The Fundamental Group's Structure of the Complement of Some Configurations of Real Line Arrangements

David Garber and Mina Teicher
Department of Mathematics and Computer Sciences
Bar-Ilan University

February 1, 2008

Abstract

In this paper, we give a fully detailed exposition of computing fundamental groups of complements of line arrangements using the Moishezon-Teicher technique for computing the braid monodromy of a curve and the Van-Kampen theorem which induces a presentation of the fundamental group of the complement from the braid monodromy of the curve. For example, we treated the cases where the arrangement has t multiple intersection points and the rest are simple intersection points. In this case, the fundamental group of the complement is a direct sum of infinite cyclic groups and t free groups. Hence, the fundamental groups in these cases is "big". These calculations will be useful in computing the fundamental group of Hirzebruch covering surfaces.

Contents

1 Introduction **3**

2 Preliminaries **4**
2.1 Some background 4
2.2 Definition of g-base 5
2.3 Braid group and braid monodromy 6
2.4 The braid monodromy of a real line arrangement 9
2.5 The algorithm of Moishezon-Teicher 11
2.6 The Van-Kampen theorem 12
2.7 An application of the Van-Kampen theorem 15
2.8 Outline of the computation of the fundamental group of the complement of line arrangements 16

3 Arrangements with t non-collinear multiple points **17**
3.1 The affine case 17
3.2 The projective case 18

4 Arrangements with t collinear multiple points **21**
4.1 The affine case 21
4.2 Proof of lemma 4.1.2 22
4.2.1 First case - with the restriction 22
4.2.2 Second case - without the restriction 27
4.3 Proof of lemma 4.1.3 35
4.4 Proof of theorem 4.1.1 40
4.5 The projective case 42

5 Arrangements with more than one equivalence class **43**
5.1 The definition of the equivalence relation 43
5.2 The affine case 44
5.3 The projective case 45

6 Results concerning the bigness of the fundamental group **46**

1 Introduction

In this paper, we give a fully detailed exposition of calculations of fundamental groups of the complements of certain configurations of real line arrangements using the Moishezon-Teicher algorithm (which calculates the braid monodromy of curves), the Van-Kampen theorem (which induces a finite presentation, in terms of generators and relations, of the fundamental group of curves' complements, from its braid monodromy), and some group computations.

In particular, we got:

1. Let $\mathcal{L}$ be a real line arrangement which is a union of t subsets of lines each of which consists of k_i+1 lines meeting in a single point, and any two lines belonging to different subsets meet in a simple point. Then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}, u_0) \cong (\bigoplus_{i=1}^{t} \mathbb{F}^{k_i}) \oplus \mathbb{Z}^t$$

   and

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}, u_0) \cong (\bigoplus_{i=1}^{t} \mathbb{F}^{k_i}) \oplus \mathbb{Z}^{t-1}$$

2. Let $\mathcal{L}$ be a real line arrangement which consists of t subsets of lines each of which consists of $k_i + 1$ lines meeting in a single point and all the t multiple points lie on the same line $L \in \mathcal{L}$. Then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}, u_0) \cong (\bigoplus_{i=1}^{t} \mathbb{F}^{k_i}) \oplus \mathbb{Z}$$

   and

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}, u_0) \cong \bigoplus_{i=1}^{t} \mathbb{F}^{k_i}$$

3. Generalizations: Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ consists of n lines. We choose the line at infinity such that all the lines are intersected in $\mathbb{C}^2$. Assume that there are k multiple intersection points $p_1, \cdots, p_k$ with multiplicities $m_1, \cdots, m_k$ respectively. Assume also that all the multiple intersection points in every equivalence class (of multiple points) are collinear, i.e. in every equivalence class (of multiple points) there is a unique line of $\mathcal{L}$ which all the multiple points of that class lie on it. Then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}, u_0) \cong \bigoplus_{i=1}^{k} \mathbb{F}^{m_i - 1} \oplus \mathbb{Z}^{n - (\sum_{i=1}^{k} (m_i - 1))}$$

   and

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}, u_0) \cong \bigoplus_{i=1}^{k} \mathbb{F}^{m_i - 1} \oplus \mathbb{Z}^{n - 1 - (\sum_{i=1}^{k} (m_i - 1))}$$

   The number of infinite cyclic groups in the affine case is a sum of two numbers: the number of equivalence classes (see definitions in section 5) and the number of lines which have only simple intersection points.

4. Therefore, in all the above cases, the fundamental group is “big”.

We will organize the paper as follows: in section 2, we introduce the needed background for the techniques which will be used, and we give a detailed description of the Moishezon-Teicher algorithm for the case of line arrangements and the Van-Kampen theorem.

In section 3, we compute the structure of the fundamental group of the complement of a line arrangement which consists of t subsets of lines and the multiple points are not collinear.

In section 4, we compute the structure of the fundamental group of the complement of a line arrangement which consists of t subsets of lines and the multiple points are collinear.

In section 5, we generalize the results of the calculations of sections 3 and 4. In section 6, we discuss the bigness of the groups which have been treated.

2 Preliminaries

2.1 Some background

This topic starts with Zariski, who proved in [Z, p. 317] that:

Proposition (Zariski)
The fundamental group of the complement of n lines in general position is abelian.

Among the modern works on this topic, one can mention [Fa1], [Fa2], [OS], [Sa], [Ra] and more.

Moishezon and Teicher developed an algorithm for computing fundamental groups of complements of branch curves of generic projection of surfaces of general type (see [MoTe1],[MoTe2]). This algorithm can be used also for computing fundamental groups of complement of line arrangements. In this paper we give a detailed exposition of this technique in some configurations of line arrangements.

Simultaneously and independently, by entirely different methods, Fan proved in [Fa1],[Fa2] the following results for the projective case:

Proposition (Fan)
Let $\Sigma = \bigcup l_i$ be a line arrangement in $\mathbb{CP}^2$ and assume that there is a line L of Σ such that for any singular point S of Σ with multiplicity ≥ 3, we have $S \in L$. Then: $\pi_1(\mathbb{CP}^2 - \Sigma)$ is isomorphic to a direct product of free groups.

Proposition (Fan)
Let Σ be an arrangement of n lines and $S = \{a_1, \cdots, a_k\}$ be the set of all singularities of Σ with multiplicity ≥ 3. Suppose that $\beta(\Sigma) = 0$, where $\beta(\Sigma)$ is the first Betti number of the subgraph of Σ which contains only the higher singularities (i.e. with multiplicity ≥ 3) and their edges. Then:

$$\pi_1(\mathbb{CP}^2 - \Sigma) \cong \mathbb{Z}^r \oplus \mathbb{F}^{m(a_1)-1} \oplus \cdots \oplus \mathbb{F}^{m(a_k)-1}$$

where $r = n + k - 1 - m(a_1) - \cdots - m(a_k)$.

It has to be noted that the assumption $\beta(\Sigma) = 0$ is equivalent to the assumption that Σ is a union of trees. The r in the last proposition is actually a sum of two combinatorial ingredients: the number of the trees in Σ minus 1 and the number of lines which are intersected only in simple intersection points.

Oka and Sakamoto proved in [OS] the following theorem, which will be a useful tool in some of our calculations:

Theorem (Oka-Sakamoto)
Let C_1 and C_2 be algebraic plane curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 \cdot d_2$ points, where d_i $(i = 1, 2)$ are the respective degrees of C_1 and C_2.
Then:

$$\pi_1(\mathbb{C}^2 - (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2)$$

Our computations on the fundamental groups of complements of line arrangements have applications to the fundamental groups of complements of branch curves, which is an important invariant of surfaces [Te2] (when we degenerate a surface to a union of planes, the branch curve degenerates to a union of lines). Moreover, the methods of this paper are important tools in the computations of the fundamental groups of Hirzebruch covering surfaces.

2.2 Definition of g-base

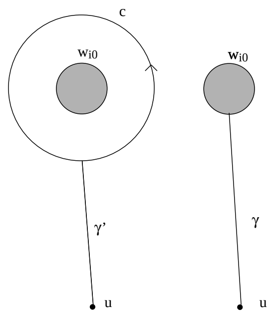
Here, we will present the required definitions and results for the presentation of the algorithm of Moishezon-Teicher. We follow the presentation of [MoTe1].

In this section, we will define the notion of *g-base (good geometric base)* for $\pi_1(D - K, *)$, where K is a finite set in a disk D. For this definition, we have to define:

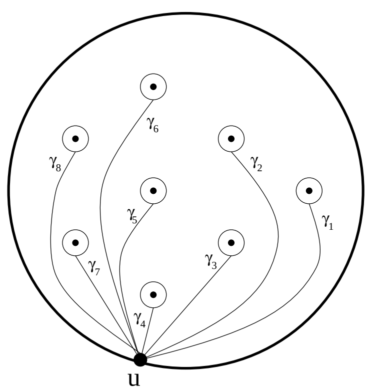
Definition 2.2.1 $l(\gamma)$
Let D be a disk. Let w_i, $i = 1, \cdots, n$, be small disks in $\mathrm{Int}(D)$ *such that:*

$$w_i \cap w_j = \emptyset, \forall i \neq j.$$

Let $u \in \partial D$. Let γ be a simple path connecting u with one of the w_i's, say w_{i_0}, which does not meet any other w_j, $j \neq i_0$.
We assign to γ a loop $l(\gamma)$ (actually an element of $\pi_1(D - K, u)$) as follows: let c be a simple loop equal to the (oriented) boundary of a small neighbourhood V of w_{i_0} chosen such that $\gamma' = \gamma - V \cap \gamma$ is a simple path.
Then: $l(\gamma) = \gamma' \cup c \cup (\gamma')^{-1}$ (we will not distinguish between $l(\gamma)$ and its representative in $\pi_1(D - K, u)$).



Definition 2.2.2 Bush, g-base (good geometric base)
Let D be a disk, $K \subset D$, $\#K < \infty$. Let $u \in D - K$. A set of simple paths $\{\gamma_i\}$ is a **bush** *in (D, K, u), if $\forall i, j,\ \gamma_i \cap \gamma_j = u$; $\forall i,\ \gamma_i \cap K$ = one point, and γ_i are ordered counterclockwise around u. Let $\Gamma_i = l(\gamma_i) \in \pi_1(D - K, u)$ be a loop around $K \cap \gamma_i$ determined by γ_i. $\{\Gamma_i\}$ is called a* **g-base** *of $\pi_1(D - K, u)$.*



2.3 Braid group and braid monodromy

Let D be a closed disk in $\mathbb{R}^2$, $K \subset D$ a finite set, $u \in \partial D$. In such a case, we can define the *braid group* $B_n[D, K]$ $(n = \#K)$:

Definition 2.3.1 Braid group - $B_n[D, K]$
Let $\mathcal{B}$ be the group of all diffeomorphisms β of D such that $\beta(K) = K$, $\beta|_{\partial D} = \mathrm{Id}|_{\partial D}$. Such diffeomorphism acts naturally on $\pi_1(D - K, u)$. We say that two such diffeomorphisms are equivalent if they define the same automorphism on $\pi_1(D - K, u)$. The quotient of $\mathcal{B}$ by this equivalence relation is called the **braid group** *$B_n[D, K]$. An element of $B_n[D, K]$ is called a* **braid**. *A composition of braids is from* **left to right**.

Let us now define the concept of a *half-twist braid*. After fixing an orientation on $\mathbb{R}^2$, we can define a simple path σ such that $[\sigma] \subseteq (D - \partial D - K) \cup \{a, b\}$, σ connects a with b $(a, b \in K)$. Choose now a small regular neighbourhood U of σ, and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{C}$ ($\mathbb{C}$ is taken with the usual "complex" orientation) such that $f(\sigma) = [-1, 1]$, $f(U) = \{z \in \mathbb{C} \mid |z| < 2\}$. Let $\alpha(x)$ be any real smooth monotone function such that

$$\alpha(x) = \begin{cases} 1 & x \in [0, \frac{3}{2}] \\ 0 & x \geq 2 \end{cases}$$

With this function, we define a diffeomorphism $h : \mathbb{C} \to \mathbb{C}$ as follows: for any $z = re^{i\varphi} \in \mathbb{C}$, we define: $h(z) = re^{i(\varphi + \alpha(r)\pi)}$. It is clear that $\forall z, |z| \leq \frac{3}{2}$, $h(z)$ is a positive rotation on $180°$ and $h(z) = \mathrm{Id}\ \forall z, |z| \geq 2$. After these preparations, we can define:

Definition 2.3.2 $H(\sigma)$ **- (positive) half-twist defined by** σ
$H(\sigma)$ is the braid defined by $(f^{-1} \cdot h \cdot f)|_D$.

We have also another way to look at braids - via *motions* of K.

Definition 2.3.3 Motion of K' to K
Let $K' = \{a'_1, a'_2, \cdots, a'_n\}$, $K = \{a_1, a_2, \cdots, a_n\}$. **A motion of K' to K in D** *is n continuous functions $m_i : [0, 1] \to D$, $i = 1, \cdots, n$, such that:*
(a) $\forall i$, $m_i(0) = a'_i$, $m_i(1) = a_i$.
(b) $\forall i \neq j$, $m_i(t) \neq m_j(t)\ \forall t \in [0, 1]$.

According to the following proposition, we can define a family of diffeomorphisms induced from the motion (under the condition that K=K').

Proposition 2.3.4 *Given a motion $\mathcal{R}$, there exists a continuous family of diffeomorphisms $D_{\mathcal{R},t} : D \to D$, $t \in [0, 1]$, such that:*
(a) $D_{\mathcal{R},t}|_{\partial D} = \mathrm{Id}|_{\partial D}$.
(b) $\forall t, i$, $D_{\mathcal{R},t}(a'_i) = m_i(t)$.

Definition 2.3.5 $b_{\mathcal{R}}$ **(braid induced from a motion** $\mathcal{R}$**)**
When K=K', $b_{\mathcal{R}}$ is the braid defined by the diffeomorphism $D_{\mathcal{R},1}$.

We define another important notion:

Definition 2.3.6 Skeleton in (D, K, K'')
Let $K'' \subset K, K'' = \{b_1, \cdots, b_m\}$. A **skeleton** *in (D, K, K'') is represented by a consecutive sequence of simple paths $(p_1, \cdots, p_{m-1})$ in $D - \partial D$ such that each p_i connects b_i to b_{i+1}. We say that two such sequences, say $(p_1, \cdots, p_{m-1}), (\tilde{p}_1, \cdots, \tilde{p}_{m-1})$, represent the same skeleton, if $H(p_i) = H(\tilde{p}_i), i = 1, \cdots, m-1$.*

Before introducing the definition of *braid monodromy*, we have to make some more constructions. From now, we will work in $\mathbb{C}^2$. Let E (resp. D) be a closed disk on x-axis (resp. y-axis), and let C be a part of an algebraic curve in $\mathbb{C}^2$ located in $E \times D$. Let $\pi_1 : E \times D \to E$ and $\pi_2 : E \times D \to D$ be the canonical projections, and let $\pi = \pi_1|_C : C \to E$. Assume π is a proper map, and $\deg \pi = n$. Let $N = \{x \in E \mid \#\pi^{-1}(x) < n\}$, and assume $N \cap \partial E = \emptyset$. Now choose $M \in \partial E$ and let $K = K(M) = \pi^{-1}(M)$. By the assumption that $\deg \pi = n \ \ (\Rightarrow \#K = n)$, we can write: $K = \{a_1, a_2, \cdots, a_n\}$. Under these constructions, from each loop in $E - N$, we can define a braid in $B_n[M \times D, K]$ in the following way:

(1) Because $\deg \pi = n$, we can lift any loop in $E - N$ with a base point M to a system of n paths in $(E - N) \times D$ which start and finish at $\{a_1, a_2, \cdots, a_n\}$.

(2) Project this system into D (by π_2), to get n paths in D which start and end at the image of K in D (under π_2). These paths actually form a motion.

(3) Induce a braid from this motion, as we did in definition 2.3.5.

To conclude, we can match a braid to each loop. Therefore, we get a map $\varphi : \pi_1(E - N, M) \to B_n[M \times D, K]$, which is also a group homomorphism which is called the **braid monodromy of C with respect to $E \times D, \pi_1, M$.**

For the next definitions, let us assume $M_0, M_1 \in E - N$ and $T : [0,1] \to E - N$ be a path which connects M_0 with M_1. We know that there exists a continuous family of diffeomorphisms $\psi_{(t)} : M_0 \times D \to T(t) \times D,\ \forall t \in [0,1]$, such that:
(a) $\psi_{(0)} = \mathrm{Id}|_{M_0 \times D}$.
(b) $\forall t \in [0,1],\ \psi_{(t)}(\pi_1^{-1}(M_0) \cap C) = \pi_1^{-1}(T(t)) \cap C$.
(c) $\forall y \in \partial D,\ \psi_{(t)}(M_0, y) = (T(t), y)$.

In this situation, we can define the *Lefschetz diffeomorphism induced by* T:

Definition 2.3.7 ψ_T, Lefschetz diffeomorphism induced by T

$$\psi_T = \psi_{(1)} : M_0 \times D \tilde{\to} M_1 \times D$$

Let $s = (x(s), y(s)) \in C$ be a singular point of π (i.e. $x(s) \in N$). Let $D'(s)$ be such a small disk on y-axis centered at $y(s)$ that $(x(s) \times D'(s)) \cap C = s$, i.e. there are no other branches of C which intersect $D'(s)$. Therefore, for any sufficiently small neighbourhood U of $x(s)$ on the x-axis centered at $x(s)$ such that $\forall x \in U - x(s)$, $\#(x \times \mathrm{Int}(D'(s))) \cap C$ is independent of x (we call this number the *local degree of* π *at* s and denote it by $\deg_s \pi$). Let $k = \deg_s \pi$ and E' be a small closed disk on the x-axis centered at $x(s)$, such that $\forall x \in E' - x(s)$, $\#(x \times \mathrm{Int}(D'(s))) \cap C = k$. Choose a point $a(s) \in \partial E'$ and let $T : [0,1] \to \mathbb{C}$ be a path in $E - N - \mathrm{Int}(E')$ connecting $a(s)$ to a point $M' \in E - N$. Let $K_{a,s} = (a(s) \times D'(s)) \cap C$.

Definition 2.3.8 $\tilde{\psi}_T$, Lefschetz embedding induced by T
Let ψ_T be the Lefschetz diffeomorphism as defined above. Let T be as above, $a = a(s)$, $D' = D'(s)$. Then:

$$\tilde{\psi}_T = \psi_T|_{a \times D'} : a \times D' \to M' \times D$$

Remark: Take k liftings of T to C starting at the different points of $K_{a,s} = (a \times D') \cap C$. These liftings are real curves in $T \times D$. We can think of $\tilde{\psi}_T$ as "pulling" of $a \times D'$ in $T \times D$ along these real curves.

