THE RELATIVE RIEMANN-ROCH THEOREM FROM HOCHSCHILD HOMOLOGY

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ABSTRACT. This write up attempts to clarify a preprint by Markarian [2] which proves the relative Riemann-Roch theorem starting from a theorem essentially describing how the Hochschild Kostant Rosenberg (HKR) map from the completed Hochschild chain complex of a smooth scheme X over a field of characteristic 0 fails to respect co multiplication. We attempt to elaborate on the core steps in [2]. In the process, we obtain a proof of the compatibility of the HKR map twisted by the square root of the Todd genus with a version of the generalized Mukai pairing defined by Caldararu [3]. This settles a part of a conjecture of Caldararu [3] regarding the equivalence between the Hochschild and Hodge structures of a smooth complex scheme.

1. INTRODUCTION

This write up is an attempted clarification of the rest of Markarian's preprint [2] , which describes in greater detail the proof of the Riemann-Roch theorem starting from a theorem similar to Theorem 3 of [1]. We start from Theorem 3 of [1] and follow Markarian [2] very closely to prove the Riemann-Roch theorem. Avoidable errors in the preprint [2] are removed in this write up and an effort is made to make it more understandable. In addition, this enables us to see what the Mukai pairing on Hochschild homology looks like after we go to the usual (Hodge) cohomology via the HKR map twisted by the square root of the Todd genus.

Throughout this write up, X is a smooth scheme over a field of characteristic 0. The second section recalls the Hochschild-Kostant-Rosenberg (HKR) quasi isomorphism and introduces a twisted version of the HKR map first seen by me in Markarian [2]. In addition we introduce the duality map which is an isomorphism between $\mathrm{R}\mathcal{H}om_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta)$ and $\mathrm{R}\mathcal{H}om_X(\Delta^*\mathcal{O}_\Delta, S_X)$ where $S_X = \wedge^n \Omega_X[n]$. Here, n is the dimension of X and Ω denotes its cotangent bundle. $\Delta : X \to X \times X$ is the diagonal map.

The third section is used to recall Theorem 3 of [1], which describes how the Hochschild-Kostant-Rosenberg (HKR) map from the complex $\oplus_i \wedge^i T_X[-i]$ with 0 differential to the complex $\mathrm{D}^{\cdot}_{poly}(X)$ of poly differential operators on X with Hochschild co boundary fails to respect multiplication. A related result of Markarian (Theorem 1 of [2]) is also described. In addition, a key computation using this result is done. This computation appears to be analogous to the computation describing the Jacobian of the inverse of the exponential map $\exp^{-1}$ from a suitable open neighborhood of the identity in a Lie group to its Lie algebra. I have however been unable as of yet to make this analogy as explicit as the analogy between Theorem 3 of [1] and the computation of $d(\exp^{-1})$ described in [1].

The fourth section is a digression into some exterior algebra that we require for the next section.

The fifth and last section introduces the main theorem in this paper. This describes the interrelation between the HKR, twisted HKR and the duality map. An erroneous version of this result appears in Markarian's preprint [2].

It proceeds to prove using this result that the HKR map twisted by the square root of the Todd genus respects a version of the generalized Mukai pairing defined by Caldararu [3] and uses this fact to prove the relative Riemann-Roch theorem. More precisely, the HKR map twisted by the square root of the Todd genus takes the Mukai pairing defined by Caldararu [3] on Hochschild homology to a pairing on $\oplus_{p,q}\mathrm{H}^p(X,\Omega^q)$ that equals the Mukai pairing when applied to Mukai vectors of elements of $K(X)$ and has the adjointness property satisfied by the Mukai pairing.

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2. The HKR, duality and twisted HKR maps

