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Noncommutative Riemann integration and singular traces for C*-algebras

Daniele Guido, Tommaso Isola
Dipartimento di Matematica,
Università di Roma "Tor Vergata",
I–00133 Roma, Italy.

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Abstract

Given a C*-algebra $\mathcal{A}$ with a semicontinuous semifinite trace τ acting on the Hilbert space $\mathcal{H}$, we define the family $\mathcal{A}^{\mathcal{R}}$ of bounded Riemann measurable elements w.r.t. τ as a suitable closure, *à la* Dedekind, of $\mathcal{A}$, in analogy with one of the classical characterizations of Riemann measurable functions [16], and show that $\mathcal{A}^{\mathcal{R}}$ is a C*-algebra, and τ extends to a semicontinuous semifinite trace on $\mathcal{A}^{\mathcal{R}}$.

Then, unbounded Riemann measurable operators are defined as the closed operators on $\mathcal{H}$ which are affiliated to $\mathcal{A}''$ and can be approximated in measure by operators in $\mathcal{A}^{\mathcal{R}}$, in analogy with improper Riemann integration. Unbounded Riemann measurable operators form a τ-a.e. bimodule on $\mathcal{A}^{\mathcal{R}}$, denoted by $\overline{\mathcal{A}^{\mathcal{R}}}$, and such bimodule contains the functional calculi of selfadjoint elements of $\mathcal{A}^{\mathcal{R}}$ under unbounded Riemann measurable functions. Besides, τ extends to a bimodule trace on $\overline{\mathcal{A}^{\mathcal{R}}}$.

As type II_1 singular traces for a semifinite von Neumann algebra $\mathcal{M}$ with a normal semifinite faithful (non-atomic) trace τ have been defined as traces on $\mathcal{M}-\mathcal{M}$-bimodules of unbounded τ-measurable operators [5], type II_1 singular traces for a C*-algebra $\mathcal{A}$ with a semicontinuous semifinite (non-atomic) trace τ are defined here as traces on $\mathcal{A}-\mathcal{A}$-bimodules of unbounded Riemann measurable operators (in $\overline{\mathcal{A}^{\mathcal{R}}}$) for any faithful representation of $\mathcal{A}$. An application of singular traces for C*-algebras is contained in [6].

1 Introduction.

In this paper we extend the notion of Riemann integrability to the non-abelian setting, namely, given a C*-algebra $\mathcal{A}$ of operators acting on a Hilbert space $\mathcal{H}$, together with a semicontinuous semifinite trace τ, we define both bounded and unbounded Riemann measurable operators with respect to τ.

The bounded Riemann measurable elements form a C*-algebra $\mathcal{A}^{\mathcal{R}}$, and τ extends to a semicontinuous semifinite trace on $\mathcal{A}^{\mathcal{R}}$. The unbounded Riemann measurable elements $\overline{\mathcal{A}^{\mathcal{R}}}$ form a τ a.e. *-bimodule on $\mathcal{A}^{\mathcal{R}}$ and τ extends to a positive bimodule-trace on $\overline{\mathcal{A}^{\mathcal{R}}}$.

On the one hand, this leads to the notion of Riemann algebra as a C*-algebra with some monotonic completion properties, and to the construction of the enveloping Riemann algebra for a pair $(\mathcal{A}, \tau)$, in analogy with what is done for the von Neumann or Borel enveloping algebras [11].

On the other hand, Riemann integration provides in fact a spatial theory of integration for C*-algebras, namely the trace τ extends to a natural class of closed (unbounded) operators on the Hilbert space $\mathcal{H}$. While L^p spaces are realized on the Hilbert space of the GNS representation w.r.t. τ, p-summable (or (p, q)-summable) Riemann measurable operators may be defined for any faithful representation of $\mathcal{A}$.

As an application, this allows us to study singular traces for (represented) C*-algebras as traces on (unbounded) Riemann measurable operators on the given Hilbert space.

In the classical case, the so-called abstract Riemann integration (cf. [10, 16, 13]) is based on a topological space X (usually Hausdorff locally compact) and a Radon measure μ, and Riemann measurable functions may be defined in two equivalent ways. Either as functions which are discontinuous in a μ-negligible set or as functions which are L^1 approximated, both from above and from below, by continuous functions. The latter characterization may be rephrased as follows: Riemann measurable functions are separating elements of a Riemann-cut ($\mathcal{R}$-cut), where an $\mathcal{R}$-cut is a pair (A^-, A^+) of bounded classes of continuous functions in $C_0(X)$ s.t.

- $f^- \leq f^+$ for any $f^\pm \in A^\pm$
- $\forall \varepsilon > 0\ \exists f^\pm_\varepsilon \in A^\pm : \int (f^+ - f^-) d\mu < \varepsilon$.

Replacing $(C_0(X), \mu)$ with $(\mathcal{A}, \tau)$, where $\mathcal{A}$ acts on $\mathcal{H}$, we define the family $\mathcal{A}^{\mathcal{R}}$ of bounded Riemann measurable elements on $\mathcal{H}$ w.r.t. τ as the linear span of the set of separating elements in $\mathcal{A}''$ for $\mathcal{R}$-cuts in $\mathcal{A}$. $\mathcal{A}^{\mathcal{R}}$ will also be called the $\mathcal{R}$-closure of $\mathcal{A}$. Clearly τ uniquely extends to a positive functional on $\mathcal{A}^{\mathcal{R}}$.

Our main results concerning bounded Riemann measurable elements are the following: $\mathcal{A}^{\mathcal{R}}$ is a C*-algebra, the extension of τ to $\mathcal{A}^{\mathcal{R}}$ is a semicontinuous semifinite trace on it, $\mathcal{A}^{\mathcal{R}}$ is closed under functional calculus w.r.t. Riemann measurable functions, and is $\mathcal{R}$-closed, namely it is stable under $\mathcal{R}$-closure. Observe that this is an order closure, and not a topological one, therefore the preceding statement is not at all obvious. We call Riemann algebra an $\mathcal{R}$-closed

C*-algebra, and enveloping Riemann algebra of $\mathcal{A}$ its $\mathcal{R}$-closure in the universal representation.

Unbounded Riemann measurable elements w.r.t τ are defined as closed operators affiliated to $\mathcal{A}''$ which may be approximated in measure by elements of $\mathcal{A}^{\mathcal{R}}$. This definition is inspired by the corresponding construction by Christensen for Borel algebras [1]. The family $\overline{\mathcal{A}^{\mathcal{R}}}$ of such elements is a τ-a.e. *-bimodule on $\mathcal{A}^{\mathcal{R}}$, namely it is closed under the *-bimodule operations and the *-bimodule properties hold τ almost everywhere, and, even if it is not necessarily closed w.r.t. the product operation (see Remark 3.30), given two operators S, $T \in \overline{\mathcal{A}^{\mathcal{R}}}$, there is $\widetilde{T} \in \overline{\mathcal{A}^{\mathcal{R}}}$, equal to T τ-a.e., s.t. $S\widetilde{T} \in \overline{\mathcal{A}^{\mathcal{R}}}$. Moreover, τ extends to a positive bimodule trace on it, satisfying $\tau(T^*eTe) = \tau(eTeT^*)$, for a suitable conull projection e. Functional calculi of selfadjoint elements of $\mathcal{A}^{\mathcal{R}}$ under unbounded Riemann measurable functions belong to $\overline{\mathcal{A}^{\mathcal{R}}}$.

As an application of this construction, we observe that the τ-a.e. bimodule $\overline{\mathcal{A}^{\mathcal{R}}}$ of unbounded Riemann measurable elements is a natural environment for the definition of singular traces, in particular for the type II_1 singular traces considered in [5].

Indeed, while type I singular traces (or singular traces at ∞) may be defined on bounded operators (and are anyhow determined by their restriction to bounded operators, cf. [5]), type II_1 singular traces need unbounded operators to be defined, since they vanish on the bounded ones.

Therefore, type II_1 singular traces for C*-algebras are defined on suitable bimodules of unbounded operators, the bimodule of unbounded measurable elements on a von Neumann algebra being replaced by the bimodule of unbounded Riemann measurable elements on a Riemann algebra.

We finally mention that type II_1 singular traces on C*-algebras furnish an asymptotic counterpart of the Wodzicki residue in the case of open manifolds, namely give the non-commutative geometric measure (*à la* Connes [2]) corresponding to the k-th asymptotic dimension (or k-th Novikov-Shubin invariant) as described in [6].

The main technical problem concerning bounded Riemann measurable elements arises in the definition already. Indeed we asked the classes (A^-, A^+) of the Riemann cut to be bounded. While such requirement is unnecessary in the abelian case, due to the simple order structure there, it was apparently not known if this was the case in the non-abelian setting either.

Such problem is equivalent to the following: suppose that an element x in the weak closure of a C*-algebra $\mathcal{A}$ with a semicontinuous semifinite trace τ may be approximated from above in the $L^1(\tau)$ norm by a sequence in $\mathcal{A}$. Is it possible to choose such a sequence uniformly bounded in norm? The answer to this question is negative in general, as it is shown in Subsection 2.1 by an explicit counterexample, therefore the two corresponding Riemann closures (with or without boundedness assumption) are different in general. It turns out that the non-uniformly bounded Riemann closure ($\mathcal{U}$-closure) is a C*-algebra if and only if it coincides with the $\mathcal{R}$-closure. More precisely, we show in a key Lemma that if the norm closure of $\mathcal{A}^{\mathcal{R}}$ is contained in the $\mathcal{U}$-closure of $\mathcal{A}$, then the former

is a C*-algebra, and all the announced results hold. When $\mathcal{A}$ is unital and τ is finite, $\mathcal{A}^{\mathcal{U}}$ is norm-closed and the Theorem follows. When the trace is densely defined, we show that $\mathcal{A}^{\mathcal{R}}$ coincides with the C*-inductive limit of the Riemann algebras corresponding to compact-support projections. Finally, in the general case, we show that the $\mathcal{R}$-closure of the maximal C*-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ for which τ is densely defined, which is a C*-algebra by the previous results, is a bimodule on $\mathcal{A}$, and that $\mathcal{A}^{\mathcal{R}} = \mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$.

We mention also that, even when the trace is faithful on $\mathcal{A}$, this is not generally the case on $\mathcal{A}^{\mathcal{R}}$.

Concerning the unbounded Riemann integration, we say that a closed, densely defined operator T on $\mathcal{H}$ is Riemann measurable if it is affiliated to $\mathcal{A}''$ and there exists an increasing sequence $\{e_n\}$ of Riemann measurable projections with conull supremum e such that both eTe_n and eT^*e_n are bounded and Riemann measurable.

The mentioned results for $\mathcal{A}^{\mathcal{R}}$ imply that (in general) there are many such operators, the set of which we denote by $\overline{\mathcal{A}^{\mathcal{R}}}$. Moreover $\overline{\mathcal{A}^{\mathcal{R}}}$ is closed under the *-bimodule operations. As explained before, $\overline{\mathcal{A}^{\mathcal{R}}}$ is not closed w.r.t. products, even though, given two operators $S, T \in \overline{\mathcal{A}^{\mathcal{R}}}$, there is $\widetilde{T} \in \overline{\mathcal{A}^{\mathcal{R}}}$, $\widetilde{T} = T$ τ-a.e., such that $S\widetilde{T}$ belongs to $\overline{\mathcal{A}^{\mathcal{R}}}$.

Due to the non-faithfulness of the trace, the *–bimodule properties may only hold almost everywhere.

Some more or less known results in the commutative setting and the classical counterpart of some non-abelian theorems are then collected in an appendix.

However, we stress that while usually the commutative Riemann integration is performed on functions on the given space, namely in the universal atomic representation of the algebra of continuous functions vanishing at infinity, one of the interesting aspects of the non-abelian Riemann integration is that it is a spatial theory, namely Riemann measurable elements may be defined in any faithful representation of the given C*-algebra.

2 Bounded Riemann integration.

Let $(\mathcal{A}, \tau)$ be a C*-algebra with a semicontinuous semifinite trace, acting on a Hilbert space $\mathcal{H}$. A pair (A^-, A^+) of sets in $\mathcal{A}_{\rm sa}$ is called an $\mathcal{R}$-cut (w.r.t. τ) if A^-, A^+ are uniformly bounded, *separated*, namely for any pair $a^\pm \in A^\pm$ we have $a^- \leq a^+$, and τ-*contiguous*, namely

$$\forall\, \varepsilon > 0\ \exists\, a_\varepsilon^\pm \in A^\pm : \tau(a_\varepsilon^+ - a_\varepsilon^-) < \varepsilon. \tag{2.1}$$

An element $x \in \mathcal{A}''$ is called separating for the $\mathcal{R}$-cut (A^-, A^+) if it is selfadjoint and for any $a^\pm \in A^\pm$ we have $a^- \leq x \leq a^+$.

Definition 2.1. Let $(\mathcal{A}, \tau)$ be a C*-algebra with a semicontinuous semifinite trace, and π a faithful representation of $\mathcal{A}$. Let us denote by $\mathcal{R}_\pi(\mathcal{A}, \tau)$ the linear span of the separating elements for the $\mathcal{R}$-cuts (w.r.t. τ) in $\pi(\mathcal{A})$. When π is the universal representation, we denote it simply by $\mathcal{R}(\mathcal{A}, \tau)$, and call it

the *enveloping Riemann algebra* of $\mathcal{A}$. For the sake of convenience we use the shorthand notation $\mathcal{A}^{\mathcal{R}} \equiv \mathcal{R}_\pi(\mathcal{A}, \tau)$, when π and τ are clear from the context. The C*-algebra $\mathcal{A}$ is called a *Riemann algebra* if it contains all the separating elements of its $\mathcal{R}$-cuts.

If $x \in \mathcal{A}^{\mathcal{R}}_{\mathrm{sa}}$ and (A^-, A^+) is a corresponding $\mathcal{R}$-cut, either all or none of the a's in $A^\pm$ have finite trace. In the first case we set $\tau(x) := \inf \tau(a^+) \equiv \sup \tau(a^-)$. In the second case, and if $x \geq 0$, we set $\tau(x) := +\infty$. This is the unique positive functional extending τ to $\mathcal{A}^{\mathcal{R}}$.

The first property of a Riemann algebra $\mathcal{A}$ is that it is closed under Riemann functional calculus. Moreover, the elements of $\mathcal{A}$ with τ-finite support are separating elements between upper and lower Riemann sums made up with projections in $\mathcal{A}$.

Lemma 2.2. *Let $(\mathcal{A}, \tau)$ be a C*-algebra with a tracial weight, and denote by $\mathcal{I}(\tau)$ its domain. Then, given a selfadjoint element x in $\mathcal{A}$,*

$$\tau(\phi_\varepsilon(|x|)) < \infty, \forall \varepsilon > 0 \Leftrightarrow x \in \overline{\mathcal{I}(\tau)},$$

where, for any $\varepsilon > 0$, ϕ_ε is an increasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}$ such that $\phi_\varepsilon = 0$ on $[0, \varepsilon/2)$ and $\phi_\varepsilon = 1$ on $(\varepsilon, +\infty)$.

Proof. ($\Rightarrow$) We have $|\tau(x\phi_\varepsilon(|x|))| \leq \|x\| \tau(\phi_\varepsilon(|x|)) < \infty$, i.e. $x\phi_\varepsilon(|x|) \in \mathcal{I}(\tau)$, and $\|x\phi_\varepsilon(|x|) - x\| < \varepsilon$, which implies $x \in \overline{\mathcal{I}(\tau)}$.
($\Leftarrow$) By definition ([11], p. 175) $|x|\phi_\varepsilon(|x|)$ belongs to the Pedersen ideal $K\left(\overline{\mathcal{I}(\tau)}\right)$. Since $\mathcal{I}(\tau)$ is a dense ideal in $\overline{\mathcal{I}(\tau)}$, $K\left(\overline{\mathcal{I}(\tau)}\right) \subset \mathcal{I}(\tau)$ by minimality, hence $\tau(|x|\phi_\varepsilon(|x|)) < \infty$. Then $\tau(\phi_\varepsilon(|x|)) < \frac{2}{\varepsilon}\tau(|x|\phi_\varepsilon(|x|)) < \infty$. □

Let us consider, for any selfadjoint x, the measure μ_x on $\sigma(x) \setminus \{0\}$ defined by

$$\int f(\lambda) d\mu_x(\lambda) = \tau(f(x)), \qquad f \in C_0(\sigma(x) \setminus \{0\}). \tag{2.2}$$

Proposition 2.3. *Let $\mathcal{A}$ be a Riemann algebra w.r.t. a semicontinuous semifinite trace τ. Then:*

(i) $\mathcal{A}$ is closed under Riemann functional calculus, namely

$$\forall x \in \mathcal{A},\ f \in \mathcal{R}_0(\sigma(x) \setminus \{0\}, \mu_x) \Rightarrow f(x) \in \mathcal{A},$$

where $\mathcal{R}_0(X, \mu)$ denotes the set of Riemann measurable functions on X w.r.t. μ and vanishing at infinity (cf. Appendix). In particular Riemann measurable spectral projections of selfadjoints elements of $\mathcal{A}$ belong to $\mathcal{A}$, where a spectral projection $e_\Omega(x)$ is Riemann measurable if $\Omega \subset\subset \sigma(x) \setminus \{0\}$ and $\mu_x(\partial\Omega) = 0$.

(ii) Let $x \in \mathcal{A}_{sa}$ have τ-finite support. Then x is the separating element between $\mathcal{R}$-cuts made by linear combinations of projections in $\mathcal{A}$. In particular, if τ is densely defined on $\mathcal{A}$, then $\mathcal{A}$ is generated by its projections as a Riemann algebra.

Proof. (i). It is sufficient to prove the assertion for a real-valued function f. Then, by Theorem 5.10 in the Appendix, there exist an open set $V \subset \sigma(x)\setminus\{0\}$, with $\mu_x(V)$ finite, and functions h, $h_\varepsilon^\pm \in C_0(\sigma(x)\setminus\{0\})$, with $h_\varepsilon^\pm$ vanishing outside V, such that $\int(h_\varepsilon^+ - h_\varepsilon^-)d\mu_x < \varepsilon$ and $h_\varepsilon^- \leq f - h \leq h_\varepsilon^+$. Then $h(x)$, $h_\varepsilon^\pm(x) \in \mathcal{A}$ by continuous functional calculus, and $f(x) - h(x) \in \mathcal{A}$ because it is a Riemann algebra.