Definition 2.3.9 $\mathcal{L}_{T,s}$**, Lefschetz injection induced by T**
Consider $\tilde{\psi}_T : a \times D' \to M' \times D$*, Lefschetz embedding induced by* T*.*
Let $K(M') = (M' \times D) \cap C$*. We have*

$$\tilde{\psi}_T(K_{a,s}) \subset K(M'), (K(M') - \tilde{\psi}_T(K_{a,s})) \cap \tilde{\psi}_T(\mathrm{Int}(D')) = \emptyset$$

Therefore, the following canonical injection is well defined:

$$\mathcal{L}_{T,s} = \psi_T^{\vee} : B_k[a \times D', K_{a,s}] \hookrightarrow B_n[M' \times D, K]$$

In order to define the *Lefschetz vanishing cycle*, we need the following definition:

Definition 2.3.10 Linear frame of a braid group $B_n[D, K]$
Let $K = \{a_1, a_2, \cdots, a_n\}$*. Let* $\{\xi_1, \xi_2, \cdots, \xi_{n-1}\}$ *be a system of straight line segments in* $D - \partial D$ *such that each* ξ_i *connects* a_i *with* a_{i+1} *(and does not intersect any other* ξ_j *except of end points). Let* $H_i = H(\sigma_i)$*. The ordered system of positive half-twists* $(H_1, H_2, \cdots, H_{n-1})$ *is called* **a linear frame of** $B_n[D, K]$ **defined by** $\{\xi_1, \xi_2, \cdots, \xi_{n-1}\}$*.*

Now, we come to one of the most important definitions:

Definition 2.3.11 $\mathcal{L}$**.V.C.**(T, H')**, Lefschetz vanishing cycle induced by** T
We call $\mathcal{L}.V.C.(T, H')$ *a skeleton* $< \xi_1, \cdots, \xi_{k-1} >$ *in* $(M' \times D, K, \tilde{\psi}_T(K_{a,s}))$ *corresponding* $\mathcal{L}_{T,s}$ *and a linear frame* $(H') = (H'_1, \cdots, H'_{k-1})$ *of* $B_k[a \times D', K_{a,s}]$*, that is* $\mathcal{L}_{T,s}(H'_i) = H(\xi_i),\ i = 1, \cdots, k-1$*.*

Because of the fact that such a linear frame is unique only when all the points of K are on a straight line in $D \subset \mathbb{R}^2$, $\mathcal{L}$.V.C.(T, H') will be well defined if all the points of $K_{a,s}$ are on a straight line in $a \times \mathbb{C}$. If all the points of $K_{a,s}$ are real, we will choose the unique linear frame $(H'_1, \cdots, H'_{k-1})$ determined by an increasing sequence of consecutive real segments on the real axis of $a \times \mathbb{C}$.

2.4 The braid monodromy of a real line arrangement

Definition 2.4.1 Line arrangement in $\mathbb{CP}^2$
A **Line arrangement in** $\mathbb{CP}^2$ *is an algebraic curve in* $\mathbb{CP}^2$ *which is a union of projective lines.*

If the lines are given by the linear forms $l_1, l_2, \cdots, l_k$, the union of the lines is the reducible curve defined by

$$l_1 l_2 \cdots l_k = 0$$

We say that the arrangement is *real* if each line can be defined by an equation with real coefficients (i.e. each linear form l_i has real coefficients).

Let $\mathbb{C}^2 = \mathbb{CP}^2 -$ (projective line) be an affine part of $\mathbb{CP}^2$. Let E (resp. D) be a closed disk on x-axis (resp. on y-axis) with the center on the real part of x-axis (resp. y-axis). Let $\pi_1 : E \times D \to E, \pi_2 : E \times D \to D$ be the canonical projections.

Definition 2.4.2 Real line arrangement in a polydisk $E \times D$
We say that C is a **real line arrangement in a polydisk** $E \times D$ *(as above), if there exists a real line arrangement $\hat{C}$ in $\mathbb{CP}^2$, such that:*
(a) $C = \hat{C} \cap (E \times D)$.
(b) $\forall x \in E, \ \pi_1^{-1}(x) \cap C \subset x \times \text{Int}(D)$.

Let $\pi = \pi_1|_C, \ n = \deg \pi$ (=number of lines in C), $N = \{x \in E \mid \#\pi^{-1}(x) < n\}, \ K_x = \pi^{-1}(x)$. Therefore, for any real $x \notin N$, we have n distinct real points $(x, y_i(x)), \ 1 \leq i \leq n$, in K_x. We choose a numeration in $\{y_1(x), \cdots, y_n(x)\}$, such that $y_1(x) < y_2(x) < \cdots < y_n(x)$.

Let $\tilde{D} = \{z \in \mathbb{C} \mid |z - \frac{n+1}{2}| \leq \frac{n+1}{2}\}, \ \tilde{K} = \{1, 2, \cdots, n\} \subset \tilde{D}$ ($\tilde{D}$ is a model which simplifies the treatment with the theoretic calculations of the braid monodromy). Let $\tilde{H} = (\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_{n-1})$ be the linear frame of $B_n[\tilde{D}, \tilde{K}]$ defined by the sequence of real segments $\tilde{\xi} = ([1,2], [2,3], \cdots, [n-1, n])$, i.e. $\tilde{H}_j = H([j, j+1])$.

For the set $E'_{\mathbb{R}} = \{x \in E - N \mid x \text{ real}\}$, we can construct a set of diffeomorphisms $\{\beta_x \mid x \times D \tilde{\rightarrow} \tilde{D}\}$ with the following properties:

(a) $\beta_x(K_x) = \tilde{K}$.

(b) $\beta_x(x \times \text{real part of } D) =$ real part of $\tilde{D}$ (order preserved).

(c) $\forall x, x' \in E'_{\mathbb{R}}, y \in \partial D, \ \beta_x(x, y) = \beta_{x'}(x', y)$.

(d) On each connected component $\tilde{\mathcal{L}}$ of $E'_{\mathbb{R}}$, $\{\beta_x \mid x \in \tilde{\mathcal{L}}\}$ is a continuous family of diffeomorphisms.

Let $\xi_x = \{\xi_{x,1}, \xi_{x,2}, \cdots, \xi_{x,n-1}\}$ ($x \in E'_{\mathbb{R}}$) be the sequence of real segments $[y_i(x), y_{i+1}(x)], \ 1 \leq i \leq n-1$, in $x \times D$ and let $H_x = (H_{x,1}, H_{x,2}, \cdots, H_{x,n-1})$ be the linear frame of $B_n[x \times D, K_x]$ defined by ξ_x.

Now, we assume that $\forall x_j \in N$, there is only one singular point of C over x_j.

Let $x_j \in N$. Choose $x'_j = x_j + \epsilon, \ \epsilon > 0$ a very small number. Let A_j be the singularity of C over x_j (i.e. $x(A_j) = x_j$), and let Y_j be the union of irreducible components of C containing A_j. In $\{y_1(x'_j), \cdots, y_n(x'_j)\}$, there is a subsequence with consecutive indices $\{y_{k_j}(x'_j), y_{k_j+1}(x'_j), \cdots, y_{l_j}(x'_j)\}$ which is equal to $K'_{x'_j} = Y_j \cap (x'_j \times D)$.

In this situation, we can define the following notions:

Definition 2.4.3 Local $\mathcal{L}$.V.C. of A_j ("Local Lefschetz vanishing cycle of A_j")
A skeleton in $(x'_j \times D, K_{x'_j}, K'_{x'_j})$ represented by the sequence of real segments

$$[y_{k_j+r-1}(x'_j), y_{k_j+r}(x'_j)], \ 1 \leq r \leq l_j - k_j$$

is called a **local $\mathcal{L}$.V.C. of** A_j.

Definition 2.4.4 (k_j, l_j), Lefschetz pair of A_j
The smallest and biggest indices k_j, l_j in the sequence considered above form a pair (k_j, l_j), which is called the **Lefschetz pair of** A_j.

Obviously, the local $\mathcal{L}$.V.C. of A_j is uniquely defined by the Lefschetz pair (k_j, l_j).

Definition 2.4.5 $< k_j, l_j >$**, skeleton representing local** $\mathcal{L}$**.V.C. of** A_j
Denote by $< k_j, l_j >$ *the skeleton in* $(\tilde{D}, \tilde{K}, (k_j, k_j+1, \cdots, l_j))$ *represented by consecutive real segments connecting points of* $(k_j, k_j+1, \cdots, l_j)$.

Lemma 2.4.6 *Let* γ *be a simple path in* $E - N$ *connecting* x_j *with* $M(\in \partial E)$, $[x_j, x'_j] \subset \gamma$. *Let* γ' *be the part of* γ *from* x'_j *to* M. *Let*

$$\varphi : \pi_1(E - N, M) \to B_n[M \times D, K_M]$$

be the braid monodromy of C *w.r.t.* $E \times D, \pi_1, M$. *Let* Γ *be the element represented by* $l(\gamma)$. *Then:*

$$\varphi(\Gamma) = \Delta^2 < \mathcal{L}.\text{V.C.}(\gamma', H(< \xi_x >)) >$$

(where, intuitively, $\Delta <$ skeleton $>$ *is a generalized half-twist which is defined according to the skeleton, and* $\Delta^2 <$ skeleton $>$ *is applying this half-twist twice).*

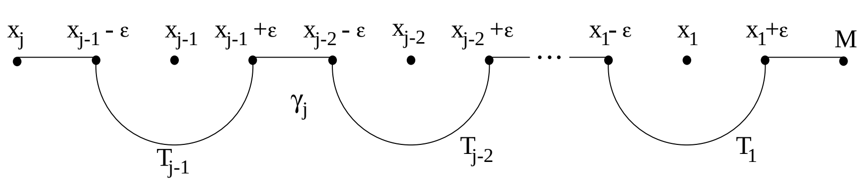
2.5 The algorithm of Moishezon-Teicher

Following lemma 2.4.6, in order to calculate the braid monodromy, we have to find the appropriate Lefschetz vanishing cycles. This is given by the following theorem [MoTe1]:

Theorem 2.5.1 (Moishezon-Teicher)
Let $N = \{x_1, x_2, \cdots, x_q\}$ *with* $x_q < x_{q-1} < \cdots < x_2 < x_1$, $M \in \partial E \cap$ (real axis), *with* $M > x_1$, *and* $\epsilon > 0$ *a very small number. Let* $T_j (1 \leq j \leq q)$ *be the path from* $x_j - \epsilon$ *to* $x_j + \epsilon$ *along the semicircle below real axis centered at* x_j.
Let γ_j *be the path from* x_j *to* M *defined by*

$$\gamma_j = [x_j, x_{j-1} - \epsilon] \cdot T_{j-1} \cdot [x_{j-1} + \epsilon, x_{j-2} - \epsilon] \cdot T_{j-2} \cdots T_1 \cdot [x_1, M]$$

$$(\gamma_j = [x_j, x_{j-1} - \epsilon] \cdot T_{j-1} \cdot (\prod_{r=j-1}^{2} [x_r + \epsilon, x_{r-1} - \epsilon] \cdot T_{r-1}) \cdot [x_1, M])$$



Considering $l(\gamma_j)$*'s, we get a g-base* $\{\delta_1, \delta_2, \cdots, \delta_q\}$ *in* $\pi_1(E - N, M)$.

Assume that for all x_j, $1 \leq j \leq q$, *there is only one singular point* A_j *with* $x(A_j) = x_j$. *Let* (k_j, l_j) *be the Lefschetz pair of* A_j, *and* $< k_j, l_j >$ *be the skeleton in* $(\tilde{D}, \tilde{K}, (k_j, k_j +$

$1, \cdots, l_j - 1, l_j))$ *representing local* $\mathcal{L}$*.V.C. of* A_j. *Let* γ_j' *be the part of* γ_j *from* $x_j' = x_j + \epsilon$ *to* M.
Then:

$$\mathcal{L}\text{.V.C.}(\gamma_j') = \beta_M^{-1}(< k_j, l_j > \cdot \prod_{m=j-1}^{1} \Delta < k_m, l_m >)$$

(where $\prod_{m=j-1}^{1} \Delta < k_m, l_m > =$

$$\Delta < k_{j-1}, l_{j-1} > \cdot \Delta < k_{j-2}, l_{j-2} > \cdots \Delta < k_1, l_1 > \in B_n[\tilde{D}, \tilde{K}])$$

and

$$\mathcal{L}\text{.V.C.}(\gamma_1') = \beta_M^{-1}(< k_1, l_1 >)$$

According to this theorem, in order to compute the braid monodromy of a line arrangement, we have to do the following steps:

1. Check that the line arrangement fulfills the assumption that there are no more than one intersection point with the same x-coordinate (so we can apply the theorem).

2. Find the Lefschetz pairs of all the intersection points.

3. Calculate the Lefschetz vanishing cycle of every intersection point according to the last theorem (2.5.1).

4. The braid monodromy is the Δ^2 of this $\mathcal{L}$.V.C.

2.6 The Van-Kampen theorem

The Van-Kampen theorem induces a finite presentation of the fundamental group of complements of curves by meaning of generators and relations. From this finite presentation, we will calculate the structure of the group in our cases (the original theorem is in [VK], other versions can be found at [Mo, pp. 127-130], [MoTe3], [MoTe4, ch. 13], [Te1]. The theorems presented here are from [MoTe3],[MoTe4] and [Te1]).

Let S be an algebraic curve in $\mathbb{C}^2$ ($p = \deg S$). Let $\pi = \pi_1 : \mathbb{C}^2 \to \mathbb{C}$ be the canonical projection on the first coordinate. Let $\mathbb{C}_x = \pi^{-1}(x)$, and now define: $K_x = \mathbb{C}_x \cap S$ (By assumption $\deg S = p$, we know $\#K_x \leq p$).

Let $N = \{x \mid \#K_x < p\}$. Choose now $u \in \mathbb{C}$, u real, such that $x \ll u$, $\forall x \in N$, and define: $B_p = B_p[\mathbb{C}_u, \mathbb{C}_u \cap S]$. Let $\varphi_u : \pi_1(\mathbb{C} - N, u) \to B_p$ be the braid monodromy of S w.r.t π, u. Also choose $u_0 \in \mathbb{C}_u$, $u_0 \notin S$, u_0 below real line far enough such that B_p does not move u_0. It is known that the group $\pi_1(\mathbb{C}_u - S, u_0)$ is free. There exists an epimorphism $\pi_1(\mathbb{C}_u - S, u_0) \to \pi_1(\mathbb{C}^2 - S, u_0)$, so a set of generators for $\pi_1(\mathbb{C}_u - S, u_0)$ determines a set of generators for $\pi_1(\mathbb{C}^2 - S, u_0)$.

In this situation, Van-Kampen's theorem says:

Theorem 2.6.1 Van-Kampen's Theorem - classic version
Let S be an algebraic curve, u, u_0, φ_u defined as above. Let $\{\delta_i\}$ be a g-base of $\pi_1(\mathbb{C} - N, u)$. Let $\{\Gamma_j \mid 1 \leq j \leq p\}$ $(p = \deg S)$ be a g-base for $\pi_1(\mathbb{C}_u - S, u_0)$.
Then, $\pi_1(\mathbb{C}^2 - S, u_0)$ is generated by the images of Γ_j in $\pi_1(\mathbb{C}^2 - S, u_0)$ and we get a complete set of relations from those induced from

$$(\varphi_u(\delta_i))(\Gamma_j) = \Gamma_j; \forall i \forall j$$

Here we present also the classic Van-Kampen theorem for the projective case. The only difference between the affine case and the projective case is that there is one additional relation in the projective case - the multiplication of all the generators is equal to the identity of the group.

Theorem 2.6.2 Van-Kampen's Theorem for projective case - classic version
Let S be an algebraic curve, u, u_0, φ_u defined as above. Let $\{\delta_i\}$ be a g-base of $\pi_1(\mathbb{C} - N, u)$. Let $\{\Gamma_j \mid 1 \leq j \leq p\}$ $(p = \deg S)$ be a g-base for $\pi_1(\mathbb{C}_u - S, u_0)$.
Then, $\pi_1(\mathbb{CP}^2 - S, u_0)$ is generated by the images of Γ_j in $\pi_1(\mathbb{C}^2 - S, u_0)$ and we get a complete set of relations from those induced from

$$(\varphi_u(\delta_i))(\Gamma_j) = \Gamma_j; \forall i \forall j$$

with one additional relation:

$$\Gamma_p \Gamma_{p-1} \cdots \Gamma_1 = 1$$

Oka [O] proved the following connection between the fundamental group of the affine case and the fundamental group of the projective case:

Theorem 2.6.3 (Oka)
Let C be a curve in $\mathbb{CP}^2$ and let L be a general line to C. Then, we have a central extension:

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{CP}^2 - (C \cup L)) \to \pi_1(\mathbb{CP}^2 - C) \to 1$$

Due to the fact that L is in a general position to C, we can say:

$$\pi_1(\mathbb{CP}^2 - (C \cup L)) \cong \pi_1((\mathbb{CP}^2 - L) - C) \cong \pi_1(\mathbb{C}^2 - C)$$

(by choosing L as the line at infinity). Therefore, we get the following short exact sequence (see also [OS]):

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 - C) \to \pi_1(\mathbb{CP}^2 - C) \to 1$$

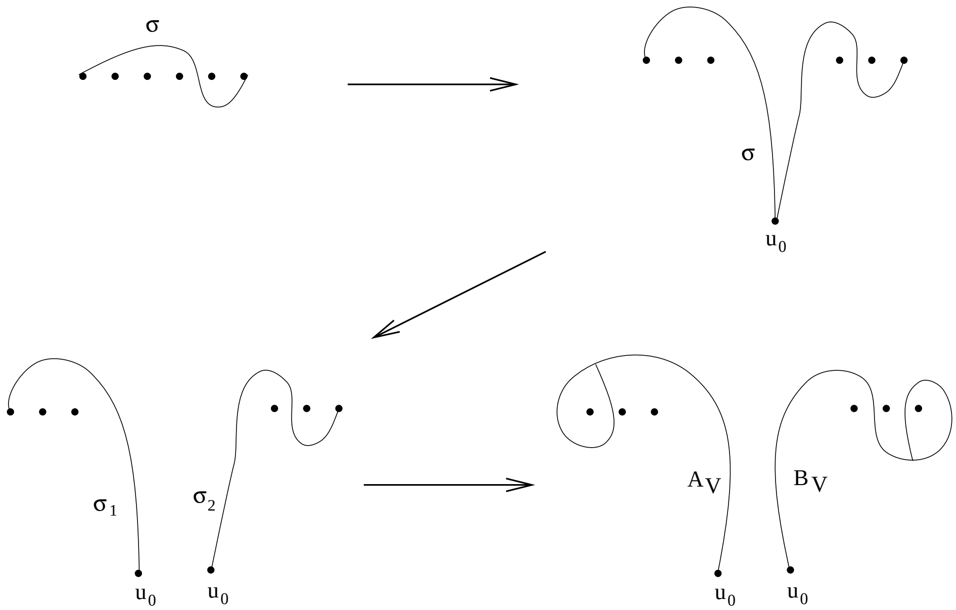
We will show that in the cases which we treat, we get:

$$\pi_1(\mathbb{C}^2 - C) \cong \pi_1(\mathbb{CP}^2 - C) \oplus \mathbb{Z}$$

and therefore, this short exact sequence splits.

