On a Poisson structure
on the space of Stokes matrices

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Abstract: In this paper we study the map associating to a linear differential operator with rational coefficients its monodromy data. The operator is of the form $\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$, with one regular and one irregular singularity of Poincaré rank 1, where U is a diagonal and V is a skewsymmetric $n \times n$ matrix. We compute the Poisson structure of the corresponding Monodromy Preserving Deformation Equations (MPDE) on the space of the monodromy data.

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0. Introduction

Monodromy preserving deformation equations (MPDE) of linear differential operators with rational coefficients are known since the beginning of the century [Fu, Schl, G]. Particularly, the famous six Painlevé equations are known [G] to be of this type. MPDE were included in the framework of the general theory of integrable systems much later, at the end of 70s [ARS, FN1, JMU]; see also [IN]). Many authors were inspired by the parallelism between the technique of soliton theory based on the spectral transform and that of the MPDE theory based on the monodromy transform. Another issue of this parallelism between soliton equations and MPDE is that, in both cases, one deals with certain classes of Hamiltonian systems, namely, with infinite-dimensional Hamiltonian structures of evolutionary equations and of their finite-dimensional invariant submanifolds in soliton theory, and with remarkable finite-dimensional time-dependent Hamiltonian systems in the MPDE theory.

Recall that one of the first steps in soliton theory was understanding of the Hamiltonian nature of the spectral transform as the transformation of the Hamiltonian system to the action-angle variables [ZF]. Further development of these ideas was very important for development of the Hamiltonian approach to the theory of solitons [FT] and for the creation of a quantum version of this theory.

In the general theory of MPDE it remains essentially an open question to understand the Hamiltonian nature of the monodromy transform, i.e., of the map associating the monodromy data to the linear differential operator with rational coefficients. This question was formulated in [FN2] and solved in an example of a MPDE of a particular second order linear differential operator. However, the general algebraic properties of the arising class of Poisson brackets on the spaces of monodromy data remained unclear. The technique of [FN2] seems not to work for more general case. The authors of the papers [AM, FR, KS, Hi] consider the important case of MPDE of Fuchsian systems in a more general setting of symplectic structures on the moduli space of flat connections (see, e.g., [A]) not writing, however, the Poisson bracket on the space of monodromy data in a closed form. MPDE of non-Fuchsian operators and Poisson structure on their monodromy data were not considered in these papers.

In the present paper we solve the problem of computing the Poisson structure of MPDE in the monodromy data coordinates for one particular example of the operators with one regular and one irregular singularity of Poincaré rank 1

$$\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$$

where U is a diagonal matrix with pairwise distinct entries and V is a skewsymmetric $n \times n$ matrix. Recently MPDE of this operators proved to play a fundamental role in the theory of Frobenius manifolds [D, D1]. The Poisson structure of MPDE for the operator Λ coincides with the standard linear Poisson bracket on the Lie algebra $\mathfrak{so}(n) \ni V$. The most important part of the monodromy data is the Stokes matrix (see the definition below). This is an upper triangular matrix $S = S(V, U)$ with all diagonal entries being equal to 1. Generically S determines other parts of the monodromy data. It turns out that, although the monodromy map

$$V \mapsto S$$

is given by complicated transcendental functions, the Poisson bracket on the space of Stokes matrices is given by very simple degree two polynomials (see formula (3.2) below). The resulting Poisson bracket does not depend on U since is involved in the Hamiltonian description of the isomonodromy deformations of the operator $\Lambda(z)$. The technique of [KS] was important in the derivation of this main result of the present paper.

We hope that this interesting new class of polynomial Poisson brackets and their quantization (cf. [R, Ha2]) deserves a further investigation that we are going to continue in subsequent publications.

The paper is organized as follows: after recalling some basic notations, in section 1.1 we describe the monodromy of the operator $\Lambda(z)$ around the two singular points; in section 1.2 we present the MPDE for this operator. In section 2.1 and 2.2 we describe the related Fuchsian system and its MPDE; in section 2.3 the Poisson structure on the space of monodromy data of the Fuchsian system is described. In section 3 we give the relation between the monodromy data of the two systems and we explicitly calculate the Poisson bracket on the space of the Stokes matrices.

Let us consider in the complex domain a differential equation with rational coefficients

$$\frac{dy}{dz} = A(z)y(z) \tag{0.1}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{pmatrix}, \qquad A(z) = \begin{pmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & .. & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & .. & .. & .. & a_{nn} \end{pmatrix}$$

An arbitrary solution $y(z)$ of (0.1) is locally holomorphic but globally multivalued; the poles of $A(z)$ are singularities of the solution. Fixing a basis $y^{(1)}, \ldots, y^{(n)}$ in the n-dimensional space of solutions we construct the *fundamental* $n \times n$ matrix

$$Y(z) = (y^{(1)}, \ldots, y^{(n)})$$

satisfying the matrix version of (0.1)

$$\frac{dY(z)}{dz} = A(z)Y(z). \tag{0.2}$$

1. Systems with irregular singularity

1.1 Stokes phenomenon

In this paper we will concentrate our attention on the linear systems

$$\frac{dY}{dz} = (U + \frac{V}{z})Y, \qquad z \in \mathbb{C}, \tag{1.1}$$

where U is a diagonal $n \times n$ matrix with distinct entries $u_1, u_2, \ldots, u_n$ and $V = (v_{ij}) \in \mathfrak{so}(n,\mathbb{C})$, with nonresonant eigenvalues $(\mu_1, \mu_2, \ldots, \mu_n)$ (i.e. $\mu_i - \mu_j \notin Z \setminus 0$). The solutions of the system (1.1) have two singular points, 0 and ∞.