(ii) Since x has τ-finite support, the measure μ_x is finite, therefore the set $S(x) = \{\alpha \in \sigma(x) : \mu_x(\alpha) \neq 0\}$ is at most countable. Consider the separated classes given by upper and lower Riemann sums of the function $f(t) = t$ on $\sigma(x)$, corresponding to subdivisions which do not intersect $S(x)$. Such classes are μ_x-contiguous and the corresponding spectral projections belong to $\mathcal{A}$ by (i). The corresponding functional calculi of x give the $\mathcal{R}$-cut for x. Concerning the last statement, we observe that $x\phi_\varepsilon(|x|)$ has τ-finite support for any $\varepsilon > 0$ and any $x \in \mathcal{A}_{\rm sa}$, by Lemma 2.2. Then the thesis easily follows by part (i). □

Then we may state the main theorem of this section.

Theorem 2.4. *Let $\mathcal{A}$ be a C^*-algebra with a semicontinuous semifinite trace τ, acting on a Hilbert space $\mathcal{H}$. Then*

(i) *$\mathcal{A}^{\mathcal{R}}$ is a C^*-algebra*

(ii) *the above described extension of τ to $\mathcal{A}^{\mathcal{R}}$ is a trace, and $(\mathcal{A}^{\mathcal{R}}, \tau)$ is a Riemann algebra*

(iii) *the GNS representation π_τ of $\mathcal{A}$ extends to a representation ρ_π of $\mathcal{A}^{\mathcal{R}}$ into $\pi_\tau(\mathcal{A})''$, and $\tau|_{\mathcal{A}^{\mathcal{R}}}$ may be identified with the pull-back of the trace on $\pi_\tau(\mathcal{A})''$ via ρ_π. As a consequence τ is semicontinuous semifinite on $\mathcal{A}^{\mathcal{R}}$.*

The rest of this section is devoted to the proof of Theorem 2.4. In order to do that we have to introduce *a priori* different kinds of Dedekind closures of $(\mathcal{A}, \tau)$, namely we shall consider unbounded cuts ($\mathcal{U}$-cuts), where the uniform boundedness property is removed, and the corresponding $\mathcal{U}$-closure $\mathcal{A}^{\mathcal{U}}$, and tight cuts ($\mathcal{T}$-cuts), where the uniform boundedness is strengthened by requiring that the $a_\varepsilon^\pm$'s in (2.1) verify

$$\sup\sigma(a_\varepsilon^+) < \sup\sigma(x) + \varepsilon, \quad \inf\sigma(a_\varepsilon^-) > \inf\sigma(x) - \varepsilon,$$

and the corresponding $\mathcal{T}$-closure, $\mathcal{A}^{\mathcal{T}}$. Of course we have $\mathcal{A} \subset \mathcal{A}^{\mathcal{T}} \subset \mathcal{A}^{\mathcal{R}} \subset \mathcal{A}^{\mathcal{U}}$, and τ extends to $\mathcal{A}^{\mathcal{U}}$ as well. We shall see that $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$ is a C*-algebra, while $\mathcal{A}^{\mathcal{U}}$ is not (cf. Example 2.16 below). It becomes a C*-algebra iff it coincides with $\mathcal{A}^{\mathcal{R}}$. This fact may be seen as a motivation for choosing $\mathcal{A}^{\mathcal{R}}$ instead of $\mathcal{A}^{\mathcal{U}}$ as the family of Riemann measurable elements. As we shall see in section 3, the possibility of taking products inside the set of Riemann measurable elements is crucial for constructing $\overline{\mathcal{A}^{\mathcal{R}}}$.

Remark 2.5. In the abelian case, $\mathcal{U}$-closure, $\mathcal{R}$-closure and $\mathcal{T}$-closure coincide, indeed in this case we may always find a very tight cut, for which $\sup\sigma(a_\varepsilon^+) = \sup\sigma(x)$ and $\inf\sigma(a_\varepsilon^-) = \inf\sigma(x)$. We conjecture that this is not always the case in the nonabelian setting.

Lemma 2.6. *The sets of separating elements for $\mathcal{R}$-cuts and $\mathcal{U}$-cuts are the selfadjoint parts of the *-bimodules $\mathcal{A}^{\mathcal{R}}$ and $\mathcal{A}^{\mathcal{U}}$ on $\mathcal{A}$.*

Proof. Linearity and *-invariance are obvious. Now let $a \in \mathcal{A}$ and $x \in \mathcal{A}''_{\mathrm{sa}}$ a separating element for the $\mathcal{R}$-cut $\{a_\varepsilon^\pm\}$. Then $a^* a_\varepsilon^\pm a$ gives an $\mathcal{R}$-cut for $a^* x a$. From the equalities

$$xy + yx = (x+1)^* y (x+1) - xyx - y \tag{2.3}$$

$$i(xy - yx) = (x+i)^* y (x+i) - xyx - y \tag{2.4}$$

which hold for any pair of selfadjoint elements we then get the bimodule property for $\mathcal{A}^{\mathcal{R}}$. The proof for $\mathcal{A}^{\mathcal{U}}$ is analogous. □

Lemma 2.7. *There is a unique linear positive extension $\rho : \mathcal{A}^{\mathcal{U}} \to \pi_\tau(\mathcal{A})''$ of the GNS representation of $\mathcal{A}$.*

Proof. Let $x \in \mathcal{A}^{\mathcal{U}}_{\mathrm{sa}}$, and $\{a_\varepsilon^\pm\}$ be a $\mathcal{U}$-cut for x. Then $\pi_\tau(a_\varepsilon^\pm)$ is a $\mathcal{U}$-cut, and, $\pi_\tau(a_\varepsilon^+)$ and $\pi_\tau(a_\varepsilon^-)$ converge to the same element in $L^1(\pi_\tau(\mathcal{A})'', \tau)$, which is the unique separating element of the $\mathcal{U}$-cut, hence belongs to $\pi_\tau(\mathcal{A})''$. Setting

$$\rho(x) := \lim_{\varepsilon \to 0} \pi_\tau(a_\varepsilon^+) = \lim_{\varepsilon \to 0} \pi_\tau(a_\varepsilon^-), \tag{2.5}$$

it follows easily that $\rho(x)$ does not depend on the $\mathcal{U}$-cut and is linear and positive. □

Lemma 2.8.

(i) ρ is a bimodule map.

(ii) $\tau|_{\mathcal{A}^{\mathcal{U}}} = \tau \circ \rho$, hence it is a trace on $\mathcal{A}^{\mathcal{U}}$ as a bimodule on $\mathcal{A}$.

Proof. (i) Follows easily from equations (2.3), (2.4).
(ii) Follows from positivity of ρ and τ. □

Lemma 2.9. *The $\mathcal{U}$-closure of $\mathcal{A}$ is $\mathcal{U}$-closed, namely the set of separating elements in $\mathcal{A}''_{sa}$ for $\mathcal{U}$-cuts in $\mathcal{A}^{\mathcal{U}}$ is contained in $\mathcal{A}^{\mathcal{U}}$.*

Proof. Indeed if $\{x_{\varepsilon'}^\pm\}_{\varepsilon'>0}$ is a $\mathcal{U}$-cut in $\mathcal{A}^{\mathcal{U}}$ for $x \in \mathcal{A}''_{\mathrm{sa}}$, and $\{a(x_{\varepsilon'}^\pm)_\varepsilon^\pm\}_{\varepsilon>0}$ is a $\mathcal{U}$-cut in $\mathcal{A}$ for $x_{\varepsilon'}^\pm$, then $\{(a(x_\varepsilon^+)_\varepsilon^+, a(x_\varepsilon^-)_\varepsilon^-)\}$ gives a $\mathcal{U}$-cut in $\mathcal{A}$ for x. □

With the above terminology, Theorem 2.4 shows that the $\mathcal{R}$-closure of a C*-algebra is $\mathcal{R}$-closed.

Lemma 2.10. *Let $\mathcal{A}$ be a C*-algebra acting on a Hilbert space $\mathcal{H}$, with a semicontinuous semifinite trace τ. If $C_0(x) := \{f(x) : f \in C_0(\sigma(x) \setminus \{0\})\}$ is contained in $\mathcal{A}^{\mathcal{U}}$ for a selfadjoint element x, then $C_0(x) \subset \mathcal{A}^{\mathcal{T}}$. In particular $\mathcal{A}^{\mathcal{T}}$, $\mathcal{A}^{\mathcal{R}}$ and $\mathcal{A}^{\mathcal{U}}$ have the same projections, and any C*-subalgebra of $\mathcal{A}^{\mathcal{U}}$ is actually contained in $\mathcal{A}^{\mathcal{T}}$.*

Proof. First we observe that any projection e in $\mathcal{A}^{\mathcal{U}}$ belongs to $\mathcal{T}$. For any $\delta > 0$ we consider the operator increasing functions (cf. [11])

$$f_\delta^+(z) = \frac{(1+\delta)z}{\delta + z} \quad ; \qquad f_\delta^-(z) = \frac{\delta z}{1+\delta - z} \quad .$$

If $\{a_\varepsilon^\pm\}_{\varepsilon>0}$ gives a $\mathcal{U}$-cut for e, $\{f_\delta^\pm(a_\varepsilon^\pm)\}_{\varepsilon>0}$ gives an $\mathcal{R}$-cut for any $\delta > 0$, indeed $f_\delta^\pm(e) = e$ together with operator monotonicity imply that $f_\delta^-(a_\varepsilon^-) \leq e \leq f_\delta^+(a_\varepsilon^+)$, and we have the estimate

$$\begin{aligned}
&\tau(f_\delta^+(a_\varepsilon^+) - f_\delta^-(a_\varepsilon^-)) \\
&\qquad = \tau(f_\delta^+(a_\varepsilon^+) - f_\delta^+(e)) + \tau(f_\delta^-(e) - f_\delta^-(a_\varepsilon^-)) \\
&\qquad \leq (1+\delta)\delta[\tau((a_\varepsilon^+ + \delta)^{-1}(a_\varepsilon^+ - e)(e+\delta)^{-1}) + \\
&\qquad\quad \tau((1+\delta-e)^{-1}(e - a_\varepsilon^-)(1+\delta-a_\varepsilon^-)^{-1})] \\
&\qquad \leq \frac{1+\delta}{\delta}\varepsilon.
\end{aligned}$$

Moreover, we have $\sup \sigma(f_\delta^+(a_\varepsilon^+)) \leq 1+\delta$ and $\inf \sigma(f_\delta^-(a_\varepsilon^-)) \geq -\delta$, therefore we may easily extract a $\mathcal{T}$-cut for e.

Now let $C_0(x) \subset \mathcal{U}$; we want to show that $x \in \mathcal{A}^{\mathcal{T}}$. First observe that $C_0(x^\pm) \subset \mathcal{U}$, where $x^\pm$ is the positive, resp. negative part of x. If 0 is in the convex hull of the spectrum of x, then $\inf \sigma(x^\pm) = 0$, hence $\mathcal{T}$-cuts for $x^\pm$ give a $\mathcal{T}$-cut for x, namely we may restrict to the case $\inf \sigma(x) = 0$. If $\inf \sigma(x) = m > 0$, then $I \in \mathcal{A}^{\mathcal{U}}$ by hypothesis, and it is sufficent to prove the statement for $x - mI$, while if $\sup \sigma(x) = m < 0$, it is sufficent to prove the statement for $-x + mI$, therefore again we may suppose $\inf \sigma(x) = 0$. Hence, by multiplying with an appropriate costant, we may assume $\inf \sigma(x) = 0$ and $\sup \sigma(x) = 1$.

Then we fix a $\delta > 0$ and apply Lemma 5.18 in the Appendix to the identity function on the spectrum of x. We may therefore find a sequence e_n of Riemann measurable spectral projections of x and a sequence of positive numbers α_n such that $\sum_n \alpha_n e_n = x$ and $\sum_n \alpha_n = 1$.

Moreover, since $\mathcal{A}^{\mathcal{U}}$ is $\mathcal{U}$-closed by Proposition 2.9, all the Riemann measurable spectral projections of x belong to $\mathcal{A}^{\mathcal{U}}$ (cf. the proof of Proposition 2.3).

Therefore, for any e_n we obtain an $\mathcal{R}$-cut $(\{a_{n,\varepsilon}^-\}_{\varepsilon>0}, \{a_{n,\varepsilon}^+\}_{\varepsilon>0})$ such that $\sup_\varepsilon \sigma(a_{n,\varepsilon}^+) \leq 1+\delta$ and $\inf_\varepsilon \sigma(a_{n,\varepsilon}^-) \geq -\delta$. Hence, setting $a_\varepsilon^\pm = \sum_n \alpha_n a_{n,\varepsilon}^\pm$, we obtain

$$\begin{aligned}
a_\varepsilon^- \leq x \leq a_\varepsilon^+, \\
\tau(a_\varepsilon^+ - a_\varepsilon^-) \leq \sum_n \alpha_n \tau(a_{n,\varepsilon}^+ - a_{n,\varepsilon}^-) \leq \varepsilon, \\
\sup \sigma(a_\varepsilon^+) \leq \sum_n \alpha_n \sup \sigma(a_{n,\varepsilon}^+) \leq 1+\delta, \\
\inf \sigma(a_\varepsilon^-) \geq \sum_n \alpha_n \inf \sigma(a_{n,\varepsilon}^-) \geq -\delta.
\end{aligned}$$

From these $\mathcal{R}$-cuts it is easy to extract a $\mathcal{T}$-cut. Finally, we observe that $C_0(f(x)) \subset \mathcal{A}^{\mathcal{U}}$ for any $f \in C_0(\sigma(x) \setminus \{0\})$, hence $C_0(x) \subset \mathcal{A}^{\mathcal{T}}$. $\square$

Lemma 2.11. *Let $X \subset \mathcal{A}^{\mathcal{T}}$ be a *-invariant vector space. Then $\overline{X} \subset \mathcal{A}^{\mathcal{R}}$.*

Proof. Let $x \in \overline{X}_{\mathrm{sa}}$, $x_n \to x$ in norm, $x_n \in X_{\mathrm{sa}}$. First, possibly passing to a subsequence, we may assume $\|x - x_n\| < 2^{-n}$. Then set $y_n = x_{n+1} - x_n \in X$ and observe that $\|y_n\| < 3 \cdot 2^{-(n+1)}$. By hypothesis we may find a $\mathcal{T}$-cut in $\mathcal{A}$ for y_n, hence elements $a^{\pm}_{n\varepsilon} \in \mathcal{A}$ such that $a^-_{n\varepsilon} \le y_n \le a^+_{n\varepsilon}$, $\|a^{\pm}_{n,\varepsilon}\| \le 2 \cdot 2^{-n}$, $\tau(a^+_{n\varepsilon} - a^-_{n\varepsilon}) \le \varepsilon\, 2^{-n}$. Then $a^{\pm}_{\varepsilon} := \sum_n a^{\pm}_{n\varepsilon}$ belongs to $\mathcal{A}$ by uniform convergence, and

$$\tau(a^+_\varepsilon - a^-_\varepsilon) \le \sum_n \tau(a^+_\varepsilon - a^-_\varepsilon) \le \varepsilon,$$

where the first inequality follows by the semicontinuity of τ. Since $\|a^{\pm}_{\varepsilon}\| \le \sum_n \|a^{\pm}_{n\varepsilon}\| \le 2$, we get $x \in \mathcal{A}^{\mathcal{R}}$. □

Lemma 2.12. *Let $x \in \mathcal{A}^{\mathcal{R}}$, and consider the functions $f_t(z) := \frac{tz}{1-tz}$, $t \in [0,1)$. Then $f_t(x) \in \mathcal{A}^{\mathcal{R}}$, and $\rho(f_t(x)) = f_t(\rho(x))$, for sufficiently small t.*

Proof. Let x be a separating element for the $\mathcal{R}$-cut $\{a^{\pm}_{\varepsilon}\} \subset \mathcal{A}$, with $r = \sup_\varepsilon \|a^{\pm}_{\varepsilon}\|$. Then $f_t(a^{\pm}_{\varepsilon}) \in \mathcal{A}$ for any $t \in [0, 1/r)$. Since, for any such t, the function f_t is operator monotone increasing (cf. [11]) on $(-\infty, r)$, we get,

$$f_t(a^-_\varepsilon) \le f_t(x) \le f_t(a^+_\varepsilon) \qquad t \in [0, 1/r)$$

and

$$\begin{aligned} f_t(a^+_\varepsilon) - f_t(a^-_\varepsilon) &= (I - tx^+_\varepsilon)^{-1}(tx^+_\varepsilon(I - tx^-_\varepsilon) - (I - tx^+_\varepsilon)tx^-_\varepsilon)(I - tx^-_\varepsilon)^{-1} \\ &= t(I - tx^+_\varepsilon)^{-1}(x^+_\varepsilon - x^-_\varepsilon)(I - tx^-_\varepsilon)^{-1}. \end{aligned}$$

Then, taking $0 \le t < \frac{1}{2r}$, we get

$$\begin{aligned} \left\|f_t(a^{\pm}_\varepsilon)\right\| &\le 1 \\ \tau\left(f_t(a^+_\varepsilon) - f_t(a^-_\varepsilon)\right) &\le \frac{2\varepsilon}{r}, \end{aligned}$$

which means that $f_t(x)$ is a separating element for the $\mathcal{R}$-cut $\{f_t(a^{\pm}_{\varepsilon})\}$, therefore, $f_t(x)$ belongs to $\mathcal{A}^{\mathcal{R}}$ for any $t \in [0, \frac{1}{2r})$. Finally, making use of equation (2.5) (where the limits are in L^1) we get

$$\rho(f_t(x)) = \lim_{\varepsilon \to 0} \pi_\tau(f_t(a^{\pm}_\varepsilon)) = \lim_{\varepsilon \to 0} f_t(\pi_\tau(a^{\pm}_\varepsilon)) = f_t(\rho(x)).$$

□

Lemma 2.13. *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$, with a semicontinuous semifinite trace τ. If the norm closure of $\mathcal{A}^{\mathcal{R}}$ is contained in $\mathcal{A}^{\mathcal{U}}$, then $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$, and Theorem 2.4 holds for $\mathcal{A}$.*

Proof. The proof requires some intermediate steps. Some of the arguments are taken form the proof of Kadison [9] that the Borel closure of a C*-algebra is a C*-algebra.
Step (i). $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$ is a Banach space and is closed under continuous functional calculus.