Now we return to the affine case. In order to give a more precise version of Van-Kampen's theorem for cuspidal curves, i.e. for curves with only nodes and cusps as singularities, we need the following two lemmas.

Lemma 2.6.4 *Let V be a half-twist in $B_p[D,K]$, $u_0 \notin K$.*
Then: there exists $A_V, B_V \in \pi_1(D-K,u_0)$, such that:
(a) *$\{A_V, B_V\}$ can be extended to a g-base of $\pi_1(D-K,u_0)$.*
(b) *$V(A_V) = B_V$.*



Let S be a cuspidal curve in $\mathbb{C}^2$ ($p = \deg S$). We assume that for every $x \in N$ (N as above), there is only one singular point over it (in $\mathbb{C}^2$). Thus, for every $x \in N$, let x' be the singular point over x. Because S is a cuspidal curve, the point x' is either a branch point, a node or a cusp.

Lemma 2.6.5 *Let $\{\delta_i\}$ be a g-base for $\pi_1(\mathbb{C}-N,u)$. For every δ_i, there exists V_i and ν_i, where V_i is a half-twist and ν_i is a number such that $\varphi_u(\delta_i) = V_i^{\nu_i}$. Moreover, $\nu_i = 1,2,3$ if c_i' (the singular point) = a branch point, a node or a cusp respectively.*

We denote:

$$[A,B] = ABA^{-1}B^{-1}$$

$$< A,B >= ABAB^{-1}A^{-1}B^{-1}$$

Now, we can give the precise version of the Van-Kampen theorem for cuspidal curves:

Theorem 2.6.6 Van-Kampen's theorem for cuspidal curves
Let S be a cuspidal curve, $u, u_0, \varphi_u, A_{V_i}, B_{V_i}$ defined as above. Let $\{\delta_i\}$ be a g-base of $\pi_1(\mathbb{C}-N,u)$. Let $\varphi_u(\delta_i) = V_i^{\nu_i}$, V_i is a half-twist, $\nu_i = 1,2,3$ (as above).

Let $\{\Gamma_j \mid 1 \leq j \leq p\}$ ($p = \deg S$) be a g-base for $\pi_1(\mathbb{C}_u - S, u_0)$.
Then: $\pi_1(\mathbb{C}^2 - S, u_0)$ is generated by the images of Γ_j in $\pi_1(\mathbb{C}^2 - S, u_0)$ and we get a complete set of relations from those induced from $\varphi_u(\delta_i) = V_i^{\nu_i}$, as follows (when A_{V_i}, B_{V_i} are expressed in terms of $\{\Gamma_j\}$):

(a) $A_{V_i} = B_{V_i}$, *when* $\nu_i = 1$.

(b) $[A_{V_i}, B_{V_i}] = 1$, *when* $\nu_i = 2$.

(c) $< A_{V_i}, B_{V_i} >= 1$, *when* $\nu_i = 3$.

What do we get from this theorem? After we calculate the appropriate braid monodromy, we can get a finite presentation of the desired fundamental group.

Note that it is easy to see that the relation, which is induced from the braid monodromy, is uniquely determined by the half-twist V, and is independent of the choice of A_V, B_V.

Now, we will present the version of Van-Kampen's theorem for an arrangement with a single multiple point, i.e. an arrangement where all the lines meet in one point (the proof is easy, and can be found, for example, in [Ga, p. 25]):

Lemma 2.6.7 Van-Kampen's theorem for a single multiple point
Let $l_1, \cdots, l_k$ *be* k *real lines in* $\mathbb{CP}^2$ *meeting in a single point* p. *Let* δ *be a loop in* $\pi_1(E-N, u_0)$ *around* $x(p)$. *Let* $\{\Gamma_1, \cdots, \Gamma_k\}$ *be a g-base of* $\pi_1(\mathbb{C}_{u_0} - \bigcup_{i=1}^{k} l_i)$.
Then, the relations which are induced from this intersection point are:

$$\Gamma_k \Gamma_{k-1} \cdots \Gamma_1 = \Gamma_1 \Gamma_k \cdots \Gamma_3 \Gamma_2 = \cdots = \Gamma_{k-1} \Gamma_{k-2} \cdots \Gamma_1 \Gamma_k$$

2.7 An application of the Van-Kampen theorem

Here, we will prove a simple proposition, which will help us in the future. We denote $[x, y] = xyx^{-1}y^{-1}$ for x, y in a group G.

Proposition 2.7.1 *Let* p *be an intersection point of* k *real lines* $l_{j_1}, \cdots, l_{j_k}$ *in* $\mathbb{CP}^2$. *Let* δ *be a loop in* $\pi_1(E - N, u_0)$ *around* $x(p)$.
Let $\{\Gamma_{j_1}, \cdots, \Gamma_{j_k}\}$ *be a g-base of* $\pi_1(\mathbb{C}_{u_0} - \bigcup_{i=1}^{k} l_{j_i})$.
Then: the relations which are induced from this intersection point are:

$$[\Gamma_{j_k} \Gamma_{j_{k-1}} \cdots \Gamma_{j_1}, \Gamma_{j_i}] = 1; \;\; 1 \leq i \leq k$$

Proof: By the Van-Kampen version for a multiple point (2.6.7), the following set of relations is induced from the intersection point p:

$$\Gamma_{j_k} \Gamma_{j_{k-1}} \cdots \Gamma_{j_1} = \Gamma_{j_{k-1}} \cdots \Gamma_{j_1} \Gamma_{j_k} = \cdots = \Gamma_{j_1} \Gamma_{j_k} \cdots \Gamma_{j_2}$$

We will prove now that this set of relations is equivalent to the set of relations in the formulation of the proposition.

($\Rightarrow$) Let $1 \leq i \leq k$. We have to show that

$$\Gamma_{j_k} \Gamma_{j_{k-1}} \cdots \Gamma_{j_1} \Gamma_{j_i} = \Gamma_{j_i} \Gamma_{j_k} \Gamma_{j_{k-1}} \cdots \Gamma_{j_1}$$

We know (from the first set of relations) that

$$(*)\ \Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1} = \Gamma_{j_i}\Gamma_{j_{i-1}}\cdots\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_{i+1}}$$

$$(**)\ \Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1} = \Gamma_{j_{i-1}}\Gamma_{j_{i-2}}\cdots\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_i}$$

Now:

$$(\Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1})\Gamma_{j_i} \overset{(*)}{=} (\Gamma_{j_i}\Gamma_{j_{i-1}}\cdots\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_{i+1}})\Gamma_{j_i} =$$

$$= \Gamma_{j_i}(\Gamma_{j_{i-1}}\cdots\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_{i+1}}\Gamma_{j_i}) \overset{(**)}{=} \Gamma_{j_i}(\Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1})$$

$(\Leftarrow)$ From the first relation we have:

$$[\Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1}, \Gamma_{j_1}] = 1$$

i.e. $\Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1}\Gamma_{j_1} = \Gamma_{j_1}\Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1}$. Now, multiply it by $\Gamma_{j_1}^{-1}$ from the right to get:

$$(***)\ \Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1} = \Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_2}$$

From the second relation we have: $(\Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1})\Gamma_{j_2} = \Gamma_{j_2}(\Gamma_{j_k}\Gamma_{j_{k-1}}\cdots\Gamma_{j_1})$, but from $(***)$ we get: $(\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_2})\Gamma_{j_2} = \Gamma_{j_2}(\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_2})$. Now, multiply it by $\Gamma_{j_2}^{-1}$ from the right to get:

$$\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_2} = \Gamma_{j_2}\Gamma_{j_1}\Gamma_{j_k}\cdots\Gamma_{j_3}.$$

Applying the same argument together with the rest of the commutative relations give us the requested cyclic relations. ■

2.8 Outline of the computation of the fundamental group of the complement of line arrangements

Let us summarize the steps we have to follow in order to compute the fundamental group of the complement of a given real line arrangement $\mathcal{L}$:

(1) Calculation of the braid monodromy of $\mathcal{L}$:

- Check that the line arrangement fulfills the assumption that there are no more than one intersection point with the same x-coordinate (so we can apply the theorem).
- Find the Lefschetz pairs of all the intersection points.
- Calculate the Lefschetz vanishing cycle of every intersection point according to the Moishezon-Teicher theorem.

(2) Calculation of the relations induced on $\pi_1(\mathbb{C}^2 - \mathcal{L})$ from the braid monodromy:

- Choose u as in section 2.6.
- Choose a g-base for $\pi_1(\mathbb{C}_u - \mathcal{L})$: $\{\Gamma_1, \cdots, \Gamma_n\}$.

- Calculate the A_{V_i}, B_{V_i} from the $\mathcal{L}$.V.C. for every singular point in terms of $\Gamma_i, i = 1, \cdots, n$.
- Find the induced relations according to the Van-Kampen theorem.

(3) Computing the structure of $\pi_1(\mathbb{C}^2 - \mathcal{L})$ from the relations in (2). This step contains some group calculations and combinatorics.

3 Arrangements with t non-collinear multiple points

In this section, we are going to calculate the fundamental group of the complement of line arrangements where there is no line on which there are two multiple points. Thus, we can divide the arrangement into t subsets of lines where all the lines in each subset intersect at a single (multiple) point and any two such subsets intersect in simple points only. We define:

Definition 3.0.1 Simple point, multiple point, multiplicity of a point
A **simple point** *in a line arrangement is a point where two lines meet. A* **multiple point** *in a line arrangement is a point where more than two lines meet. The* **multiplicity of a point** *is the number of lines which meet in the point.*

Definition 3.0.2 An arrangement with t non-collinear multiple points
An **arrangement with t non-collinear multiple points** *is an arrangement where there is no line on which there are two multiple points and we can divide it into t subsets of lines where all the lines in each subset intersect in a single multiple point.*

We denote by $\mathbb{F}^k$ the free group with k generators.

3.1 The affine case

We calculate the affine case:

Theorem 3.1.1 *Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ with t non-collinear multiple points. Let $k_i + 1$ be the multiplicity of the multiple point P_i, $1 \le i \le t$.*
Then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong (\bigoplus_{i=1}^{t} \mathbb{F}^{k_i}) \oplus \mathbb{Z}^t$$

Proof: Randell [Ra] showed that the fundamental group of the complement of a real line arrangement which consists of n lines meet in a single point is $\mathbb{F}^{n-1} \oplus \mathbb{Z}$.

We can observe $\mathcal{L}$ as a union of t subsets of lines $\mathcal{L}_i, 1 \le i \le t$, where every such subset $\mathcal{L}_i$, $1 \le i \le t$, consists of $k_i + 1$ lines which are passing through the multiple point P_i (there is no $l \in \mathcal{L}_i \cap \mathcal{L}_j$, because then l connects P_i and P_j, a contradiction to the assumption). The degree of each $\mathcal{L}_i$ is exactly $k_i + 1$, because there are $k_i + 1$ lines which pass through the point P_i. Moreover, $\mathcal{L}_i \cap \mathcal{L}_j = (k_i + 1)(k_j + 1)$ points, because every line in $\mathcal{L}_i$ meets every line in $\mathcal{L}_j$.

Every $\mathcal{L}_i,\ 1 \leq i \leq t$, consists of $k_i + 1$ lines which pass through the multiple point P_i. This is the configuration of Randell. Therefore:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}_i) = \mathbb{F}^{k_i} \oplus \mathbb{Z}$$

Now we can use the Oka-Sakamoto theorem (see section 2.1), in order to compute the fundamental group of the complement of $\mathcal{L}$:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) = \pi_1(\mathbb{C}^2 - \bigcup_{i=1}^{t} \mathcal{L}_i) \overset{\text{(O−S)}}{\cong} \bigoplus_{i=1}^{t} (\pi_1(\mathbb{C}^2 - \mathcal{L}_i)) =$$

$$= \bigoplus_{i=1}^{t} (\mathbb{F}^{k_i} \oplus \mathbb{Z}) = (\bigoplus_{i=1}^{t} \mathbb{F}^{k_i}) \oplus \mathbb{Z}^t$$

■

The Oka-Sakamoto theorem gives us a new inductive approach to prove Zariski's proposition:

Proposition 3.1.2 (Zariski)
The fundamental group of the complement of n lines in general position is abelian.

Proof: It is known that for a line L:

$$\pi_1(\mathbb{C}^2 - L) \cong \mathbb{Z}$$

Due to the general position of the lines in the arrangement, we can use the Oka-Sakamoto theorem (see section 2.1) inductively in the following way:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) = \pi_1(\mathbb{C}^2 - \bigcup_{i=1}^{n} l_i) \overset{\text{(O−S)}}{\cong} \bigoplus_{i=1}^{n} (\pi_1(\mathbb{C}^2 - l_i)) \cong \bigoplus_{i=1}^{n} \mathbb{Z} \cong \mathbb{Z}^n$$

And $\mathbb{Z}^n$ is an abelian group (see [O] too). ■

3.2 The projective case

Now, we will investigate the projective case.

Theorem 3.2.1 *Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ with t non-collinear multiple points. Let $k_i + 1$ be the multiplicity of the multiple point P_i, $1 \leq i \leq t$.*
Then:

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong (\bigoplus_{i=1}^{t} \mathbb{F}^{k_i}) \oplus \mathbb{Z}^{t-1}$$

Proof: First, we will prove this theorem for $t = 1$, i.e. if $\mathcal{L}$ is a real line arrangement in $\mathbb{CP}^2$ which consists of $k + 1$ lines meeting in one point P, then $\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{F}^k$.

Let $\{\Gamma_1, \cdots, \Gamma_{k+1}\}$ be a g-base of $\pi_1(\mathbb{C}_u - \mathcal{L})$ (see section 2.8). In this line arrangement, we have only one singular point - P. Therefore, according to lemma 2.6.7 and proposition 2.7.1, this singular point induced the following set of relations:

$$[\Gamma_{k+1}\Gamma_k \cdots \Gamma_1, \Gamma_i] = 1, i = 1, \cdots, k+1$$

Hence, the fundamental group of its affine complement has the following presentation:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \; = < \Gamma_1, \cdots, \Gamma_{k+1} \mid [\Gamma_{k+1}\Gamma_k \cdots \Gamma_1, \Gamma_i] = 1, i = 1, \cdots, k+1 >$$

We will compute now another presentation for this group.

Let us modify the set of generators $g = \{\Gamma_1, \cdots, \Gamma_{k+1}\}$ by replacing the generator Γ_1 by the generator

$$\Gamma' = \Gamma_{k+1}\Gamma_k \cdots \Gamma_1$$

Then, we have to check that after the modifications we get an equivalent set of generators, and we have to calculate the new set of relations.

Claim 3.2.2 *After replacing Γ_1 by Γ' (which was defined above) in g, we again get a set of generators. We denote this set of generators by $\tilde{g}$.*

Proof: we have to show that $\Gamma_1 \in < \tilde{g} >$. But this is obvious, because:

$$\Gamma_1 = \Gamma_2^{-1}\Gamma_3^{-1} \cdots \Gamma_{k+1}^{-1}\Gamma'$$

■

The next step is the calculation of the new set of relations for $\tilde{g}$.

Claim 3.2.3 *The set of relations:*

$$\{[\Gamma', \Gamma] = 1 \mid \forall \Gamma \in \tilde{g}\}$$

is a complete set of relations for $\tilde{g}$.

Proof: We have to show that

$$(*) \quad \{[\Gamma', \Gamma] = 1 \mid \forall \Gamma \in \tilde{g}\}$$

is an equivalent set of relations to

$$(**) \quad \{[\Gamma_{k+1}\Gamma_k \cdots \Gamma_1, \Gamma_i] = 1 \mid 1 \leq i \leq k+1\}$$

under the assignment: $\Gamma' = \Gamma_{k+1} \cdots \Gamma_1$.

Let us assume $(*)$. All the relations are equal except the first one. We have to show that:

$$[\Gamma_{k+1} \cdots \Gamma_1, \Gamma_1] = 1$$

But:

$$\Gamma'\Gamma_1 = \Gamma'(\Gamma_2^{-1}\cdots\Gamma_{k+1}^{-1}\Gamma') \overset{(*)\ +\ ab=ba\Rightarrow ab^{-1}=b^{-1}a}{=} (\Gamma_2^{-1}\cdots\Gamma_{k+1}^{-1}\Gamma')\Gamma' = \Gamma_1\Gamma'$$

Now, if we assume $(**)$, all the relations in $(*)$ are equal except of $\Gamma'\Gamma' = \Gamma'\Gamma'$ which is trivial. ■

Hence we got the following presentation for the fundamental group of the affine complement of $\mathcal{L}$:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) =< \Gamma', \Gamma_2, \cdots, \Gamma_{k+1} \mid [\Gamma_i, \Gamma'] = 1, 2 \leq i \leq k+1 >$$

Now, when we are going to the projective case, we add one additional relation, according to theorem 2.6.2:

$$\Gamma_{k+1}\cdots\Gamma_1 = 1$$

In terms of the new generator Γ', this relation gets the following form:

$$\Gamma' = 1$$

Therefore, we can copmute the structure of the fundamental group in the projective case with $t = 1$:
$\pi_1(\mathbb{CP}^2 - \mathcal{L}) =< \Gamma', \Gamma_2, \cdots, \Gamma_{k+1} \mid [\Gamma_i, \Gamma'] = 1, 2 \leq i \leq k+1; \Gamma' = 1 > \cong$
$\cong < \Gamma_2, \cdots, \Gamma_{k+1} > \oplus < \Gamma' \mid \Gamma' = 1 > \cong \mathbb{F}^k$.