• Near the point $z = 0$ a fundamental matrix of solutions $Y_0(z)$ exists such that

$$Y_0(z) = W(z)z^\theta = [W_0 + W_1 z, \ldots]z^\theta, \tag{1.2}$$

where θ is the diagonalization of V, $\theta = W_0^{-1}VW_0 = diag(\mu_1, \mu_2, \ldots, \mu_n)$, and $W(z)$ converges for small $|z|$. Such kind of singularities is called *Fuchsian*.

If one continues $Y_0(z)$ along a path encircling the point $z = 0$, the columns of the resulting matrix are linear combinations of the columns of $Y_0(z)$; hence there exists a matrix M_0 such that

$$Y_0(z) \mapsto Y_0(z)M_0.$$

The matrix M_0 is called *monodromy* matrix around zero. In our case $M_0 = exp(2\pi i\theta)$.

• At ∞ the solution has an *irregular* singularity of Poincaré rank 1. This means that it is possible to construct a formal series

$$\Gamma(z) = \mathbb{1} + \frac{\Gamma_1}{z} + \frac{\Gamma_2}{z^2} + \ldots$$

where $V = [\Gamma_1, U]+$ *diagonal*, i.e. $\Gamma_1 = (\gamma_{ij}) = (\frac{v_{ij}}{u_j-u_i})$ for $i \neq j$, and to define certain sectors $\mathfrak{S}_i$ in which a fundamental matrix of solutions Y_i exists with asymptotic behavior

$$Y_i \sim Y_\infty = \Gamma(z)e^{zU}, \tag{1.3}$$

for $|z| \to \infty$ in $\mathfrak{S}_i$. This means that $\Gamma(z)$ is the asymptotic expansion of $Y_i e^{-zU}$.

In different sectors one has different solutions, and this fact is known as *Stokes phenomenon*. The matrices connecting the solutions in different sectors are called *Stokes matrices*.

A complete and detailed description of the phenomenon can be found in [BJL1], [Si], [IN],[U]; here we will concentrate our attention on the particular operator $\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$ (see also [D]).

Following [D] we define an *admissible* line for the system (1.1) as a line l through the origin on the z-plane such that

$$\mathrm{Re}\, z(u_i - u_j)|_{z \in l} \neq 0 \qquad \forall i \neq j.$$

We denote the half-lines

$$l_+ = z : \arg z = \psi \qquad l_- = z : \arg z = \psi - \pi,$$

where ψ is a fixed real value of the argument.

The line l lies in the intersection $\mathfrak{S}_+ \cup \mathfrak{S}_-$ of the two sectors

$$\mathfrak{S}_R : \ \psi - \pi - \epsilon < \arg z < \psi + \epsilon$$

and

$$\mathfrak{S}_L : \ \psi - \epsilon < \arg z < \psi + \pi + \epsilon.$$

Here ϵ is a sufficiently small positive number.

Theorem 1.1 : *There exists a unique solution $Y_L(z)$ analytic in the sector $\mathfrak{S}_L$ with the asymptotic behavior*

$$Y_L(z) \sim Y_\infty;$$

the same holds for $Y_R(z)$ in $\mathfrak{S}_L$.

Proof: See [BJL1].

S_+ and S_- are the Stokes matrices connecting the two solutions in $\mathfrak{S}_+$, resp. in $\mathfrak{S}_-$, i.e.

$$Y_L(z) = Y_R(z)S_+, \qquad z \in \mathfrak{S}_+$$

and

$$Y_L(z) = Y_R(z)S_-, \qquad z \in \mathfrak{S}_-.$$

From the skew-symmetry $V^T = -V$ it follows

$$S_- = S_+^T.$$

Moreover, one can prove that, given an admissible line, it is possible to order the elements u_i, i.e., to perform a conjugation

$$\Lambda(z) \mapsto P^{-1}\Lambda(z)P,$$

where P is the matrix of the permutation in such a way that the Stokes matrix $S \equiv S_+$ is upper triangular.

Remark: The full set of monodromy data for the operator Λ consists of the Stokes matrix S but also of the monodromy matrix at the point 0 and of the matrix C connecting the solution (1.2) near zero with a solution near the infinity:

$$Y_0(z) = Y_L(z)C.$$

The monodromy data $\{S, M_0, C\}$ satisfy certain constraints described in [D1]. Particularly,

$$C^{-1}S^{T^{-1}}S\,C = M_0.$$

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So, in the generic case (i.e., the diagonalizable and nonresonant one) under consideration the diagonal entries of M_0 $e^{2\pi i \mu_1}$, ..., $e^{2\pi i \mu_n}$ are the eigenvalues of $S^{T^{-1}}S$ and C is the diagonalizing transformation for this matrix. The ambiguity in the choice of the diagonalizing transformation does not affect the operator Λ. So, the $\frac{n(n-1)}{2}$ entries of the Stokes matrix $S = S(V; U)$ can serve as local coordinates near a generic point of the space of monodromy data of the operator Λ (see details in [D], [D1]).

