} \rightleftarrows \mathcal{S}pectra \colon Sing$, and the category $U_* \mathcal{C}$ is isomorphic to the category of Γ-spaces as defined in [BF]. The realization and singular functors are what Segal calls B and A, respectively. These functors are clearly not a Quillen equivalence, but let us see if we can somehow turn them into one.

Let $S^{\times k}$ denote the representable object in $U_* \mathcal{C}$ corresponding to $S \times \cdots \times S$ (k times). There are obvious maps $S^{\times k} \vee S^{\times l} \to S^{\times (k+l)}$, restricting to the inclusions on each wedge-summand—these maps certainly become equivalences after applying Re. Consider also the 'shearing map' $sh \colon S^{\times 1} \vee S^{\times 1} \to S^{\times 2}$ which on the first wedge-summand is the inclusion i_1 and on the second wedge-summand is the diagonal map. This map becomes a weak equivalence under realization as well. If W denotes the set of all these maps, then after localizing at W our Quillen pair descends to give $U_* \mathcal{C}/W \rightleftarrows \mathcal{S}pectra$.

The model category $U_* \mathcal{C}/W$ turns out to be precisely one of the well-known model structures for the category of Γ-spaces: it is the one used by Schwede [Sch], and the identification of the appropriate maps to localize is implicit in that paper. The fibrant objects can be seen to be the 'very special' Γ-spaces (see [Sch, bottom paragraph on Page 349] for an argument). Of course it's still not quite true that $U_* \mathcal{C}/W \rightleftarrows \mathcal{S}pectra$ is a Quillen equivalence: the image of the realization functor consists only of spectra which can be built from finite products of spheres, which up to homotopy are the *connective* spectra. But it's well-known that this is the only problem, and that Γ-spaces model the homotopy theory of connective spectra.

To summarize: if one starts with a 'formal sphere object' S and its finite products $S^{\times k}$, builds the universal pointed homotopy theory determined by these, and imposes the relations

$$S^{\times k} \vee S^{\times l} \xrightarrow{\sim} S^{\times (k+l)}, \qquad sh \colon S^{\times 1} \vee S^{\times 1} \xrightarrow{\sim} S^{\times 2}$$

then one recovers the homotopy theory of connective spectra. In the language of section 6 this is a *presentation* for that homotopy theory.

5.8. **Further applications.** We conclude this section with a result suggesting that model categories of the form $U\mathcal{C}/S$ behave something like the *cofibrant objects* among model categories. For another result along these lines, see Corollary 6.5.

Definition 5.9. *Let $L_1, L_2 \colon \mathcal{M} \to \mathcal{N}$ be two maps between model categories.*

(a) *A **Quillen homotopy** between the maps L_1 and L_2 is a natural transformation $L_1 \to L_2$ with the property that $L_1 X \to L_2 X$ is a weak equivalence whenever X is cofibrant.*

(b) *As expected, two maps are **Quillen-homotopic** if they can be connected by a zig-zag of Quillen homotopies.*

Proposition 5.10. *Let $P \colon \mathcal{M} \to \mathcal{N}$ be a Quillen equivalence of model categories, and let $F \colon U\mathcal{C}/S \to \mathcal{N}$ be any map. Then there is a map $l \colon U\mathcal{C}/S \to \mathcal{M}$ such that the composite Pl is Quillen-homotopic to F.*

Proof. See Section 9.6. □

6. Presentations for model categories

In this section we consider model categories which can be obtained—up to Quillen equivalence—by starting with a universal model category $U\mathcal{C}$ and then localizing at some set of maps S. We refer to these as model categories with *presentations*, since the category $\mathcal{C}$ can be thought of as a category of 'generators', and the set S a collection of 'relations'.

We begin with the basic definition:

Definition 6.1. *Let* $\mathcal{M}$ *be a model category. A* ***small presentation*** *of* $\mathcal{M}$ *consists of the following data:*
(1) a small category $\mathcal{C}$,
(2) a choice of Quillen pair $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$,
(3) a set of maps S *in* $U\mathcal{C}$,
and we require the properties that
(a) The left derived functor of Re *takes maps in* S *to weak equivalences;*
(b) The induced Quillen pair $U\mathcal{C}/S \rightleftarrows \mathcal{M}$ *is a Quillen equivalence.*

One may think of a small presentation as giving 'generators' and 'relations' for the model category $\mathcal{M}$—see (5.5) in the preceding section for some examples. It is not true that every model category will have a small presentation, but many examples of interest do. In fact there is a very large class called the *combinatorial model categories* which have been introduced by Jeff Smith, and such model categories turn out to have small presentations. Combinatorial model categories include essentially any model category of algebraic origin, as well as any model category built-up in some way from simplicial sets. We recall the basic definition:

Definition 6.2. *A model category* $\mathcal{M}$ *is called* ***combinatorial*** *if it is cofibrantly-generated and the underlying category is locally presentable. The latter means that there is a regular cardinal* λ *and a set of objects* $\mathcal{A}$ *in* $\mathcal{M}$ *such that*
(i) Every object in $\mathcal{A}$ *is small with respect to* λ*-filtered colimits, and*
(ii) Every object of $\mathcal{M}$ *can be expressed as a* λ*-filtered colimit of elements of* $\mathcal{A}$.