Now we continue to the general case $(t > 1)$. For simplicity of the proof, we will prove it for two multiple points and the proof for t multiple points uses exactly the same arguments.

From the last theorem, we get for a line arrangement $\mathcal{L}$ with two multiple points:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F}^{k_1} \oplus \mathbb{F}^{k_2} \oplus \mathbb{Z}^2$$

Let $l_1, \cdots, l_{k_1+1}$ be k_1+1 lines which pass through P_1 and let $l_{k_1+2}, \cdots, l_{k_1+k_2+2}$ be k_2+1 lines which pass through P_2. We choose $\{\Gamma_1, \cdots, \Gamma_{k_1+k_2+2}\}$, a g-base of $\pi_1(\mathbb{C}_u - \mathcal{L})$ (see section 2.8) where Γ_i corresponds to the line l_i.

Similarly to the first part of the proof, we can write the following presentation for $\pi_1(\mathbb{C}^2 - \mathcal{L})$:
Generators: $g = \{\Gamma_1, \cdots, \Gamma_{k_1}, \Gamma', \Gamma_{k_1+2}, \cdots, \Gamma_{k_1+k_2+1}, \Gamma''\}$.
Relations: $\mathcal{R} = \{\Gamma_i\Gamma_j = \Gamma_j\Gamma_i, 1 \leq i \leq k_1, k_1+2 \leq j \leq k_1+k_2+1;$
$[\Gamma', \Gamma] = 1, \forall\Gamma \in g;\ [\Gamma'', \Gamma] = 1, \forall\Gamma \in g\}$, where:

$$\Gamma' = \Gamma_{k_1+1}\cdots\Gamma_1; \quad \Gamma'' = \Gamma_{k_1+k_2+2}\cdots\Gamma_{k_1+2}$$

Now, when we are going to the projective case, we add one additional relation, according to theorem 2.6.2:

$$\Gamma_{k_1+k_2+2}\cdots\Gamma_1 = 1$$

In terms of the new generators Γ', Γ'', this relation gets the following form:

$$\Gamma''\Gamma' = 1$$

Now, we can finish to compute the structure the fundamental group in the projective case:
$\pi_1(\mathbb{CP}^2 - \mathcal{L}) =< g \mid \mathcal{R}, \Gamma''\Gamma' = 1 > \cong$
$\cong < \Gamma_1, \cdots, \Gamma_{k_1} > \oplus < \Gamma_{k_1+2}, \cdots, \Gamma_{k_1+k_2+1} > \oplus < \Gamma', \Gamma'' \mid \Gamma''\Gamma' = 1 > \cong \mathbb{F}^{k_1} \oplus \mathbb{F}^{k_2} \oplus \mathbb{Z}.$ ∎

As a consequence of the last theorem, we get:

Corollary 3.2.4

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}) \oplus \mathbb{Z}$$

Therefore, the short exact sequence which was proved by Oka (theorem 2.6.3):

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 - \mathcal{L}) \to \pi_1(\mathbb{CP}^2 - \mathcal{L}) \to 1$$

splits.

4 Arrangements with t collinear multiple points

In this section, we are going to calculate the fundamental group of the complement of line arrangements which consist of t subsets of lines where all the lines in each subset intersect at a single (multiple) point, all the t multiple intersection points lie on a single line which belongs to all the subsets and any two subsets of lines intersect in that line and in simple points out of that line. We define:

Definition 4.0.5 An arrangement with t collinear multiple points
An **arrangement with t collinear multiple points** *is a line arrangement which contains a line where all the t multiple points lie on it.*

4.1 The affine case

Theorem 4.1.1 *Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ with t collinear multiple points $P_1, \cdots, P_t$ with multiplicities $k_1 + 1, \cdots, k_t + 1$, respectively. Then:*

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \bigoplus_{i=1}^{t} \mathbb{F}^{k_i} \oplus \mathbb{Z}$$

It has to be noted that this theorem has a similar result to what we have got in the previous section in the non-collinear case. In both cases, the multiple points induced the free groups. The difference between the cases is that the connected line of the collinear case degenerates all the infinite cyclic groups of the non-collinear case into one infinite cyclic group.

Let L be the line on which all the multiple points lie. We choose $\{\Gamma_1, \cdots, \Gamma_n\}$ ($n = \#\{l \in \mathcal{L}\}$), a g-base of $\pi_1(\mathbb{C}_{u_0} - \mathcal{L})$ (see section 2.8), where Γ_i corresponds to the line l_i in $\mathcal{L}$. The proof of the theorem is based on the following two lemmas:

Lemma 4.1.2 *In the situation of the theorem, let $\mathcal{L}_i$ be the subset of lines meet in P_i apart from L. Then: $[\Gamma_i, \Gamma_j] = 1$ where $l_i \in \mathcal{L}_i, l_j \in \mathcal{L}_j$ and $1 \leq i < j \leq t$.*

Lemma 4.1.3 *Let $\mathcal{L}_i \cup L = \{l_{p_1}, \cdots, l_{p_{k_i+1}}\}$ be the k_i+1 lines that meet in the multiple point P_i. Then, the relations that are induced from this multiple point are:*

$$[\Gamma_{p_{k_i+1}} \cdots \Gamma_{p_1}, \Gamma_{p_j}] = 1, \quad 1 \leq j \leq k_i + 1$$

The proof of lemma 4.1.2 is in section 4.2. The proof of lemma 4.1.3 is in section 4.3. The proof of the theorem (4.1.1) is in section 4.4.

4.2 Proof of lemma 4.1.2

For simplicity, we prove the lemma only for two multiple points, and the proof for t multiple points uses exactly the same arguments.

We will split the proof of this lemma into two cases: with the restriction that all the simple intersection points are to the right of the multiple points, and without this restriction. This restriction simplifies the proof significantly, and help to understand the proof of the general case.

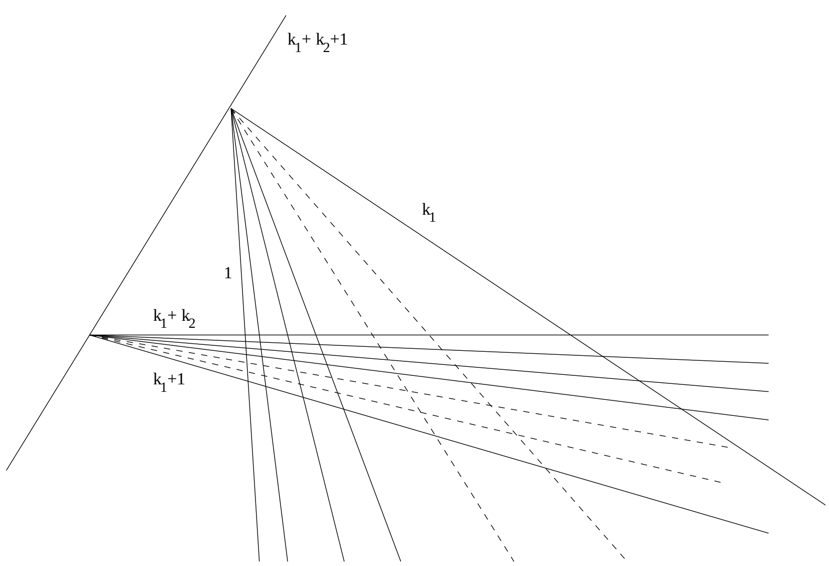
4.2.1 First case - with the restriction

In this case, all the simple points are to the right of the multiple points.

Let $N = \{x \in \mathbb{C} \mid (x, y) \text{ is an intersection point}\}$, and let $u_0 \in \mathbb{R}$ such that $x \ll u_0$ for all $x \in N$. Let $\mathbb{C}_{u_0} = \{(u_0, y) \mid y \in \mathbb{C}\}$. We numerate the lines according to their intersection with $\mathbb{C}_{u_0}$. By a proper choosing of the line in infinity and homotopic movements of the lines, we can assume that the line arrangement has the following property: for $1 \leq i < j \leq k_1$,

$$x(l_i \cap l_t) < x(l_j \cap l_s), \quad k_1 + 1 \leq t, s \leq k_1 + k_2$$

Therefore, we get the following line arrangement:



Let $g = \{\Gamma_1, \cdots, \Gamma_{k_1+k_2+1}\}$ be a g-base of $\pi_1(\mathbb{C}_{u_0} - \mathcal{L})$. By abuse of notations, let us denote the images of Γ_i in $\pi_1(\mathbb{C}^2 - \mathcal{L})$ by the same notation.

Now, we prove this lemma using the braid monodromy techniques (2.5.1) and the Van-Kampen theorem (2.6.6). First, let us calculate the skeletons representing the $\mathcal{L}$.V.C.s of the braid monodromy.

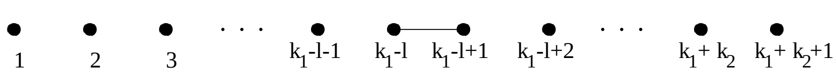
According to this line arrangement, we have the following set of Lefschetz pairs:

j	λ_{x_j}
1	(k_1, k_1+1)
2	(k_1+1, k_1+2)
3	(k_1+2, k_1+3)
$\vdots$	$\vdots$
k_2	(k_1+k_2-1, k_1+k_2)
k_2+1	(k_1-1, k_1)
k_2+2	(k_1, k_1+1)
$\vdots$	$\vdots$
$2k_2$	(k_1+k_2-2, k_1+k_2-1)
$\vdots$	$\vdots$
$(k_1-1)k_2+1$	$(1,2)$
$(k_1-1)k_2+2$	$(2,3)$
$\vdots$	$\vdots$
k_1k_2	(k_2, k_2+1)
k_1k_2+1	(k_2+1, k_1+k_2+1)
k_1k_2+2	$(1, k_2+1)$

Let $\{\delta_i \mid 1 \le i \le k_1k_2+2\}$ be a g-base for $\pi_1(\mathbb{C}^X - N, u_0)$ (where $\mathbb{C}^X$ is the x-axis). Let φ be the braid monodromy of $\mathcal{L}$ w.r.t. π_1, u_0.

Now, using the table of Lefschetz pairs, we can calculate the skeletons representing the $\mathcal{L}$.V.C.s for the braids $\varphi(\delta_i)$ (according to Moishezon-Teicher's algorithm (2.5.1)). Here, we will calculate the $\mathcal{L}$.V.C.s of the two general cases.

Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{lk_2+1}), 0 \le l \le k_1-1$: The Lefschetz pair is (k_1-l, k_1-l+1). So the skeleton representing the local $\mathcal{L}$.V.C. is:



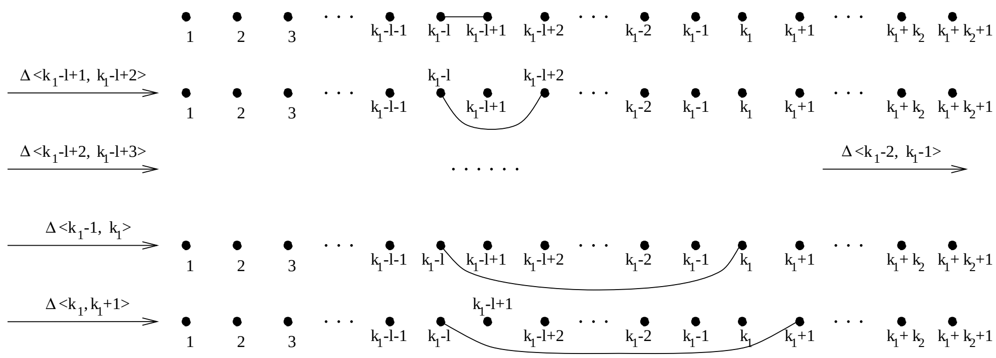
According to the algorithm, we have to apply on the skeleton the composition of the following l sequences of braids:

$$\Delta < k_1+k_2-l, k_1+k_2-l+1 > \Delta < k_1+k_2-l-1, k_1+k_2-l > \cdots$$

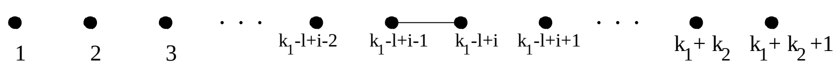
$$\Delta < k_1-l+2, k_1-l+3 > \Delta < k_1-l+1, k_1-l+2 >$$

$$\Delta < k_1+k_2-l+1, k_1+k_2-l+2 > \Delta < k_1+k_2-l, k_1+k_2-l+1 > \cdots$$
$$\Delta < k_1-l+3, k_1-l+4 > \Delta < k_1-l+2, k_1-l+3 >$$
$$\vdots$$
$$\Delta < k_1+k_2-1, k_1+k_2 > \Delta < k_1+k_2-2, k_1+k_2-1 > \cdots \Delta < k_1, k_1+1 >$$

In every sequence, only the last braid of the sequence affects the skeleton (because the region of the others has no intersection with the region of the skeleton). Therefore, we get the following skeleton:



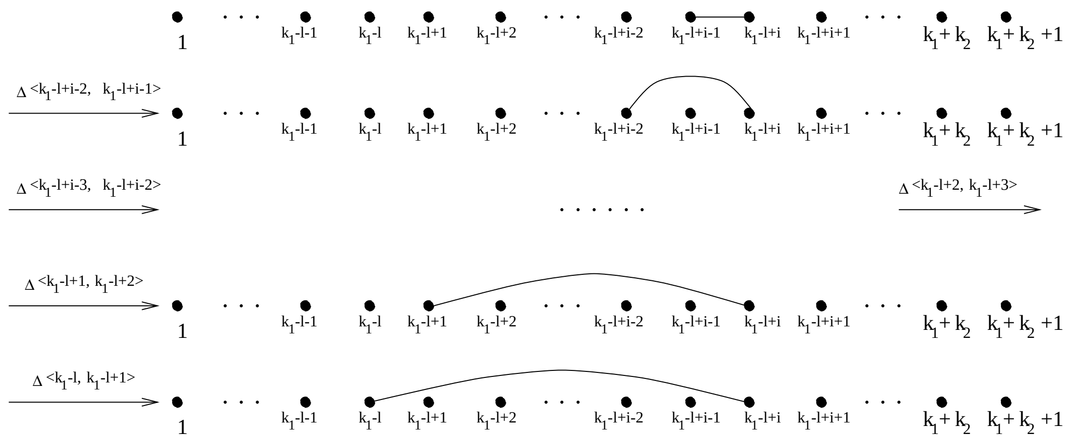
Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{lk_2+i}), 0 \leq l \leq k_1-1, 2 \leq i \leq k_2$: The Lefschetz pair is $(k_1-l+i-1, k_1-l+i)$. So the skeleton representing local $\mathcal{L}$.V.C. is:



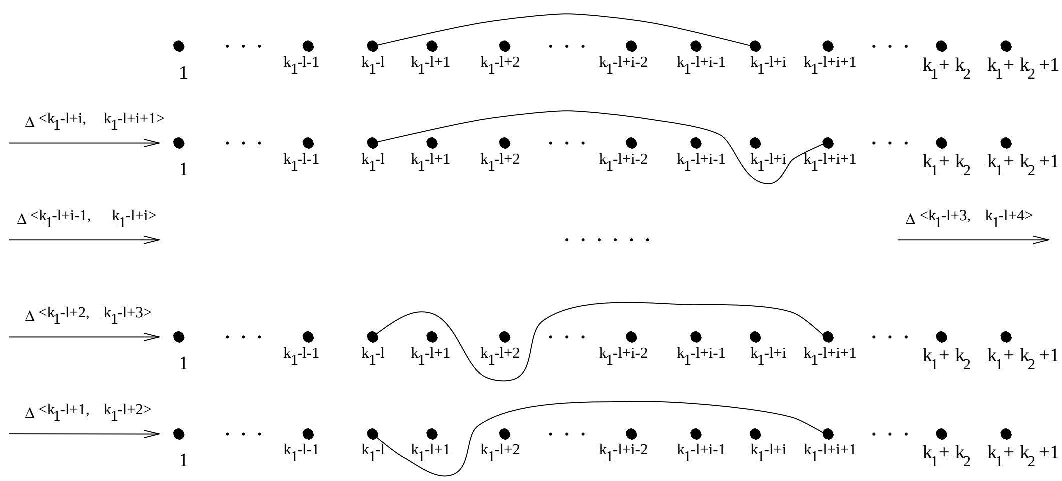
According to the algorithm, we have to apply on the skeleton the composition of the following $l+1$ sequences of braids:

$$\Delta < k_1-l+i-2, k_1-l+i-1 > \Delta < k_1-l+i-3, k_1-l+i-2 > \cdots$$
$$\Delta < k_1-l+1, k_1-l+2 > \Delta < k_1-l, k_1-l+1 >$$
$$\Delta < k_1+k_2-l, k_1+k_2-l+1 > \Delta < k_1+k_2-l-1, k_1+k_2-l > \cdots$$
$$\Delta < k_1-l+2, k_1-l+3 > \Delta < k_1-l+1, k_1-l+2 >$$
$$\Delta < k_1+k_2-l+1, k_1+k_2-l+2 > \Delta < k_1+k_2-l, k_1+k_2-l+1 > \cdots$$
$$\Delta < k_1-l+3, k_1-l+4 > \Delta < k_1-l+2, k_1-l+3 >$$
$$\vdots$$
$$\Delta < k_1+k_2-1, k_1+k_2 > \Delta < k_1+k_2-2, k_1+k_2-1 > \cdots \Delta < k_1, k_1+1 >$$

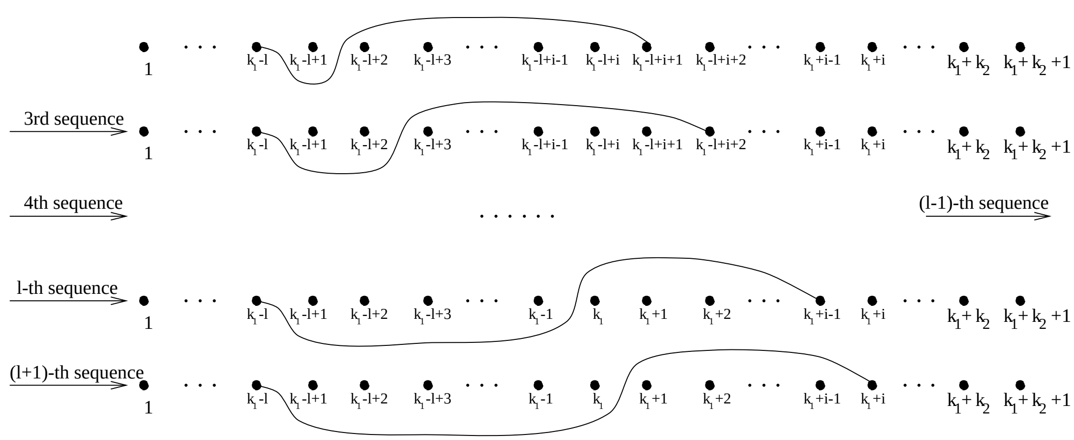
The first sequence causes the following effect to the skeleton:



Only the last part of the second sequence affects the skeleton as follows:

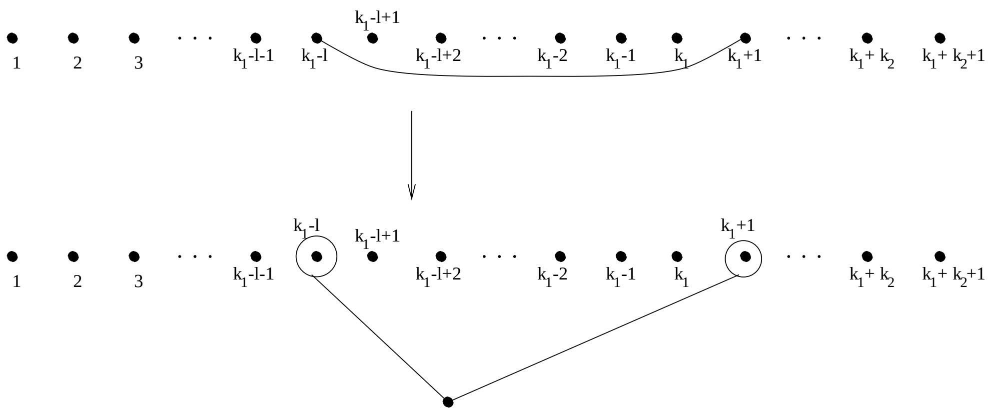


In the other $l-1$ sequences of braids, only the second part of the sequence affects, i.e. only the braids whose region intersects the region of the skeleton. Therefore, we get the following skeleton representing the $\mathcal{L}$.V.C.:



After we have calculated the skeletons representing $\mathcal{L}$.V.C.s for the braid monodromy, we can calculate the relations that they induced. As we have introduced in the previous section, according to Van-Kampen theorem (2.6.6), every $\mathcal{L}$.V.C. induces a relation. Now, we will calculate the general relations which are induced from the general $\mathcal{L}$.V.C.s .