1.2 Monodromy Preserving Deformation Equations

MPDE describe how should the matrix V be deformed, as a function of the "coordinates" u_i, in order to preserve the monodromy data. MPDE are the analogue of the isospectral equations in soliton theory. The MPDE for the operator $\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$ are obtained (see [U], [D]) as compatibility equations of the system (1.1) with the system

$$\frac{\partial Y}{\partial u_i} = (zE_i - V_i)Y,$$

where $V_i = [\Gamma_1, E_i] = ad_{E_i} ad_U^{-1} V$ and $(E_i)_b^a = \delta_i^a \delta_b^i$. These equations admit the Lax form

$$\frac{\partial V}{\partial u_i} = [V, V_i]. \tag{1.4}$$

In the generic case (see Remark above) the solution $V = V(U)$ of the MPDE can be locally written in implicit form

$$S(V; U) = S \tag{1.5}$$

for a given constant Stokes matrix S. In other words, the entries of the Stokes matrix can serve as a complete system of first integrals of the MPDE (1.4). To explicitly resolve the system (1.5) one has to solve an appropriate Riemann–Hilbert boundary value problem. Although this can be explicitly done in a very few cases, one can extract certain important information regarding the analytic properties of the solution; see more detailed discussions of these properties in [IN], [JM], [JMU], [Si].

One can write the MPDE as a Hamiltonian system on the space of the skewsymmetric matrices V with the standard linear Poisson bracket for $V = (v_{ab}) \in \mathfrak{so}(n)$:

$$\{v_{ab}, v_{cd}\} = v_{ad}\delta_{bc} + v_{bc}\delta_{ad} - v_{bd}\delta_{ac} - v_{ac}\delta_{bd}. \tag{1.6}$$

Indeed, the Lax equation (1.4) can be rewritten as

$$\frac{\partial V}{\partial u_i} = \{V, H_i(V, u)\},$$

for the Hamiltonian function

$$H_i = \frac{1}{2} \sum_{j \neq i} \frac{v_{ij}^2}{u_i - u_j}. \tag{1.7}$$

In this case, the Poisson bracket is linear but the dynamic of the problem is very complicated; in the following we will show how, very much as in the case of isospectral equations, it is possible to find a different coordinate system (the entries of the Stokes matrix) in which the dynamic of the evolution is trivial, but the Poisson structure is quadratic. The technique developed here consists in building up the monodromy map $V \to S$ passing through an auxiliary Fuchsian system. The MPDE for the system (1.1) can be represented also as MPDE for an appropriate Fuchsian system

$$\frac{d\chi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} \chi,$$

which we shall describe in the next section. The basic idea to construct the Poisson bracket on the space of Stokes matrices is to include the map from $V \in \mathfrak{so}(n)$ to $S \in \mathcal{S}$ into the following commutative diagram of Poisson maps

$$
\begin{array}{ccc}
\mathfrak{so}(n) & \longrightarrow & \mathcal{S} \\
\downarrow & & \downarrow \\
\mathcal{A}/\mathcal{G} & \longrightarrow & \mathfrak{M}/GL(n,\mathbb{C})
\end{array}
\tag{1.8}
$$

where $\mathcal{A}/\mathcal{G}$ is the space of residues $\{A_i\}$ of the connection $A = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} d\lambda$ modulo the action of the gauge group $\mathcal{G}$, as we will explain in section 2.2, and $\mathfrak{M}/GL(n,\mathbb{C})$ is the space of the monodromy data of the Fuchsian system (section 2.3), i.e. the space of n-dimensional representations of the free group with n generators.

2. Related Fuchsian system

2.1 Fuchsian system

One can relate the system (1.1), with one regular and one irregular singularity to a system with $n+1$ Fuchsian singularities:

$$
\frac{d\Phi}{d\lambda} = \sum_{i=1}^{n} \frac{B_i}{\lambda - u_i} \Phi,
\tag{2.1}
$$

where

$$
B_i = -E_i \left(V + \frac{1}{2}\mathbb{1} \right), \quad i = 1, \dots, n
$$

and

$$
B_\infty = V + \frac{1}{2}\mathbb{1}.
$$

Such a relation is well known in the domain of differential equations, see, e.g. [BJL], [Sch].

Now we will briefly describe the monodromy data of the system (2.1).

In this case u_j is a Fuchsian singular points and, as in (1.2), the general solution near u_j can be expressed as

$$
\Phi_j(\lambda) = W^{(j)}(\lambda)(\lambda - u_j)^{\hat{B}_j},
$$

where $W^{(j)}(\lambda) = W_0^{(j)} + (\lambda - u_j)W_1^{(j)} + \dots$ converges for small $|\lambda - u_j|$ and $\hat{B}_j = -\frac{1}{2}E_j$ is the diagonalization of B_j.

We denote M_j the monodromy matrix along the path γ_j encircling the point u_j w.r.t. the basis Φ_∞ we define in (2.2) below. The matrix M_j is conjugated with the matrix $exp(2\pi i \hat{B}_j)$.

Also the point ∞ is Fuchsian; the general solution can be expressed as

$$
\Phi_\infty(\lambda) = W^{(\infty)}(\lambda)\left(\frac{1}{\lambda}\right)^{\hat{B}_\infty},
\tag{2.2}
$$

where $W^{(\infty)}(\lambda) = W_0^{(\infty)} + \frac{W_1^{(\infty)}}{\lambda} + \dots$ converges at $|\lambda| \to \infty$ and $\hat{B}_\infty = diag(\frac{1}{2} + \mu_1, \dots, \frac{1}{2} + \mu_n)$ is the diagonalization of B_∞. Indeed, the following relation holds in the space of the residues :

$$
-\sum_{i=1}^{n} B_i = B_\infty = \frac{1}{2}\mathbb{1} + V.
$$

In this basis the monodromy matrix $M_\infty = -e^{2\pi i \theta}$. We assume that the loops $\gamma_1, \ldots, \gamma_n$ and γ_∞ are chosen in such a way that

$$M_1 M_2 \ldots M_n M_\infty = 1. \tag{2.3}$$

2.2 Monodromy Preserving Deformation equations

We now want to deduce the MPDE for the system (2.1). This amounts to find how can the matrix B_j be deformed as function of $u_1, u_2, \ldots, u_n$ in order to preserve the monodromy matrices $M_1, \ldots, M_n, M_\infty$. The answer is given by

Theorem 2.1 (Schlesinger): *If the fundamental solution near infinity is normalized as in (2.2) and A_∞ is a constant diagonal matrix with nonresonant elements, then the dependence of the A_j on the position of the poles of the Fuchsian system*

$$\frac{d\Phi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} \Phi$$

is given, in order to preserve the monodromy, by

$$\begin{cases} \frac{\partial A_i}{\partial u_j} = \frac{1}{u_i - u_j}[A_i, A_j] & i \neq j \\ \frac{\partial A_j}{\partial u_j} = -\sum_{i \neq j} \frac{[A_i, A_j]}{u_i - u_j}, \end{cases}$$

Proof: it can be found in [Si].