Let $x \in \mathcal{A}^{\mathcal{R}}$, then by Lemma 2.12 we have $f_t(x) \in \mathcal{A}^{\mathcal{R}}$, and the equality

$$x^n = \lim_{t\to 0}\left(f_t(x) - \sum_{k=1}^{n-1} t^k x^k\right) t^{-n}, \tag{2.6}$$

inductively implies $x^n \in \mathcal{A}^{\mathcal{U}}$, hence $C_0(x) \subset \mathcal{A}^{\mathcal{U}}$. Then Lemma 2.10 shows that $C_0(x) \subset \mathcal{A}^{\mathcal{T}}$, namely $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$ and it is closed under functional calculus. Then, since $\mathcal{A}^{\mathcal{R}}$ is a vector space, we may apply Lemma 2.11 to $\mathcal{A}^{\mathcal{T}}$, obtaining that $\mathcal{A}^{\mathcal{R}}$ is norm closed.
Step (ii). $\mathcal{A}^{\mathcal{R}}$ is $\mathcal{R}$-closed, namely if $x \in \mathcal{A}^{\mathcal{U}}$ is a separating element for an $\mathcal{R}$-cut $\{x_\varepsilon^\pm\} \subset \mathcal{A}^{\mathcal{R}}$, then $x \in \mathcal{A}^{\mathcal{R}}$.

Choose a $\mathcal{T}$-cut $\{(a_\varepsilon^\pm)_\delta^\pm\} \subset \mathcal{A}$ for $x_\varepsilon^\pm$. Then $\{(a_\varepsilon^-)_\varepsilon^-, (a_\varepsilon^+)_\varepsilon^+\}$ is an $\mathcal{R}$-cut for x, and $x \in \mathcal{A}^{\mathcal{R}}$.
Step (iii). ρ is norm-continuous and $\rho(x^2) = \rho(x)^2$.

Indeed, adjoining the identity to $\mathcal{A}$ if necessary, if $x \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, then, as ρ is a linear and positive map, we get $\|\rho(x)\| \le \|x\|\|\pi(1)\|$, and by linearity of ρ we get norm continuity. The last equation follows by applying ρ to equation (2.6) for $n = 2$ and using Lemma 2.12
Step (iv). If $x, y \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, then $xy+yx \in \mathcal{A}^{\mathcal{R}}$ and $\rho(xy+yx) = \rho(x)\rho(y)+\rho(y)\rho(x)$.

Immediate by steps (i) and (iii), and

$$xy + yx = (x + y)^2 - x^2 - y^2. \tag{2.7}$$

Step (v). If $x, y \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, then $xyx \in \mathcal{A}^{\mathcal{R}}$ and $\rho(xyx) = \rho(x)\rho(y)\rho(x)$.

Immediate by step (iv) and

$$2xyx = (xy + yx)x + x(xy + yx) - (yx^2 + x^2y). \tag{2.8}$$

Step (vi). If $x \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, $y \in \mathcal{A}_{\mathrm{sa}}$ then $(x+i)^*y(x+i) \in \mathcal{A}^{\mathcal{R}}$ and $\rho((x+i)^*y(x+i)) = (\rho(x) + i)^*\rho(y)(\rho(x) + i)$.

Follows by equations (2.4) and (2.8) for $x \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, $y \in \mathcal{A}_{\mathrm{sa}}$, and by the bimodule property.
Step (vii). If $x, y \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, then $(x + i)^*y(x + i) \in \mathcal{A}^{\mathcal{R}}$ and $\rho((x + i)^*y(x + i)) = (\rho(x) + i)^*\rho(y)(\rho(x) + i)$.

Let $\{b_\varepsilon^\pm\}$ be an $\mathcal{R}$-cut for y, then we have

$$(x + i)^*b_\varepsilon^-(x + i) \le (x + i)^*y(x + i) \le (x + i)^*b_\varepsilon^+(x + i),$$

and, by step (vi),

$$\begin{aligned}\tau((x + i)^*(b_\varepsilon^+ - b_\varepsilon^-)(x + i)) &= \tau \circ \rho((x + i)^*(b_\varepsilon^+ - b_\varepsilon^-)(x + i)) \\ &\le \|x + i\|^2\tau(b_\varepsilon^+ - b_\varepsilon^-) \le \varepsilon\|x + i\|^2\end{aligned}$$

so that $\{(x+i)^* b_\varepsilon^\pm (x+i)\} \subset \mathcal{A}^{\mathcal{R}}$ is an $\mathcal{R}$-cut for $(x+i)^* y (x+i)$. Then the thesis follows by step (ii) and equation (2.5).
Step $(viii)$. $\mathcal{A}^{\mathcal{R}}$ is a C*-algebra and ρ is a representation.

The only missing property is multiplicativity, which directly follows from steps (iv) and (vii), and equation (2.4).
Step $(viii)$. ρ is the GNS representation of $\mathcal{A}^{\mathcal{R}}$ w.r.t. τ.

Let us denote by $(\pi_\tau, \mathcal{H}_\tau, \eta_\tau)$ the GNS triple of $\mathcal{A}$ w.r.t. τ. Set $\mathfrak{N} := \{x \in \mathcal{A}^{\mathcal{R}} : \tau(x^* x) < \infty\}$, and define the linear map $\eta : \mathfrak{N} \to \mathcal{H}_\tau$ as follows. If $x \in \mathfrak{N}_+$, and $\{a_\varepsilon\} \subset \mathfrak{N} \cap \mathcal{A}_+$ is s.t. $x \le a_\varepsilon$ and $\tau(|a_\varepsilon - x|^2) < \varepsilon$, let us set $\eta(x) := \lim_{\varepsilon \to 0} \eta_\tau(a_\varepsilon)$, where the limit is independent of $\{a_\varepsilon\}$, and extend by linearity to all of $\mathfrak{N}$, which is generated by its positive elements. Then it is easy to see that $(\eta(x), \eta(y)) = \tau(x^* y)$, for $x, y \in \mathfrak{N}$, and all that remains to show is that $\rho(x)\eta(y) = \eta(xy)$ for $x \in \mathcal{A}^{\mathcal{R}}$, $y \in \mathfrak{N}$, and it sufficies to show it for $x, y \ge 0$. Let us prove it first for $y \in \mathfrak{N} \cap \mathcal{A}_+$. Observe that, using equations (2.3) and (2.4), we can write

$$\begin{aligned} xy &= \frac{1}{2}(y+1)x(y+1) + \frac{i}{2}(y+i)^* x (y+i) - \frac{1+i}{2} yxy - \frac{1+i}{2} x \\ &\equiv \sum_{k=1}^{4} \lambda_k b_k^* x b_k, \end{aligned} \tag{2.9}$$

where $\lambda_k \in \mathbb{C}$, $b_k \in \mathcal{A} + \mathbb{C}$. Let now $\{a_\varepsilon^\pm\} \subset \mathcal{A}$ be an $\mathcal{R}$-cut, with x as separating element. Then, using equation (2.9), we get $a_\varepsilon^+ y = \sum_{k=1}^4 \lambda_k b_k^* a_\varepsilon^+ b_k$, so that $b_k^* x b_k \le b_k^* a_\varepsilon^+ b_k$, and

$$\tau(|b_k^*(a_\varepsilon^+ - x) b_k|^2) \le \|b_k\|^2 \|a_\varepsilon^+ - x\| \tau(a_\varepsilon^+ - x) \to 0, \ \varepsilon \to 0,$$

hence,

$$\begin{aligned} \lim_{\varepsilon \to 0} \pi_\tau(a_\varepsilon^+)\eta_\tau(y) = \lim_{\varepsilon \to 0} \eta_\tau(a_\varepsilon^+ y) &= \lim_{\varepsilon \to 0} \sum_{k=1}^{4} \lambda_k \eta(b_k^* a_\varepsilon^+ b_k) \\ &= \sum_{k=1}^{4} \lambda_k \eta(b_k^* x b_k) = \eta(xy). \end{aligned}$$

But we also have

$$\|[\pi_\tau(a_\varepsilon^+) - \rho(x)]\eta_\tau(y)\|^2 \le \|\pi_\tau(a_\varepsilon^+) - \rho(x)\| \, \|\pi_\tau(a_\varepsilon^+) - \rho(x)\|_1 \, \|y\|^2 \to 0,$$

as $\varepsilon \to 0$, so that $\rho(x)\eta(y) = \eta(xy)$. Finally for $x \in \mathcal{A}_+^{\mathcal{R}}$, $y \in \mathfrak{N}_+$, $a \in \mathfrak{N} \cap \mathcal{A}_+$, we get

$$(\eta(a), \rho(x)\eta(y)) = (\rho(x)\eta(a), \eta(y)) = (\eta(xa), \eta(y)) = \tau(axy) = (\eta(a), \eta(xy))$$

and the thesis follows. □

In the following subsections we shall prove that the closure of $\mathcal{A}^{\mathcal{R}}$ is contained in $\mathcal{A}^{\mathcal{U}}$ in the cases in which $\mathcal{A}$ is unital and the trace is finite, $\mathcal{A}$ is non-unital and the trace is densely defined, and finally in the non-densely defined case, thus completing the proof of Theorem 2.4.

Corollary 2.14. *$\mathcal{A}^{\mathcal{R}}$ is the unique Riemann algebra between $\mathcal{A}$ and $\mathcal{A}^{\mathcal{U}}$. $\mathcal{A}^{\mathcal{U}}$ is a C*-algebra if and only if it coincides with $\mathcal{A}^{\mathcal{R}}$.*

Proof. By Lemma 2.13 and Theorem 2.4 we obtain that $\mathcal{A}^{\mathcal{R}}$, being the minimal Riemann algebra between $\mathcal{A}$ and $\mathcal{A}^{\mathcal{U}}$ and being the maximal C*-algebra between $\mathcal{A}$ and $\mathcal{A}^{\mathcal{U}}$, is indeed the unique Riemann algebra between $\mathcal{A}$ and $\mathcal{A}^{\mathcal{U}}$. As a consequence, either $\mathcal{A}^{\mathcal{R}} = \mathcal{A}^{\mathcal{U}}$, or $\mathcal{A}^{\mathcal{U}}$ is not a C*-algebra. □

2.1 Finite trace on a unital C*-algebra

In this subsection we prove Theorem 2.4 in the case of a unital C*-algebra with a finite trace. This will be an immediate corollary of the following theorem.

Theorem 2.15. *Let $(\mathcal{A}, \tau)$ be a unital C*-algebra acting on a Hilbert space $\mathcal{H}$, τ a tracial state on $\mathcal{A}$, then $\mathcal{A}^{\mathcal{U}}$ is norm closed, therefore Theorem 2.4 holds for $\mathcal{A}$.*

Proof. Let $x \in \overline{\mathcal{A}^{\mathcal{U}}}$ and $x_n \in \mathcal{A}^{\mathcal{U}}$, $\|x_n - x\| < \frac{1}{n}$. Then x is a separating element for the $\mathcal{U}$-cut $(\{x_n - \frac{1}{n}\}, \{x_n + \frac{1}{n}\})$, so that $x \in \mathcal{A}^{\mathcal{U}}$ since $\mathcal{A}^{\mathcal{U}}$ is $\mathcal{U}$-closed. Then the result follows by Lemma 2.13. □

Since $\mathcal{A}^{\mathcal{U}}$ is a norm closed *-bimodule, $\mathcal{A}^{\mathcal{R}} \neq \mathcal{A}^{\mathcal{U}} \iff \mathcal{A}^{\mathcal{U}}$ is not an algebra. We conclude this subsection with an example showing that this is indeed possible. Apparently such phenomenon depends on τ being non-faithful on $\mathcal{A}^{\mathcal{R}}$, which can happen even if τ is faithful on $\mathcal{A}$.

Example 2.16. Let μ be the sum of the Lebesgue measure on $[0,1]$ plus the Dirac measure in $\{0\}$, and consider the C*-algebra $\mathcal{A} = \{f \in C([0,1]) \otimes M(2) : f_{12}(0) = f_{21}(0) = 0\}$ acting on the Hilbert space $\mathcal{H} = L^2([0,1], d\mu) \otimes \mathbb{C}^2$. Let τ be the state on $\mathcal{B}(\mathcal{H})$ defined by

$$\tau(f) := \frac{1}{2} \int \mathrm{tr} f(t) dt + \frac{1}{2} f_{11}(0).$$

Then τ is a trace on $\mathcal{A}''$ and is faithful on $\mathcal{A}$, $\mathcal{A}^{\mathcal{R}}$ is given by the $M(2)$-valued Riemann measurable functions f on $[0,1]$ s.t. f_{11} is continuous in 0 and f_{12}, f_{21} vanish and are continuous in 0, $\mathcal{A}^{\mathcal{U}}$ is given by the $M(2)$-valued Riemann measurable functions f on $[0,1]$ s.t. f_{11} is continuous in 0 and f_{12}, f_{21} vanish in 0. In particular $\mathcal{A}^{\mathcal{U}}$ is not an algebra.

Proof. Since $\mathcal{A}''$ is given by $\{f \in L^\infty([0,1], d\mu) \otimes M(2) : f_{12}(0) = f_{21}(0) = 0\}$, the trace property follows.

Since elements in $\mathcal{A}^{\mathcal{U}}$ are separating elements for $\mathcal{U}$-cuts in $\mathcal{A}$, we obtain that f_{11} is continuous in 0. Since $\mathcal{A}^{\mathcal{R}}$ is a *-algebra, $f \in \mathcal{A}^{\mathcal{R}}$ implies $|f|^2 \in \mathcal{A}^{\mathcal{R}}$, hence

$|f_{11}|^2 + |f_{12}|^2$ is continuous in 0. Since f_{12} vanishes in 0, f_{12} (and analogously f_{21}) is continuous in 0. On the other hand, let g be a real valued Riemann measurable function on $[0,1]$ w.r.t. Lebesgue measure s.t. $g(0) = 0$ and $|g| \leq 1$ and consider the matrix $f := \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}$. For any $\varepsilon > 0$ set

$$f_\varepsilon^\pm(t) := \begin{cases} \begin{pmatrix} 0 & g(t) \\ g(t) & 0 \end{pmatrix} & t > \varepsilon \\ \\ \begin{pmatrix} \pm\varepsilon^2 & 0 \\ 0 & \pm\varepsilon^{-2} \end{pmatrix} & t \leq \varepsilon. \end{cases}$$

We easily get $f_\varepsilon^\pm \in \mathcal{A}^\mathcal{R}$, $f_\varepsilon^- \leq f \leq f_\varepsilon^+$ and $\tau(f_\varepsilon^+ - f_\varepsilon^-) = 2\varepsilon + \varepsilon^3$, namely f is separating for $\mathcal{U}$-cuts in $\mathcal{A}^\mathcal{R}$, hence $f \in \mathcal{A}^\mathcal{U}$. This shows at once that $\mathcal{A}^\mathcal{R}$ is strictly smaller then $\mathcal{A}^\mathcal{U}$ and that the latter is not an algebra, since, if g is not continuous in 0, $(f^2)_{11}$ is not continuous in 0 hence does not belong to $\mathcal{A}^\mathcal{U}$. The rest of the proof follows with analogous arguments. □

2.2 Densely defined trace on a non-unital C*-algebra

The construction considered in this subsection corresponds to the most general case described in the abelian setting, namely the case of a Radon measure on a non-compact space. There the local structure, namely the structure given by the compact support functions, plays a crucial role, and the same will be in our construction.

In the classical case, a function is said Riemann integrable if it has compact (or τ-finite) support, and is continuous but for a null set. Given a C*-algebra with a semicontinuous densely defined trace, we define a suitable analogue of this family, and show that its norm closure coincides with the $\mathcal{R}$-closure of the given C*-algebra. Therefore on such elements, the integral may be defined in two equivalent ways. Either as a separating element, as explained before, or as a limit (when it exists) of the τ-finite restrictions (see below).

In the nonabelian setting, the local structure is determined by a suitable family of projections which has to be closed w.r.t. the "$\vee$" operation. We present here a family of projections with these properties; we shall show at the end of this section that any other reasonable family would have produced the same result.

In this subsection $\mathcal{A}$ denotes a non unital C*-algebra acting on a Hilbert space $\mathcal{H}$, equipped with a semicontinuous densely defined trace τ.

Lemma 2.17. *Any projection in $\mathcal{A}^\mathcal{R}$ has finite trace.*

Proof. Let e be a projection in $\mathcal{A}^\mathcal{R}$, $a \in \mathcal{A}$ such that $e \leq a$, and $a_n \in \mathcal{A}_+$ with finite trace with $a_n \to a$ in norm. Then $ea_ne \to eae$, hence there exists an n such that $ea_ne \geq eae - 1/2e \geq 1/2e$, from which the thesis follows. □

Definition 2.18. Let us denote by $\mathcal{E}$ the minimal family of projections in $\mathcal{A}''$ containing all the Riemann measurable spectral projections of the selfadjoint elements in $\mathcal{A}$, and closed w.r.t. the "$\vee$" operation.