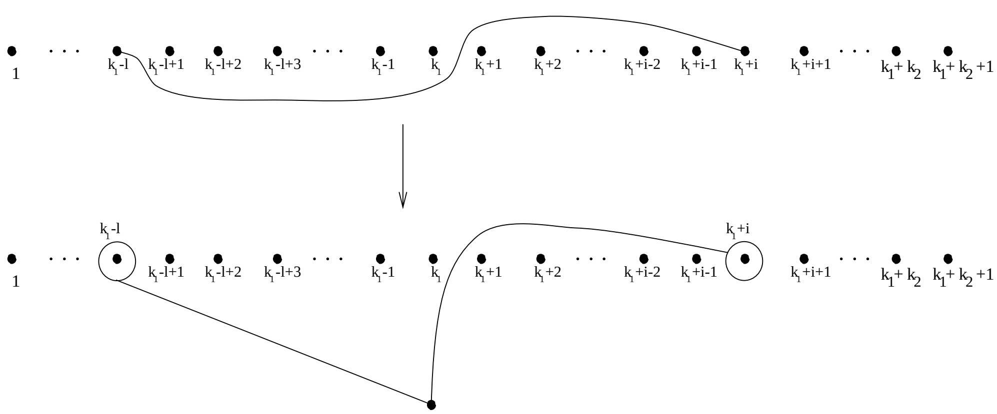
The relation which is induced from $\varphi(\delta_{lk_2+1}),\ 0 \leq l \leq k_1 - 1$:



Therefore, the relation is:

$$\Gamma_{k_1-l}\Gamma_{k_1+1} = \Gamma_{k_1+1}\Gamma_{k_1-l}$$

The relation which is induced from $\varphi(\delta_{lk_2+i}),\ 0 \leq l \leq k_1 - 1,\ 2 \leq i \leq k_2$:



Therefore, the relation is:

$$\Gamma_{k_1-l}\Gamma^{-1}_{k_1+1}\cdots\Gamma^{-1}_{k_1+i-1}\Gamma_{k_1+i}\Gamma_{k_1+i-1}\cdots\Gamma_{k_1+1} =$$

$$\Gamma^{-1}_{k_1+1}\cdots\Gamma^{-1}_{k_1+i-1}\Gamma_{k_1+i}\Gamma_{k_1+i-1}\cdots\Gamma_{k_1+1}\Gamma_{k_1-l}$$

Therefore, we got the following set of relations:
for all $0 \leq l \leq k_1 - 1,\ 1 \leq i \leq k_2$,

$$\Gamma_{k_1-l}\Gamma^{-1}_{k_1+1}\cdots\Gamma^{-1}_{k_1+i-1}\Gamma_{k_1+i}\Gamma_{k_1+i-1}\cdots\Gamma_{k_1+1} =$$

$$\Gamma^{-1}_{k_1+1}\cdots\Gamma^{-1}_{k_1+i-1}\Gamma_{k_1+i}\Gamma_{k_1+i-1}\cdots\Gamma_{k_1+1}\Gamma_{k_1-l}$$

Now, it is easy to see that this set of relations is equivalent to the following set of relations (see [Ga]):

$$\Gamma_i\Gamma_j = \Gamma_j\Gamma_i;\ 1 \leq i \leq k_1,\ k_1+1 \leq j \leq k_1+k_2$$

and this finished the proof of the first case of the first lemma (4.1.2). ∎

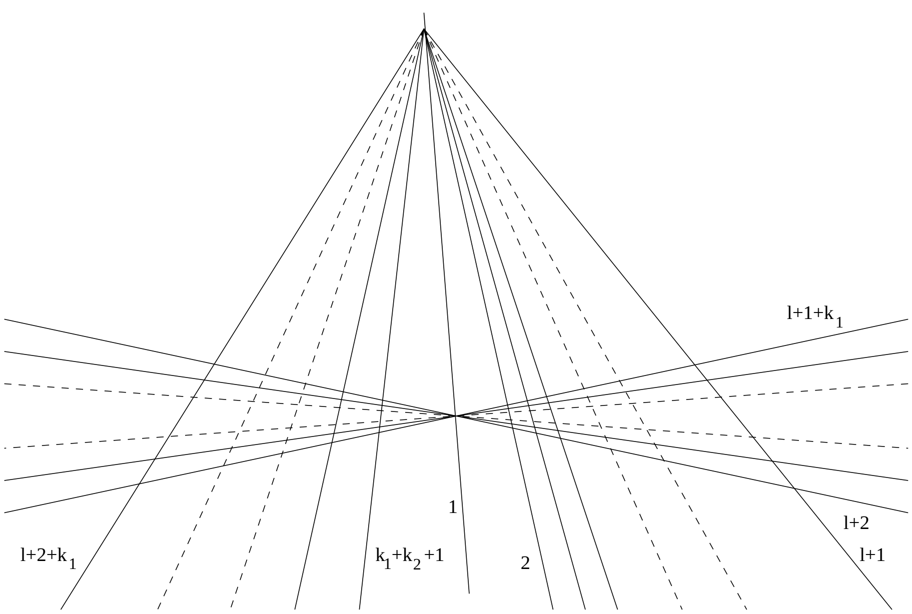
4.2.2 Second case - without the restriction

Let $N = \{x \in \mathbb{C} \mid (x,y) \text{ is an intersection point}\}$, and let $u_0 \in \mathbb{R}$ such that $x \ll u_0$ for all $x \in N$. Let $\mathbb{C}_{u_0} = \{(u_0, y) \mid y \in \mathbb{C}\}$. We numerate the lines according to their intersection with $\mathbb{C}_{u_0}$. We organized this line arrangement in such a way that the following property holds:
for $1 \leq i < j \leq l$ and $k_1+l+1 \leq i < j \leq k_1+k_2$,

$$x(L_i \cap L_t) < x(L_j \cap L_s), \quad l+1 \leq s,t \leq k_1+l$$

It is easy to see that this is the general case, i.e. every line arrangement is homotopic to this situation by rotations and a proper choosing of the line at infinity.

Therefore, we get the following line arrangement:



Let $g = \{\Gamma_1, \cdots, \Gamma_{k_1+k_2+1}\}$ be a g-base of $\pi_1(\mathbb{C}_{u_0} - \mathcal{L})$. By abuse of notations, let us denote the images of Γ_i in $\pi_1(\mathbb{C}^2 - \mathcal{L})$ by the same notation.

Now, we prove this lemma using the braid monodromy techniques (2.5.1) and the Van-Kampen theorem (2.6.6). First, let us calculate the skeletons representing the $\mathcal{L}$.V.C.s of the braid monodromy.

According to this line arrangement, we have the following set of Lefschetz pairs:

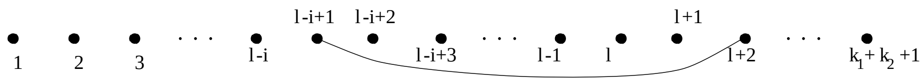
j	λ_{x_j}
1	$(l+1, l+2)$
2	$(l+2, l+3)$
$\vdots$	$\vdots$
k_1	(k_1+l, k_1+l+1)
k_1+1	$(l, l+1)$
k_1+2	$(l+1, l+2)$
$\vdots$	$\vdots$
$2k_1$	(k_1+l-1, k_1+l)
$\vdots$	$\vdots$
$(l-1)k_1+1$	$(2,3)$
$(l-1)k_1+2$	$(3,4)$
$\vdots$	$\vdots$
lk_1	(k_1+1, k_1+2)
lk_1+1	$(1, k_1+1)$
lk_1+2	(k_1+1, k_1+k_2+1)
$(lk_1+2)+1$	(k_1, k_1+1)
$(lk_1+2)+2$	(k_1-1, k_1)
$\vdots$	$\vdots$
$(lk_1+2)+k_1$	$(1,2)$
$(lk_1+2)+k_1+1$	(k_1+1, k_1+2)
$(lk_1+2)+k_1+2$	(k_1, k_1+1)
$\vdots$	$\vdots$
$(lk_1+2)+2k_1$	$(2,3)$
$\vdots$	$\vdots$
$(lk_1+2)+(k_2-l-1)k_1+1$	(k_1+k_2-l-1, k_1+k_2-l)
$(lk_1+2)+(k_2-l-1)k_1+2$	$(k_1+k_2-l-2, k_1+k_2-l-1)$
$\vdots$	$\vdots$
$(lk_1+2)+(k_2-l)k_1$ $[= k_1k_2+2]$	(k_2-l, k_2-l+1)

Let $\{\delta_i \mid 1 \le i \le k_1k_2+2\}$ be a g-base for $\pi_1(\mathbb{C}^X - N, u_0)$ (where $\mathbb{C}^X$ is the x-axis). Let φ be the braid monodromy of $\mathcal{L}$ w.r.t. π_1, u_0.

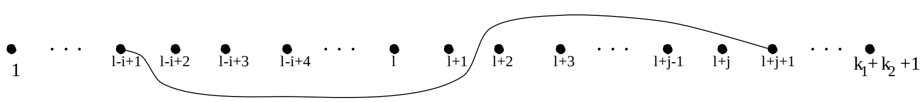
Now, using the table of the Lefschetz pairs, we can calculate the skeletons representing $\mathcal{L}$.V.C.s for the braids $\varphi(\delta_i)$ (according to the Moishezon-Teicher algorithm (2.5.1)).

Until singular point number lk_1 we have almost the same configuration as in the first case of the lemma, hence the general skeleton, which represents the $\mathcal{L}$.V.C., which we have found there is identical (but its center is shifted one point left) to the general skeleton in this case of the lemma until point number lk_1. Therefore:

Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{ik_1+1}), 0 \leq i \leq l-1$:



Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{ik_1+j}),\ 0 \leq i \leq l-1,\ 2 \leq j \leq k_1$:

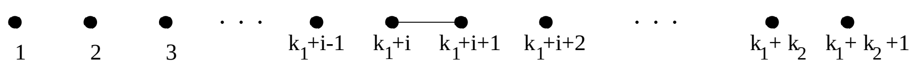


We skip the calculations of the braid monodromy of the two multiple points (which will be done in the proof of the next lemma (4.1.3)), and we continue with the rest of the simple points and we pass directly to the general case:

Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{(lk_1+2)+ik_1+1}), 0 \leq i \leq (k_2-l-1)$: The Lefschetz pair is

$$(k_1+i, k_1+i+1),$$

therefore the skeleton representing the local $\mathcal{L}$.V.C. is:



We have to apply on this skeleton the following sequences of braids:

$$\Delta < i, i+1 > \Delta < i+1, i+2 > \cdots \Delta < k_1+i-1, k_1+i >$$

$$\vdots$$

$$\Delta < 1, 2 > \Delta < 2, 3 > \cdots \Delta < k_1, k_1+1 >$$

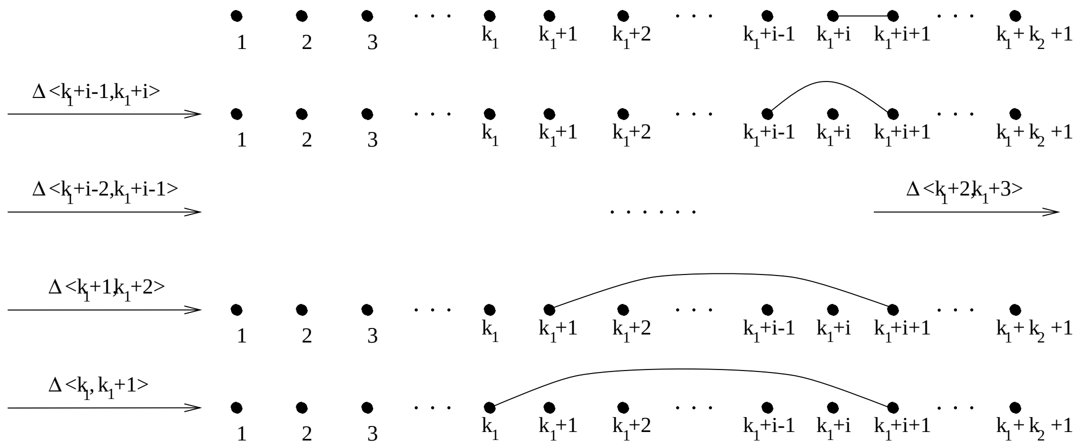
$$\Delta < k_1+1, k_1+k_2+1 > \Delta < 1, k_1+1 >$$

$$\Delta < k_1+1, k_1+2 > \Delta < k_1, k_1+1 > \cdots \Delta < 2, 3 >$$

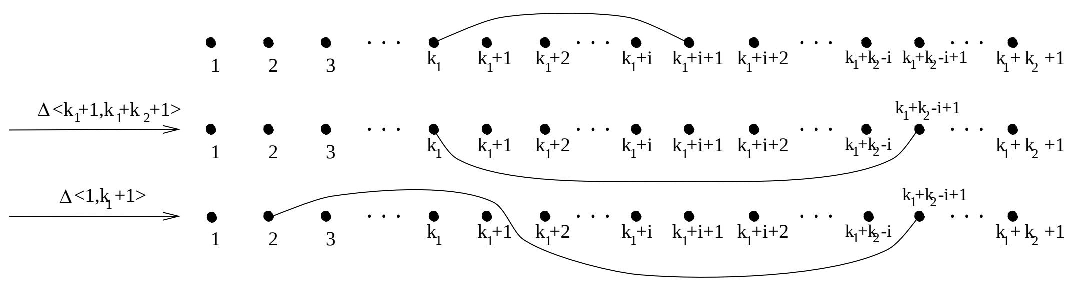
$$\vdots$$

$$\Delta < k_1+l, k_1+l+1 > \Delta < k_1+l-1, k_1+l > \cdots \Delta < l+1, l+2 >$$

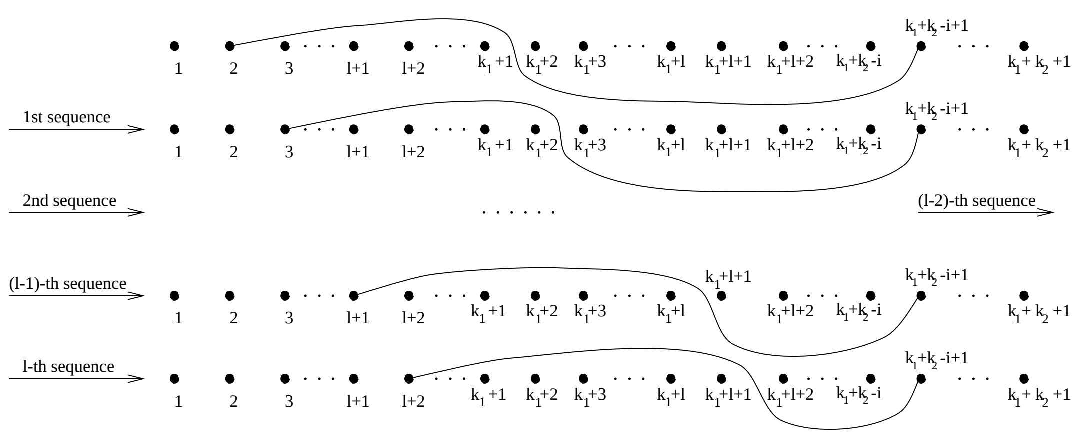
In the first $i-1$ sequences, only the last braid in each sequence affects the skeleton, hence we get:



Next, the action of the braids $\Delta < k_1+1, k_1+k_2+1 >$ and $\Delta < 1, k_1+1 >$ is as follows:



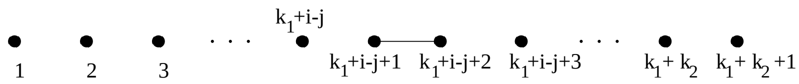
Then, the l sequences of braids move the leftest side of the skeleton l points right:



Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{(lk_1+2)+ik_1+j})$; $0 \leq i \leq k_2-l-1$, $2 \leq j \leq k_1$: The Lefschetz pair is

$$(k_1+i-j+1, k_1+i-j+2)$$

therefore the skeleton representing the local $\mathcal{L}$.V.C. is:



We have to apply on this skeleton the following sequences of braids:

$$\Delta < k_1+i-j+2, k_1+i-j+3 > \Delta < k_1+i-j+3, k_1+i-j+4 > \cdots$$

$$\Delta < k_1+i-1, k_1+i > \Delta < k_1+i, k_1+i+1 >$$

$$\Delta < i, i+1 > \Delta < i+1, i+2 > \cdots \Delta < k_1+i-1, k_1+i >$$

$$\vdots$$

$$\Delta < 1, 2 > \Delta < 2, 3 > \cdots \Delta < k_1, k_1+1 >$$

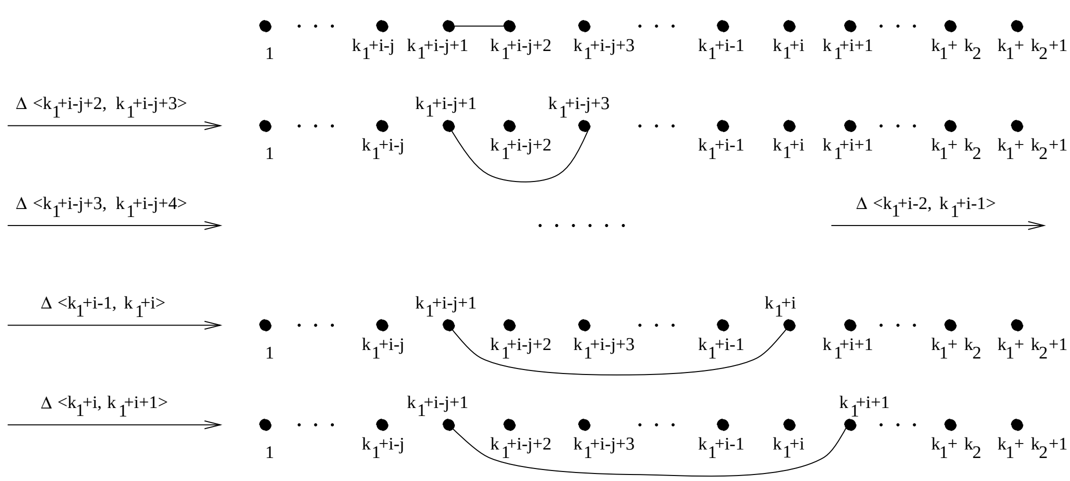
$$\Delta < k_1+1, k_1+k_2+1 > \Delta < 1, k_1+1 >$$

$$\Delta < k_1+1, k_1+2 > \Delta < k_1, k_1+1 > \cdots \Delta < 2, 3 >$$

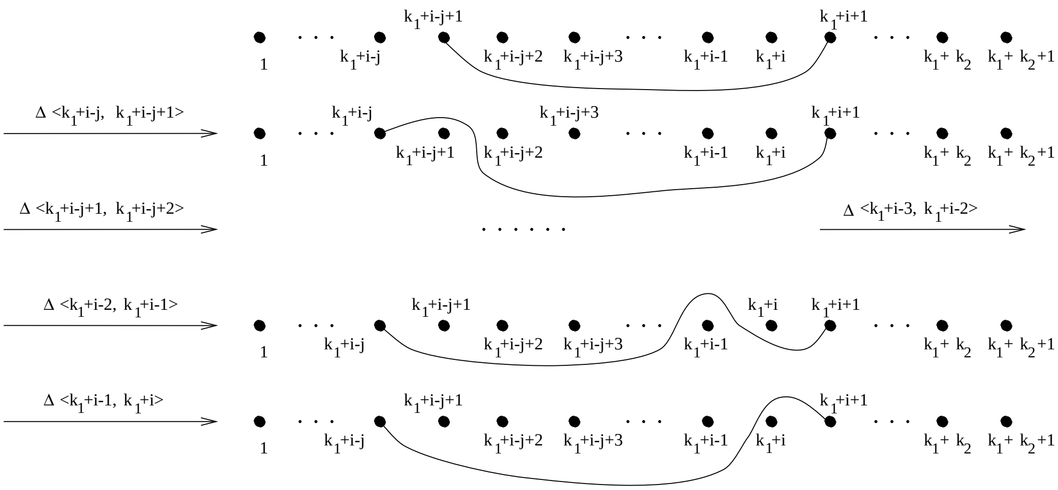
$$\vdots$$

$$\Delta < k_1+l, k_1+l+1 > \Delta < k_1+l-1, k_1+l > \cdots \Delta < l+1, l+2 >$$

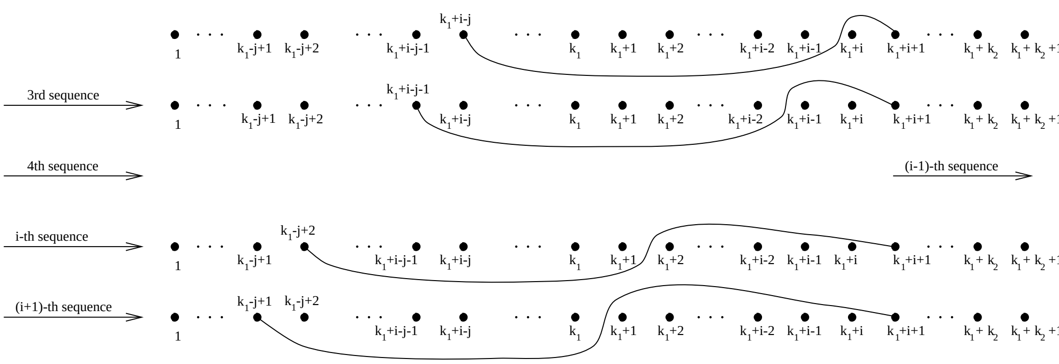
The first sequence acts as follows:



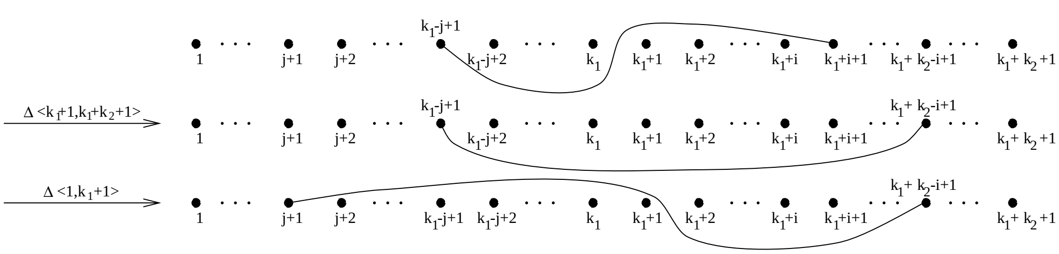
The second sequence moves the left side of the skeleton one point left (the first part of the sequence does not affect the skeleton):



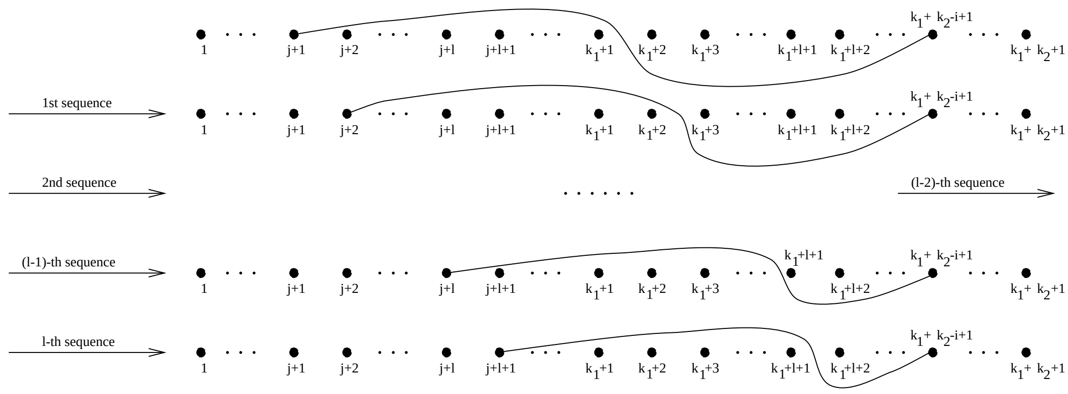
Each of the next $i-1$ sequences moves the left side of the skeleton another step left, so we get the following:



Next, the action of the braids $\Delta < k_1+1, k_1+k_2+1 >$ and $\Delta < 1, k_1+1 >$ is as follows:

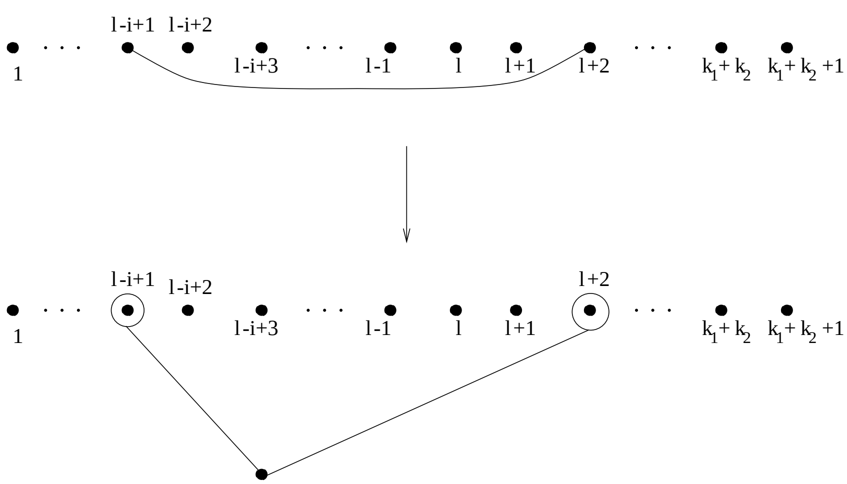


Then, the l sequences of braids move the leftest side of the skeleton l points right:



After we have calculated the skeletons representing $\mathcal{L}$.V.C.s for the braid monodromy, we can calculate the relations that they induced. As we have introduced in the previous section, according to Van-Kampen's theorem (2.6.6), every $\mathcal{L}$.V.C. induces a relation. Now, we will calculate the general relations which are induced from the general $\mathcal{L}$.V.C.s .

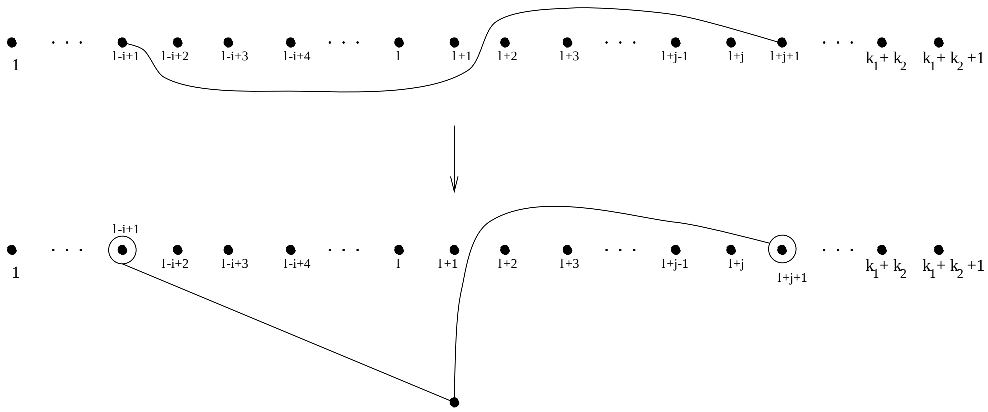
The relation which is induced from $\varphi(\delta_{ik_1+1}), 0 \leq i \leq l-1$:



Therefore, the relation is:

$$\Gamma_{l-i+1}\Gamma_{l+2} = \Gamma_{l+2}\Gamma_{l-i+1}$$

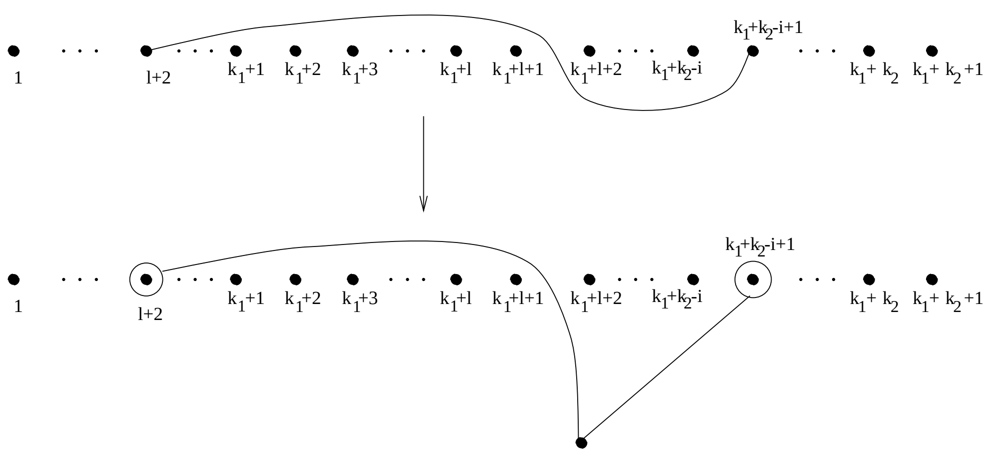
The relation which is induced from $\varphi(\delta_{ik_1+j}),\ 0 \leq i \leq l-1,\ 2 \leq j \leq k_1$:



Therefore, the relation is:

$$\Gamma_{l-i+1}\Gamma_{l+2}^{-1}\cdots\Gamma_{l+j}^{-1}\Gamma_{l+j+1}\Gamma_{l+j}\cdots\Gamma_{l+2} = \Gamma_{l+2}^{-1}\cdots\Gamma_{l+j}^{-1}\Gamma_{l+j+1}\Gamma_{l+j}\cdots\Gamma_{l+2}\Gamma_{l-i+1}$$

The relation which is induced from $\varphi(\delta_{(lk_1+2)+ik_1+1}),\ 0 \leq i \leq (k_2-l-1)$:

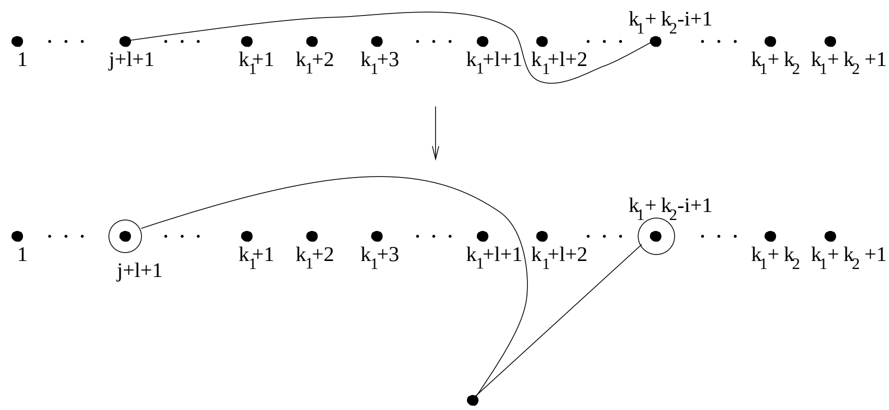


Therefore, the relation is:

$$\Gamma_{k_1+l+1}\cdots\Gamma_{l+3}\Gamma_{l+2}\Gamma_{l+3}^{-1}\cdots\Gamma_{k_1+l+1}^{-1}\Gamma_{k_1+k_2-i+1} =$$

$$\Gamma_{k_1+k_2-i+1}\Gamma_{k_1+l+1}\cdots\Gamma_{l+3}\Gamma_{l+2}\Gamma_{l+3}^{-1}\cdots\Gamma_{k_1+l+1}^{-1}$$

The relation which is induced from $\varphi(\delta_{(lk_1+2)+ik_1+j}), 0 \leq i \leq (k_2-l-1)$, $2 \leq j \leq k_1$:



Therefore, the relation is:

$$\Gamma_{k_1+l+1}\cdots\Gamma_{l+j+2}\Gamma_{l+j+1}\Gamma^{-1}_{l+j+2}\cdots\Gamma^{-1}_{k_1+l+1}\Gamma_{k_1+k_2-i+1} =$$

$$\Gamma_{k_1+k_2-i+1}\Gamma_{k_1+l+1}\cdots\Gamma_{l+j+2}\Gamma_{l+j+1}\Gamma^{-1}_{l+j+2}\cdots\Gamma^{-1}_{k_1+l+1}$$

Therefore, we got the following two sets of relations:
for all $0 \leq i \leq l-1,\ 1 \leq j \leq k_1$:

$$\Gamma_{l-i+1}\Gamma^{-1}_{l+2}\cdots\Gamma^{-1}_{l+j}\Gamma_{l+j+1}\Gamma_{l+j}\cdots\Gamma_{l+2} =$$

$$\Gamma^{-1}_{l+2}\cdots\Gamma^{-1}_{l+j}\Gamma_{l+j+1}\Gamma_{l+j}\cdots\Gamma_{l+2}\Gamma_{l-i+1}$$

and for all $0 \leq i \leq k_2-l-1,\ 1 \leq j \leq k_1$:

$$\Gamma_{k_1+l+1}\cdots\Gamma_{l+j+2}\Gamma_{l+j+1}\Gamma^{-1}_{l+j+2}\cdots\Gamma^{-1}_{k_1+l+1}\Gamma_{k_1+k_2-i+1} =$$

$$\Gamma_{k_1+k_2-i+1}\Gamma_{k_1+l+1}\cdots\Gamma_{l+j+2}\Gamma_{l+j+1}\Gamma^{-1}_{l+j+2}\cdots\Gamma^{-1}_{k_1+l+1}$$

Now, it is easy to see that these two sets of relations are equivalent to the following two sets of relations:

$$\Gamma_i\Gamma_j = \Gamma_j\Gamma_i;\ 2 \leq i \leq l+1,\ l+2 \leq j \leq l+k_1+1$$

and

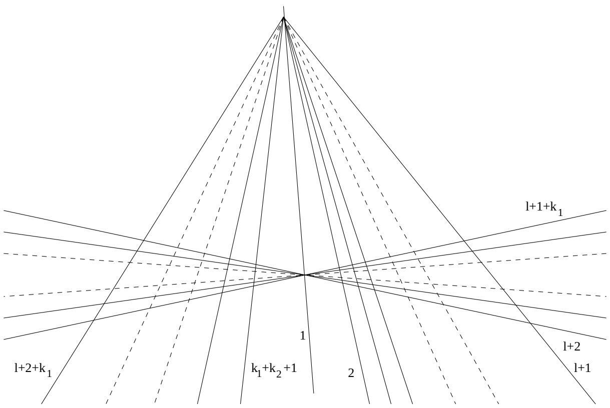
$$\Gamma_i\Gamma_j = \Gamma_j\Gamma_i;\ l+2 \leq i \leq l+k_1+1,\ l+k_1+2 \leq j \leq k_1+k_2+1$$

and this finished the proof of the second case of the first lemma (4.1.2). ■

4.3 Proof of lemma 4.1.3

As in the first lemma, we prove this lemma only for two multiple points, and the proof for t multiple points uses exactly the same arguments.