2.1. The HKR map. Throughout this write up $\mathrm{DG}(\mathrm{mod}-\mathcal{O}_X)$ is the category of differential graded $\mathcal{O}_X$ modules, and $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$ is the subcategory of the derived category of the category of $\mathcal{O}_X$ modules whose objects are quasi isomorphic to a finite complex of locally free coherent sheaves. We recall that is X is a smooth scheme over a field of characteristic 0, then we have the completed Hochschild chain complex $\widehat{C}^{\cdot}(X)$ which gives us a resolution of $\Delta^*\mathcal{O}_\Delta$ by free $\mathcal{O}_X$ modules, where $\mathcal{O}_\Delta = \Delta_*\mathcal{O}_X$. We also have the complex $\mathrm{D}^{\cdot}_{poly}(X)$ of polydifferential operators on X with Hochschild co boundary. Yekutieli [4] showed that $\mathrm{D}^{\cdot}_{poly}(X) = \mathrm{RD}(\widehat{C}^{\cdot}(X))$. Further , in [1], we saw that $\widehat{C}^{\cdot}(X)$ and $\mathrm{D}^{\cdot}_{poly}(X)$ are equipped with Hopf algebra structures in $\mathrm{DG}(\mathrm{mod}-\mathcal{O}_X)$ in such a way that each is the Hopf dual of the other. The multiplication on $\widehat{C}^{\cdot}(X)$ is induced by the (signed) shuffle product and the comultiplication is induced by the cut coproduct. The multiplication on $\mathrm{D}^{\cdot}_{poly}(X)$ is induced by the tensor product and the co-multiplication is induced by the (signed) shuffle-cut co-product.

Recall that we have the Hochschild-Kostant-Rosenberg quasi-isomorphisms $\mathrm{I}_{HKR} : \widehat{C}^{\cdot}(X) \to \oplus_i \wedge^i \Omega_X[i]$ and $\mathrm{I}_{HKR} : \oplus_i \wedge^i T_X[-i] \to \mathrm{D}^{\cdot}_{poly}(X)$. If $U = \mathrm{Spec}R$ is an open subscheme of X, then $\mathrm{I}_{HKR} : \widehat{C}^k(R) \to \wedge^k\Omega_R[k]$ is the map $r_0 \otimes .... \otimes r_k \to \frac{1}{k!}r_0dr_1....dr_k$ and $\mathrm{I}_{HKR} : \wedge^k T_U[-k] \to \mathrm{D}^k_{poly}(U)$ is the map $f\frac{\partial}{\partial x_1} \wedge .... \wedge \frac{\partial}{\partial x_k} \to (r_1 \otimes .... \otimes r_k \to \sum_\sigma \mathrm{sgn}(\sigma) f\frac{\partial r_1}{\partial x_{\sigma(1)}}....\frac{\partial r_k}{\partial x_{\sigma(k)}})$

2.2. The duality map. Let $\mathcal{F}$ and $\mathcal{G}$ be objects in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$. We write $\mathrm{R}\mathcal{H}om_X(\mathcal{F},\mathcal{G})$ and $\mathrm{RHom}_X(\mathcal{F},\mathcal{G})$ for $\mathrm{R}\mathcal{H}om_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{F},\mathcal{G})$ and $\mathrm{RHom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{F},\mathcal{G})$ respectively. Note that if $\Delta_!$ is the left adjoint to Δ^*, $\mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta) =$

$\mathrm{RHom}_{X\times X}(\Delta_!\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. Further , since tensoring by S_X is the Serre duality functor, $\Delta_! = S^{-1}_{X\times X}\Delta_* S_X$ where we identify a line bundle with the functor of tensoring by that bundle (see Caldararu [4]). Thus, $\mathrm{R}\Gamma(\widehat{C}^{\cdot}(X)) = \mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta)$ is identified with $\mathrm{RHom}_{X\times X}(S^{-1}_{X\times X}\Delta_* S_X, \mathcal{O}_\Delta)$. Tensoring with $p_2^* S_X$ gives us an isomorphism from $\mathrm{RHom}_{X\times X}(S^{-1}_{X\times X}\Delta_* S_X \otimes p_2^* S_X, \mathcal{O}_\Delta \otimes p_2^* S_X)$. But we can see that $S^{-1}_{X\times X}\Delta_* S_X \otimes p_2^* S_X = \mathcal{O}_\Delta$ and $\mathcal{O}_\Delta \otimes p_2^* S_X = \Delta_* S_X$. Further $\mathrm{RHom}_{X\times X}(\mathcal{O}_\Delta, \Delta_* S_X) = \mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X)$. We therefore obtain an identification between the Hochschild homology $\mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta)$ and $\mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X)$. We follow Markarian [2] and denote this identification by D_Δ and call this map the duality map.