Lemma 2.19. *The family $\mathcal{E}$ is contained in $\mathcal{A}^{\mathcal{R}}$.*

Proof. Let us consider the family $\mathcal{E}_1$ of projections $e \in \mathcal{A}^{\mathcal{R}}$ which are separating for $\mathcal{R}$-cuts $\{a_\varepsilon^\pm\}_{\varepsilon>0}$ such that $a_\varepsilon^- \geq 0$ and $\tau(\mathrm{supp}(a_\varepsilon^+) - e) < \varepsilon$. The τ-finite Riemann measurable spectral projections of a selfadjoint element $x \in \mathcal{A}$ belong to $\mathcal{E}_1$ by Lemma 5.19 applied to $(\sigma(x), \mu_x)$.
Now we show that $\mathcal{E}_1$ is closed w.r.t. the $\vee$ operation. Let e, f be in $\mathcal{E}_1$, with $\{a_\varepsilon^\pm\}_{\varepsilon>0}$, $\{b_\varepsilon^\pm\}_{\varepsilon>0}$ the corresponding cuts. Then $\tau(e - \mathrm{supp}(a_\varepsilon^-)) \leq \varepsilon$ and the same for f, hence $\tau(e \vee f - \mathrm{supp}(a_\varepsilon^-) \vee \mathrm{supp}(b_\varepsilon^-)) \leq 2\varepsilon$. Since for any positive element y with τ-finite support we have $\tau(\mathrm{supp}(y) - y^{1/n}) \to 0$ and if y_1, y_2 are positive elements then $\mathrm{supp}(y_1) \vee \mathrm{supp}(y_2) = \mathrm{supp}(y_1 + y_2)$, we get an n such that $\tau(e \vee f - ((a_\varepsilon^- + b_\varepsilon^-)/2)^{1/n}) \leq 3\varepsilon$. As for the approximation from above, we observe that $\tau(\mathrm{supp}(a_\varepsilon^+) \vee \mathrm{supp}(b_\varepsilon^+) - e \vee f) \leq 2\varepsilon$, and there exists an n such that $\tau((a_\varepsilon^+ + b_\varepsilon^+)^{1/n} - \mathrm{supp}(a_\varepsilon^+ + b_\varepsilon^+)) \leq \varepsilon$, hence $\tau((a_\varepsilon^+ + b_\varepsilon^+)^{1/n} - e \vee f) \leq 3\varepsilon$. This shows that $e \vee f \in \mathcal{E}_1$. Since $\mathcal{E}_1$ is contained in $\mathcal{A}^{\mathcal{R}}$, the thesis follows by the minimality of $\mathcal{E}$. □

Let e be a projection in $\mathcal{A}^{\mathcal{R}}$, and consider the C*-algebra $\mathcal{A}_e := \{a \in \mathcal{A} : ae = ea = a\}$. Since $\mathcal{A}^{\mathcal{R}}$ is a bimodule and contains e, we have $\mathcal{A}_e + \mathbb{C}e \subset \mathcal{A}^{\mathcal{R}}$, hence $(\mathcal{A}_e + \mathbb{C}e)^{\mathcal{R}} \subset \mathcal{A}^{\mathcal{U}}$. By Lemma 2.17, projections in $\mathcal{A}^{\mathcal{R}}$ are τ-finite. Therefore, by the results of the previous subsection, $(\mathcal{A}_e + \mathbb{C}e)^{\mathcal{R}}$ is an $\mathcal{R}$-closed C*-algebra in $\mathcal{A}_e''$. By Lemma 2.10 we conclude that $(\mathcal{A}_e + \mathbb{C}e)^{\mathcal{R}}$ is indeed contained in $\mathcal{A}^{\mathcal{R}}$. We may therefore consider the minimal $\mathcal{R}$-closed C*-algebra $\mathcal{R}_e$ verifying $\mathcal{A}_e \subseteq \mathcal{R}_e \subset \mathcal{A}^{\mathcal{R}}$. By definition, the map $e \to \mathcal{R}_e$ is order preserving, therefore the union $\cup_{e\in\mathcal{E}}\mathcal{R}_e$ is a *-algebra, since $(\mathcal{E}, \vee)$ is a directed set. We shall denote by $\mathcal{R}_{\mathcal{E}}$ the C*-algebra given by the norm closure of $\cup_{e\in\mathcal{E}}\mathcal{R}_e$. Then the following holds.

Theorem 2.20. *Let $(\mathcal{A}, \tau)$ be a C*-algebra with a semicontinuous densely defined trace, acting on a Hilbert space $\mathcal{H}$. Then $\mathcal{R}_{\mathcal{E}} = \mathcal{A}^{\mathcal{R}}$.*

First we observe that such result immediately implies Theorem 2.4 in the densely defined case, by Lemma 2.13. Before proving the Theorem, we need a simple Lemma.

Lemma 2.21. *Let e be a projection on $\mathcal{H}$, and x a selfadjoint operator s.t. $0 \leq x \leq \alpha e + \beta e^\perp$, $\alpha, \beta \geq 0$. Then $\|e^\perp x e\| \leq \sqrt{\alpha\beta}$*

Proof. Let η_1 be a unit vector in $e\mathcal{H}$, η_2 a unit vector in $e^\perp\mathcal{H}$, $\vartheta \in (0, \pi/2)$. Then

$$\begin{aligned} 2\sin\vartheta\cos\vartheta Re(\eta_1, x\eta_2) &= \\ &= ((\cos\vartheta\ \eta_1 + \sin\vartheta\ \eta_2), (e^\perp x e + e x e^\perp)(\cos\vartheta\ \eta_1 + \sin\vartheta\ \eta_2)) \\ &\leq (\alpha\ \cos^2\vartheta + \beta\ \sin^2\vartheta) \end{aligned}$$

which gives $Re(\eta_1, x\eta_2) \leq \alpha \ \cot\vartheta + \beta \ \tan\vartheta$. Taking the supremum of the left hand side w.r.t. η_1, η_2 and the infimum of the right hand side w.r.t. $\vartheta \in (0, \pi/2)$ we immediately get the thesis. □

Proof. (of Theorem 2.20). ($\mathcal{R}_\mathcal{E} \subset \mathcal{A}^\mathcal{R}$). By the results of the preceding subsection, it follows that $\cup_{e\in\mathcal{E}}\mathcal{R}_e$ is a vector space in $\mathcal{A}^\mathcal{T}$, hence its norm closure $\mathcal{R}_\mathcal{E}$ is contained in $\mathcal{A}^\mathcal{R}$ by Lemma 2.11.

($\mathcal{A}^\mathcal{R} \subset \mathcal{R}_\mathcal{E}$). Let $x \in \mathcal{A}^\mathcal{R}_{\rm sa}$, and $\{a^\pm_\varepsilon\}_{\varepsilon>0}$ the corresponding $\mathcal{R}$-cut in $\mathcal{A}$. We observe that it is not restrictive to assume $x \geq 0$, possibly replacing x with $x - a_1^-$. Then we set $x_\delta := \phi_\delta x \phi_\delta$, where $\phi_\delta := \phi_\delta(a_1^+)$ denotes the mollified spectral projection defined in Lemma 2.2, and observe that $\{\phi_\delta a^\pm_\varepsilon \phi_\delta\}_{\varepsilon>0}$ gives an $\mathcal{R}$-cut for x_δ. Since $\phi_\delta a^\pm_\varepsilon \phi_\delta \in \mathcal{A}_{e_\delta}$, with $e_\delta = \chi_{[\delta/2,\infty)}(a_1^+)$, we have $x_\delta \in \cup_{e\in\mathcal{E}}\mathcal{R}_e$, for any $\delta > 0$. The theorem is proved if we show that $x_\delta \to x$ when $\delta \to 0$. Indeed,

$$\begin{aligned}\|x - x_\delta\| &\leq \|(1-\phi_\delta)x(1-\phi_\delta)\| + 2\|(1-\phi_\delta)x\| \leq 3\|e_\delta^\perp x\| \\ &\leq 3\|e_\delta^\perp x e_\delta^\perp\| + 3\|e_\delta^\perp x e_\delta\| \leq 3\frac{\delta}{2} + 6\sqrt{\frac{\delta}{2}}\sqrt{\|a_1^+\|},\end{aligned}$$

where the last inequality relies on the estimates $\|e_\delta^\perp x e_\delta^\perp\| \leq \|e_\delta^\perp a_1^+ e_\delta^\perp\| \leq \frac{\delta}{2}$ and $x \leq a_1^+ \leq \frac{\delta}{2} e_\delta^\perp + \|a_1^+\| e_\delta$, together with Lemma 2.21. □

We say that $a \in \mathcal{A}^\mathcal{R}$ is $\mathcal{R}$-summable, if at least one of a^+ or a^- is τ-finite. Then we may prove

Proposition 2.22. *a is $\mathcal{R}$-summable iff there exists $\lim_{e\in\mathcal{A}^\mathcal{R}} \tau(eae)$. When a is positive or τ-finite, the above limit coincides with $\tau(a)$.*

As explained before, τ coincides with the pull-back via ρ_π of the trace on the GNS representation, hence it is a semicontinuous semifinite trace on $\mathcal{A}^\mathcal{R}$.

Proposition 2.23. *If τ is densely defined on $\mathcal{A}$, then $(\mathcal{A}^\mathcal{R}, \tau)$ is a Riemann algebra with a semicontinuous, densely defined trace.*

Proof. τ is densely defined on each $\mathcal{R}_e$, hence the thesis follows. □

Remark 2.24. Previous Theorem shows that the algebra $\mathcal{R}_\mathcal{E}$ may be defined independently of the family $\mathcal{E}$. We note that, replacing $\mathcal{E} := \mathcal{E}_0$ with other reasonable families, the same algebra is obtained. Indeed we may consider the set $\mathcal{E}_1$ considered in the proof of Lemma 2.19, the set $\mathcal{E}_2$ of all projections in $\mathcal{A}^\mathcal{R}$, the set $\mathcal{E}_3$ of all compact support projections in $\mathcal{A}''$, namely projections which are majorized by an element in $\mathcal{A}$ (cf. [11]), or the set $\mathcal{E}_4$ of all τ-finite projections in $\mathcal{A}''$. It is easy to see that $\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_3 \subseteq \mathcal{E}_4$, and all such sets are directed. If e is a τ-finite projection, $\mathcal{R}_e$ can be defined as the minimal Riemann algebra containing $\mathcal{A}_e$, the set of such algebras being non empty, containing at least $\mathcal{A}^\mathcal{R}$. Then for any family $\mathcal{E}_i$, we may define the C*-algebra $\mathcal{R}_i := \overline{\cup_{e\in\mathcal{E}_i}\mathcal{R}_e}$, and obtain $\mathcal{A}^\mathcal{R} = \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \mathcal{R}_4$. Since on

the other hand $\mathcal{R}_4$ is contained in $\mathcal{A}^{\mathcal{R}}$, we have proved that all these algebras coincide.
Let e be a τ-finite projection in $\mathcal{A}''$. We say that an $\mathcal{R}$-cut is e-supported if all elements in A^- and A^+ belong to $\mathcal{A}_e$. Then we proved that, for any $i = 0, 1, 2, 3, 4$, $\mathcal{A}^{\mathcal{R}}$ is the norm closure of the Dedekind completion of $\mathcal{A}$ w.r.t. $\mathcal{E}_i$-supported $\mathcal{R}$-cuts.

2.3 Semifinite non-densely defined trace

In the rest of the section we consider semifinite non-densely defined traces, namely pairs $(\mathcal{A}, \tau)$ for which the closure $\mathcal{A}_0$ of the domain of the trace is a proper ideal in $\mathcal{A}$.

Theorem 2.25. *The $\mathcal{R}$-closure $\mathcal{A}_0^{\mathcal{R}}$ of $\mathcal{A}_0$ is a bimodule on $\mathcal{A}$.*

Proof. Let $x \in (\mathcal{A}_0^{\mathcal{R}})_{\mathrm{sa}}$, then there is an $\mathcal{R}$-cut $\{a_\varepsilon^\pm\}$ in $\mathcal{A}_0$ for which x is a separating element. Let $a \in \mathcal{A}$, then a^*xa is a separating element for the $\mathcal{R}$-cut $\{a^*a_\varepsilon^\pm a\}$ in $\mathcal{A}_0$, so that $a^*xa \in (\mathcal{A}_0^{\mathcal{R}})_{\mathrm{sa}}$. Now we have to show that if $x \in (\mathcal{A}_0^{\mathcal{R}})_{\mathrm{sa}}$, $y \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, then $xy + yx$ and $i(xy - yx) \in (\mathcal{A}_0^{\mathcal{R}})_{\mathrm{sa}}$. Because of equations (2.3) and (2.4) the thesis follows by the previous result. □

Lemma 2.26. *The set $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$ is a C^*-algebra, therefore coincides with $\mathcal{A}^{\mathcal{R}}$ and Theorem 2.4 follows for $\mathcal{A}$.*

Proof. Since $\mathcal{A}_0^{\mathcal{R}}$ is a *-bimodule on $\mathcal{A}$, $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$ is a *-algebra. Moreover, $\mathcal{A}_0^{\mathcal{R}}$ is a closed ideal in the closure of $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$, hence, by Corollary 1.5.8 in [11] $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$ is a C*-algebra. Finally we show that $\mathcal{A}^{\mathcal{R}} \subset \mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$, and the thesis will follow from Lemma 2.13. Indeed, if $x \in \mathcal{A}_{\mathrm{sa}}^{\mathcal{R}}$, it is a separating element for an $\mathcal{R}$-cut $\{a_\varepsilon^\pm\}$ in $\mathcal{A}_{\mathrm{sa}}$. Then $x - a_1^-$ is a separating element for the $\mathcal{R}$-cut $\{a_\varepsilon^\pm - a_1^-\}$ in $(\mathcal{A}_0)_{\mathrm{sa}}$, because $-\varepsilon \leq \tau(a_\varepsilon^- - a_1^-) < 1$, and $0 \leq \tau(a_\varepsilon^+ - a_1^-) < 1 + \varepsilon$. □

Lemma 2.27. *If $x \in (\mathcal{A}_0^{\mathcal{R}} + \mathcal{A})_+$ and $\tau(x)$ is finite, then $x \in \mathcal{A}_0^{\mathcal{R}}$.*

Proof. Let $x = r + a$, $r \in (\mathcal{A}_0^{\mathcal{R}})_{\mathrm{sa}}$, $a \in \mathcal{A}_{\mathrm{sa}}$. Since τ is densely defined on $\mathcal{A}_0^{\mathcal{R}}$ by Theorem 2.23, $r \in \overline{\mathcal{I}(\tau|_{\mathcal{A}_0^{\mathcal{R}}})} \subset \overline{\mathcal{I}(\tau|_{\mathcal{A}_0^{\mathcal{R}}+\mathcal{A}})}$, hence $a \in \overline{\mathcal{I}(\tau|_{\mathcal{A}_0^{\mathcal{R}}+\mathcal{A}})}$. By Lemma 2.2 we get $\tau(\phi_\varepsilon(|a|)) < \infty$ for any $\varepsilon > 0$, hence $a \in \overline{\mathcal{I}(\tau|_{\mathcal{A}})} = \mathcal{A}_0$, and the thesis follows. □

Corollary 2.28. *The $\mathcal{R}$-closure of a C^*-algebra $\mathcal{A}$ with a semicontinuous semifinite trace coincides with $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$. The closure of the domain of $\tau|_{\mathcal{A}^{\mathcal{R}}}$ coincides with $\mathcal{A}_0^{\mathcal{R}}$.*

Proof. The first statement follows by Lemma 2.26. The second statement is an immediate consequence of Lemma 2.27. □

We now mention that the Riemann enveloping algebra of a C*-algebra with a semifinite, non-densely defined trace, which is described as a subalgebra of $\mathcal{A}^{**}$, may also be seen as a subalgebra of the von Neumann enveloping algebra of $\mathcal{A}_0$. The classical counterpart of this result is described in the Appendix.

Lemma 2.29. *$\mathcal{A}_0$ is an essential ideal in $\mathcal{A}$, therefore $\mathcal{A}$ may be identified with a subalgebra of the multipliers of $\mathcal{A}_0$.*

Proposition 2.30. *The Riemann subalgebra $\mathcal{R}(\mathcal{A}_0, \tau) + \mathcal{A}$ in $\mathcal{A}_0^{**}$, where $\mathcal{A}$ is identified with a subalgebra of the multipliers of $\mathcal{A}_0$ as in the previous Lemma, is isomorphic to the Riemann enveloping algebra $\mathcal{R}(\mathcal{A}, \tau)$ of $\mathcal{A}$.*

2.4 Remarks on complete measurability

In this subsection we describe a property of $\mathcal{R}(\mathcal{A}, \tau)$, the $\mathcal{R}$-closure of $\mathcal{A}$ in the universal representation, which we call complete measurability.

Definition 2.31. A linear subspace X in $\mathcal{A}^{**}$ is said completely τ-measurable if, for any faithful representation π of $\mathcal{A}$, the map $\rho_\pi := \pi_\tau \cdot \pi^{-1} : \pi(\mathcal{A}) \to \pi_\tau(\mathcal{A})$ extends to a linear map from $\pi(X)$ in such a way that the following diagram is commutative.

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{A}^{**} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_\tau} \\ \pi(X) & \xrightarrow{\rho_\pi} & \pi_\tau(\mathcal{A})'' \end{array}$$

Remark 2.32. When π is not normal, τ cannot be extended to $\pi(\mathcal{A})''$. However, if $\mathcal{R}$ is a completely measurable C*-algebra, $\mathcal{A} \subset \mathcal{R} \subset \mathcal{A}^{**}$, τ uniquely extends to a trace on $\pi(\mathcal{R})$. Completely measurable "sufficiently large" C*-algebras can be used, in the following section, as a starting point to extend the trace to bimodules of unbounded elements. As Proposition 2.34 shows, $\mathcal{A}^{\mathcal{U}}$ (hence $\mathcal{A}^{\mathcal{R}}$) in the universal representation is completely measurable, however the algebra generated by $\mathcal{A}^{\mathcal{U}}$ may fail to be completely measurable, as in Example 2.16.

Lemma 2.33. *Let us consider the following properties of a linear space X in $\mathcal{A}^{**}$.*

(i) X is completely τ-measurable

(ii) $\pi(x) = 0$ for some faithful representation π of $\mathcal{A}$ implies $\pi_\tau(x) = 0$

(iii) $\pi(x) \geq 0$ for some faithful representation π of $\mathcal{A}$ implies $\pi_\tau(x) \geq 0$.
Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i). If X is linearly generated by its positive elements, (iii) is equivalent to (i) and (ii). If (i) holds, ρ_π is unique, and if in addition X is a *-algebra, ρ_π is a *-algebra morphism.

Proof. $(i) \Leftrightarrow (ii)$ Easily follows by a diagram chasing argument.
$(iii) \Rightarrow (ii)$ obvious.
Now let X be generated by its positive elements. Since $\pi(x) \geq 0$ is equivalent to $\pi(x_-) = 0$, we get $(ii) \Rightarrow (iii)$. Finally if X is completely τ-measurable, uniqueness of ρ_π follows observing that ρ_π is determined by $\rho_\pi(\pi(x)) = \pi_\tau(x)$ for any $x \in X$. Uniqueness of ρ_π implies the last statement. □

Proposition 2.34. *$\mathcal{U}(\mathcal{A}, \tau)$, namely the $\mathcal{U}$-closure of $\mathcal{A}$ in the universal representation, is completely τ-measurable.*

Proof. Indeed ρ defined in Lemma 2.7 gives the required extension. □

The set of completely measurable C*-algebras in $\mathcal{A}^{**}$ containing $\mathcal{A}$ has a maximal element (by Zorn lemma), which is $\mathcal{R}$-closed (by maximality), hence contains $\mathcal{A}^{\mathcal{R}}$. In general, $\mathcal{A}^{\mathcal{R}}$ is not maximal, as the following example shows.

Example 2.35. Let $X := \mathbb{N} \cup \{\infty\}$, $\mathcal{A} := C(X)$, τ the functional induced by some weighted faithful counting measure on X. Clearly $\mathcal{A}^{**}$ consists of the bounded functions on X. Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$, and set $\mathcal{A}^{\mathcal{F}} := \{f \in \mathcal{A}^{**} : \lim_{\mathcal{F}} f = f(\infty)\}$. Then $\mathcal{A}^{\mathcal{F}}$ is completely measurable. Indeed, if π is a faithful representation and $x \in \mathcal{A}^{\mathcal{F}}$, define $f(n)$ by $x\pi(\delta_n) = f(n)\pi(\delta_n)$ and set $\rho_\pi(x)(n) := f(n)$, and $\rho_\pi(x)(\infty) := \lim_{\mathcal{F}} f$.