We can now state the main theorem of this section:

Theorem 6.3. *Any combinatorial model category has a small presentation.*

For background on locally presentable categories we refer the reader to [AR, Section 1.B]. It is a standard result from category theory that any locally presentable category is equivalent to a full, reflective subcategory of a category of diagrams $\mathcal{S}et^{\mathcal{A}}$ (where $\mathcal{A}$ is some small category)—see [AR, Prop. 1.46]. The above theorem is the homotopy-theoretic analog of this result. The reflecting functor corresponds to the fibrant-replacement functor for the localized model category.

The proof of Theorem 6.3 is too involved to give here, but can be found in the companion paper [D3]. Here we can at least note two immediate corollaries. It was proven in [D1] that any left proper, combinatorial model category is Quillen equivalent to a simplicial one. Using the above theorem we can give a completely different proof of this result, and in fact we do slightly better in that we eliminate the left properness assumption:

Corollary 6.4. *Any combinatorial model category is Quillen equivalent to one which is both simplicial and left proper.*

Proof. The point is that the model categories $U\mathcal{C}$ are simplicial and left proper, and these properties are inherited by the localizations $U\mathcal{C}/S$. □

The second corollary is another instance of the 'cofibrant-like' behavior of the model categories $U\mathcal{C}/S$—we offer it mainly as an intriguing curiosity:

Corollary 6.5. *Suppose one has a zig-zag of Quillen equivalences*

$$\mathcal{M}_1 \xrightarrow{\sim} \mathcal{M}_2 \xleftarrow{\sim} \mathcal{M}_3 \xrightarrow{\sim} \cdots \xleftarrow{\sim} \mathcal{M}_n$$

in which $\mathcal{M}_1$ is a combinatorial model category. Then there is a combinatorial model category $\mathcal{N}$ and a simple zig-zag of Quillen equivalences

$$\mathcal{M}_1 \xleftarrow{\sim} \mathcal{N} \xrightarrow{\sim} \mathcal{M}_n.$$

In fact, $\mathcal{N}$ may be taken to be of the form $U\mathcal{C}/S$ where both $\mathcal{C}$ and S are small.

Proof. One simply chooses a presentation $U\mathcal{C}/S \xrightarrow{\sim} M_1$ and then uses Proposition 5.10 to lift this map across the Quillen equivalences. □

7. Applications to sheaf theory

Over the years several people have realized that one can construct model categories which serve as natural settings for 'homotopical' generalizations of sheaf cohomology [BG, Jo, J2]. What we mean is that sheaf cohomology appears in these settings as homotopy classes of maps to certain abelian group objects, but one is allowed to consider maps to non-additive objects as well. This 'homotopical sheaf theory' has been very important in applications to algebraic K-theory [Th, J1], and recently to motivic homotopy theory [MV]. In this section we explain a very direct way for recovering the same homotopy theory via our universal constructions (7.3).

Recall that a Grothendieck site is a small category $\mathcal{C}$ equipped with finite limits, together with a *topology*: a collection of families $\{U_\alpha \to X\}$ called **covering families**, which are required to satisfy various reasonable properties [Ar]. (There is also a more general approach involving *covering sieves*, which we have foregone only for ease of presentation). The prototype for all Grothendieck sites is the category of topological spaces (contained in a certain universe, say) where the covering families are just the usual open covers.

If $f\colon E \to B$ is a map between presheaves, where both E and B are coproducts of representables, one says that f is a **cover** if it has the following property: for any map $rX \to B$, there is a covering family $\{U_\alpha \to X\}$ for which the compositions $rU_\alpha \to rX \to B$ lift through f.

Definition 7.1. *Let $X \in \mathcal{C}$ and suppose that U_* is a simplicial presheaf with a map $U_* \to rX$. This map is called a **hypercover** of X if*

  (i) Each U_n is a coproduct of representables,
  (ii) $U_0 \to rX$ is a cover, and
  (iii) For every integer $n \geq 1$, the component of $U^{\Delta^n} \to U^{\partial \Delta^n}$ in degree 0 is a cover.

This definition is not particularly enlightening, but it's easy to explain. The easiest examples of hypercovers are the Čech covers, which have the form

$$\cdots \coprod U_{\alpha\beta\gamma} \Rrightarrow \coprod U_{\alpha\beta} \rightrightarrows \coprod U_\alpha \longrightarrow X$$

for some chosen covering family $\{U_\alpha \to X\}$. Here $U_{\alpha\beta} = U_\alpha \times_X U_\beta$, etc. The Čech covers are the hypercovers in which the maps $U^{\Delta^n} \to U^{\partial\Delta^n}$ are *isomorphisms* in degree 0. In a general hypercover one takes the iterated fibred-products at each level but then is allowed to refine that object further, by taking a cover of it. We refer the reader to [AM, Section 8] for further discussion of hypercovers.