We will prove it directly in the general case. By homotopic rotations and movements and a proper choosing of the line at infinity, we can get the following line arrangement from any line arrangement with two multiple points:

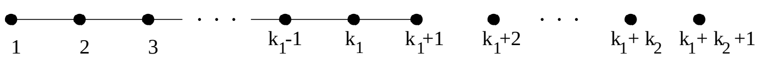


In the first lemma (4.1.2), we already wrote down the set of Lefschetz pairs of this line arrangement. In order to calculate the induced relations of the multiple points, we have to compute their braid monodromy according to the Moishezon-Teicher algorithm (2.5.1) and then we have to use the Van-Kampen theorem (2.6.6) to get their induced relations.

Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{lk_1+1})$: The Lefschetz pair is

$$(1, k_1 + 1),$$

then the skeleton representing the local $\mathcal{L}$.V.C. is:



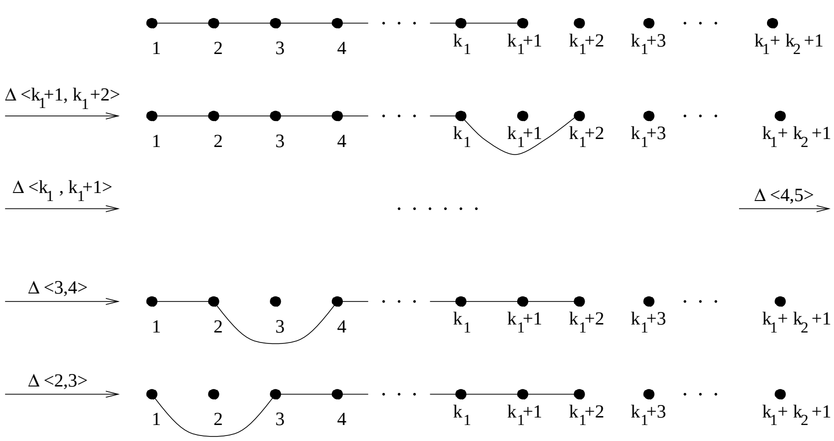
According to the algorithm, we have to apply on the skeleton the following sequence of braids:

$$\Delta < k_1 + 1, k_1 + 2 > \Delta < k_1, k_1 + 1 > \cdots \Delta < 2, 3 >$$

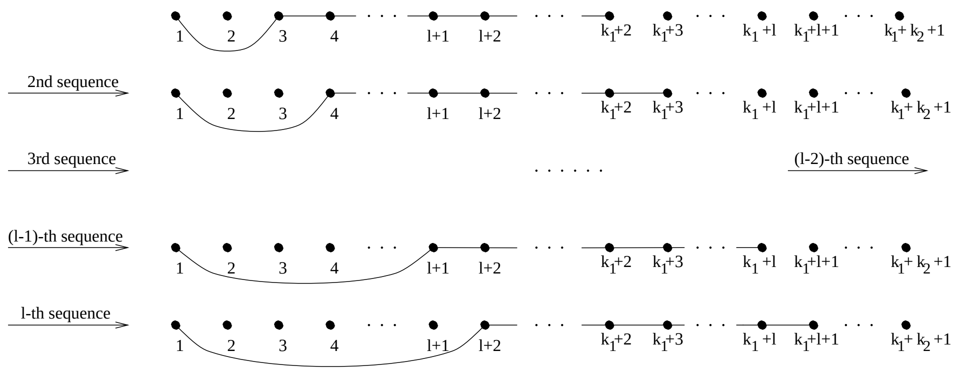
$$\vdots$$

$$\Delta < k_1 + l, k_1 + l + 1 > \Delta < k_1 + l - 1, k_1 + l > \cdots \Delta < l + 1, l + 2 >$$

The first sequence acts as follows:



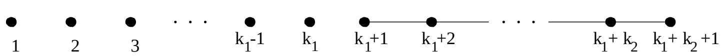
Each of the next $l-1$ sequences moves the right side of the skeleton one step right, so we get the following:



Skeleton representing the $\mathcal{L}$.V.C. of $\varphi(\delta_{lk_1+2})$: The Lefschetz pair is

$$(k_1+1, k_1+k_2+1),$$

therefore the skeleton representing the local $\mathcal{L}$.V.C. is:



According to the algorithm, we have to apply on the skeleton the following sequence of braids:

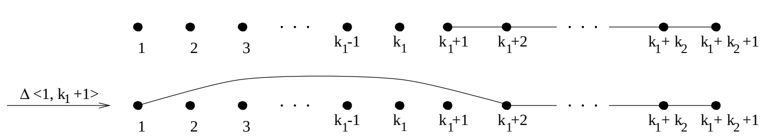
$$\Delta < 1, k_1+1 >$$

$$\Delta < k_1+1, k_1+2 > \Delta < k_1, k_1+1 > \cdots \Delta < 2, 3 >$$

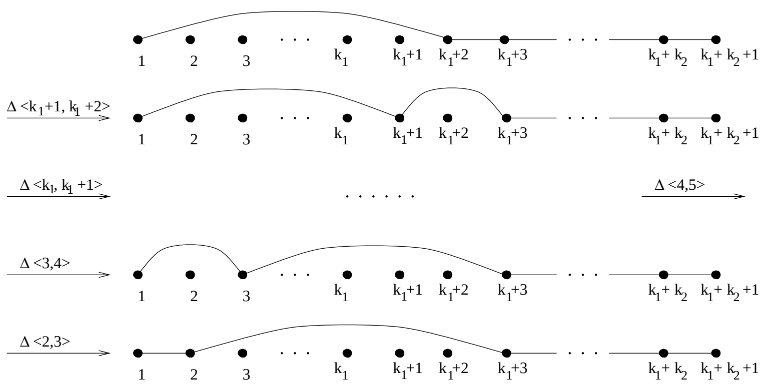
$$\vdots$$

$$\Delta < k_1+l, k_1+l+1 > \Delta < k_1+l-1, k_1+l > \cdots \Delta < l+1, l+2 >$$

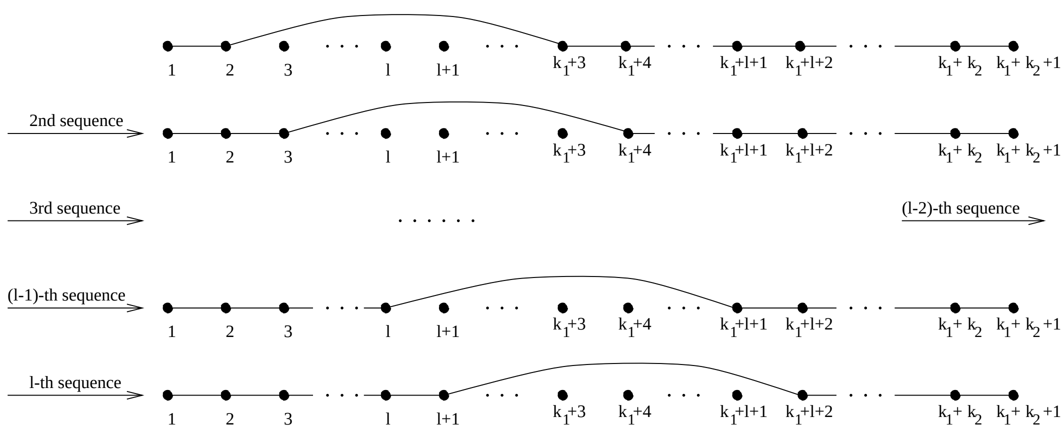
The effect of the braid $\Delta < 1, k_1+1 >$ is:



The first sequence acts as follows:

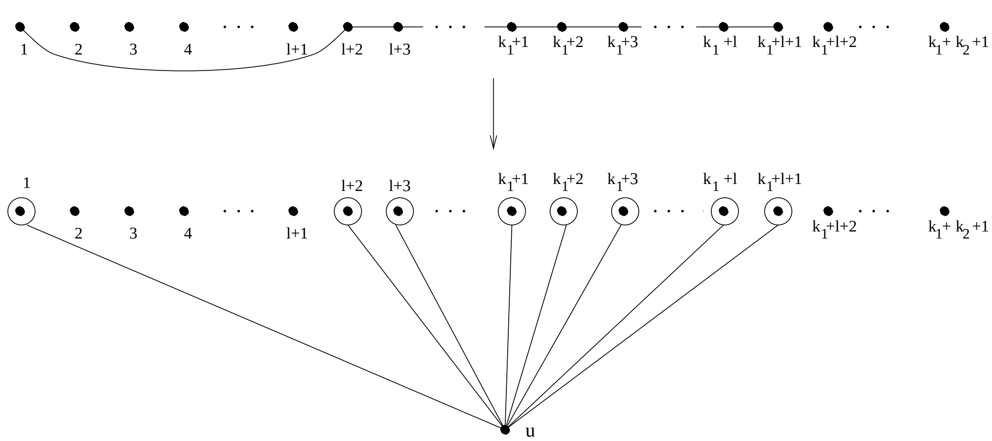


Each of the next $l-1$ sequences moves the left side of the skeleton one step left, so we get the following:



After we have calculated the skeletons representing $\mathcal{L}$.V.C.s for the braid monodromy, we can calculate the relations which they induced.

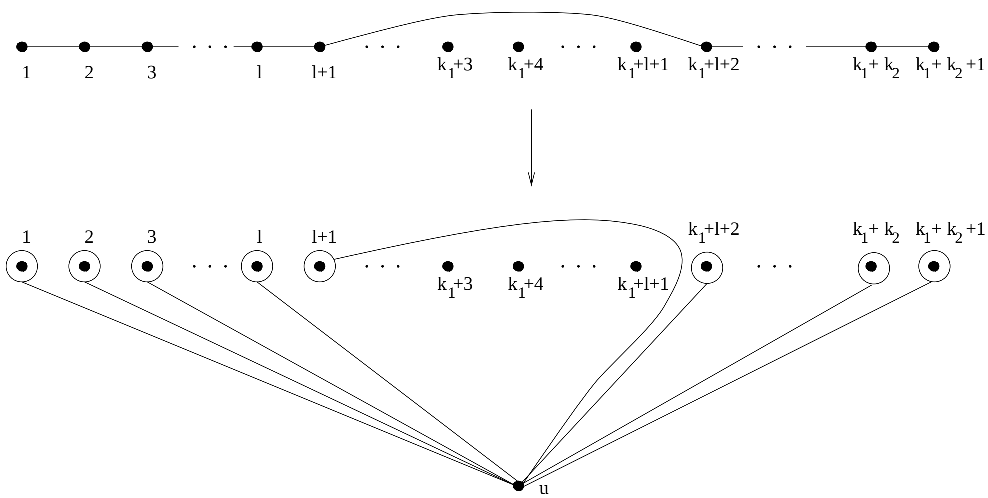
The relations which are induced from $\varphi(\delta_{lk_1+1})$:



Therefore, according to lemma 2.6.7, the relations are:

$$\Gamma_{k_1+l+1}\Gamma_{k_1+l}\cdots\Gamma_{l+2}\Gamma_1 = \Gamma_{k_1+l}\cdots\Gamma_{l+2}\Gamma_1\Gamma_{k_1+l+1} = \cdots = \Gamma_1\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}$$

The relations which are induced from $\varphi(\delta_{lk_1+2})$:



Therefore, according to lemma 2.6.7, the relations are:

$$\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}(\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{l+1}\Gamma_{l+2}^{-1}\cdots\Gamma_{k_1+l+1}^{-1})\Gamma_l\cdots\Gamma_1 =$$

$$\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}(\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{l+1}\Gamma_{l+2}^{-1}\cdots\Gamma_{k_1+l+1}^{-1})\Gamma_l\cdots\Gamma_1\Gamma_{k_1+k_2+1} = \cdots =$$

$$\Gamma_1\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}(\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{l+1}\Gamma_{l+2}^{-1}\cdots\Gamma_{k_1+l+1}^{-1})\Gamma_l\cdots\Gamma_2$$

Now, according to the first lemma (4.1.2), second case, Γ_{l+1} commutes with all $\Gamma_j,\ l+2 \leq j \leq k_1+l+1$, therefore:

$$\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{l+1}\Gamma_{l+2}^{-1}\cdots\Gamma_{k_1+l+1}^{-1} = \Gamma_{l+1}.$$

Hence, the last set of relations comes to the following simplified form:

$$\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\Gamma_l\cdots\Gamma_1 = \Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\Gamma_l\cdots\Gamma_1\Gamma_{k_1+k_2+1} =$$

$$= \cdots = \Gamma_1\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\Gamma_l\cdots\Gamma_2$$

According to the proof of proposition 2.7.1, these two sets of relations (of the two multiple points) are equivalent to the following two sets, respectively:

$$[\Gamma_{k_1+l+1}\Gamma_{k_1+l}\cdots\Gamma_{l+2}\Gamma_1, \Gamma_i] = 1,\ \forall i \in \{1, l+2, \cdots, k_1+l+1\}$$

$$[\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\Gamma_l\cdots\Gamma_1, \Gamma_i] = 1,\ \forall i \in \{1, \cdots, l+1, k_1+l+2, \cdots, k_1+k_2+1\}$$

And the second lemma (4.1.3) is proved. ■

4.4 Proof of theorem 4.1.1

For simplicity, we prove the theorem only for two multiple points, and the proof for t multiple points uses exactly the same arguments.

Till now, we got the following set of generators:

$$g = \{\Gamma_1, \Gamma_2, \cdots, \Gamma_{k_1+k_2+1}\}$$

and the following sets of relations:

(1) $\Gamma_i\Gamma_j = \Gamma_j\Gamma_i;\ 2 \leq i \leq l+1,\ l+2 \leq j \leq k_1+l+1$

(2) $\Gamma_i\Gamma_j = \Gamma_j\Gamma_i;\ l+2 \leq i \leq k_1+l+1,\ k_1+l+2 \leq j \leq k_1+k_2+1$

(3) $[\Gamma_{k_1+l+1}\Gamma_{k_1+l}\cdots\Gamma_{l+2}\Gamma_1, \Gamma_i] = 1,\ \forall i \in \{1, l+2, \cdots, k_1+l+1\}$

(4) $[\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\Gamma_l\cdots\Gamma_1, \Gamma_i] = 1,$
$\forall i \in \{1, \cdots, l+1, k_1+l+2, \cdots, k_1+k_2+1\}$

We have to show that this finitely presented group is isomorphic to

$$\mathbb{F}^{k_1} \oplus \mathbb{F}^{k_2} \oplus \mathbb{Z}$$

Let us modify the set of generators by replacing the generator Γ_1 by the generator

$$\Gamma' = \Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_{k_1+1}\Gamma_{k_1}\cdots\Gamma_2\Gamma_1$$

Now, we have to check that after the modifications we get an equivalent set of generators, and then we have to calculate the new set of relations.

Claim 4.4.1 *After replacing Γ_1 by Γ' (which was defined above) in g, we again get a set of generators. We denote this set of generators by $\tilde{g}$.*

Proof: We have to show that $\Gamma_1 \in < \tilde{g} >$. But this is obvious, because:

$$\Gamma_1 = \Gamma_2^{-1}\Gamma_3^{-1}\cdots\Gamma_{k_1+k_2+1}^{-1}\Gamma'$$

■

The next step is the calculation of the new set of relations for $\tilde{g}$. The sets (1) and (2) of the old sets of relations have not been changed (because these generators in the relations have not been replaced). We have to deal with the sets (3) and (4).

Claim 4.4.2 $[\Gamma', \Gamma] = 1, \forall \Gamma \in \tilde{g}$.

Proof: Obviously, $\Gamma'\Gamma' = \Gamma'\Gamma'$. We will split the rest of the proof into two cases:
(a) $\Gamma \in \{\Gamma_2, \cdots, \Gamma_{l+1}, \Gamma_{k_1+l+2}, \cdots, \Gamma_{k_1+k_2+1}\}$:
$\Gamma'\Gamma \overset{\text{Def}}{=} \Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{l+1}\cdots\Gamma_1\Gamma \overset{(2)}{=}$
$\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_1\Gamma \overset{(4)}{=}$
$\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_1 \overset{(1)(2)}{=}$
$\Gamma\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_1 \overset{(2)+\text{Def}}{=} \Gamma\Gamma'$
(b) $\Gamma \in \{\Gamma_{l+2}, \cdots, \Gamma_{k_1+l+1}\}$:
$\Gamma'\Gamma \overset{\text{Def}}{=} \Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_{l+1}\cdots\Gamma_1\Gamma \overset{(1)}{=}$
$\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_2\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_1\Gamma \overset{(3)}{=}$
$\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_2\Gamma\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_1 \overset{(1)+(2)}{=}$
$\Gamma\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_2\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_1 \overset{(1)+\text{Def}}{=} \Gamma\Gamma'$ ■

Now, we can claim:

Claim 4.4.3 *The following set is a complete set of relations for* $\tilde{g}$ *(we denote it by* $\mathcal{R}'$*):*

(1') $\Gamma_i\Gamma_j = \Gamma_j\Gamma_i;\ 2 \le i \le l+1,\ l+2 \le j \le k_1+l+1$

(2') $\Gamma_i\Gamma_j = \Gamma_j\Gamma_i;\ l+2 \le i \le k_1+l+1,\ k_1+l+2 \le j \le k_1+k_2+1$

(3') $[\Gamma', \Gamma] = 1;\ \forall\Gamma \in \tilde{g}$.