Remark 2.36. Let π be a faithful representation of $\mathcal{A}$. Then $\pi(\mathcal{U}(\mathcal{A}, \tau)) \subset \mathcal{A}^{\mathcal{U}}$ and $\pi(\mathcal{R}(\mathcal{A}, \tau)) \subset \mathcal{A}^{\mathcal{R}}$, whereas it is not known to the authors if equalities hold.

Proposition 2.37. *The enveloping Riemann algebra of a commutative C*-algebra $\mathcal{A}$ is universal, namely its image under any faithful representation of $\mathcal{A}$ is a Riemann algebra, therefore it coincides with the $\mathcal{R}$-closure in such representation.*

Proof. Because of Remark 2.36, we need only prove that $\mathcal{A}^{\mathcal{U}} \subset \pi(\mathcal{U}(\mathcal{A}, \tau))$, for any faithful representation π of $\mathcal{A}$. Let then $x \in \mathcal{A}^{\mathcal{U}}_{\mathrm{sa}}$, and $\{a^{\pm}_{\varepsilon}\} \subset \mathcal{A}_{\mathrm{sa}}$ s.t. $\pi(a^-_\varepsilon) \leq x \leq \pi(a^+_\varepsilon)$, $\tau(a^+_\varepsilon - a^-_\varepsilon) < \varepsilon$. Then $b^- := \limsup_n a^-_{1/n}$, $b^+ := \liminf_n a^+_{1/n} \in \mathcal{A}^{**}$, and $\pi(b^-) \leq x \leq \pi(b^+)$. From ([17], Proposition I.8.5) there is $y \in (\mathcal{A}^{**})_{\mathrm{sa}}$ s.t. $\pi(y) = x$ and $a^-_{1/n} \leq b^- \leq y \leq b^+ \leq a^+_{1/n}$. Therefore $y \in \mathcal{A}^{\mathcal{U}}$. □

In the classical setting, complete measurability is described as follows.

Proposition 2.38. *Let X be a locally compact Hausdorff space with a Radon measure μ, and let ν be a full measure, namely a measure which is strictly positive on open sets. Then, if a function f is Riemann μ-measurable, and ν-null, it is also μ-null. As a consequence, the integration w.r.t. μ is well defined on the image of the Riemann μ-measurable functions in $L^\infty(X, \nu)$. Equivalently, there is a well defined map from the image of the Riemann μ-measurable functions in $L^\infty(X, \nu)$ into $L^\infty(X, \mu)$.*

Proof. Observe that if f is ν-null, it is indeed null on a dense set, since ν is full. This implies that f is equal to zero on each point where it is continuous and, since f is discontinuous on a μ-negligible set, it is μ-null. □

According to the mentioned universality property of the enveloping Riemann algebra, the image of the previous map is the algebra of Riemann μ-measurable functions in the representation on $L^2(X, d\nu)$.

3 Unbounded Riemann integration

In this Section we give the construction of the unbounded Riemann measurable elements. They form a family which is closed under the $\mathcal{A}^{\mathcal{R}} - \mathcal{A}^{\mathcal{R}}$ *-bimodule operations, even if the *-bimodule properties only hold $\tau - a.e.$ The product of two unbounded Riemann measurable elements is not Riemann measurable in general, but, if S, T are Riemann measurable, there is $\widetilde{T}$, τ-a.e. equivalent to T, s.t. $S\widetilde{T}$ is Riemann measurable.

As a matter of fact, this construction will be performed on a general C*-algebra with a tracial weight, and then particularized to the $\mathcal{R}$-closure of a C*-algebra. We begin with technical results on unbounded operators and operations on them, needed in the sequel.

Let us denote by $\mathcal{L}(\mathcal{H})$ the set of linear operators on $\mathcal{H}$, neither necessarily bounded nor closed, with $\mathcal{C}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the set of closed, densely defined operators, and with $\mathcal{B}(\mathcal{H})$, as usual, the C*-algebra of bounded operators. Let $T \in \mathcal{L}(\mathcal{H})$, then T is densely defined as an operator from $\mathcal{K} := \overline{\mathcal{D}(T)}$ to $\mathcal{H}$, and we can take its adjoint $T^* : \mathcal{D}(T^*) \subset \mathcal{H} \to \mathcal{K}$. Set

$$T^+\xi := \begin{cases} T^*\xi & \xi \in \mathcal{D}(T^*) \\ 0 & \xi \in \mathcal{D}(T^*)^\perp, \end{cases}$$

extended by linearity.

Lemma 3.1. $T^+ \in \mathcal{C}(\mathcal{H})$.

Proof. Let $\{\xi_n\} \subset \mathcal{D}(T^+)$, $\xi_n \to \xi$, $T^+\xi_n \to \eta$, and let p be the projection onto $\overline{\mathcal{D}(T^*)}$. Then $p\xi_n \in \mathcal{D}(T^*)$, $p\xi_n \to p\xi$ and $p^\perp\xi_n \to p^\perp\xi \in \mathcal{D}(T^*)^\perp$, so that $T^*p\xi_n = T^+p\xi_n + T^+p^\perp\xi_n = T^+\xi_n \to \eta$. As T^* is closed, we get $p\xi \in \mathcal{D}(T^*)$ and $T^*p\xi = \eta$, so that $\xi = p\xi + p^\perp\xi \in \mathcal{D}(T^+)$ and $T^+\xi = T^+p\xi = T^*p\xi = \eta$. □

Definition 3.2. For any linear operator T its *natural extension* is the closed operator $T^\natural := (T^+)^*$. Introduce the set of *locally bounded* operators, w.r.t. a projection e, $\mathcal{L}_0(e) := \{T \in \mathcal{L}(\mathcal{H}) :$ there is an increasing sequence of projections $e_n \nearrow e$ s.t. $\mathcal{H}_0 := \cup e_n\mathcal{H} \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$, eTe_n, $e_nTe \in \mathcal{B}(\mathcal{H})\}$.

We want to show that the natural extension of a locally bounded linear operator is locally bounded as well. For the elementary rules of calculus with unbounded operators we refer the reader to [12] (in particular chap. 8).

Lemma 3.3. *$T \in \mathcal{L}_0(e) \Rightarrow \mathcal{D}(Te)$ is dense in $\mathcal{H}$.*

Proof. As $\mathcal{D}(Te) = \{\xi \in \mathcal{H} : e\xi \in \mathcal{D}(T)\} \equiv \mathcal{D}(T) \cap e\mathcal{H} \oplus e^{\perp}\mathcal{H}$ and $\mathcal{D}(T) \cap e\mathcal{H} \supset \mathcal{H}_0$ is dense in $e\mathcal{H}$, we get the thesis. □

Lemma 3.4. *$T \in \mathcal{L}_0(e) \Rightarrow eTe$ is closable and densely defined.*

Proof. eTe is densely defined because of the previous Lemma. Let $\{\xi_n\} \subset \mathcal{D}(eTe)$, $\xi_n \to 0$, $eTe\xi_n \to \eta$. Then $e_m eTe\xi_n = e_m Te\xi_n \to 0$, because $e_m Te$ is bounded. On the other hand $e_m eTe\xi_n \to e_m\eta$, so that $e_m\eta = 0$ for all $m \in \mathbb{N}$. Therefore $\eta = e\eta = 0$. □

Lemma 3.5. *$T \in \mathcal{L}_0(e) \Rightarrow (eTe)^*|_{\mathcal{H}_0} = (eT^+e)|_{\mathcal{H}_0}$.*

Proof. Let us set $\mathcal{K} := \overline{\mathcal{D}(T)}$ and observe that $eT^* \equiv e^*T^* : \mathcal{D}(T^*) \subset \mathcal{H} \to \mathcal{K} \hookrightarrow \mathcal{H}$ is extended by $(Te)^*$, so that $(e_nTe)^* = (Te)^*e_n \supset eT^*e_n$, where the equality holds because Te is densely defined. Therefore $eT^+e_n \subset (e_nTe)^*$, so that equality holds because eT^+e_n is bounded and everywhere defined. Finally $(eTe)^*e_n = (Te)^*ee_n = (Te)^*e_n = eT^+e_n$ which implies the thesis. □

Lemma 3.6. *$T \in \mathcal{L}_0(e) \Rightarrow e_nT^+e \subset (eTe_n)^*$.*

Proof. Let $\xi \in \mathcal{D}(T^*) \cap e\mathcal{H}$, $\eta \in \mathcal{H}$, then

$$\begin{aligned}(\eta, e_nT^+e\xi) &= (\eta, e_nT^+\xi) = (\eta, e_nT^*\xi) = (e_n\eta, T^*\xi)\\ &= (Te_n\eta, \xi) = (Te_n\eta, e\xi) = (eTe_n\eta, \xi) = (\eta, (eTe_n)^*\xi)\end{aligned}$$

because $e_n\eta \in \mathcal{D}(T)$ and eTe_n is bounded. □

Lemma 3.7. *$T \in \mathcal{L}_0(e) \Rightarrow eTe|_{\mathcal{H}_0} = eT^\natural e|_{\mathcal{H}_0}$.*

Proof. As $e_nT^+e \subset (eTe_n)^* \Rightarrow eTe_n \subset (e_nT^+e)^*$, we get $eTe_n = (e_nT^+e)^* \supset e(e_nT^+)^* = eT^\natural e_n$. But $e_n\mathcal{H} \subset \mathcal{D}(T^\natural)$, implies $eT^\natural e_n \in \mathcal{B}(\mathcal{H})$, and the thesis follows. □

Proposition 3.8. *Let T be a locally bounded operator w.r.t. e. Then T^+ and $T^\natural$ are locally bounded w.r.t. e, too. Moreover, setting $T_m := \overline{(eTe)|_{\mathcal{H}_0}}$ and $T_M := \overline{((eTe)^*|_{\mathcal{H}_0})^*}$, T_m, T_M are closed operators s.t. $(T_M)^* = (T^*)_m$, and $T_m \subset \overline{eT^\natural e} \subset T_M$.*

Proof. The first statement follows from $eT^+e_n = (e_nTe)^*$ and $e_nT^+e \subset (eTe_n)^*$. From Lemma 3.4 it follows that eT^+e is closable, therefore $\overline{eT^+e} \supset \overline{(eT^+e)|_{\mathcal{H}_0}} = \overline{(eT^*e)|_{\mathcal{H}_0}} = T_M^*$, so that $T_M \supset (eT^+e)^* \supset e(eT^+)^* = eT^\natural e$. Finally $\overline{eT^\natural e} \supset \overline{eT^\natural e|_{\mathcal{H}_0}} = \overline{eTe|_{\mathcal{H}_0}} = T_m$. □

Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a Riemann algebra w.r.t. a semicontinuous semifinite trace τ. Inspired by Christensen [1] we now introduce the set of essentially τ-measurable operators (a subset of the locally bounded ones). Recall that $T\widehat{\in}\mathcal{R}''$ stands for $u'T \subset Tu'$ for any unitary operator $u' \in \mathcal{R}'$.

Definition 3.9. A sequence $\{e_n\}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a *Strongly Dense Domain* (SDD) w.r.t. $(\mathcal{R}, \tau)$, if $e_n^\perp \in \mathcal{R}$ is τ-finite, $\tau(e_n^\perp) \to 0$. Then $e^\perp := \inf_n e_n^\perp \in \mathcal{R}$, because of $\mathcal{R}$-closedness. If $T\widehat{\in}\mathcal{R}''$, we say $\{e_n\}$ is a SDD for T if $\mathcal{H}_0 := \cup e_n\mathcal{H} \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$ and eTe_n, $e_nTe \in \mathcal{R}$. Let us introduce the set of *essentially τ-measurable* operators $\overline{\mathcal{R}}_0 := \{T\widehat{\in}\mathcal{R}'' : \text{there is a SDD for } T\}$, and the set of *$\tau$-measurable* operators $\overline{\mathcal{R}} := \overline{\mathcal{R}}_0 \cap \mathcal{C}(\mathcal{H})$. Let S, $T\widehat{\in}\mathcal{R}''$, we say that $S = T$ almost everywhere if there is a common SDD $\{e_n\}$ for S and T s.t. $eSe|_{\mathcal{H}_0} = eTe|_{\mathcal{H}_0}$.

Remark 3.10. In order for $\overline{\mathcal{R}}$ to be larger than $\mathcal{R}$, one needs SDD's for which $\tau(e_n^\perp) > 0$ for any n.

Lemma 3.11. $T \in \mathcal{L}(\mathcal{H})$, $T\widehat{\in}\mathcal{R}'' \Rightarrow T^+$, $T^\natural\widehat{\in}\mathcal{R}''$.

Proof. Let $u' \in \mathcal{U}(\mathcal{R}')$, then $u'T \subset Tu'$ so that $u'T^* \subset T^*u'$, therefore $u'\mathcal{D}(T^*) = \mathcal{D}(T^*)$. Moreover for $\eta \in \mathcal{D}(T^*)^\perp = u'\mathcal{D}(T^*)^\perp$ we get $T^+u'\eta = 0 = u'T^+\eta$. In all $T^+\widehat{\in}\mathcal{R}''$, so that $T^\natural\widehat{\in}\mathcal{R}''$ as well. □

Lemma 3.12. *Let $\mathcal{R}$ be a C^*-algebra, e, f projections in $\mathcal{R}$. Then $e \vee f$, $e \wedge f \in \mathcal{R}$.*

Proof. As $e \wedge f \leq e + f \leq 2(e \wedge f)$ and $t \to t^{1/n}$ is operator-increasing, it follows $e \wedge f \leq (e+f)^{1/n} \leq 2^{1/n}(e \wedge f)$. Therefore $(e+f)^{1/n} \to e \wedge f$ and $e \wedge f \in \mathcal{R}$. Besides, in the unitalization of $\mathcal{R}$, $e \vee f = 1 - (1-e) \wedge (1-f) = \lim_n 1 - (2 - e - f)^{1/n} = \lim_n 1 - 2^{1/n} \sum_{k=0}^\infty (-1)^k \binom{1/n}{k} \left(\frac{e+f}{2}\right)^{1/n} = \lim_n e_n$, where $e_n := 2^{1/n} \sum_{k=1}^\infty (-1)^{k-1} \binom{1/n}{k} \left(\frac{e+f}{2}\right)^{1/n} \in \mathcal{R}$. □

Lemma 3.13. *Let $T \in \overline{\mathcal{R}}_0$, $\{e_n\}$, $\{f_n\}$ SDD for T. Then $g_n := e_n \wedge f_n$ is an SDD for T.*

Proof. Because of Lemma 3.12, $g_n^\perp \in \mathcal{R}$, and $\tau(g_n^\perp) \to 0$, that is $\{g_n\}$ is an SDD. Besides $g_nTg = g_ne_nTeg = e_nTeg - g_n^\perp e_nTeg = e_nTe - e_nTeg^\perp - g_n^\perp e_nTe + g_n^\perp e_nTeg^\perp \in \mathcal{R}$, and the thesis follows. □

Proposition 3.14.

(*i*) *Equality almost everywhere is an equivalence relation.*

(*ii*) $T \in \overline{\mathcal{R}}_0 \;\Rightarrow\; T = T^\natural$ *almost everywhere and* $T^\natural \in \overline{\mathcal{R}}_0$.

Proof. (i) is obvious.
(ii) As a SDD for T is also a SDD for $T^\natural$, the thesis follows from Lemmas 3.11 and 3.7. □

In particular we have just proved that the equivalence class of an operator in $\overline{\mathcal{R}}_0$ contains a closed operator, hence $\overline{\mathcal{R}}_0/{\sim}=\overline{\mathcal{R}}/{\sim}$.

Lemma 3.15. *$S, T \in \overline{\mathcal{R}}_0 \Rightarrow S+T \in \overline{\mathcal{R}}_0$ and $(S+T)^\natural = S^\natural + T^\natural$ almost everywhere.*

Proof. Assume $\{e_n\}$, $\{f_n\}$ are SDD for S and T respectively, and set $g_n := e_n \wedge f_n$. Then $\{g_n\}$ is a common SDD for S and T, as in Lemma 3.13. Besides $g_n(S+T)g = g_n e_n S e g + g_n f_n T f g \in \mathcal{R}$, $g(S+T)g_n = g e S e_n g_n + g f T f_n g_n \in \mathcal{R}$, as in Lemma 3.13. Finally $g(S+T)^\natural g|_{\mathcal{H}_0} = g(S+T)g|_{\mathcal{H}_0} = gSg|_{\mathcal{H}_0} + gTg|_{\mathcal{H}_0} = g(S^\natural + T^\natural)g|_{\mathcal{H}_0}$. □

Lemma 3.16. *$T \in \overline{\mathcal{R}}_0$, $a \in \mathcal{R} \Rightarrow aT, Ta \in \overline{\mathcal{R}}_0$, and $(aT)^\natural = aT^\natural$, $(Ta)^\natural = T^\natural a$ almost everywhere.*

Proof. Using the previous Lemma, we need only prove it for $a = u$ a unitary operator. Set $g_n := e_n \wedge u^* e_n u \wedge u e_n u^*$. Then g_n is a common SDD for T, uT, Tu, as in Lemma 3.13, and $g_n T u g = g_n e_n T u u^* e u g \in \mathcal{R}$, $g T u g_n = g e T u u^* e_n u g_n \in \mathcal{R}$, as in Lemma 3.13. Analogously $g_n u T g \in \mathcal{R}$, $g u T g_n \in \mathcal{R}$. The last two statements are proved as in the previous Lemma. □

Definition 3.17. For $S,\ T \in \overline{\mathcal{R}}$, $a \in \mathcal{R}$ define $S \oplus T := (S+T)^\natural$, $a \odot T := (aT)^\natural$, $T \odot a := (Ta)^\natural$.

Theorem 3.18. *$\overline{\mathcal{R}}$ is an almost everywhere *-bimodule over $\mathcal{R}$, w.r.t. strong sense operations, namely $S, T \in \overline{\mathcal{R}}$, $a \in \mathcal{R} \Rightarrow S \oplus T \in \overline{\mathcal{R}}$, $a \odot T$, $T \odot a \in \overline{\mathcal{R}}$, and the bimodule properties hold almost everywhere.*

Proof. Follows by a repeated application of Lemmas 3.15, 3.16, and Proposition 3.14. □

Observe that the previous Theorem generalises results by Segal [14], indeed, if $\mathcal{R}$ is a von Neumann algebra and τ a normal semifinite faithful trace on it, equality almost everywhere turns out to be equality (cf. [14] Corollary 5.1) and the two notions of strong sense operations coincide.