For the category of topological spaces one has the following very useful property: if $U_* \to X$ is any hypercover of the space X—where in this context we now consider the U_n's as spaces, not representable presheaves—then the natural map

$$\operatorname{hocolim} U_* \to X$$

is a weak equivalence. (This is not that difficult to prove: if one has a homotopy element $\partial\Delta^n \to X$ then by subdividing the domain enough times one can gradually lift the map up through the hypercover). Based on this observation, it is natural to make the following construction for any Grothendieck site:

Definition 7.2. *Suppose $\mathcal{C}$ is a Grothendieck site with topology $\mathcal{T}$. Then $U\mathcal{C}/\mathcal{T}$ denotes the model category obtained by forming the universal model category $U\mathcal{C}$ and then localizing at the set of maps $\{\operatorname{hocolim} U_* \to X\}$, where X runs through all objects of $\mathcal{C}$ and U_* runs through all hypercovers of X.*

In words, we have freely added homotopy colimits to $\mathcal{C}$ and then imposed relations telling us that any object X may be homotopically decomposed by taking covers. Of course sheaf theory is, in the end, precisely this study of how objects decompose in terms of covers.

In [J2] Jardine introduced a model structure on simplicial presheaves $sPre(\mathcal{C})$ in which cofibrations are monomorphisms and weak equivalences are maps inducing isomorphisms on sheaves of homotopy groups. We will denote this model category by $sPre(\mathcal{C})_{Jardine}$. Since one has the obvious functor $r\colon \mathcal{C} \to sPre(\mathcal{C})$ sending an object to the corresponding representable presheaf, our general machinery can be seen to give a map $U\mathcal{C}/\mathcal{T} \to sPre(\mathcal{C})_{Jardine}$ (one must of course check that the maps we are localizing are weak equivalences in Jardine's sense, but this is easy). The essence of the following proposition could almost be considered folklore—a proof can be found in [D2]:

Proposition 7.3. *The above map $U\mathcal{C}/\mathcal{T} \to sPre(\mathcal{C})_{Jardine}$ is a Quillen equivalence.*

Remark 7.4. The model categories $U\mathcal{C}/\mathcal{T}$ and $sPre(\mathcal{C})_{Jardine}$ are of course not that different: they share the same underlying category and (it turns out) the same weak equivalences, but the notions of cofibration and fibration differ. These two different model structures can already be seen at the level of $U\mathcal{C}$, before we localize: in this paper we have consistently used the Bousfield-Kan model structure, in which the fibrations and weak equivalences are detected objectwise, but there is also a **Heller model structure** [He] in which the *cofibrations* and weak equivalences are detected objectwise. The Heller structure doesn't seem to enjoy any kind of universal property, however.

It is sometimes considered more 'natural' to work with simplicial *sheaves* than with simplicial presheaves, although they give rise to the same homotopy theory— this was what Joyal [Jo] originally did, and simplicial sheaves were also used in [MV]. But from the viewpoint of universal model categories simplicial presheaves

are very natural. By working with sheaves one allows oneself to recover any object
as a colimit of the objects in a covering, but if you're doing homotopy theory and
only care about *homotopy* colimits then working in the category of sheaves is not
so important.

For more on the rich subject of 'homotopical' sheaf theory we refer the reader
to the papers of Jardine [J1, J2] together with [Be, D2, Th].

8. Applications to the homotopy theory of schemes

Fix a field k. Morel and Voevodsky [MV] have shown that studying the alge-
braic K-theory and motivic cohomology of smooth k-schemes is part of a much
larger subject which they call the $\mathbb{A}^1$-*homotopy theory* of such schemes. They have
produced various Quillen equivalent model categories representing this homotopy
theory. In this section we describe how their procedures relate to our framework of
universal model categories.

Let Sm_k denote the category of smooth schemes of finite type over k. Let $\mathcal{T}$
be a Grothendieck topology on this category. Morel and Voevodsky consider the
category of simplicial sheaves $s\mathrm{Shv}(Sm_k)$ on this site, with the model structure of
[Jo] in which

 (i) The cofibrations are the monomorphisms,
 (ii) The weak equivalences are maps which induce weak equivalences on all stalks
      (in the case where the site has enough points, which we will assume for con-
      venience), and
 (iii) The fibrations are the maps with the appropriate lifting property.

They then define the associated $\mathbb{A}^1$-local structure as the localization of this model
category with respect to the projections $X \times \mathbb{A}^1 \to X$, for all $X \in Sm_k$. We'll use
the notation $\mathcal{MV}_k$ for this localized model category.

The point we would like to make is that we can recover the same homotopy
theory from our methods for universal constructions, and in fact this is not so far
from what Morel and Voevodsky actually do. Based on what we have learned in
this paper, it is natural to construct a homotopy theory for schemes by taking Sm_k
and expanding it into the universal model category $U(Sm_k)$ by formally adjoining
homotopy colimits. We will then impose two types of relations:

 (i) The homotopy-colimit-type relations coming from the Grothendieck topology,
      as we saw in the previous section, and
 (ii) The relations $X \times \mathbb{A}^1 \xrightarrow{\sim} X$.