2.3. **The twisted HKR map.** It follows from the fact that the HKR map I_{HKR} is a quasi-isomorphism and the fact that D_Δ is an identification that $\mathrm{R}\Gamma(\oplus_i \wedge^i \Omega_X[i])$ is isomorphic to $\mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X)$. We intend to independently introduce another way of identifying these two spaces.

We note that we have the following composition of morphisms in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$

$$\oplus_i \wedge^i \Omega_X[i] \otimes \Delta^*\mathcal{O}_\Delta \xrightarrow{\mathrm{id}\otimes \mathrm{I}_{HKR}} \oplus_i \wedge^i \Omega_X[i] \otimes \oplus_i \wedge^i \Omega_X[i] \longrightarrow \oplus_i \wedge^i \Omega_X[i] \longrightarrow S_X$$

The arrow in the middle is the wedge product and the last arrow is the natural projection. This yields us a map from $\mathrm{R}\Gamma(\oplus_i \wedge^i \Omega_X[i])$ to $\mathrm{RHom}_X(\mathcal{O}_\Delta, S_X)$. We call this map the twisted HKR map and denote this map by $\widehat{\mathrm{I}_{HKR}}$. The following lemma is evident from the definitions.

Lemma 1. *If $a \in RHom_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta)$ and $b \in R\Gamma \oplus_i \wedge^i\Omega_X[i]$, then*

$$tr(\widehat{I_{HKR}}(b) \circ a) = \int_X b \wedge I_{HKR}(a)$$

where $tr : RHom_X(\mathcal{O}_X, S_X) \to k$ is the trace map, and $\int_X$ is 0 on $H^p(X, \Omega^q)$if $(p, q) \neq (n, n)$ and is the map identifying $H^n(X, \Omega^n)$ with the base field otherwise.

The main focus of this write up is to find the interrelation between the HKR , twisted HKR and duality maps. In other words, we want to fill in the arrow denoted by ? in the following commutative diagram

$$\begin{array}{ccc}
\mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta) & \xrightarrow{D_\Delta} & \mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X) \\
\Big\downarrow \mathrm{I}_{HKR} & & \widehat{\mathrm{I}_{HKR}} \Big\uparrow \\
\oplus_{p,q}\mathrm{H}^p(X, \Omega^q) & \xrightarrow{?} & \oplus_{p,q}\mathrm{H}^p(X, \Omega^q)
\end{array}$$

It is here that the Todd genus enters the picture. To see this we need to make an important calculation arising as a corollary of a theorem of Markarian [2]. This is done in the next section.

3. The left and right Markarian maps

3.1. **Definitions of φ_L and φ_R.** In this section Δ shall denote comultiplication for any Hopf algebra in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$ that we may come across. This section describes the left and right Markarian maps and makes a key calculation which we use later to relate I_{HKR} to the duality map and $\widehat{\mathrm{I}_{HKR}}$. We showed in [1] that $T_X[-1]$ is a Lie algebra in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$ and that $\mathrm{D}^{\cdot}_{poly}(X)$ is its universal enveloping algebra. The Lie bracket on $T_X[-1]$ is given by the Atiyah class of T_X, $\alpha_{T_X} : T_X[-1] \otimes T_X[-1] \to T_X[-1]$. We recall from [1] (Theorem 3 Section 10) the following theorem

Theorem 1. *The following diagram commutes in* $D(mod - \mathcal{O}_X)$

$$\begin{tikzcd} D^{\cdot}_{poly}(X) \otimes T_X[-1] \arrow[r, "\mu"] & D^{\cdot}_{poly}(X) \\ \oplus_i \wedge^i T_X[-i] \otimes T_X[-1] \arrow[u, "I_{HKR} \otimes id"] \arrow[r, "\mu \circ \frac{ad}{1-e^{-ad}}"] & \oplus_i \wedge^i T_X[-i] \arrow[u, "I_{HKR}"'] \end{tikzcd}$$