Note that system (2.1) does not satisfy the hypotheses of the Schlesinger theorem, because $B_\infty = \left(V + \frac{1}{2}\mathbb{1}\right)$ is not diagonal.

In order to apply the Schlesinger theorem it is sufficient to perform the gauge transformation

$$B_i \mapsto A_i = W_0^{-1} B_i W_0, \tag{2.4}$$

where W_0 is the matrix of eigenvectors of V normalized in such a way that

$$\frac{\partial W_0}{\partial u_i} = \mathrm{ad}_{\mathrm{E}_i}\mathrm{ad}_{\mathrm{U}}^{-1}\mathrm{V}. \tag{2.5}$$

Indeed, substituting $\Phi = W_0 \chi$, the system (2.1) transforms into

$$\frac{d\chi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} \chi \tag{2.6}$$

and the Schlesinger system follows from the compatibility of (2.6) with

$$\frac{\partial \chi}{\partial u_i} = -\frac{A_i}{\lambda - u_i} \chi.$$

(See [D]).

The Schlesinger system can be rewritten in the Hamiltonian form

$$\frac{dA_i}{du_j} = \{A_i, \mathfrak{H}_j\}$$

with the Hamiltonians

$$\mathfrak{H}_j = -\sum_{k \neq j} \frac{\mathrm{Tr}(A_j A_k)}{u_j - u_k}$$

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w.r.t. the linear Poisson bracket

$$\{(A_i)_b^a, (A_j)_d^c\} = \delta_{ij}\big(\delta_d^a(A_i)_b^c - \delta_b^c(A_j)_d^a\big). \tag{2.7}$$

This corresponds to taking, for every u_i, the residue $A_i \in \mathfrak{gl}(n,\mathbb{C})$ with the natural Poisson bracket on $\mathfrak{gl}(n,\mathbb{C})$. The residues relative to different singular points commute. In other words (see [KS],[FR],[A]) this corresponds to read the matrices A_i as residues of a flat connection (with values in the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$) on the Riemann surface with $n+1$ punctures:

$$A = \sum_{i=0}^{n} \frac{A_i}{\lambda - u_i} d\lambda$$

(in our case $u_0 = \infty$). On the space of flat connections modulo gauge transformations it is defined the Poisson bracket

$$\{A^a(\mu), A^b(\nu)\} = -f_c^{ab} \frac{A^c(\mu) - A^c(\nu)}{\mu - \nu},$$

where f_c^{ab} are the structure constants of $\mathfrak{g}$ w.r.t. the basis $\{e_a\}$ and

$$A^a(\mu) = \sum_{i=0}^{n} \frac{A_i^a}{\mu - u_i}, \qquad A(\mu) = A^a(\mu)e_a.$$

This Poisson bracket gives (2.6).

Now we can perform the first step in the construction of the map between V and S, that is we pass from $\mathfrak{so}(n)$ to the space $\mathcal{A}/\mathcal{G}$, where

$$\mathcal{A} = \big\{V, A_1, \ldots, A_n \big| \sum_{i=0}^{n} A_i = 0\big\}$$

is the family of the residues of $A(\lambda)$ and $\mathcal{G}$ is the gauge group.

Lemma 2.1 : *The map $V \in \mathfrak{so}(n) \mapsto (V, A_1, \ldots, A_n) \in \mathcal{A}/\mathcal{G}$ is a Poisson map.* (Cf. [Ha1],[Hi])

Proof: We must compare the Poisson brackets on the two spaces. In $\mathfrak{so}(n)$ one has the natural coordinates $\{v_{ab}\}$, with the Poisson bracket (1.6). The natural coordinates in the quotient space $\mathcal{A}/\mathcal{G}$ are the traces of the products of the matrices A_i, so that we consider the brackets

$$\begin{aligned}
\{\mathrm{Tr}(A_iA_k), \mathrm{Tr}(A_jA_l)\} &= \{(A_i)_b^a(A_k)_a^b, (A_j)_d^c(A_l)_c^d\} = \\
&= (A_i)_b^a(A_j)_d^c\{(A_k)_a^b, (A_l)_c^d\} + (A_i)_b^a(A_l)_c^d\{(A_k)_a^b, (A_j)_d^c\} + \\
&+ (A_k)_a^b(A_j)_d^c\{(A_i)_b^a, (A_l)_c^d\} + (A_k)_a^b(A_l)_c^d\{(A_i)_b^a, (A_j)_d^c\}
\end{aligned} \tag{2.8a}$$

and

$$\begin{aligned}
\{\mathrm{Tr}(A_iV), \mathrm{Tr}(A_jV)\} &= \{(A_i)_b^aV_a^b, (A_j)_d^cV_c^d\} = \\
&= (A_i)_b^a(A_j)_d^c\{V_a^b, V_c^d\} + V_a^bV_c^d\{(A_i)_b^a, (A_j)_d^c\}.
\end{aligned} \tag{2.8b}$$