Call the resulting model structure $U(Sm_k)_{\mathbb{A}^1}$. The following proposition is essen-
tially routine:

Proposition 8.1. *There is a Quillen equivalence $U(Sm_k)_{\mathbb{A}^1} \xrightarrow{\sim} \mathcal{MV}_k$.*

Before giving the proof we need to recall one useful fact. If $L\colon M \to N$ is a
map between localizable model categories, and S is a set of maps in M, then there
is of course an induced map $M/S \to N/(L^{cof}S)$. The fact we need is that if L
was a Quillen equivalence then the induced map on localizations is also a Quillen
equivalence (see [H] for a proof).

Proof of (8.1). The Yoneda embedding $Sm_k \to \mathcal{MV}_k$ will extend to a map
$U(Sm_k) \to \mathcal{MV}_k$ (and for convenience we choose the extension induced by the

standard cosimplicial resolution, using the fact that $\mathcal{MV}_k$ is a simplicial model cat-
egory). The relations we are imposing in $U(Sm_k)$ clearly hold in $\mathcal{MV}_k$, and so this
map descends to $U(Sm_k)_{\mathbb{A}^1} \to \mathcal{MV}_k$. It's easy to check that the left adjoint is the
sheafification functor and the right adjoint is the inclusion of simplicial sheaves into
simplicial presheaves.

Perhaps the easiest way to see that this is a Quillen equivalence is to factor the
map into two pieces. In fact, to start with let's forget about the $\mathbb{A}^1$-homotopy
relations; the map we're considering factors as follows:

$$U(Sm_k)/\mathcal{T} \rightleftarrows sPre(Sm_k)_{Jardine} \rightleftarrows s\mathrm{Shv}(Sm_k)_{Joyal}.$$

Here $U(Sm_k)/\mathcal{T}$ is the model structure constructed in the last section, and
$s\mathrm{Shv}(Sm_k)_{Joyal}$ is the model structure of [Jo] mentioned above.

The first Quillen pair is an equivalence by Proposition 7.3. That the second is
a Quillen equivalence is essentially [J2, Prop. 2.8]. By the above observation these
also give Quillen equivalences after we localize at the maps $X \times \mathbb{A}^1 \to X$. $\square$

For Grothendieck topologies like the Zariski and Nisnevich topologies one can
get by with a much smaller class of relations than the hypercovers we used above.
In these cases one only has to consider the Čech hypercovers coming from certain
two-fold covers $\{U_1, U_2 \to X\}$. Of course the more manageable the set of relations
is, the better chance one has of understanding the localized model category. More
information about all this can be found in [MV].

Remark 8.2. Here is one simple instance in which the model category $U(Sm_k)_{\mathbb{A}^1}$
is more handy than $\mathcal{MV}_k$. Consider the case where the field is $\mathbb{C}$, so that one has
a functor

$$Sm_k \to \mathcal{T}op, \qquad X \to X(\mathbb{C}).$$

The functor sends a scheme X to the topological space of its complex-valued points.
This map immediately induces a Quillen pair $U(Sm/\mathbb{C}) \rightleftarrows \mathcal{T}op$, and since the
relations that we are localizing hold in $\mathcal{T}op$ the Quillen pair descends to

$$U(Sm/\mathbb{C})_{\mathbb{A}^1} \rightleftarrows \mathcal{T}op.$$

One cannot get a similar Quillen pair when using the Morel-Voevodsky construc-
tion, one only gets an adjoint pair on the homotopy categories—in essence, the
model category $\mathcal{MV}_k$ has too many cofibrations. Having an actual Quillen pair can
be useful, though.

Concerning our construction of the model category $U(Sm_k)_{\mathbb{A}^1}$, the natural ques-
tion is how do we know that we have 'enough' relations to give an interesting
homotopy theory? The prototype for this situation is the case of topological man-
ifolds, in which case these relations really do generate the usual homotopy theory
of topological spaces:

Let $\mathcal{M}an$ denote the category of all topological manifolds which are contained
in $\mathbb{R}^\infty$ (the embedding is not part of the data, it is just a convenient condition to
ensure that we have a *small* subcategory of manifolds which contains everything we
will be interested in). This category has a Grothendieck topology consisting of the
usual open covers. Consider the model category $U(\mathcal{M}an)_{\mathbb{R}}$ obtained by imposing
on $U(\mathcal{M}an)$ the same relations we used in constructing $U(Sm_k)_{\mathbb{A}^1}$ (the analog of
$\mathbb{A}^1$ is the manifold $\mathbb{R}$). Note that the obvious map $\mathcal{M}an \to \mathcal{T}op$ induces a map of
model categories $U(\mathcal{M}an)_{\mathbb{R}} \to \mathcal{T}op$.