Proof: We have to show that {(1),(2),(3),(4)} is equivalent to {(1'),(2'),(3')} (with respect to the required replacements). In the previous claim, we proved that {(1),(2),(3),(4)} $\Rightarrow$ {(1'),(2'),(3')}. We have to prove the opposite direction. Assume the set of relations {(1'),(2'),(3')}, and prove the relations {(1),(2),(3),(4)}:
(1) and (2): this is the same as (1') and (2'), respectively.
(3): We have to prove that

$$[\Gamma_{k_1+l+1}\Gamma_{k_1+l}\cdots\Gamma_{l+2}\Gamma_1, \Gamma_i] = 1,\ \forall i \in \{1, l+2, \cdots, k_1+l+1\}$$

From (3') we know that $[\Gamma', \Gamma_i] = 1$. Therefore, we have:
$\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_1\Gamma_i = \Gamma_i\Gamma_{k_1+k_2+1}\Gamma_{k_1+k_2}\cdots\Gamma_1 \overset{(1')}{\Rightarrow}$
$\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_2\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_i =$
$= \Gamma_i\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_2\Gamma_{k_1+l+1}\cdots\Gamma_{l+2} \overset{(1')+(2')+(l+2\le i\le k_1+l+1)}{\Longrightarrow}$
$\Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_2\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_i =$
$= \Gamma_{k_1+k_2+1}\cdots\Gamma_{k_1+l+2}\Gamma_{l+1}\cdots\Gamma_2\Gamma_i\Gamma_{k_1+l+1}\cdots\Gamma_{l+2} \overset{\Gamma_2^{-1}\cdots\Gamma_{l+1}^{-1}\Gamma_{k_1+l+2}^{-1}\cdots\Gamma_{k_1+k_2+1}^{-1}\cdot}{\Longrightarrow}$
$[\Gamma_{k_1+l+1}\Gamma_{k_1+l}\cdots\Gamma_{l+2}\Gamma_1, \Gamma_i] = 1,\ \forall i \in \{1, l+2, \cdots, k_1+l+1\}$

Now, it remains to prove that:

$$[\Gamma_{k_1+l+1}\Gamma_{k_1+l}\cdots\Gamma_{l+2}\Gamma_1, \Gamma_1] = 1$$

$\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_1\Gamma_1 =$
$\Gamma_{k_1+l+1}\cdots\Gamma_{l+2}\Gamma_1(\Gamma_{l+2}^{-1}\cdots\Gamma_{k_1+l+1}^{-1}\Gamma_{k_1+l+1}\cdots\Gamma_{l+2})\Gamma_1 \overset{(3)\ +\ ab=ba\Rightarrow ab^{-1}=b^{-1}a}{=}$

$(\Gamma_{l+2}^{-1} \cdots \Gamma_{k_1+l+1}^{-1})\Gamma_{k_1+l+1} \cdots \Gamma_{l+2}\Gamma_1(\Gamma_{k_1+l+1} \cdots \Gamma_{l+2})\Gamma_1 = \Gamma_1\Gamma_{k_1+l+1} \cdots \Gamma_{l+2}\Gamma_1$
(4) Same arguments as (3). ■

We return to the proof of the theorem. Using the above claim, we can find the structure of the calculated group:

$G = \langle g|\mathcal{R}\rangle \cong \langle \tilde{g}|\mathcal{R}'\rangle = \langle \Gamma_2, \cdots, \Gamma_{k_1+k_2+1}, \Gamma' \ \mid \mathcal{R}'\rangle \cong$
$\cong \langle \Gamma' \rangle \oplus \langle \Gamma_2, \cdots, \Gamma_{k_1+k_2+1} \mid \mathcal{R}'\rangle \cong$
$\cong \langle \Gamma' \rangle \oplus \langle \Gamma_2, \cdots, \Gamma_{l+1}, \Gamma_{k_1+l+2}, \cdots, \Gamma_{k_1+k_2+1}\rangle \oplus \langle \Gamma_{l+2}, \cdots, \Gamma_{k_1+l+1}\rangle \cong$
$\cong \mathbb{Z} \oplus \mathbb{F}^{k_2} \oplus \mathbb{F}^{k_1}$
Hence, we finished the proof of theorem 4.1.1. ■

4.5 The projective case

Now, we will investigate the projective case.

Theorem 4.5.1 *Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ where all the t multiple points are on the same line $L \in \mathcal{L}$. Let $k_i + 1$ be the multiplicity of the multiple point P_i, $1 \leq i \leq t$. Then:*

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \bigoplus_{i=1}^{t} \mathbb{F}^{k_i}$$

Proof: For simplicity, we will prove it for two multiple points and the proof for t multiple points uses exactly the same arguments.

From the last theorem, we get:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F}^{k_1} \oplus \mathbb{F}^{k_2} \oplus \mathbb{Z}$$

According to claim 4.4.3, we get the following presentation for this group:
Generators: $g = \{\Gamma', \Gamma_2, \cdots, \Gamma_{k_1+k_2+1}\}$.
Relations: $\mathcal{R} = \{\Gamma_i\Gamma_j = \Gamma_j\Gamma_i,\ 2 \leq i \leq l+1,\ l+2 \leq j \leq k_1+l+1;$
$\Gamma_i\Gamma_j = \Gamma_j\Gamma_i,\ l+2 \leq i \leq k_1+l+1,\ k_1+l+2 \leq j \leq k_1+k_2+1; [\Gamma', \Gamma] = 1,\ \forall \Gamma \in g\}$
where $\Gamma' = \Gamma_{k_1+k_2+1} \cdots \Gamma_1$.

Now, when we are going to the projective case, we add one additional relation, according to theorem 2.6.2:

$$\Gamma_{k_1+k_2+1} \cdots \Gamma_1 = 1$$

In terms of Γ', this relation gets the following form:

$$\Gamma' = 1$$

Now,

$\pi_1(\mathbb{CP}^2 - \mathcal{L}) = \langle g \mid \mathcal{R}, \Gamma' = 1\rangle \cong$
$\langle \Gamma' \mid \Gamma' = 1\rangle \oplus \langle \Gamma_2, \cdots, \Gamma_{l+1}, \Gamma_{k_1+l+2}, \cdots, \Gamma_{k_1+k_2+1}\rangle \oplus \langle \Gamma_{l+2}, \cdots, \Gamma_{k_1+l+1}\rangle \cong$
$\cong \mathbb{F}^{k_2} \oplus \mathbb{F}^{k_1}$ ■

As a consequence of the last theorem, we get:

Corollary 4.5.2

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}) \oplus \mathbb{Z}$$

Therefore, the short exact sequence which was proved by Oka (theorem 2.6.3):

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 - \mathcal{L}) \to \pi_1(\mathbb{CP}^2 - \mathcal{L}) \to 1$$

splits.

5 Arrangements with more than one equivalence class

5.1 The definition of the equivalence relation

The above results can be generalized more. Let us define the following relation on the set of multiple intersection points:

Definition 5.1.1 *Let p_1, p_2 be two multiple intersection points. We say that $p_1 \sim p_2$ if p_1 is connected to p_2 by a "path" which its vertices are multiple intersection points.*

Claim 5.1.2 *$\sim$ is an equivalence relation on the set of multiple intersection points.*

Proof: **Reflexive:** each point is connected to itself by the empty path.
Symmetry: if p_1 is connected to p_2 by a path P, p_2 is connected to p_1 by P^{-1} - the opposite path of P (which is also a path of multiple points).
Transitive: if p_1 is connected to p_2 by P, and p_2 is connected to p_3 by Q, p_1 is connected to p_3 by $P \cdot Q$, which is the concatenation of P and Q and therefore it is a path of multiple points, because p_2 itself is a multiple point too. ∎

This equivalence relation induces equivalence classes on the set of multiple intersection points. We also want to show that this equivalence relation induces a partition on the lines of the arrangement:

Claim 5.1.3 *Let $C_1 = \{p_1, \cdots, p_k\}$ be the multiple points of one equivalence class and $C_2 = \{q_1, \cdots, q_l\}$ be the multiple points of another equivalence class. Let $\mathcal{L}_i$ be the set of lines which pass through one of the multiple points in C_i.*
Then: $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$.

Proof: Assume, on the contrary, that there exists a line L, such that $L \in \mathcal{L}_1 \cap \mathcal{L}_2$. Therefore, $L \in \mathcal{L}_1$ and $L \in \mathcal{L}_2$. From the definitions of $\mathcal{L}_1$ and $\mathcal{L}_2$, there exist points $p \in C_1$ and $q \in C_2$ such that L passes through p and q. Therefore, $p \sim q$, and hence $C_1 = C_2$, a contradiction to the assumption that C_1 and C_2 are distinct equivalence classes. ∎

5.2 The affine case

Now, we can claim the following:

Theorem 5.2.1 *Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ consists of n lines. We choose the line at infinity such that all the lines are intersected in $\mathbb{C}^2$. Assume that there are k multiple intersection points $p_1, \cdots, p_k$ with multiplicities $m_1, \cdots, m_k$ respectively. Assume also that all the multiple intersection points in every equivalence class are collinear, i.e. every equivalence class contains a unique line which connects all the multiple points of that class. Then:*

$$\pi_1(\mathbb{C}^2 - \mathcal{L}, u_0) \cong \bigoplus_{i=1}^{k} \mathbb{F}^{m_i - 1} \oplus \mathbb{Z}^{n - (\sum_{i=1}^{k}(m_i - 1))}$$

The number of infinite cyclic groups is a sum of two numbers: the number of equivalence classes and the number of lines which have only simple intersection points.

Proof: Let $C_i, 1 \leq i \leq t$ be the different equivalence classes of multiple points. According to the last claim, we define $\mathcal{L}_i$ to be the lines which pass through points in C_i. Let $l_1, \cdots, l_r$ be lines which are not in any $\mathcal{L}_i$ (which means that they do not pass through any multiple point, or equivalently, they intersect all the other lines at simple points only).

In every $\mathcal{L}_i$, we have a line L_i which connects all the multiple points in C_i. Therefore, according to theorem 4.1.1, we have:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}_i) = (\bigoplus_{j=1}^{n_i} \mathbb{F}^{m_{P_{i,j}} - 1}) \oplus \mathbb{Z}$$

where $n_i = \#C_i$ and $m_{P_{i,j}}$ is the multiplicity of the j-th point in C_i, $1 \leq j \leq n_i$.

For l_i, we know:

$$\pi_1(\mathbb{C}^2 - l_i) = \mathbb{Z}$$

Now, we use the Oka-Sakamoto theorem (see section 2.1) to get:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) = \pi_1(\mathbb{C}^2 - (\bigcup_{i=1}^{t} \mathcal{L}_i \cup \bigcup_{i=1}^{r} l_i)) \cong$$

$$\cong (\bigoplus_{i=1}^{t} \pi_1(\mathbb{C}^2 - \mathcal{L}_i)) \oplus (\bigoplus_{i=1}^{r} \pi_1(\mathbb{C}^2 - l_i)) \cong$$

$$\cong (\bigoplus_{i=1}^{t} (\bigoplus_{j=1}^{n_i} \mathbb{F}^{m_{P_{i,j}} - 1}) \oplus \mathbb{Z}) \oplus (\bigoplus_{i=1}^{r} \mathbb{Z}) \cong (\bigoplus_{i=1}^{t} \bigoplus_{j=1}^{n_i} \mathbb{F}^{m_{P_{i,j}} - 1}) \oplus \mathbb{Z}^{t+r}$$

It remains to show that this group is equal to the group mentioned in the formulation of the theorem. First, in the double sum, every multiple point appears exactly once, because it appears in only one equivalence class. Therefore:

$$(\bigoplus_{i=1}^{t} \bigoplus_{j=1}^{n_i} \mathbb{F}^{m_{P_{i,j}} - 1}) = \bigoplus_{i=1}^{k} \mathbb{F}^{m_i - 1}$$

Now we have to show that:

$$t + r = n - (\sum_{i=1}^{k}(m_i - 1))$$

Let o_i be the number of lines in $\mathcal{L}_i$. We know that

$$(\sum_{i=1}^{t} o_i) + r = n$$

It is easy to see that:

$$o_i = (\sum_{j=1}^{n_i}(m_{P_{i,j}} - 1)) + 1,$$

because there is a unique line which connects all the multiple points in every equivalence class.

When we combine the last two equations, we get:

$$\sum_{i=1}^{t}\sum_{j=1}^{n_i}(m_{P_{i,j}} - 1) + t + r = n$$

As before, due to the fact that every multiple point appears exactly in one equivalence class, we get:

$$\sum_{i=1}^{t}\sum_{j=1}^{n_i}(m_{P_{i,j}} - 1) = \sum_{i=1}^{k}(m_i - 1)$$

and therefore, we get:

$$t + r = n - (\sum_{i=1}^{k}(m_i - 1))$$

■

5.3 The projective case

Now, we will investigate the projective case.

Theorem 5.3.1 *Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ consists of n lines. We choose the line at infinity such that all the lines are intersected in $\mathbb{C}^2$. Assume that there are k multiple intersection points $p_1, \cdots, p_k$ with multiplicities $m_1, \cdots, m_k$ respectively. Assume also that all the multiple intersection points in every equivalence class are collinear, i.e. every equivalence class contains a unique line which connects all the multiple points of that class. Then:*

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}, u_0) \cong \bigoplus_{i=1}^{k} \mathbb{F}^{m_i - 1} \oplus \mathbb{Z}^{n-1-(\sum_{i=1}^{k}(m_i - 1))}$$

The number of infinite cyclic groups is a sum of two numbers: the number of equivalence classes minus 1 and the number of lines which have only simple intersection points.

Proof: This is the projective analogue of theorem 5.2.1. We induce it using the same techniques as we induced theorem 3.2.1 from theorem 3.1.1. ∎

As a consequence of the last theorem, we get:

Corollary 5.3.2

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}) \oplus \mathbb{Z}$$

Therefore, the short exact sequence which was proved by Oka (theorem 2.6.3):

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 - \mathcal{L}) \to \pi_1(\mathbb{CP}^2 - \mathcal{L}) \to 1$$

splits.

Remark: Simultaneously and independently, Fan [Fa2] got similar results (see section 2.1), with entirely different methods, in even more general case, when there is no equivalence class which has a cycle of multiple points in it.

6 Results concerning the bigness of the fundamental group

Definition 6.1.1 *A group G is called* **big** *if $\mathbb{F}^2 \subset G$.*

As a result from the general theorems (5.2.1,5.3.1), we can say the following:

Corollary 6.1.2 *Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ consisting of n lines which satisfies the conditions of theorem 5.2.1. Then, the fundamental groups of its complement, $\pi_1(\mathbb{C}^2 - \mathcal{L}, u_0)$ and $\pi_1(\mathbb{CP}^2 - \mathcal{L}, u_0)$, are big.*

Proof: According to theorem 5.2.1, the fundamental group of its affine complement is of the form:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}, u_0) \cong \bigoplus_{i=1}^{k} \mathbb{F}^{m_i - 1} \oplus \mathbb{Z}^{n - (\sum_{i=1}^{k}(m_i - 1))}$$

Now, $m_i \geq 3$ in every multiple point, and hence $\mathbb{F}^2$ is contained in this group. Therefore, the fundamental group of its affine complement is big. The proof for the projective case is the same. ∎

In fact, this result has been recently proven [DOZ] for any arrangement which has at least one multiple intersection point:

Theorem 6.1.3 (Dethloff, Orevkov, Zaidenberg)
Let $\mathcal{L}$ be a real line arrangement in $\mathbb{CP}^2$ consisting of n lines. We choose the line at infinity such that all the lines are intersected in $\mathbb{C}^2$. Assume that there exists in $\mathcal{L}$ at least one multiple intersection point.
Then, $\pi_1(\mathbb{CP}^2 - \mathcal{L}, u_0)$ is big.

Remark: It seems that this phenomena is not happen for branch curves of surfaces, unlike previous expectations which followed earlier results of Zariski and Moishezon. Most fundamental groups of complements of branch curves are "almost solvable", i.e. they contain a solvable subgroup of finite index and they are not "big" (see [Te2]).

Acknowledgments. We thank Prof. Leonid Makar-Limanov for suggestions which led to the crucial part of the proof of proposition 2.7.1.

References

[CS] Cohen, D.C. and Suciu, A.I., *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helv. **72(2)**, 285-315 (1997).

[DOZ] Dethloff, G., Orevkov, S. and Zaidenberg, M., *Plane curves with a big fundamental group of the complement*, *in:* Voronezh Winter Mathematical Schools: Dedicated to Selim Krein (P. Kuchment, V. Lin, eds.), American Mathematical Society Translations–Series 2 **184** (1998).

[Fa1] Fan, K.M., *Position of singularities and fundamental group of the complement of a union of lines*, Proc. Amer. Math. Soc. **124(11)**, 3299-3303 (1996).

[Fa2] Fan, K.M., *Direct product of free groups as the fundamental group of the complement of a union of lines*, Michigan Math. J. **44(2)**, 283-291 (1997).

[Ga] Garber, D., *On the fundamental group of complement of real line arrangements*, M.Sc. thesis, Bar-Ilan University (1997).

[Mo] Moishezon, B., *Stable branch curves and braid monodromies*, Lect. Notes in Math. **862**, 107-192 (1981).

[MoTe1] Moishezon, B. and Teicher, M., *Braid group techniques in complex geometry I, Line arrangements in* $\mathbb{CP}^2$, Contemporary Math. **78**, 425-555 (1988).

[MoTe2] Moishezon, B. and Teicher, M., *Braid group techniques in complex geometry II, From arrangements of lines and conics to cuspidal curves*, Algebraic Geometry, Lect. Notes in Math. **1479** (1990).

[MoTe3] Moishezon, B. and Teicher, M., *Braid group techniques in complex geometry V: The fundamental group of a complement of a branched curve of a Veronese generic projection*, Com. in Analysis and Geometry **4(1)**, 1-120 (1996).

[MoTe4] Moishezon, B. and Teicher, M., *Braid groups, singularities, and algebraic surfaces*, Academic Press, to appear.

[O] Oka, M., *On the fundamental group of a reducible curve in* $\mathbb{P}^2$, J. London Math. Soc. (2) **12**, 239-252 (1976).

[OS] Oka, M. and Sakamoto, K., *Product theorem of the fundamental group of a reducible curve*, J. Math. Soc. Japan **30(4)**, 599-602 (1978).

[OT] Orlik, P. and Terao, H., *Arrangements of hyperplanes*, Grundlehren **300**, Springer-Verlag (1992).

[Ra] Randell, R., *The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. **69**, 103-108 (1982). *Correction*, Invent. Math. **80**, 467-468 (1985).

[Sa] Salvetti, M., *Topology of the complement of real hyperplanes in* $\mathbb{C}^N$, Invent. Math. **88**, 603-618 (1987).

[Te1] Teicher, M., *Braid groups, algebraic surfaces and fundamental groups of complement of branch curves*, Proc. Symp. Pure Math. **62(1)**, 127-150 (1997).

[Te2] Teicher, M., *New invariants of surfaces*, Contemp. Math., to appear.

[VK] Van Kampen, E.R., *On the fundamental group of an algebraic curve*, Amer. J. Math. **55**, 255-260 (1933).

[Z] Zariski, O., *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51**, 305-328 (1929).