Here, we must make the meaning of ad more precise. $\mathrm{ad} : \oplus_i \wedge^i T_X[-i] \otimes T_X[-1] \to \oplus_i \wedge^i T_X[-i] \otimes T_X[-1]$ is the composite

$$\oplus_i \wedge^i T_X[-i] \otimes T_X[-1] \longrightarrow \oplus_i T_X^{\otimes i}[-i] \otimes T_X[-1] \xrightarrow{\mathrm{ad}} \oplus_i T_X^{\otimes i}[-i] \otimes T_X[-1]$$

$$\oplus_i T_X^{\otimes i}[-i] \otimes T_X[-1] \longrightarrow \oplus_i \wedge^i T_X[-i] \otimes T_X[-1]$$

where all unnamed arrows are identity on the second factor and $\mathrm{ad} : \oplus_i T_X^{\otimes i}[-i] \otimes T_X[-1] \to \oplus_i T_X^{\otimes i}[-i] \otimes T_X[-1]$ is the map $\sum_i (-1)^{i-1} \alpha_{T_X} \circ s_i$ where $s_i : T_X^{\otimes j}[-j] \otimes T_X[-1] \to T_X^{\otimes j-1}[-j+1] \bigotimes T_X[-1] \otimes T_X[-1]$ is the map from $T_X^{\otimes j+1}$ to itself obtained by shifting the $i+1$ st factor from the right to the position of the factor second from the right.

But we must note that in [1] the map $\mathrm{I}_{HKR} : \widehat{C}^{\cdot}(X) \to \oplus_i \wedge^i \Omega_X[i]$ was defined without factorials in the denominators. We need to use factorials in the denominators so that I_{HKR} is a rind homomorphism in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$. We must therefore, make the necessary adjustments here. Let $Q : \oplus_i \wedge^i \Omega_X[i] \to \oplus_i \wedge^i \Omega_X[i]$ act on $\wedge^k \Omega_X[k]$ by multiplication by $\frac{1}{k!}$.

Applying the functor RD to the commutative diagram described in Theorem 1 , we get the following commutative diagram in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$

$$\begin{tikzcd} \widehat{C}^{\cdot}(X) \arrow[r, "\Delta"] \arrow[d, "\mathrm{I}_{HKR}"] & \widehat{C}^{\cdot}(X) \otimes \Omega[1] \arrow[d, "\mathrm{I}_{HKR} \otimes id"'] \\ \oplus_i \wedge^i \Omega_X[i] \arrow[r, "Q \circ \frac{\mathrm{ad}}{1-e^{-\mathrm{ad}}} \circ \Delta \circ Q^{-1}"] & \oplus_i \wedge^i \Omega_X[i] \otimes \Omega[1] \end{tikzcd}$$

We need to look at the map Δ more carefully. It is in fact , the composite $(\mathrm{id} \otimes \pi) \circ \Delta$ where $\pi : \widehat{C}^{\cdot}(X) \to \Omega[1]$ is the HKR map followed by projection to $\Omega[1]$ and Δ here denotes the comultiplication on $\widehat{C}^{\cdot}(X)$. We call the map in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$ from $\oplus_i \wedge^i \Omega_X[i]$ to $\oplus_i \wedge^i \Omega_X[i] \otimes \Omega[1]$ seen in the above commutative diagram the right Markarian map and denote it by φ_R.

In the same spirit we can ask how the map $(\pi \otimes \mathrm{id}) \circ \Delta$ looks like after applying the HKR map to $\widehat{C}^{\cdot}(X)$ throughout. We denote this map by φ_L. Markarian's main theorem (Theorem 1 of [2]) was an explicit description of this map along the lines of our description of φ_R. It can easily be derived from our description of φ_R though we do not need such a description for our purpose.

In addition we define $\bar{\varphi_R}$ to be φ_R composed with the morphism in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$ from $\oplus_i \wedge^i \Omega_X[i] \otimes \Omega[1]$ to $\Omega[1] \otimes \oplus_i \wedge^i \Omega_X[i]$ given by the (signed) swap map.

3.2. **Some observations and remarks.** 1. $\widehat{C}^{\cdot}(X)$ and $\oplus_i \wedge^i \Omega_X[i]$ are algebras in $\mathrm{DG}(\mathrm{mod} - \mathcal{O}_X)$ and thus, in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$ as well. Further, $\mathrm{I}_{HKR} : \widehat{C}^{\cdot}(X) \to \oplus_i \wedge^i \Omega_X[i]$ is a morphism of algebras in $\mathrm{DG}(\mathrm{mod} - \mathcal{O}_X)$ and therefore, in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$ as well.