Proposition 3.19. *If $S, T \in \overline{\mathcal{R}}$ and $\{e_n\}$ is an SDD both for S and T then SeT and TeS belong to $\overline{\mathcal{R}}$. In particular, there exists $\widetilde{T}$ τ-a.e. equivalent to T such that $S\widetilde{T}$ and $\widetilde{T}S$ belong to $\overline{\mathcal{R}}$.*

Proof. Let $eTe_n = \sum_i \lambda_{ni} u_{ni}$, $e_n Se = \sum_i \mu_{ni} v_{ni}$ be decompositions of eTe_n and $e_n Se$ into linear combinations of unitaries in $\mathcal{R}$, and set $f_n := \wedge_i u_{ni}^* e_n u_{ni} \wedge e_n \wedge_i v_{ni} e_n v_{ni}^*$. Then $f_n \mathcal{H} \subset \mathcal{D}(T)$ and

$$\begin{aligned} eTf_n = eTe_n f_n &= \sum_i \lambda_{ni} u_{ni} f_n = \sum_i \lambda_{ni} e_n u_{ni} f_n \\ &= e_n (eTe_n) f_n, \end{aligned}$$

as a consequence $eTf_n\mathcal{H} \subset \mathcal{D}(S)$, then $fSeTf_n = f(eSe_n)(eTe_n)f_n \in \mathcal{R}$. Repeating this argument for $f_n SeTf$ we show that $\{f_n\}$ is an SDD for SeT. Since $\widetilde{T} := \overline{eTe}$ is τ-a.e. equivalent to T the last statement follows. □

From now on we denote by ρ the GNS representation of τ, and with $\mathcal{M} := \rho(\mathcal{R})''$. We want to extend ρ to a morphism of $\overline{\mathcal{R}}$ to the *-algebra $\overline{\mathcal{M}}$ of τ-measurable operators affiliated with $\mathcal{M}$ [14]. Let us recall that the topology of convergence in measure in $\overline{\mathcal{M}}$ is generated by the neighborhood basis $\{V(\varepsilon,\delta)\}_{\varepsilon,\delta>0}$, where $V(\varepsilon,\delta) := \{T \in \overline{\mathcal{M}}$: there is a projection $p \in \mathcal{M}$ s.t. $\tau(p^\perp) < \delta$, $\|Tp\| < \varepsilon\}$. Let us set $T = \tau \lim T_n$ for $T_n \to T$ in measure.

Lemma 3.20. *Let $T \in \overline{\mathcal{R}}$. Then $\rho(T) := \tau\, lim\, \rho(eTe_n)$ does not depend on the SDD $\{e_n\}$ and $\rho(T) \in \overline{\mathcal{M}}$.*

Proof. Let $\{f_n\}$ be another SDD for T, and set $g_n := e_n \wedge f_n$. Then $\tau(g_n^\perp) \to 0$, and $\|(\rho(eTe_n) - \rho(fTf_n))(1-\rho(g_n^\perp))\| \le \|\rho(e-g)\rho(eTe_n)(1-\rho(g_n^\perp))\| + \|\rho(f-g)\rho(fTf_n)(1-\rho(g_n^\perp))\| = 0$, because $\rho(e^\perp) = \rho(f^\perp) = \rho(g^\perp) = 0$, as, for example, $\rho(e^\perp) = \tau \lim \rho(e_n^\perp)$. Indeed for all $\delta > 0$, let $k \in \mathbb{N}$ be s.t. $\tau(e_k^\perp) < \delta$, then for $n > k$ one has $\|\rho(e_n^\perp)(1-\rho(e_k^\perp))\| = 0$. Finally $\|[\rho(eTe_m) - \rho(eTe_n)](1-\rho(e_k^\perp))\| = 0$, therefore $\rho(T) \in \overline{\mathcal{M}}$. □

Lemma 3.21. $T \in \overline{\mathcal{R}} \;\Rightarrow\; \tau\, lim\, \rho(eTe_n) = \tau\, lim\, \rho(e_nTe)$.

Proof. Observe that from ([15], Theorem 3.7) $\tau \lim \rho(e_nTe_n) = \tau \lim (1-\rho(e_n^\perp))\rho(eTe_n) = \rho(T)$. Finally $\tau \lim \rho(e_nTe) = \tau \lim \rho(e_nTe_n)$, because for all $\delta > 0$, let $k \in \mathbb{N}$ be s.t. $\tau(e_k^\perp) < \delta$, and $n > k$, then $\|[\rho(e_nTe) - \rho(e_nTe_n)](1-\rho(e_k^\perp))\| = \|\rho(e_nTe_k) - \rho(e_nTe_k)\| = 0$. The thesis follows. □

Observe that $\rho(\overline{eTe}) = \rho(T)$, because $\rho(\overline{eTe}) = \tau \lim \rho(eTee_n) = \rho(T)$.

Lemma 3.22. $T \in \overline{\mathcal{R}} \;\Rightarrow\; \rho(T^*) = \rho(T)^*$.

Proof.

$$\begin{aligned} \rho(T^*) &= \tau \lim \rho(eT^*e_n) = \tau \lim \rho(eT^+e_n) = \tau \lim \rho(e_nT^+e) \\ &= \tau \lim \rho((eTe_n)^*) = \tau \lim \rho(eTe_n)^* = \rho(T)^*. \end{aligned}$$

□

Lemma 3.23. *$S, T \in \overline{\mathcal{R}} \;\Rightarrow\; \rho(S \oplus T) = \rho(S) \oplus \rho(T)$.*

Proof. Using Theorem 3.3 in [15] one has

$$\begin{aligned}\rho(S \oplus T) &= \tau \lim \rho(g(S+T)^\natural g_n) = \tau \lim \rho(g(S+T)g_n) \\ &= \tau \lim \rho(gSg_n) \oplus \tau \lim \rho(gTg_n) = \rho(S) \oplus \rho(T).\end{aligned}$$

□

Lemma 3.24. *$a \in \mathcal{R}$, $T \in \overline{\mathcal{R}} \;\Rightarrow\; \rho(a \odot T) = \rho(a) \odot \rho(T)$, and $\rho(T \odot a) = \rho(T) \odot \rho(a)$.*

Proof. First we prove it for $a \in \mathcal{U}(\mathcal{R})$. Let f_n be an SDD for T, and set $g_n := f_n \wedge a f_n a^*$. Then, by ([15], Theorem 3.7)

$$\begin{aligned}\rho(a \odot T) &= \tau \lim \rho(g(aT)^\natural g_n) = \tau \lim \rho(gaTg_n) = \tau \lim \rho(gafa^*aTf_ng_n) \\ &= \tau \lim (1 - \rho(g^\perp))\rho(a)\rho(fTf_n)(1 - \rho(g_n^\perp)) = \rho(a) \odot \rho(T).\end{aligned}$$

The general case follows from Lemmas 3.15, 3.16, 3.23. The proof of $\rho(T \odot a) = \rho(T) \odot \rho(a)$ is analogous. □

Theorem 3.25. *The map $\rho : \overline{\mathcal{R}} \to \overline{\mathcal{M}}$ is a morphism of almost everywhere bimodules. Therefore $\tau \circ \rho$ is a trace on $\overline{\mathcal{R}}$ as an almost everywhere bimodule on $\mathcal{R}$, extending τ on $\mathcal{R}$. Besides, if $S, T \in \overline{\mathcal{R}}$ and $\{e_n\}$ is a common SDD, then $\rho(SeT) = \rho(S)\rho(T)$.*

Proof. The first part easily follows by the Lemmas. Concerning the last statement, we have

$$\rho(SeT) = \tau \lim \rho(e_nSeTe_n) = \tau \lim \rho(e_nSe)\tau \lim \rho(eTe_n) = \rho(S)\rho(T)$$

□

Remark 3.26. We proved in Theorem 3.18 that $\overline{\mathcal{R}}/\sim$ is a *-bimodule over $\mathcal{R}$, and it immediately follows from the definition of ρ contained in Lemma 3.20, that if $T \in \overline{\mathcal{R}}$, $T = 0$ τ-a.e., then $\rho(T) = 0$. Therefore we get a bimodule map from $\overline{\mathcal{R}}/\sim$ to $\overline{\mathcal{M}}$, and we could ask if this is an isomorphism. That this is not always the case is illustrated by the following example, where we exhibit a pair $(\mathcal{A}, \tau)$, and an element $a \in \mathcal{A}_+$ s.t. $\rho(a) \equiv \pi_\tau(a) = 0$, but $a \neq 0$ τ-a.e. Indeed let $\mathcal{A} := C[0,1]$, $\tau :=$ the Dirac measure in 0, and $a(t) := t$, $t \in [0,1]$. Clearly $\pi_\tau(a) = 0$, whereas, since any SDD $e_n = \chi_{\Omega_n}$ is s.t. $0 \in \Omega_n$ eventually, we get that $eae_n \neq 0$ eventually.

As the previous Remark shows, the kernel of ρ is larger than the set of τ-a.e. zero operators, and the following Proposition characterises it.

Proposition 3.27. *Let $T \in \overline{\mathcal{R}}$, then $\rho(T) = 0$ iff for any $\varepsilon > 0$, there is an SDD $\{e_n\}$ for T s.t. $\|eTe\| < \varepsilon$.*

Proof. This condition is clearly sufficient, since $\|\rho(T)\| = \rho(eTe)\| \leq \|eTe\| < \varepsilon$, and the thesis follows from the arbitrariness of ε.
Conversely, $\rho(T) = 0$ means that $\tau \lim \rho(eTe_n) = 0$, which implies that for any $\varepsilon, \delta > 0$ there are $n_0 \in \mathbb{N}$ and a projection $p \in \mathcal{M}$ s.t. $\tau(p^\perp) < \delta$ and $\|\rho(e_nT^*e)p\| < \varepsilon$, for any $n > n_0$. Therefore for any $i \in \mathbb{N}$, choosing $n > \max\{i, n_0\}$, we get $\|\rho(e_iT^*e)p\| = \|\rho(e_i)\rho(e_nT^*e)p\| < \varepsilon$, that is $\rho(e_iT^*e)$ belongs to any neighbourhood of 0 in the measure topology, hence $\rho(e_iT^*e) = 0$ for any $i \in \mathbb{N}$. This implies $\tau(e_iT^*eTe_i) = 0$, so that for any $\varepsilon > 0$ there is a spectral projection $e_{i\varepsilon}$ of $e_iT^*eTe_i$ s.t. $\|e_iT^*eTe_i\| < \varepsilon$, $e_{i\varepsilon}^\perp \in \mathcal{A}^\mathcal{R}$ and $\tau(e_{i\varepsilon}^\perp) = 0$. Then $\{q_n\}$, where $q_n := e_n \wedge e_{n\varepsilon}$, is an SDD for T, and, with $q := \sup q_n$, we get $\|qTq_n\| \leq \|eTe_ne_{n\varepsilon}\| < \varepsilon^{1/2}$. Since $\|qTq\| = \sup \|qTq_n\|$, we get the thesis. □

Finally we show how the previous construction can be applied in order to extend a semicontinuous semifinite trace on a concrete C*-algebra to a suitable family of unbounded operators. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C*-algebra with a semicontinuous semifinite trace τ, and let $\mathcal{A}^\mathcal{R}$ be the algebra of bounded Riemann measurable elements, then we call $\overline{\mathcal{A}^\mathcal{R}}$ the bimodule of unbounded Riemann measurable elements, and the extension of τ from $\mathcal{A}^\mathcal{R}$ to $\overline{\mathcal{A}^\mathcal{R}}$, provided by the previous Theorem, the (noncommutative) unbounded Riemann integral.

Proposition 3.28. *Let $(\mathcal{A}, \tau)$, $\mathcal{A}^\mathcal{R}$ and $\overline{\mathcal{A}^\mathcal{R}}$ be as above. Then τ extends to a trace on $\overline{\mathcal{A}^\mathcal{R}}$ as an almost everywhere bimodule on $\mathcal{A}^\mathcal{R}$, namely $\tau(Ax) = \tau(xA)$ for any $x \in \mathcal{A}^\mathcal{R}$, $A \in \overline{\mathcal{A}^\mathcal{R}}$. Moreover, unbounded Riemann functional calculi of bounded Riemann measurable elements are Riemann measurable, namely for any $x \in \mathcal{A}^\mathcal{R}_{sa}$ and $f \in \overline{\mathcal{R}}_0(\sigma(x) \setminus \{0\}, \mu_x)$, $f(x) \in \overline{\mathcal{A}^\mathcal{R}}$.*

Proof. The first statement follows by the previous results in this section. Let f be in $\overline{\mathcal{R}}_0(\sigma(x) \setminus \{0\}, \mu_x)$. By Proposition 5.20 in the Appendix, there exists an SDD given by the characteristic functions of Riemann measurable sets G_n s.t. $f|_{G_n}$ is Riemann measurable. As a consequence $\chi_{G_n}(x)$ give an SDD for $f(x)$, and the second statement follows. □

Proposition 3.29. *Let $(\mathcal{A}, \tau)$ be a C*-algebra with a semicontinuous semifinite trace τ acting on a Hilbert space $\mathcal{H}$ and let $x \in \overline{\mathcal{A}^\mathcal{R}} \cap \mathcal{A}''$. Then there is a projection $p \in \mathcal{A}^\mathcal{R}$ s.t. $\tau(p) = 0$, and $p^\perp xp^\perp \in \mathcal{A}^\mathcal{R}$.*

Proof. Let $\{e_n\}$ be an SDD for x. Then exe_n and e_nxe belong to $\mathcal{A}^\mathcal{R}$, and we may set $x_n^\pm = exe_n + e_nxe - e_nxe_n \pm \|x\|e_n^\perp$. Since $\tau(x_n^+ - x_n^-) \leq 2\|x\|\tau(e_n^\perp) \to 0$, $x_n^\pm$ give an $\mathcal{R}$-cut for exe, and the thesis follows with $p := e^\perp$. □

Remark 3.30. (*i*) $\overline{\mathcal{A}^\mathcal{R}}$ is not an algebra in general, and the following counterexample shows that even $\overline{\mathcal{A}^\mathcal{R}} \cap \mathcal{A}''$ is not an algebra. Let $(\mathcal{A}, \tau)$ be as in Example 2.16, denote by χ the characteristic function of $\{\infty\}$ and set

$$v = \begin{pmatrix} 0 & \chi \\ 0 & 0 \end{pmatrix}, \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 1-\chi \end{pmatrix}.$$

We easily get $e \in \mathcal{A}^{\mathcal{R}}$, $\tau(e^{\perp}) = 0$, $eve = ev^*e = 0$, and

$$vv^* = \begin{pmatrix} \chi & 0 \\ 0 & 0 \end{pmatrix} \notin \mathcal{A}^{\mathcal{R}}, \qquad v^*v = \begin{pmatrix} 0 & 0 \\ 0 & \chi \end{pmatrix} = e^{\perp} \in \mathcal{A}^{\mathcal{R}}.$$

Clearly $e_n \equiv e$ is an SDD for v, hence $v \in \overline{\mathcal{A}^{\mathcal{R}}} \cap \mathcal{A}''$, and the same for v^*, but vv^* does not belong to $\overline{\mathcal{A}^{\mathcal{R}}} \cap \mathcal{A}''$. Indeed, since vv^* is a minimal central projection in $\mathcal{A}''$ with $\tau(vv^*) = 1$, the characterization in the previous Proposition implies that $vv^* \in \overline{\mathcal{A}^{\mathcal{R}}} \cap \mathcal{A}''$ if and only if it belongs to $\mathcal{A}^{\mathcal{R}}$, which is false. It is also possible to show that the C*-algebra generated by $\overline{\mathcal{A}^{\mathcal{R}}} \cap \mathcal{A}''$ in the universal representation is not completely measurable in this case.
(ii) If $A \in \overline{\mathcal{A}^{\mathcal{R}}}$ has finite trace or is positive, its trace may be computed as $\lim_{e \in \mathcal{A}^{\mathcal{R}}} \tau(eAe)$, where the projections e satisfy $eAe \in \mathcal{A}^{\mathcal{R}}$.

4 Singular traces on C*-algebras

In this Section we construct singular traces on a C*-algebra with a semicontinuous semifinite trace. Let us first recall that, if $\mathcal{M}$ is a von Neumann algebra with a normal semifinite faithful trace τ, $\overline{\mathcal{M}}$ the *-algebra of τ-measurable operators, and $T \in \overline{\mathcal{M}}$, its distribution function and non-increasing rearrangement, the basic building blocks for the construction of singular traces [5], are defined as follows (cf. e.g. [4, 5])

$$\lambda_T(t) := \tau(\chi_{(t,+\infty)}(|T|))$$
$$\mu_T(t) := \inf\{s \geq 0 : \lambda_T(s) \leq t\}.$$

Let now $\mathcal{A}$ be a C*-algebra with a semicontinuous semifinite trace τ acting on a Hilbert space $\mathcal{H}$. As follows from the previous Section, the GNS representation of $\mathcal{A}$ extends to a *-bimodule map from the unbounded Riemann measurable operators in $\overline{\mathcal{A}^{\mathcal{R}}}$ into the measurable operators of $\overline{\mathcal{M}}$, so that we may define the distribution function (and therefore the associated non-increasing rearrangement) w.r.t. τ of an operator $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ as $\lambda_T = \lambda_{\pi(T)}$, and we get $\mu_T = \mu_{\rho(T)}$. Let us observe that, if $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ is a positive (unbounded) continuous functional calculus of an element in $\mathcal{A}^{\mathcal{R}}$, then $\chi_{(t,+\infty)}(T)$ belongs to $\mathcal{A}^{\mathcal{R}}$ a.e., therefore its distribution function may be defined without using the representation ρ as $\lambda_T(t) = \tau(\chi_{(t,+\infty)}(T))$. With these preliminaries out of the way, we may carry out the construction of singular traces (with respect to τ) as it has been done in [5].