Proposition 8.3. *The Quillen pair $U(\mathcal{M}an)_\mathbb{R} \rightleftarrows \mathcal{T}op$ is a Quillen equivalence.*

Proof. We only give a sketch. The reader can also consult [MV, Prop. 3.3.3] for a similar statement.

Consider the subcategory $pt \hookrightarrow \mathcal{M}an$ whose unique object is the one-point manifold. This inclusion induces a Quillen map $U(pt) \to U(\mathcal{M}an)_\mathbb{R}$. The composition

$$s\mathcal{S}et = U(pt) \to U(\mathcal{M}an)_\mathbb{R} \to \mathcal{T}op$$

is the usual realization/singular functor pair, and is therefore a Quillen equivalence. So the homotopy theory of topological spaces is a retract of that of $U(\mathcal{M}an)_\mathbb{R}$, and what we have to show is that $U(\mathcal{M}an)_\mathbb{R}$ doesn't contain anything more. This is where our relations come in, because they are enough to allow us to unravel any manifold into a simplicial set. If M is a manifold we may choose a cover U_α whose elements are homeomorphic to open balls in Euclidean space, hence contractible. For each intersection $U_\alpha \cap U_\beta$ we may do the same, and so on for all the multiple intersections—in this way we build a hypercover for M in which all the open sets are contractible. Relation (i) allows us to express M (the object in $U(\mathcal{M}an)_\mathbb{R}$) as a homotopy colimit of these contractible pieces, and relation (ii) allows us to replace each contractible piece by a point, up to weak equivalence. So we find that any representable object in $U(\mathcal{M}an)_\mathbb{R}$ may be expressed as a homotopy colimit of points, which of course is just the data in a simplicial set. In addition we know that every object of $U(\mathcal{M}an)_\mathbb{R}$ is canonically a homotopy colimit of representables, so it follows that *every* object can be decomposed into just a simplicial set. $\square$

9. The proofs

This section contains the more technical proofs that were deferred in the body of the paper.

9.1. **Section 2: Cofibrant replacement in $U\mathcal{C}$.** Our first goal in this section is to prove Lemma 2.7 and Proposition 2.8. We must show that given a simplicial presheaf F, the construction QF is a cofibrant-replacement for F in $U\mathcal{C}$. We then prove Proposition 2.9, which is the same statement for the construction $\mathcal{Q}F$.

Roughly speaking, a simplicial presheaf F will be said to have 'free degeneracies' if there exist presheaves N_k such that F is isomorphic to the simplicial presheaf

$$\cdots N_2 \amalg (N_1 \amalg N_1 \amalg N_0) \Rrightarrow N_1 \amalg (N_0) \rightrightarrows N_0.$$

Here the terms in parentheses in degree k are called the *degenerate part* of F_k, and the idea is that these degenerate parts are as free as possible. For instance the degenerate part in degree 2 consists of a term corresponding to $s_0(N_1)$, a term corresponding to $s_1(N_1)$, and a term corresponding to $s_1 s_0(N_0) = s_0 s_0(N_0)$, and we are requiring that there be no overlap between these parts. The following gives a precise definition:

Definition 9.2. *A simplicial presheaf F has **free degeneracies** if there exist subpresheaves $N_k \hookrightarrow F_k$ such that the canonical map*

$$\coprod_\sigma N_\sigma \to F_k$$

*is an isomorphism: here the variable σ ranges over all surjective maps in Δ of the form $[k] \to [n]$, N_σ denotes a copy of N_n, and the map $N_\sigma \to F_k$ is the one induced by $\sigma^* \colon F_n \to F_k$. (This is called a **splitting** of F in [AM, Def. 8.1]).*

Lemma 9.3. *If F has free degeneracies then F is the colimit of the maps*

$$\mathrm{sk}_0 F \to \mathrm{sk}_1 F \to \mathrm{sk}_2 F \to \cdots$$

where $\mathrm{sk}_0 F = N_0$ and $\mathrm{sk}_n F$ is defined by a pushout-square

$$
\begin{array}{ccc}
N_n \otimes \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1} F \\
\downarrow & & \vdots \\
N_n \otimes \Delta^n & \dashrightarrow & \mathrm{sk}_n F.
\end{array}
$$

Proof. Left to the reader. $\qquad\square$

Corollary 9.4. *If F has a free degeneracy decomposition in which the N_k are cofibrant in $U\mathcal{C}$, then F is itself cofibrant.*

Proof. The fact that N_k is cofibrant implies that $N_k \otimes \partial\Delta^k \to N_k \otimes \Delta^k$ is a cofibration, and so the map $\mathrm{sk}_{k-1} F \to \mathrm{sk}_k F$ is also a cofibration. Then F is a sequential colimit of cofibrations beginning with $\emptyset \rightarrowtail \mathrm{sk}_0 F$, hence cofibrant. $\qquad\square$

We now prove that if F is a *discrete* simplicial presheaf then $\tilde{Q}F$ is a cofibrant replacement for F:

Proof of Lemma 2.7. First observe that $\tilde{Q}F$ has a free degeneracy decomposition: we take N_k to be the coproduct

$$\coprod_{rX_k \to \cdots rX_0 \to F} (rX_k)$$

in which no map $X_{i+1} \to X_i$ is an identity map. Each N_k is a coproduct of representables, hence cofibrant. So $\tilde{Q}F$ is itself cofibrant by the above corollary.