On $\mathcal{A}/\mathcal{G}$ by direct calculation, using the bracket (2.7), one obtains

$$\begin{aligned}
\{\mathrm{Tr}(A_iA_k), \mathrm{Tr}(A_jA_l)\} &= \delta_{kl}\mathrm{Tr}\big(A_iA_jA_k - A_kA_jA_i\big) + \delta_{kj}\mathrm{Tr}\big(A_iA_lA_k - A_kA_lA_i\big) + \\
&+ \delta_{il}\mathrm{Tr}\big(A_kA_jA_i - A_iA_jA_k\big) + \delta_{ij}\mathrm{Tr}\big(A_kA_lA_i - A_iA_lA_k\big) = \\
&= 2\Big(\delta_{kl}\mathrm{Tr}\big(A_iA_jA_k\big) + \delta_{kj}\mathrm{Tr}\big(A_iA_lA_k\big) + \delta_{il}\mathrm{Tr}\big(A_kA_jA_i\big) + \delta_{ij}\mathrm{Tr}\big(A_kA_lA_i\big)\Big).(2.9)
\end{aligned}$$

Indeed, $A_i = -E_i\left(V + \frac{1}{2}\mathbb{1}\right)$ implies

$$\text{Tr}\left(A_i A_j A_k\right) = -\text{Tr}\left(A_k A_j A_i\right) = v_{ij} v_{jk} v_{ki}. \tag{2.10}$$

On the other hand, $\text{Tr}(A_i A_j) = -v_{ij}^2$, hence

$$
\begin{aligned}
\{\text{Tr}(A_i A_k), \text{Tr}(A_j A_l)\} &= 4 v_{ik} v_{jl} \{v_{ik}, v_{jl}\} \\
&= 4\left(\delta_{kl} v_{ij} v_{jk} v_{ki} - \delta_{kj} v_{ik} v_{kl} v_{li} + \delta_{il} v_{ik} v_{kj} v_{ji} + \delta_{ij} v_{ik} v_{kl} v_{li}\right) \tag{2.11}
\end{aligned}
$$

where we have used the bracket (1.6). By means of (2.10) it is easy to check that it coincides with (2.9).

The same can be done for equation (2.8b). Indeed, using the bracket on the A_i matrices and observing that

$$\text{Tr}\left(A_i A_j V\right) = -\text{Tr}\left(V A_j A_i\right) = \sum_{k \neq i \neq j} v_{ij} v_{ki} v_{jk},$$

one finds

$$\{\text{Tr}(A_i V), \text{Tr}(A_j V)\} = 2\text{Tr}(A_i A_j V).$$

On the other hand $\text{Tr}(A_i V) = -\sum_{k \neq i} v_{ki}^2$, that gives

$$\{\text{Tr}(A_i V), \text{Tr}(A_j V)\} = 4 \sum_{k \neq i} \sum_{l \neq j} v_{ki} v_{lj} \{v_{ki}, v_{lj}\} = -4 \sum_{k \neq i} v_{ki} v_{kj} v_{ij}$$

which coincides with (2.8b).   Q.E.D

Lemma 2.2 : *The MPDE for the system (1.1) and its related Fuchsian system coincide.*

Proof: It follows immediately from Lemma 2.1 by a straightforward calculation using (2.5), that MPDE for the Fuchsian system (2.6) after the gauge transformation (2.4) coincide with (1.4). Actually, one can see that the pull back of the Hamiltonian

$$\mathfrak{H}_j = -\sum_{k \neq j} \frac{\text{Tr}(A_j A_k)}{u_j - u_k} = -\sum_{k \neq j} \frac{\text{Tr}(B_j B_k)}{u_j - u_k}$$

is exactly equal to H_j, as defined in (1.7).

2.3 Poisson structure on monodromy data

In this section we will perform the second step of our construction, that is we will map the Poisson structure of $\mathcal{A}/\mathcal{G}$ into the space of monodromy data of the Fuchsian system; this is shown in the following well-known (see, e.g., [Hi])

Theorem 2.2: *The monodromy map*

$$\mathcal{A}/\mathcal{G} \to \mathfrak{M}/SL(n,\mathbb{C})$$

where $\mathfrak{M} = \left\{M_0, M_1, \ldots, M_n | M_1 M_2 \ldots M_n M_0 = \mathbb{1}\right\}$, is a Poisson map.

To actually compute the Poisson bracket on the space of monodromy data, i.e., on the space of n-dimensional representations of the free group with n generators we will use, following [KS] (Th. 4.2), the following technique. We construct the skewsymmetric bracket

$$\left\{(M_i)_b^a, (M_j)_d^c\right\} = i\pi \left((M_j M_i)_b^c \delta_d^a + (M_i M_j)_d^a \delta_b^c - (M_i)_b^c (M_j)_d^a - (M_j)_b^c (M_i)_d^a\right) \qquad i < j \quad (2.12a)$$

$$\left\{(M_i)_b^a, (M_i)_d^c\right\} = i\pi \left((M_i^2)_b^c \delta_d^a - (M_i^2)_d^a \delta_b^c\right). \tag{2.12b}$$

on the space $\mathfrak{M}$ of the monodromy matrices. As it was proved in [KS], when restricted to the space of representations $\mathfrak{M}/SL(n,\mathbb{C})$, this bracket defines a Poisson structure on the quotient induced by the monodromy map. Observe that the eigenvalues of the matrices M_i are the Casimirs of the Poisson bracket, i.e., the functions Poisson commuting with all others (see [KS]).