Definition 4.1. An operator $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ is called *eccentric* at 0 if either

$$\int_0^1 \mu_T(t)dt < \infty \quad \text{and} \quad \liminf_{t \to 0} \frac{\int_0^{2t} \mu_T(s)ds}{\int_0^t \mu_T(s)ds} = 1$$

or

$$\int_0^1 \mu_T(t)dt = \infty \quad \text{and} \quad \limsup_{t \to 0} \frac{\int_{2t}^1 \mu_T(s)ds}{\int_t^1 \mu_T(s)ds} = 1.$$

It is called *eccentric* at ∞ if either

$$\int_1^\infty \mu_T(t)dt < \infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_{2t}^\infty \mu_T(s)ds}{\int_t^\infty \mu_T(s)ds} = 1$$

or

$$\int_1^\infty \mu_T(t)dt = \infty \quad \text{and} \quad \liminf_{t\to\infty} \frac{\int_1^{2t} \mu_T(s)ds}{\int_1^t \mu_T(s)ds} = 1.$$

The following proposition trivially holds

Proposition 4.2. *Let $(\mathcal{A},\tau)$ be a C^*-algebra with a semicontinuous semifinite trace, $T \in \overline{\mathcal{A}^{\mathcal{R}}}$, and let $X(T)$ denote the *-bimodule over $\mathcal{A}^{\mathcal{R}}$ generated by T in $\overline{\mathcal{A}^{\mathcal{R}}}$, while $X(\rho(T))$ denotes the *-bimodule over $\mathcal{M} := \pi(\mathcal{A})''$ generated by $\rho(T)$ in $\overline{\mathcal{M}}$. Then*

(i) T is eccentric if and only if $\rho(T)$ is

(ii) $\rho(X(T)) \subset X(\rho(T))$.

As in the case of von Neumann algebras, with any eccentric operator (at 0 or at ∞) in $\overline{\mathcal{A}^{\mathcal{R}}}$ we may associate a singular trace, where the word singular refers to the original trace τ. Indeed such singular traces vanish on τ-finite operators, and those associated to 0-eccentric operators even vanish on all bounded operators. Of course singular traces may be described as the pull-back of the singular traces on $\mathcal{M}$ via the (extended) GNS representation. On the other hand, explicit formulas may be written in terms of the non-increasing rearrangement. Since Riemann integration is crucial in the extension of the trace to unbounded operators, we write these formulas only in case of 0-eccentric operators. Moreover this is the case occuring in [6]. First we observe that, by definition, if $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ is eccentric (at 0) there exists a pure state ω on $C_b(0,\infty)$ which is a generalized limit in 0, namely is an extension of the Dirac measure concentrated in 0 from $C_b[0,\infty)$ to $C_b(0,\infty)$, such that

$$\text{if} \quad \int_0^1 \mu_T(t)dt < \infty \quad \text{then} \quad \omega\left(\frac{\int_0^{2t} \mu_T(s)ds}{\int_0^t \mu_T(s)ds}\right) = 1$$

$$\text{if} \quad \int_0^1 \mu_T(t)dt = \infty \quad \text{then} \quad \omega\left(\frac{\int_{2t}^1 \mu_T(s)ds}{\int_t^1 \mu_T(s)ds}\right) = 1.$$

Then the singular trace associated with τ, T and ω may be written as follows on the a.e. positive elements of $X(T)$, i.e. elements whose image under ρ is positive

$$\begin{aligned} \tau_\omega(A) &:= \omega\left(\frac{\int_0^t \mu_A(s)ds}{\int_0^t \mu_T(s)ds}\right), \quad A \in X(T)_+ \quad \text{if} \int_0^1 \mu_T(t)dt < \infty \\ \tau_\omega(A) &:= \omega\left(\frac{\int_t^1 \mu_A(s)ds}{\int_t^1 \mu_T(s)ds}\right), \quad A \in X(T)_+ \quad \text{if} \int_0^1 \mu_T(t)dt = \infty \end{aligned} \tag{4.1}$$

According to the previous analysis, some results in [5] may be rephrased as follows

Theorem 4.3. *The functionals defined in formula* (4.1) *linearly extend to traces on $X(T)$, the a.e. *-bimodule over $\mathcal{A}^{\mathcal{R}}$ generated by T. They also naturally extend to traces on $X(T) + \mathcal{A}^{\mathcal{R}}$, setting $\tau_\omega|_{\mathcal{A}^{\mathcal{R}}} = 0$.*

We conclude this section mentioning that the proof of Lemma 2.5 in [5] contains a gap, even though the statement is correct. We thank B. De Pagter and F. Sukochev for having noticed this gap and having also furnished a correct proof. Since the Lemma is quite standard, stating that the unique positive dilation invariant functional on the cone of bounded, right continuous functions compact support on $(0, +\infty)$ is the Lebesgue measure (up to a positive constant), we do not include the proof here.

5 Appendix

Here we present some more or less known results relating weights on $C_0(X)$, with the measures on X they define, then we collect some results on the Riemann measurable functions on X which are needed in the previous sections. Finally we discuss the relations between our non-abelian Riemann integration and the classical theory. Throughout this appendix X will denote a locally compact Hausdorff space.

5.1 Weights on $C_0(X)$

Let φ be a weight on $C_0(X)$, and set dom $\varphi :=$ span $\{f \in C_0(X)_+ : \varphi(f) < \infty\}$.

Definition 5.1. We say $x \in X$ is φ-finite if there is $f \in C_c(X)_+$ s.t. $f(x) > 0$ and $\varphi(f) < \infty$. Set $X_0 := \{x \in X : x \text{ is } \varphi\text{-finite }\}$.

Proposition 5.2.

(i) X_0 is an open subset of X

(ii) on identifying $C_0(X_0)$ with $\{f \in C_0(X) : f|_{X_0^c} \equiv 0\}$ (see next Lemma), one has $C_c(X_0) \subset$ dom φ

(iii) X_0 is the largest open set in X satisfying (ii)

(iv) dom $\varphi \subset C_0(X_0)$ so that $\overline{\text{dom } \varphi} = C_0(X_0)$

(v) if μ denotes the Radon measure associated, by Riesz theorem, to $\varphi|_{C_c(X_0)}$, then $\int f d\mu \leq \varphi(f)$, for all $f \in C_0(X_0)$.

Proof. (i) Let $x_0 \in X_0$, and $f \in C_c(X)_+$ be s.t. $f(x_0) = 1$, $\varphi(f) < \infty$. Then $\{x \in X : f(x) > \frac{1}{2}\}$ is an open neighbourhood of x_0, and is contained in X_0.
(ii) Let $f \in C_c(X_0)_+$, then any $x \in$ supp$f$ belongs to X_0 and there is $g_x \in C_c(X)_+$ s.t. $g_x(x) = 1$, $\varphi(g_x) < \infty$. Let $\Omega_x := \{y \in X : g_x(y) > \frac{1}{2}\}$, then, as supp$f$ is compact, there are $x_1, \ldots, x_n \in$ supp$f$ s.t. supp$f \subset \cup_{i=1}^n \Omega_{x_i}$.

Therefore $f \leq 2\|f\| \sum_{i=1}^{n} g_{x_i}$ and the thesis follows.
(iii) Let V be an open subset of X s.t. $V \supset X_0$, and let $x_0 \in V \setminus X_0$. Then for any open set $U \ni x_0$, $U \subset V$, there is, by Urysohn's lemma, $f \in C_c(V)_+$ s.t. $f(x_0) = 1$, and $\mathrm{supp} f \subset U$, but $\varphi(f) = \infty$, because $x_0 \notin X_0$. Therefore φ is not finite on $C_c(V)$.
(iv) Let $f \in C_0(X)_+ \setminus C_0(X_0)$. Then there is $x_0 \in X \setminus X_0$ s.t. $f(x_0) > 0$. Then $\varphi(f) = \infty$ and $f \notin \mathrm{dom}\, \varphi$. Therefore $\mathrm{dom}\, \varphi \subset C_0(X_0)$ and the thesis follows from (ii).
(v) Let $f \in C_0(X_0)_+$. As $C_c(X_0)_+$ is dense in $C_0(X_0)_+$, there is $\{g_n\} \subset C_c(X_0)_+$ s.t. $g_n \to f$. Setting $f_n := f \wedge \sup_{1 \leq k \leq n} g_n \in C_c(X_0)_+$, we get $f_n \nearrow f$. Then $\int f d\mu = \lim \int f_n d\mu = \lim \varphi(f_n) \leq \varphi(f)$. □

Lemma 5.3. *The map $\psi : f \in C_0(X_0) \to \tilde{f} \in \{f \in C_0(X) : f|_{X_0^c} \equiv 0\}$ is an isomorphism, where*

$$\tilde{f}(x) := \begin{cases} f(x) & x \in X_0 \\ 0 & x \notin X_0. \end{cases}$$

Proof. We need only prove surjectivity of ψ. Observe that $f \in C_0(X)$ iff $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact for any $\varepsilon > 0$. Let now $f \in C_0(X)$ with $f|_{X_0^c} = 0$. Then $\{x \in X : |f(x)| \geq \varepsilon\}$ is a compact subset of X_0 for any $\varepsilon > 0$, and the thesis follows. □

Proposition 5.4. *Let φ be a weight on $C_0(X)$, and μ be the Radon measure associated to $\varphi|_{C_c(X_0)}$. Then*

(i) φ is lower semicontinuous iff $\varphi(f) = \int f d\mu$, for any $f \in C_0(X_0)$

(ii) if φ is lower semicontinuous, setting $\mu(x) := \infty$, for $x \notin X_0$, one gets $\varphi(f) = \int f d\mu$, for any $f \in C_0(X)_+$.

Proof. (i) Let φ be lower semicontinuous. Then, because of Proposition 5.2, we have only to prove equality, which follows from $\varphi(f) \leq \liminf \varphi(f_n)$. The reverse implication follows by Fatou's lemma.

(ii) If $f \in C_0(X_0)_+$ the equality holds because of (i). If $f \in C_0(X)_+ \setminus C_0(X_0)$, then there is $x_0 \in X \setminus X_0$ s.t. $f(x_0) > 0$. Therefore $\varphi(f) = \infty$ and $\int f d\mu = \infty$. □

Corollary 5.5. *Let φ be a weight on $C_0(X)$. Then φ is lower semicontinuous iff there are a complete positive Borel measure μ on X and an open subset X_0 of X s.t. $\mu(x) = \infty$, $x \in X_0^c$, $\mu|_{X_0}$ is a Radon measure, and $\varphi(f) = \int f d\mu$, $f \in C_0(X)_+$.*

Proof. $(\Rightarrow)$ follows from the previous Propositions.
$(\Leftarrow)$ we must show that $\varphi^{-1}([0, \alpha])$ is closed in $C_0(X)_+$, for any $\alpha > 0$. But $\varphi(f) < \infty$ implies $f \in C_0(X_0)_+$, as follows from Proposition 5.2 (iv), so that $\varphi^{-1}([0, \alpha]) \subset C_0(X_0)_+$ which is closed because $\mu|_{X_0}$ is a Radon measure. □

The following Proposition gives a characterisation of the measures corresponding to lower semicontinuous weights on $C_0(X)$.

Proposition 5.6. *Let μ be a complete positive Borel measure, then the following are equivalent*

(i) *there is an open subset X_0 of X s.t. $\mu|_{X_0}$ is a Radon measure and $\mu(x) = \infty$ for any $x \in X \setminus X_0$*

(ii) *μ is outer regular on Borel sets, and inner regular on open sets and Borel sets of finite μ-measure.*

Proof. $(ii) \Rightarrow (i)$ Let us set $X_0 := \{x \in X : \mu(x) < \infty\}$; then, as μ is outer regular, for any $x_0 \in X_0$, there is an open set $V \subset X$ s.t. $x_0 \in V$ and $\mu(V) < \infty$. Then $V \subset X_0$ and X_0 is open. Besides $\mu|_{X_0}$ is a Radon measure because

(a) it is outer regular, as for any Borel set $E \subset X_0$ we have $\mu(E) = \inf\{\mu(V) : V$ open $\supset E\} = \inf\{\mu(V \cap X_0) : V$ open $\supset E\}$;

(b) it is inner regular on Borel subsets of X_0 of finite measure, and open subsets of X_0, because if E is any such set, for any K compact $\subset E$, we have $K \subset X_0$, so that the thesis follows by the same property satisfied by μ;

(c) any compact set $K \subset X_0$ has finite μ-measure; indeed for any $x \in K \subset X_0$ there is an open set $V_x \subset X_0$ s.t. $x \in V_x$, $\mu(V_x) < \infty$, and $K \subset \cup_{i=1}^n V_{x_i}$ because of compactness. Then $\mu(K) < \infty$.
$(i) \Rightarrow (ii)$

(a) μ is outer regular, because for any Borel set $E \subset X$, if $E \cap X_0^c \neq \emptyset$, we have $\mu(E) = \infty = \inf\{\mu(V) : V$ open $\supset E\}$, otherwise $E \subset X_0$ and the thesis follows;

(b) μ is inner regular on open sets and Borel sets of finite μ-measure, as if E is such a set, and $E \subset X_0$ the thesis follows, while if $E \cap X_0^c \supset \{x_0\}$, then $\mu(E) = \infty = \sup\{\mu(K) : K$ compact $\subset E\}$, because $\{x_0\}$ is one such compact set. □

Given a lower semicontinuous weight φ and the measure μ corresponding to it, let us set $X_1 := \{x \in X \setminus X_0 :$ for any open subset $\Omega \ni x,\ \mu(\Omega \cap X_0) = \infty\}$ and $X_s := (X_0 \cup X_1)^o$.

Proposition 5.7. *Let φ be a lower semicontinuous weight on $C_0(X)$. Then*

(i) *$\varphi|_{C_0(X_s)}$ is semifinite*

(ii) *X_s is the largest open set in X satisfying (i).*

Proof. (i) Let $f \in C_0(X_s)_+ \setminus C_0(X_0)$, then there is $x_1 \in X_1$ s.t. $\lambda := f(x_1) > 0$. Let $\Omega := \{x \in X : f(x) > \frac{\lambda}{2}\} \ni x_1$, then $\mu(\Omega \cap X_0) = \infty$. As μ is inner regular on open subsets of X_0, there is a sequence of compact sets K_n increasing to $\Omega \cap X_0$, and s.t. $\mu(K_n) \geq n$. By Urysohn's lemma, there is $g_n \in C_c(X_0)$ s.t. $0 \leq g_n \leq 1$, $\mathrm{supp} g_n \subset \Omega \cap X_0$, $g_n|_{K_n} = 1$. Then $f \geq \frac{\lambda}{2} g_n$ and $\varphi(f) \geq \frac{\lambda}{2}\varphi(g_n) \geq \frac{\lambda}{2}\mu(K_n) \geq \frac{\lambda}{2}n$, so that $f_n := \frac{\lambda}{2}g_n \in \mathrm{dom}\ \varphi$ and $\varphi(f_n) \nearrow \varphi(f)$.
(ii) Let V be an open subset of X s.t. $V \supset X_s$. Then there is $x_0 \in V \setminus (X_0 \cup X_1)$,

because X_s is the largest open set contained in $X_0 \cup X_1$. Let Ω be an open subset of V s.t. $x_0 \in \Omega$ and $\mu(\Omega \cap X_0) < \infty$. Let $f \in C_c(\Omega)$ be s.t. $0 \leq f \leq 1$, $f(x_0) = 1$, then $\varphi(f) = \infty$. We want to show that for any $g \in C_0(V)$, $0 \leq g \leq f$, $\varphi(g) < \infty$, we have $\varphi(g) \leq \mu(\Omega \cap X_0)$ so that φ cannot be semifinite on $C_0(V)$. Indeed, because of Proposition 5.2 (iv), $g \in C_0(X_0)$, and by Proposition 5.4 (i), $\varphi(g) = \int g d\mu \leq \|g\| \mu(\Omega \cap X_0)$. □

Corollary 5.8. *Let φ be a lower semicontinuous weight on $C_0(X)$. Then φ is semifinite iff there are an open subset X_0 and a complete positive Borel measure μ on X s.t.*

(i) $\varphi(f) = \int f d\mu$, for any $f \in C_0(X)_+$,

(ii) $\mu|_{X_0}$ is a Radon measure,

(iii) for any $x \in X_0^c$, and any open subset $\Omega \ni x$ one has $\mu(x) = \infty$ and $\mu(\Omega \cap X_0) = \infty$.

Corollary 5.9. *Let φ be a lower semicontinuous weight on $C_0(X)$. Then φ is densely defined iff there is complete positive Radon measure on X s.t. $\varphi(f) = \int f d\mu$, for any $f \in C_0(X)_+$.*

Proof. If φ is densely defined, from Proposition 5.2 (iv), it follows that $C_0(X) = C_0(X_0)$, so that $X_0 = X$ and the thesis follows. The converse is easy. □

5.2 Riemann measurable functions

Let us now introduce $\ell_0^\infty(X) := \{f : X \to \mathbb{C} : f \text{ is bounded and } \lim_{x\to\infty} f(x) = 0\}$, and, for μ an outer regular, complete, positive Borel measure, $\mathcal{R}_0(X, \mu) := \{f \in \ell_0^\infty(X) : f \text{ is continuous but for a set of zero } \mu\text{-measure }\}$, the set of Riemann μ-measurable functions.

A different description of $\mathcal{R}_0(X, \mu)$ is contained in the following

Proposition 5.10. *Let $f : X \to \mathbb{R}$, then the following are equivalent*

(i) $f \in \mathcal{R}_0(X, \mu)$

(ii) for any $\varepsilon > 0$ there are $f_\varepsilon^\pm \in C_0(X)$ s.t. $f_\varepsilon^- \leq f \leq f_\varepsilon^+$ and $\int (f_\varepsilon^+ - f_\varepsilon^-) d\mu < \varepsilon$

(iii) there are $h \in C_0(X)$, an open subset V of finite μ-measure, and, for any $\varepsilon > 0$, $h_\varepsilon^\pm \in C_0(V)$ s.t. $h_\varepsilon^- \leq f - h \leq h_\varepsilon^+$ and $\int (h_\varepsilon^+ - h_\varepsilon^-) d\mu < \varepsilon$.