We must next show that $\tilde{Q}F \to F$ is a weak equivalence in $U\mathcal{C}$—that is, we must show that $(\tilde{Q}F)(X) \to F(X)$ is a weak equivalence of simplicial sets, for every $X \in \mathcal{C}$. Let $\mathcal{A}$ denote the subcategory of $\mathcal{C}$ consisting of the same objects but only identity maps. Consider the adjoint pair

$$T : \mathcal{S}et^{\mathcal{A}^{op}} \rightleftarrows \mathcal{S}et^{\mathcal{C}^{op}} : U,$$

where U is the restriction functor and T is its left adjoint. Then TU is a cotriple, and the cotriple resolution

$$\cdots (TU)^3 F \Rrightarrow (TU)^2 F \rightrightarrows (TU)F \longrightarrow F$$

can be seen to exactly coincide with $\tilde{Q}F$. Now of course if we apply U again then we pick up an extra degeneracy, and the map $U[(TU)^*F] \to UF$ is a weak equivalence in $s\mathcal{S}et^{\mathcal{A}^{op}}$. But applying U to a simplicial presheaf gives precisely the collection of all its values, and so we have that $(\tilde{Q}F)(X) \to F(X)$ is a weak equivalence for every X. $\qquad\square$

Now we move on to handle arbitrary simplicial presheaves:

Proof of Proposition 2.8. One again shows that QF has a free degeneracy decomposition in which the N_k are coproducts of representables. This takes a little more work than for $\tilde{Q}F$, but we will leave it to the reader. The fact that QF is cofibrant follows from Corollary 9.4.

To see that $QF \to F$ is a weak equivalence we consider the bisimplicial object $Q_{**}F$ whose nth row is $\tilde{Q}(F_n)$, as well as the 'constant' bisimplicial object F_{**} whose

nth row is the discrete simplicial presheaf consisting of F_n in every level. The map $QF \to F$ is the diagonal of a map $Q_{**}F \to F_{**}$. But we have already shown that $\tilde{Q}F_n \to F_n$ is a weak equivalence for every n, which says that $Q_{**}F \to F_{**}$ is a weak equivalence on each row. It follows that the map yields a weak equivalence on the diagonal as well.   □

The last thing we must do is prove Proposition 2.9, which concerned a different functor QF—we are to show that this is another cofibrant-replacement functor for $U\mathcal{C}$. The proof is an unpleasant calculation of a homotopy colimit.

Proof of Proposition 2.9. We explained in section (2.6) why QF was cofibrant, therefore the only thing to prove is that the natural map $QF \to F$ is a weak equivalence in $U\mathcal{C}$. So we need to show that for every $X \in \mathcal{C}$ the map $QF(X) \to F(X)$ is a weak equivalence of simplicial sets.

For brevity let I denote the category $(\mathcal{C} \times \Delta \downarrow F)$. The object QF is the homotopy colimit of the diagram $I \to U\mathcal{C}$ which sends $[A \times [n], rA \otimes \Delta^n \to F]$ to $rA \otimes \Delta^n$. Because the simplicial structure in $U\mathcal{C}$ is the objectwise structure, homotopy colimits are also computed objectwise. This says that $QF(X)$ is equal to the homotopy colimit of the diagram $\mathcal{D} \colon I \to s\mathcal{S}et$ sending $[A \times [n], rA \otimes \Delta^n \to F]$ to $(rA \otimes \Delta^n)(X)$. This latter object may be identified with $rA(X) \otimes \Delta^n$, which is $\mathcal{C}(X, A) \otimes \Delta^n$—it is a coproduct of copies of Δ^n, one for each map $X \to A$.

Consider the functor $\Theta \colon I \to \mathcal{S}et$ which sends $[A \times [n], rA \otimes \Delta^n \to F]$ to the set $\mathcal{C}(X, A)$. From this functor we may form its Grothendieck construction $\mathrm{Gr}\,\Theta$: this is the category whose objects are pairs (i, σ) where $i \in I$ and $\sigma \in \Theta(i)$, and a map $(i, \sigma) \to (j, \alpha)$ is a map $i \to j$ in I such that $(\Theta i)(\sigma) = \alpha$. An object of $\mathrm{Gr}\,\Theta$ corresponds to the data $[A \times [n], rA \otimes \Delta^n \to F, X \to A]$, so define a functor $\mathcal{E} \colon \mathrm{Gr}\,\Theta \to s\mathcal{S}et$ which sends this object to the simplicial set Δ^n.