3. Poisson structure on the Stokes matrices

3.1 Connecting the monodromy data of the two systems

In the previous section we have seen that the space of monodromy data of a Fuchsian system carries a natural Poisson stucture. In this section we will show that this structure induces a Poisson bracket on the space of Stokes matrices of the related system we studied in chapter 1. To this end we consider the relation between the monodromy matrices $M_1, M_2, \ldots, M_n$ of the Fuchsian system and the Stokes matrix S.

In section 2.1 we claimed that the two systems

$$\frac{dY}{dz} = (U + \frac{V}{z})Y$$

and

$$\frac{d\Phi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} \Phi$$

are related, in the sense that, (see Lemma 2.3), the MPDE for the operator $\Lambda(z) = \frac{d}{dz} - U - \frac{V}{z}$ can be represented also as MPDE for the operator $A(\lambda) = \frac{d}{d\lambda} - \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}$.

For a detailed analysis of the transform connecting the two system see [D1]; here we will concentrate our attention on the relation between the monodromy data of the two systems.
Following Theorem 2.2, we are interested in the quotient of the space of the monodromy data of the Fuchsian system w.r.t. the $GL(n,\mathbb{C})$ conjugations. So, we can choose a particular basis of solution of the system and work with the corresponding monodromy matrices.

Theorem 3.1 : *Suppose that $\left(S + S^T\right)$ is nondegenerate; then there exists a unique basis of solutions (which depends on the particular choice of the branchcuts in the complex λ–plane) $\{\Phi^{(j)}(\lambda)\}$ of the Fuchsian system (2.1), such that*
- Near u_i the solution has the behavior

$$\Phi_a^{(i)} \sim \frac{1}{\sqrt{u_i - \lambda}} \delta_a^i.$$

- the monodromy matrices are reflections, i.e., going around the singularity u_i the solutions transform
as

$$\Phi^{(i)} \to -\Phi^{(i)}$$
$$\Phi^{(j)} \to \Phi^{(j)} - 2g_{ij}\Phi^{(i)}$$

where $G = (g_{ij}) = \frac{1}{2}\left(S + S^T\right)$ is the Gram matrix of the following invariant bilinear form w.r.t. the chosen basis

$$g_{ij} = \left(\Phi^{(i)}, \Phi^{(j)}\right) := \Phi^{(i)^T}\left(U - \lambda\right)\Phi^{(j)}.$$

Invariance means that g_{ij} does not depend on λ neither on $u_1, \ldots, u_n$.

Proof: See [D1], Th.5.3.

Remark: Φ and Y_L are related by the Laplace transform

$$(Y_L)_a^{(j)}(z) = \frac{-\sqrt{z}}{2\sqrt{\pi}} \int_{\gamma_{(j)}} \Phi_a^{(j)}(\lambda)e^{\lambda z}d\lambda$$

where $\gamma_{(j)}$ is a fixed path in the λ–plane; analogously for Y_R.

In the $\{\Phi^{(j)}(\lambda)\}$ basis the i–th monodromy matrix M_i has the form

$$
M_i = \begin{pmatrix}
1 & 0 & \dots & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
-2g_{1i} & -2g_{2i} & \dots & -1 & \dots & -2g_{ni} \\
\vdots & \vdots & \dots & \vdots & \dots & \vdots \\
0 & 0 & \dots & 0 & \dots & 1
\end{pmatrix},
$$

$$
2g_{ij} = 2g_{ji} = s_{ij} \quad \text{for } i < j.
$$

This is a reflection w.r.t. the hyperplane normal to the vector $\Phi^{(i)}$.

The Coxeter identity ([B]) gives

$$
M_1 M_2 \dots M_n = -S^{-1}S^T.
$$

Lemma 3.1: *The following relations hold (all the indices are pairwise distinct)*

$$
\text{Tr}(M_i M_j) = n - 4 + s_{ij}^2 \tag{3.1a}
$$

$$
\text{Tr}(M_k M_i M_j M_i) = n - 4 + (s_{kj} - s_{ij}s_{ik})^2 \tag{3.1b}
$$

$$
\text{Tr}(M_i M_j M_l M_k) = n - 8 + s_{ij}^2 + s_{ik}^2 + s_{il}^2 + s_{jk}^2 + s_{jl}^2 + s_{kl}^2 - s_{ij}s_{ik}s_{jk} + \\
- s_{ik}s_{il}s_{kl} - s_{jk}s_{jl}s_{kl} - s_{ij}s_{il}s_{jl} + s_{ij}s_{il}s_{jk}s_{kl}. \tag{3.1c}
$$

Proof: The fact that the M_i are reflections and that $S + S^T = 2G$ geometrically reads into

$$
-2\cos\alpha_{ij} = s_{ij}
$$

where α_{ij} is the angle between the two hyperplanes normal to $\Phi^{(i)}$ and $\Phi^{(j)}$.
On the other hand, the products $M_i M_j$ are rotations by the angle $2\alpha_{ij}$ and this provides the relation (3.1a), indeed

$$
\text{Tr}(M_i M_j) = n - 2 + 2\cos(2\alpha_{ij}) = n - 4 + s_{ij}^2.
$$

To obtain relation (3.1b) we observe that the product $M_i M_j M_i$ is still a reflection, w.r.t. the mirror normal to the vector $M_i(\Phi^{(j)})$. This means that the product $M_k M_i M_j M_i$ is a rotation by the angle 2β, where

$$
-2\cos\beta = \left(M_i(\Phi^{(j)}), \Phi^{(k)} \right) = (\Phi^{(j)} - s_{ij}\Phi^{(i)}, \Phi^{(k)}) = s_{kj} - s_{ij}s_{ik}
$$

so that $\text{Tr}(M_k M_i M_j M_i) = n - 2 + 2\cos(2\beta) = n - 4 + (s_{kj} - s_{ij}s_{ik})^2$.