Lemma 5.11. *Let X be a compact Hausdorff space, $x_0 \in X$, $f : X \to [0, 1]$, $f(x_0) = 0$, f continuous in x_0. Then there is $g \in C(X)$ s.t. $0 \leq f \leq g \leq 1$, and $g(x_0) = 0$.*

Proof. Let U_n be an open neighbourhood of x_0 s.t. $f|_{U_n} < 2^{-n}$, for any $n \in \mathbb{N}$, and $\overline{U_n} \subset U_{n-1}$, $n \geq 2$. Use Tietze's extension theorem to get a continuous function g_1 on U_2^c s.t. $g_1|_{U_1^c} = 1$, $g_1|_{\partial U_2} = \frac{1}{2}$, and $\frac{1}{2} \leq g_1 \leq 1$, so that $g_1 \geq f$

on U_2^c. For any $n \geq 2$, use Tietze's theorem to get a continuous function g_n on $\overline{U_n} \setminus U_{n+1}$ s.t. $g_n|_{\partial U_n} = 2^{-n+1}$, $g_n|_{\partial U_{n+1}} = 2^{-n}$, and $2^{-n} \leq g_n \leq 2^{-n+1}$ on $\overline{U_n} \setminus U_{n+1}$, so that $g_n \geq f$ on $\overline{U_n} \setminus U_{n+1}$. Let g be defined as

$$g(x) := \begin{cases} g_1(x) & x \in U_2^c \\ g_n(x) & x \in \overline{U_n} \setminus U_{n+1} \\ 0 & x \in \cap U_n, \end{cases}$$

then it satisfies all the requests. □

Lemma 5.12. *Let Y be a locally compact Hausdorff space, $f : Y \to [a, b]$, $\lim_{x\to\infty} f(x) = 0$. Then there are $g_\pm \in C_0(Y)$ s.t. $a \leq g_- \leq f \leq g_+ \leq b$.*

Proof. Denote with X the one-point compactification of Y and extend f to X setting $f(\infty) = 0$. As $a \leq 0 \leq b$, we can consider $f = f_+ - f_-$, with $0 \leq f_+ \leq b$, $0 \leq f_- \leq -a$, $\lim_{x\to\infty} f_\pm(x) = 0$. Using the previous Lemma, we obtain $g_\pm \in C_0(Y)$ s.t. $0 \leq f_+ \leq g_+ \leq b$, $0 \leq f_- \leq -g_- \leq -a$, so that $a \leq g_- \leq -f_- \leq f \leq f_+ \leq g_+ \leq b$. □

Proof of Proposition 5.10 $(i) \Rightarrow (iii)$ Let us set $D_f := \{x \in X : f \text{ is discontinuous at } x\}$. As μ is outer regular, for any $\varepsilon > 0$ there is an open set $V_\varepsilon \supset D_f$ s.t. $\mu(V_\varepsilon) < \frac{\varepsilon}{4\|f\|}$, and $V_\varepsilon \searrow D_f$. By Tietze's extension Theorem, $f|_{V_\varepsilon^c}$ extends to $g_\varepsilon \in C_0(X)$, with $\|g_\varepsilon\| \leq \|f\|$. Then $f - g_\varepsilon$ is continuous and zero on V_ε^c, and $\|f - g_\varepsilon\| \leq 2\|f\|$, that is $f - g_\varepsilon \in \ell_0^\infty(V_\varepsilon)$. Lemma 5.12 gives us $g_\varepsilon^\pm \in C_0(V_\varepsilon)$ s.t. $g_\varepsilon^- \leq f - g_\varepsilon \leq g_\varepsilon^+$ and $\|g_\varepsilon^\pm\| \leq 2\|f\|$. Setting $h := g_1$, $V := V_1$, $h_\varepsilon^\pm := g_\varepsilon - g_1 + g_\varepsilon^\pm \in C_0(V)$, we get $h_\varepsilon^- \leq f - h \leq h_\varepsilon^+$, $\int (h_\varepsilon^+ - h_\varepsilon^-) d\mu = \int (g_\varepsilon^+ - g_\varepsilon^-) d\mu \leq 4\mu(V_\varepsilon)\|f\| < \varepsilon$.
$(iii) \Rightarrow (ii)$ is trivial.
$(ii) \Rightarrow (i)$ Let $f_\varepsilon^- \leq f \leq f_\varepsilon^+$ be as in the statement, and set $f^+ := \inf_\varepsilon f_\varepsilon^+$, $f^- := \sup_\varepsilon f_\varepsilon^-$. Then $f^- \leq f \leq f^+$, f^- is upper semicontinuous, f^+ is lower semicontinuous, and $\int (f^+ - f^-) d\mu = 0$, so that $f^+ = f^-$ μ-a.e., that is, for μ-almost all $x \in X$, we have $f^-(x) = f^+(x) = f(x)$, and f is continuous for those x. □

The following results do not follow from the general theory because we do not assume the semifiniteness of the weight on $C_0(X)$ associated to the measure μ, or because may not be true in the noncommutative setting.

Proposition 5.13. *$\mathcal{R}_0(X, \mu)$ is a C^*-algebra.*

Proof. Let $f_n \in \mathcal{R}_0(X, \mu)$, $f_n \to f$ in norm. Then clearly $f \in \ell_0^\infty(X)$. Moreover, setting $D_n := \{x \in X : f_n \text{ is not continuous in } x\}$, then f can be discontinuous at most on $D := \cup D_n$, which is μ-null by σ-additivity of the measure. Then $f \in \mathcal{R}_0(X, \mu)$. □

Proposition 5.14. *$\mathcal{R}_0(X, \mu)$ is the closure of $C_0(X)$ w.r.t. μ-contiguous classes with μ-finite support. In particular if μ is as in Corollary 5.8, then $C_0(X)^{\mathcal{R}} = \mathcal{R}_0(X, \mu)$.*

Proof. Because of Proposition 5.10 (*iii*), for any $f \in \mathcal{R}_0(X,\mu)_{sa}$ we can find an open subset V of finite μ-measure and functions $h \in C_0(X)$, and, for any $\varepsilon > 0$, $h_\varepsilon^\pm \in C_0(V)$ s.t. $h_\varepsilon^- \leq f - h \leq h_\varepsilon^+$ and $\int (h_\varepsilon^+ - h_\varepsilon^-) d\mu \leq \varepsilon$. Therefore $f - h$ belongs to the completion of $C_0(X)$ w.r.t. μ-contiguous classes with μ-finite support. Since $h \in C_0(X)$, this ends the proof. □

Proposition 5.15. *Any separating element in $\ell_0^\infty(X)$ between separated, μ-contiguous classes in $\mathcal{R}_0(X,\mu)$ belongs to $\mathcal{R}_0(X,\mu)$. In particular if μ is as in Corollary 5.8, then $C_0(X)^{\mathcal{U}} = \mathcal{R}_0(X,\mu)$.*

Proof. Let $\{f_\varepsilon^\pm\} \subset \mathcal{R}_0(X,\mu)_{sa}$, $\int (f_\varepsilon^+ - f_\varepsilon^-) d\mu < \varepsilon$ and $f \in \ell_0^\infty(X)$ be s.t. $f_\varepsilon^- \leq f \leq f_\varepsilon^+$. Then we find $(f_\varepsilon^\pm)_\delta^\pm \in C_0(X)$ s.t. $\int ((f_\varepsilon^\pm)_\delta^+ - (f_\varepsilon^\pm)_\delta^-) d\mu < \delta$ and $(f_\varepsilon^\pm)_\delta^- \leq f_\varepsilon^\pm \leq (f_\varepsilon^\pm)_\delta^+$. Therefore $(f_\varepsilon^-)_\delta^- \leq f_\varepsilon^- \leq f \leq f_\varepsilon^+ \leq (f_\varepsilon^+)_\delta^+$ and $\int ((f_\varepsilon^+)_\delta^+ - (f_\varepsilon^-)_\delta^-) d\mu = \int [(f_\varepsilon^+)_\delta^+ - f_\varepsilon^+ + f_\varepsilon^+ - f_\varepsilon^- + f_\varepsilon^- - (f_\varepsilon^-)_\delta^-] d\mu \leq 2\delta + \varepsilon$, and the thesis follows. □

Proposition 5.16. $\mathcal{R}_c(X,\mu) := \{f \in \mathcal{R}_0(X,\mu) : supp f \subset\subset X\}$ *is norm-dense in* $\mathcal{R}_0(X,\mu)$.

Proof. If $f \in \mathcal{R}_0(X,\mu)$, as $f(x) \to 0$, for $x \to \infty$, there is, for any $\varepsilon > 0$, a compact set $K_\varepsilon \subset X$ s.t. $|f(x)| < \varepsilon$, for $x \in K_\varepsilon^c$. Then by Urysohn's Lemma there is $g_\varepsilon \in C_c(X)$ s.t. $0 \leq g_\varepsilon \leq 1$ and $g_\varepsilon|_{K_\varepsilon} = 1$, so that $g_\varepsilon f \in \mathcal{R}_c(X,\mu)$ and $g_\varepsilon f \to f$. □

We now prove some Lemmas used in Section 2.

A measurable set $\Omega \subset X$ is said Riemann μ-measurable if its characteristic function is Riemann μ-measurable, which is equivalent to saying $\mu(\partial\Omega) = 0$.

Lemma 5.17. *Let f be a positive Riemann μ-measurable function on X such that $\int f < \infty$ and set $O_y = \{x \in X : f(x) > y\}$, $C_y = \{x \in X : f(x) \geq y\}$, $y \geq 0$. Then, for any $0 < \alpha < \beta$ there are uncountably many $y \in (\alpha,\beta)$ for which O_y and C_y are Riemann μ-measurable.*

Proof. Let $\varepsilon > 0$ and $f_\varepsilon^\pm \in C_0(X)$ be s.t. $f_\varepsilon^- \leq f \leq f_\varepsilon^+$ and $\int (f_\varepsilon^+ - f_\varepsilon^-) d\mu < \varepsilon$. Then $f^+ := \inf_{\varepsilon>0} f_\varepsilon^+$ is upper semicontinuous, $f^- := \sup_{\varepsilon>0} f_\varepsilon^-$ is lower semicontinuous, $f^- \leq f \leq f^+$ and $\int (f^+ - f^-) d\mu = 0$. Then $\{x \in X : f^-(x) > y\} \subset O_y \subset C_y \subset \{x \in X : f^+(x) \geq y\}$. Therefore $\partial O_y, \partial C_y \subset \Omega_y := \{x \in X : f^+(x) \geq y\} \setminus \{x \in X : f^-(x) > y\}$, and it suffices to show that $\mu(\Omega_y) = 0$ for uncountably many $y \in (\alpha,\beta)$. Now observe that $\Omega_y = \{x \in X : f^-(x) < y, f^+(x) > y\} \cup \{x \in X : f^-(x) = y, f^+(x) \geq y\} \cup \{x \in X : f^-(x) \leq y, f^+(x) = y\}$. Besides $\{x \in X : f^-(x) < y, f^+(x) > y\} = \cup_n \Omega_n$, where $\Omega_n := \{x \in X : f^-(x) \leq y - \frac{1}{n}, f^+(x) \geq y + \frac{1}{n}\}$, is μ-null because $0 = \int_{\Omega_n} (f^+ - f^-) d\mu \geq \int_{\Omega_n} \frac{2}{n} d\mu = \frac{2}{n}\mu(\Omega_n)$. Moreover $\{x \in X : f^-(x) = y, f^+(x) \geq y\} \subset \{x \in X : f^-(x) = y\}$ and $\{x \in X : f^-(x) \leq y, f^+(x) = y\} \subset \{x \in X : f^+(x) = y\}$, so that they are μ-null. Indeed, letting g denote one of $f^\pm$, $g \in L^1(X,\mu)_+$, and $\{x \in X : g(x) = y\} = \{x \in X : g(x) \geq y\} \setminus \{x \in X : g(x) > y\}$, so that $\{x \in X : g(x) = y\}$ is μ-null as soon as $\mu(\{x \in X : g(x) \geq y\}) = \mu(\{x \in X : g(x) > y\})$.

Observe that $y \to \mu(\{x \in X : g(x) \geq y\})$ is decreasing and left continuous, hence it has only countably many points of discontinuity, and in a point y of continuity we have $\mu(\{x \in X : g(x) \geq y\}) = \mu(\{x \in X : g(x) > y\})$. The thesis follows. □

Lemma 5.18. *Let f be a positive Riemann measurable function such that $\int f d\mu < \infty$. Then, for any $\delta > 0$, we may find a sequence of Riemann measurable characteristic functions χ_n and a sequence of positive numbers α_n such that $\sum_n \alpha_n = \|f\|$ and $f = \sum_n \alpha_n \chi_n$.*

Proof. First we construct by induction the sequences $\chi_n := \chi_{\Omega_n}$ and $\alpha_n > 0$, $n \geq 1$, such that, $\forall n \geq 0$,

- $f \geq \sum_1^n \alpha_k \chi_k$,
- $\beta_n := \|f - \sum_1^n \alpha_k \chi_k\| > 0$,
- $\beta_n/4 \leq \alpha_{n+1} \leq \beta_n/2$,
- $\Omega_{n+1} := \{x \in X : f(x) - \sum_1^n \alpha_k \chi_k(x) \geq \alpha_{n+1}\}$ is Riemann measurable.

Indeed, given χ_k, α_k for $k \leq n$, as prescribed, the existence of α_{n+1} satisfying the last two properties follows by Lemma 5.17, while the first two inequalities (hence $\alpha_n > 0$) follow by the definition of α_{n+1} and Ω_{n+1}.

We now observe that, since $\sup_X (f - \sum_1^n \alpha_k \chi_k) = \sup_{\Omega_{n+1}} (f - \sum_1^n \alpha_k \chi_k)$, we have

$$\sup_{\Omega_{n+1}} \left(f - \sum_{k=1}^{n+1} \alpha_k \chi_k \right) = \sup_{\Omega_{n+1}} \left(f - \sum_{k=1}^{n} \alpha_k \chi_k - \alpha_{n+1} \right) = \beta_n - \alpha_{n+1} \geq \alpha_{n+1}$$

$$\sup_{\Omega_{n+1}^c} \left(f - \sum_{k=1}^{n+1} \alpha_k \chi_k \right) = \sup_{\Omega_{n+1}^c} \left(f - \sum_{k=1}^{n} \alpha_k \chi_k \right) \leq \alpha_{n+1}.$$

hence

$$\beta_{n+1} = \max \left(\sup_{\Omega_{n+1}} (f - \sum_{k=1}^{n+1} \alpha_k \chi_k), \sup_{\Omega_{n+1}^c} (f - \sum_{k=1}^{n+1} \alpha_k \chi_k) \right) = \beta_n - \alpha_{n+1} \leq \frac{3}{4} \beta_n.$$

This shows at once that $\beta_n \leq (3/4)^n \|f\| \to 0$, namely $\sum \alpha_n \chi_n$ converges to f uniformly and $\sum_0^\infty \alpha_{n+1} = \sum_0^\infty (\beta_n - \beta_{n+1}) = \|f\|$, which concludes the proof. □

Lemma 5.19. *Let $\Omega \subset\subset X$ with $\mu(\partial\Omega) = 0$ and $\mu(\Omega) < \infty$. Then for any $\varepsilon > 0$ $\exists f_\varepsilon^\pm \in C_0(X)$ such that $0 \leq f_\varepsilon^- \leq \chi_\Omega \leq f_\varepsilon^+ \leq 1$, $\int (f_\varepsilon^+ - f_\varepsilon^-) d\mu \leq \varepsilon$ and $\mu(supp(f_\varepsilon^+) \setminus \Omega) \leq \varepsilon$.*

Proof. Since χ_Ω is Riemann measurable (cf. Proposition 5.10), we may find $f_\varepsilon^\pm$ satisfying all the properties above, except possibly the last one. Then, choosing a continuous increasing function ψ_δ on $[0,1]$ s.t. $\psi_\delta(t) = 0$ when $t \in [0, 1-\delta]$ and $\psi_\delta(1) = 1$, we may replace f_ε^+ with $\psi_\delta \circ f_\varepsilon^+$. We have

$$\begin{aligned}\mu(\mathrm{supp}(\psi_\delta \circ f_\varepsilon^+)) &\leq \mu(\{x \in X : f_\varepsilon^+(t) > 1-\delta\}) \\ &\leq \frac{1}{1-\delta}\int f_\varepsilon^+ d\mu \leq \frac{1}{1-\delta}(\varepsilon + \mu(\Omega))\end{aligned}$$

from which the thesis follows. □

We conclude this Appendix giving a characterization, in the commutative case, of the unbounded Riemann μ-measurable functions.

Proposition 5.20. *Setting $\overline{\mathcal{R}}_0(X,\mu) := \{f : X \to \mathbb{C} : f$ is μ-a.e. defined and continuous, and there is a compact, Riemann μ-measurable subset K of finite μ-measure s.t. $f|_{K^c} \in \ell_0^\infty(X)\}$, we have $\overline{\mathcal{R}}_0(X,\mu) = \overline{(C_0(X))^{\mathcal{R}}}$, in the universal atomic representation.*

Proof. Let $f \in \overline{(C_0(X))^{\mathcal{R}}}$. Then there is an increasing sequence of Riemann μ-measurable subsets G_n s.t. $G_n^c \subset\subset X$, $\mu(G_n^c) \searrow 0$ and $f\chi_{G_n} \in \mathcal{R}_0(X,\mu)$. Therefore $\lim_{x\to\infty} f(x) = 0$, and, for any $n \in \mathbb{N}$ there is $E_n \subset G_n$ s.t. $\mu(E_n) = 0$ and $f|_{E_n^c \cap G_n}$ is continuous. Setting $G := \cup_n G_n$, $E := \cup_n E_n$, we get $\mu(E) = \mu(G^c) = 0$ and $f|_{E^c \cap G}$ is (defined and) continuous. Finally choose an $n \in \mathbb{N}$ and set $K := \overline{G_n^c}$. Then $f \in \overline{\mathcal{R}}_0(X,\mu)$.
Let now $f \in \overline{\mathcal{R}}_0(X,\mu)$, and let $E \subset X$ be s.t. $f|_{E^c}$ is defined and continuous. Then, because of outer regularity of μ, there are open sets $\Omega_\varepsilon \searrow E$, with $\mu(\Omega_\varepsilon) < \varepsilon$. Let us fix $\varepsilon > 0$ and set, for any $\lambda > 0$, $V_{\varepsilon,\lambda} := \{x \in K \cap \Omega_\varepsilon^c : |f(x)| \leq \lambda\}$. Then, with $M := \sup_{K^c} |f|$, from Lemma 5.17 we conclude the existence of uncountably may $\lambda > M$ s.t. $V_{\varepsilon,\lambda}$ is Riemann μ-measurable. For any $\varepsilon > 0$ choose one such $\lambda \equiv \lambda_\varepsilon$ so large that, with $A_\varepsilon := V^o_{\varepsilon,\lambda_\varepsilon}$, we get $\mu(K \cap A_\varepsilon^c) = \mu(K) - \mu(V_{\varepsilon,\lambda_\varepsilon}) < 2\varepsilon$. Finally set $G_n := \cup_{k=1}^n A_{1/k} \cup K^c$, which is an increasing family of Riemann μ-measurable subsets, s.t. $\mu(G_n^c) \leq \mu(K \cap A_{1/n}^c) \leq \frac{2}{n}$, and $\sup_{G_n} |f| = \max\{M, \max_{k\leq n} \sup_{A_{1/k}} |f|\} \leq \max_{k\leq n} \lambda_{1/k}$. Therefore $\{G_n\}$ is an SDD for f, and $f \in \overline{(C_0(X))^{\mathcal{R}}}$. □

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