Thomason has a theorem about homotopy colimits over Grothendieck constructions [CS, Cor. 24.6], and in our situation it gives us a weak equivalence

$$\operatorname*{hocolim}_{\mathrm{Gr}\,\Theta} \mathcal{E} \xrightarrow{\sim} \operatorname*{hocolim}_{i \in I} \Big[\operatorname*{hocolim}_{\sigma \in \Theta(i)} \mathcal{E}(i, \sigma) \Big].$$

If $i \in I$ corresponds to the data $[A \times [n], rA \otimes \Delta^n \to F]$, then the homotopy colimit inside the brackets is just a coproduct of copies of Δ^n, one for each element of $\Theta(i) = \mathcal{C}(X, A)$. In this way the double homotopy colimit on the right is readily identified with $\operatorname{hocolim}_I \mathcal{D}$, and we have already seen that this is $QF(X)$.
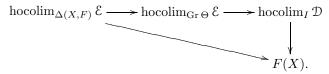
Now consider the category $\Delta(X, F)$, defined so that the objects consist of the data $[[n], rX \otimes \Delta^n \to F]$—this is equal to the *category of simplices* of the simplicial set $F(X)$ (defined in [Ho, p. 75], for instance). We again let $\mathcal{E} \colon \Delta(X, F) \to s\mathcal{S}et$ denote the diagram which sends $[[n], rX \otimes \Delta^n \to F]$ to Δ^n. The colimit of this diagram is just $F(X)$, and the natural map $\operatorname{hocolim}_{\Delta(X,F)} \mathcal{E} \to F(X)$ is a weak equivalence of simplicial sets.

There is a functor $\Delta(X, F) \to \mathrm{Gr}\,\Theta$ which sends

$$[[n], rX \otimes \Delta^n \to F] \mapsto [X \times [n], X \otimes \Delta^n \to F, Id \colon X \to X],$$

and this induces a map of homotopy colimits $\operatorname{hocolim}_{\Delta(X,F)} \mathcal{E} \to \operatorname{hocolim}_{\mathrm{Gr}\,\Theta} \mathcal{E}$. The map of categories has a retraction which is easily checked to be homotopy-cofinal, so it follows that the map of homotopy colimits is a weak equivalence.

All-in-all what we have is the following diagram:

$$\text{hocolim}_{\Delta(X,F)} \, \mathcal{E} \longrightarrow \text{hocolim}_{\text{Gr}\,\Theta} \, \mathcal{E} \longrightarrow \text{hocolim}_I \, \mathcal{D}$$

$$F(X).$$

We have shown that every map is a weak equivalence except the vertical one, but then the vertical map must be one as well. This is the statement that $\mathcal{Q}F(X) \to F(X)$ is a weak equivalence, which was our goal. $\qquad\qquad\qquad\qquad\blacksquare$

9.5. **Section 3: Cosimplicial resolutions and maps from universal model categories.** In this section we prove Proposition 3.4, which said that extending a map $\gamma \colon \mathcal{C} \to \mathcal{M}$ to the universal model category $U\mathcal{C}$ was equivalent to giving a cosimplicial resolution on γ.

To begin with we will need some machinery. If $K \in s\mathcal{S}et$ and $X^* \in c\mathcal{M}$ one can define a tensor product $X \otimes K \in \mathcal{M}$ (see [Ho, Prop. 3.1.5]). Start with some general notation: For a set S and an object $W \in \mathcal{M}$, let $W \cdot S$ denote a coproduct of copies of W, one for each element of S. Then $X \otimes K$ can be defined as a coend:

$$X \otimes K = \text{coeq}\left[\coprod_{[k]\to[m]} X_k \cdot K_m \rightrightarrows \coprod_n X_n \cdot K_n \right].$$

This construction has the adjointness property that

$$\mathcal{M}(X \otimes K, W) \cong s\mathcal{S}et(K, \mathcal{M}(X^*, W))$$

where $\mathcal{M}(X^*, W)$ is the simplicial set whose n-simplices are the hom-set $\mathcal{M}(X^n, W)$.

Now if we have diagrams $\Gamma \colon \mathcal{C} \to c\mathcal{M}$ and $F \colon \mathcal{C}^{op} \to s\mathcal{S}et$ then we can again form a coend

$$\Gamma \otimes_{\mathcal{C}} F = \text{coeq}\left[\coprod_{a\to b} \Gamma(a) \otimes F(b) \rightrightarrows \coprod_{c\in\mathcal{C}} \Gamma(c) \otimes F(c) \right].$$

For this construction we have that

(9.5.1) $$\mathcal{M}(\Gamma \otimes_{\mathcal{C}} F, W) = s\mathcal{S}et^{\mathcal{C}^{op}}(F, \mathcal{M}(\Gamma, W))$$

where $\mathcal{M}(\Gamma, W)$ is the simplicial presheaf defined by $c \mapsto \mathcal{M}(\Gamma^*c, W)$.