Finally, (3.1c) can be obtained directly in the case of the 4×4 reflection matrices M_i. Indeed, for ordered indices i, j, k, l, the Coxeter identity gives

$$
M_i M_j M_k M_l = -S_{ijkl}^{-1} S_{ijkl}^T,
$$

where

$$
S_{ijkl} = \begin{pmatrix}
1 & s_{ij} & s_{ik} & s_{il} \\
0 & 1 & s_{jk} & s_{jl} \\
0 & 0 & 1 & s_{kl} \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

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An easy calculation gives the result.

The same result holds also in dimension $n > 4$. Indeed, one can observe that, for every n, the product of four matrices acts non trivially in the 4–dimensional subspace spanned by the vectors normal to the mirrors of the reflections M_i, M_k, M_l, M_j. It is equal to the identity in the orthogonal complement to the 4-dimensional subspace.

<div align="right">Q.E.D.</div>

Combining all the above facts, we can conclude our construction proving the following main

Theorem 3.2: *1. The following formulae*

$$\{s_{ik}, s_{il}\} = \frac{i\pi}{2}(2s_{kl} - s_{ik}s_{il}) \qquad i < k < l \tag{3.2a}$$

$$\{s_{ik}, s_{jk}\} = \frac{i\pi}{2}(2s_{ij} - s_{ik}s_{jk}) \qquad i < j < k \tag{3.2b}$$

$$\{s_{ik}, s_{kl}\} = \frac{i\pi}{2}(s_{ik}s_{kl} - 2s_{il}) \qquad i < k < l \tag{3.2c}$$

$$\{s_{ik}, s_{jl}\} = 0 \qquad i < k < j < l \tag{3.2d}$$

$$\{s_{ik}, s_{jl}\} = 0 \qquad i < j < l < k \tag{3.2e}$$

$$\{s_{ik}, s_{jl}\} = i\pi(s_{ij}s_{kl} - s_{il}s_{kj}) \qquad i < j < k < l. \tag{3.2f}$$

define a Poisson bracket on the space $\mathcal{S}$ of Stokes matrices.

2. The monodromy map

$$\mathfrak{so}(n) \to \mathcal{S}$$

associating the Stokes matrix $S \in \mathcal{S}$ to the operator $\Lambda = \frac{d}{dz} - U - \frac{V}{z}$, $V \in \mathfrak{so}(n)$, is a Poisson map.

3. The eigenvalues of $S^{-1}S^T$ are the Casimir functions of the Poisson bracket.

4. The Poisson bracket (3.2) is invariant w.r.t. the action of the braid group B_n on the space of braid matrices.

Observe that the Poisson bracket (3.2) does not depend on the times $u_1, u_2, \ldots, u_n$, although the monodromy map does.

Proof: *1.* As a first step we explicitly write the restriction of the bracket (2.12) to the space of representations. By direct calculation one obtains

$$\{\operatorname{Tr}(M_i M_k), \operatorname{Tr}(M_j M_l)\} = \{(M_i)_b^a (M_k)_a^b, (M_j)_d^c (M_l)_c^d\} =$$
$$= (M_i)_b^a (M_j)_d^c \{(M_k)_a^b, (M_l)_c^d\} + (M_i)_b^a (M_l)_c^d \{(M_k)_a^b, (M_j)_d^c\} +$$
$$+ (M_k)_a^b (M_j)_d^c \{(M_i)_b^a, (M_l)_c^d\} + (M_k)_a^b (M_l)_c^d \{(M_i)_b^a, (M_j)_d^c\}. \tag{3.3}$$

where we mean summation over repeated indices; using (3.1a), one can rewrite the left hand sides of (3.3) as

$$\{\operatorname{Tr}(M_i M_k), \operatorname{Tr}(M_j M_l)\} = \{n - 4 + s_{ik}^2, n - 4 + s_{jl}^2\} = 4s_{ik}s_{ij}\{s_{ik}, s_{ij}\}. \tag{3.4}$$

Now one has to distinguish between three essentially different cases, in correspondence with the different order of the indices.

- $i < k < j < l$ or $i < j < l < k$:

For $i < k < j < l$ all the addenda in the right hand side of (3.3) involve a Poisson bracket of the form (2.12) with correctly ordered indices. Here we write explicitly only the first one:

$$i\pi \operatorname{Tr}\left(M_i M_j M_l M_k + M_i M_k M_l M_j - M_i M_l M_j M_k - M_i M_k M_j M_l\right).$$

<div align="center">12</div>

The others have a similar form, and it is easy to see that they cancel pairwise (the first with the second and the third with the fourth).

The same happens when $i < j < l < k$, since the only difference is a change of sign in the two last elements. Hence it follows

$$\{\mathrm{Tr}(M_i M_k), \mathrm{Tr}(M_j M_l)\} = 0 \qquad i < k < j < l \qquad (3.5a)$$

$$\{\mathrm{Tr}(M_i M_k), \mathrm{Tr}(M_j M_l)\} = 0 \qquad i < j < l < k \qquad (3.5b)$$

Using (3.1b) one immediately obtains equations (3.2d/e)

$\bullet\ i < j < k < l$

Here the different order of the indices induces a change of sign in the second addendum, which becomes equal to the first. Equation (3.3) gives

$$\{\mathrm{Tr}(M_i M_k), \mathrm{Tr}(M_j M_l)\} = 2i\pi \mathrm{Tr}\Big(M_i M_j M_l M_k + M_i M_k M_l M_j - M_i M_l M_j M_k - M_i M_k M_j M_l \Big)$$

$$= 4i\pi s_{ik} s_{jl} (s_{ij} s_{kl} - s_{il} s_{kj}),$$

where the last equality follows from Lemma 3.1. Using eq.(3.4) we obtain immediately eq. (3.2f)