The above is all that's necessary to prove our result:

Proof of Proposition 3.4. Suppose we have a factorization of $\gamma \colon \mathcal{C} \to \mathcal{M}$ through $U\mathcal{C}$: so we have a Quillen pair $Re : U\mathcal{C} \rightleftarrows \mathcal{M} : Sing$ and a natural weak equivalence $Re(rX) \xrightarrow{\sim} \gamma(X)$. Then for each $X \in \mathcal{C}$ we get a cosimplicial resolution of γX by taking $\Gamma(X)$ to be

$$[n] \mapsto Re(rX \otimes \Delta^n).$$

This is clearly functorial in X, and so gives a resolution of γ.

Conversely, suppose we start with a resolution $\Gamma \colon \mathcal{C} \to c\mathcal{M}$ for the functor γ. Define the functors $Re : U\mathcal{C} \to \mathcal{M}$ and $Sing : \mathcal{M} \to U\mathcal{C}$ by the formulas

$$Re(F) = \Gamma \otimes_{\mathcal{C}} F, \qquad Sing(X) = [c \mapsto \mathcal{M}(\Gamma^*(c), X)].$$

(9.5.1) says that these are an adjoint pair.

To see that these are a Quillen pair we will check that $Sing$ preserves fibrations and trivial fibrations. For this we need to know that if A^* is a cosimplicial resolution

and $X \to Y$ is a fibration (resp. trivial fibration) then $\mathcal{M}(A^*, X) \to \mathcal{M}(A^*, Y)$ is a fibration (resp. trivial fibration) of simplicial sets. But this is [Ho, Cor. 5.1.4].

The last thing is to give a natural weak equivalence $Re(rX) \to \gamma(X)$. But $Re(rX)$ is isomorphic to the object of $\Gamma(X)$ in degree 0, and our cosimplicial resolution came with a weak equivalence from this object to $\gamma(X)$. So we're done.

Checking the equivalence of categories $\mathrm{Fact}(\gamma) \simeq \mathrm{coRes}(\gamma)$ is fairly routine at this point: we have given the functors in either direction.   □

9.6. **Section 5: Lifting maps from the model categories $U\mathcal{C}/S$.** Here we fill in the proof of Proposition 5.10. We must show that a map from a model category $U\mathcal{C}/S$ may be lifted, up to homotopy, across a Quillen equivalence.

It will be useful to isolate the following lemma:

Lemma 9.7. *Let $\mathcal{M}$ be a model category, and let $\gamma_1, \gamma_2 \colon \mathcal{C} \to \mathcal{M}$ be two functors whose images lie in the cofibrant objects. Suppose there is a natural weak equivalence $\gamma_1 \xrightarrow{\sim} \gamma_2$. Then any two extensions $L_1, L_2 \colon U\mathcal{C} \to \mathcal{M}$ of γ_1 and γ_2 are Quillen-homotopic.*

Proof. Recall that there exists an equivalence between maps of model categories $U\mathcal{C} \to \mathcal{M}$ and the following data:

- A functor $\gamma \colon \mathcal{C} \to \mathcal{M}$ whose image lies in the cofibrant objects, and
- A cosimplicial resolution on γ.

Giving a Quillen homotopy between two maps $L_1, L_2 \colon U\mathcal{C} \to \mathcal{M}$ corresponds to giving a natural weak equivalence $\gamma_1 \to \gamma_2$ and a lifting of this to a natural weak equivalence between the resolutions. Using these facts, proving the lemma is just a matter of getting zig-zags between the resolutions. But this is standard—see [H].   □

Proof of Proposition 5.10. Let $\tilde{F}$ be the composite $U\mathcal{C} \to U\mathcal{C}/S \to \mathcal{N}$. We will begin by lifting $\tilde{F}$, and this can be accomplished just by lifting $\gamma \colon \mathcal{C} \to U\mathcal{C} \to \mathcal{N}$.

Define $\epsilon : \mathcal{C} \to \mathcal{M}$ by

$$\epsilon(X) = [Q^{fib}(\gamma(X))]^{cof}$$

where Q is the right-adjoint to P. We may extend ϵ to a map $\tilde{l} : U\mathcal{C} \to \mathcal{M}$.

Claim 1: The composite $P\tilde{l}$ is Quillen-homotopic to $\tilde{F}$.

To see this, observe that there are natural weak equivalences

$$P\epsilon(X) \xrightarrow{\sim} [\gamma X]^{fib} \xleftarrow{\sim} \gamma(X).$$

Since $P\tilde{l}$ is an extension of $P\epsilon$ and F is an extension of γ, the claim follows directly from the above lemma.

Claim 2: The map $\tilde{l}$ takes elements of S to weak equivalences.

For this, note that by hypothesis the derived functor of $\tilde{F}$ takes elements of S to weak equivalences. The same must be true for $P\tilde{l}$, since $P\tilde{l}$ is homotopic to $\tilde{F}$ (maps which are Quillen-homotopic will have isomorphic derived functors on the homotopy categories). But P was a Quillen equivalence, and so the derived functor of $\tilde{l}$ must also take elements of S to weak equivalences.

From Claim 2 it follows that $\tilde{l}$ descends to a map $l : U\mathcal{C}/S \to \mathcal{M}$. The fact that Pl is homotopic to F is just a restatement of Claim 1.   □

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