$\bullet\ i = j < k < l$ or $i < j = k < l$ or $i < j < k = l$

If two indices coincide, for instance $i = j < k < l$, the other two cases are analogous, we find

$$\{\mathrm{Tr}(M_i M_k), \mathrm{Tr}(M_i M_l)\} = \{(M_i)_b^a (M_k)_a^b, (M_i)_d^c (M_l)_c^d\} =$$
$$= (M_i)_b^a (M_i)_d^c \{(M_k)_a^b, (M_l)_c^d\} + (M_i)_b^a (M_l)_c^d \{(M_k)_a^b, (M_i)_d^c\} +$$
$$+ (M_k)_a^b (M_i)_d^c \{(M_i)_b^a, (M_l)_c^d\} + (M_k)_a^b (M_l)_c^d \{(M_i)_b^a, (M_i)_d^c\}.$$

The first and the third addendum cancel, the last is zero (because $M_i^2 = \mathbb{1}$), and it remains:

$$\{\mathrm{Tr}(M_i M_k), \mathrm{Tr}(M_i M_l)\} = 2i\pi \Big((\mathrm{Tr}(M_i^2 M_l M_k) - \mathrm{Tr}(M_i M_k M_i M_l) \Big)$$

$$= 2i\pi[(n - 4 + s_{kl}^2) - (n - 4 + s_{kl}^2 + s_{ik}^2 s_{il}^2 - 2s_{kl} s_{ik} s_{il})]$$

$$= 2i\pi s_{ik} s_{il} (2s_{kl} - s_{ik} s_{il}),$$

where the second equality follows from (3.1a) and (3.1b). Using (3.4) this leads to (3.2a/b/c).

2. It follows from the commutativity of the diagram (1.7), where all the arrows are Poisson maps

3. As we have said above, the eigenvalues of the monodromy matrices are the Casimir functions for this Poisson structure. Particularly, applying to M_∞ we obtain, due to (2.3), the needed statement. Practically it is more convenient to use the coefficients of the characteristic polynomial $det(S^{-1}S^T - \mu\mathbb{1})$ as the basic Casimirs.

4. Recall [D], that the natural action of the braid group B_n with n strands on the space of Stokes matrices is generated by the following transformations corresponding to the standard generators σ_1, ..., σ_{n-1}

$$\sigma_i : \ S \mapsto K_i S K_i$$

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where the matrix $K_i = K_i(S)$ has the form

$$K_{jj} = 1, \quad j = 1, \ldots, n; \quad j \neq i, i+1$$
$$K_{ii} = -s_{ii+1}, \quad K_{ii+1} = K_{i+1i} = 1, \quad K_{i+1i+1} = 0.$$

Other matrix entries of K_i vanish. According to [D] this action describes the structure of analytic continuation of the solutions of MPDE. Our Poisson bracket is obviously invariant w.r.t. analytic continuation.

Q.E.D.

Example 1. $n = 3$. In this case the space of Stokes matrices has dimension 3. Denoting $x = s_{12}$, $y = s_{13}$, $z = s_{23}$ we obtain,

$$\{x, y\} = \frac{i\pi}{2}(2z - xy)$$
$$\{y, z\} = \frac{i\pi}{2}(2x - yz)$$
$$\{z, x\} = \frac{i\pi}{2}(2y - zx).$$

Our Poisson bracket coincides, within the constant factor $-\frac{i\pi}{2}$, with that of [D].

Example 2. $n = 4$. For convenience of the reader we write here down, omitting the constant factor $\frac{i\pi}{2}$, the Poisson bracket on the six-dimensional space of the Stokes matrices of the form

$$S = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & x & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\{p, q\} = (2x - pq) \qquad \{x, y\} = (2z - xy)$$
$$\{p, r\} = (2y - pr) \qquad \{y, z\} = (2x - yz)$$
$$\{q, r\} = (2z - qr) \qquad \{z, x\} = (2y - zx)$$

$$\{x, p\} = (2q - xp) \qquad \{q, x\} = (2p - qx) \qquad \{r, x\} = 0$$
$$\{y, p\} = (2r - yp) \qquad \{q, y\} = 2(pz - rx) \qquad \{r, y\} = (2p - ry)$$
$$\{p, z\} = 0 \qquad \{z, q\} = (2r - zq) \qquad \{r, z\} = (2q - rz) \qquad (3.6)$$

The Casimirs of this Poisson bracket are

$$C_1 = -4 + p^2 + q^2 + r^2 + x^2 + y^2 + z^2 - pqx - pry - qrz - xyz + prxz$$

and

$$C_2 = 6 - 2(+p^2 + q^2 + r^2 + x^2 + y^2 + z^2) + 2(-pqx - pry - qrz - xyz) - 2(pqyz + qrxy) + p^2r^2 + q^2y^2 + r^2x^2$$

On the 4-dimensional level surfaces of the Casimirs the Poisson bracket (3.6) induces a symplectic structure. These surfaces and the symplectic structures on them are invariant w.r.t. the following action of the braid group B_4:

$$\sigma_1 : (p, q, r, x, y, z) \mapsto (-p, x - pq, y - pr, q, r, z)$$
$$\sigma_2 : (p, q, r, x, y, z) \mapsto (q - px, p, r, -x, z - xy, y)$$
$$\sigma_3 : (p, q, r, x, y, z) \mapsto (p, r - qz, q, y - xz, x, -z)$$

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