2. We recall that $\widehat{C}^{\cdot}(X)$ is a Hopf algebra in $\mathrm{DG}(\mathrm{mod} - \mathcal{O}_X)$. It follows that its comultiplication is an algebra homomorphism. If we set $\alpha_L = (\pi \otimes \mathrm{id}) \circ \Delta$, this implies that if a and b are sections of $\widehat{C}^{\cdot}(X)$ over an open subscheme $\mathrm{Spec}R$, then $\alpha_L(a.b) = \alpha_L(a).b + (-1)^{|a|}a.\alpha_L(b)$. This is a Leibniz identity satisfied by α_L . Here $|a|$ denotes the degree of a. We observe that the HKR map from $\widehat{C}^{\cdot}(X)$ to $\oplus_i \wedge^i \Omega_X[i]$ is an algebra homomorphism in $\mathrm{DG}(\mathrm{mod} - \mathcal{O}_X)$. It follows that φ_L satisfied the same Leibniz identity in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$.

Also, if we set $\alpha_R = S \circ (\mathrm{id} \otimes \pi) \circ \Delta$, where S is the signed swap, then we can check that $\alpha_R(a.b) = \alpha_R(a).b + (-1)^{|a|}a.\alpha_R(b)$. It follows that $\bar{\varphi_R}$ also satisfies the same Leibniz identity in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$.

We need to be a little careful regarding these sign conventions. Here, by $a.\alpha_L(b)$ or by $a.\alpha_R(b)$ we mean the section of $\Omega[1] \otimes \widehat{C}^{\cdot}(X)$ obtained by swapping a with the $\Omega[1]$ *without* taking signs into account. This is strictly not the correct sign convention. The "differential operator" parts of α_L and α_R are degree -1 but there is an additional $\Omega[1]$ "part" of these operators. Their total degree as elements of $\mathrm{RHom}_X(\mathcal{O}_X, \Omega[1] \otimes \mathrm{R}\mathcal{H}om_X(\widehat{C}^{\cdot}(X), \widehat{C}^{\cdot}(X)))$ is 0. If we take the signs of the swaps into account, as should be done, then what we have just observed would read as $\alpha_L(a.b) = \alpha_L(a).b + a.\alpha_L(b)$ and $\alpha_R(a.b) = \alpha_R(a).b + a.\alpha_R(b)$ respectively.

3. In other words, α_L and α_R are "connections" on the ring of functions of a super manifold whose ring of functions is $\widehat{C}^{\cdot}(X)$. In the same spirit, φ_L and $\bar{\varphi_R}$ are "connections" on the algebra $\oplus_i \wedge^i \Omega_X[i]$ in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$.

4. We note that $\mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \Omega[1] \otimes \wedge^i \Omega_X[i]) = \Omega[1] \otimes \mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])$.

5. The following composition of morphisms yields an isomorphism in $\mathrm{DG}(\mathrm{mod} - \mathcal{O}_X)$, and hence, in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$

$$\oplus_i \wedge^i \Omega_X[i] \otimes \oplus_j \wedge^j T_X[-j] \xrightarrow{\iota \otimes \iota} \mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])^{\otimes 2}$$

$$\mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])^{\otimes 2} \xrightarrow{\circ} \mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])$$

Here, $\iota : \oplus_i \wedge^i \Omega_X[i] \to \mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])$ is the morphism which takes a section of $\oplus_i \wedge^i \Omega_X[i]$ to wedge product by that section on the left and $\iota : \oplus_i \wedge^i T_X[-i] \to \mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])$ is the morphism taking a section of $\oplus_i \wedge^i T_X[-i]$ to the "differential operator" on $\oplus_i \wedge^i \Omega_X[i]$ corresponding to that section.

6. If D_i is the image of $\oplus_i \wedge^i \Omega_X[i] \otimes \wedge^i T_X[-i]$ in $\mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])$ under the morphism described in the previous observation, then $\mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i]) = \oplus_i D_i$. This is true in $\mathrm{DG}(\mathrm{mod} - \mathcal{O}_X)$, and therefore in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$ as well. Observations 2 and 3 tell us that $\bar{\varphi_R}$ and φ_L are in $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \Omega[1] \otimes D_1)$. In fact D_i can be thought of as the "complex of i th order differential operators on $\oplus_i \wedge^i \Omega_X[i]$" .

7. D_1 is isomorphic to $\oplus_i \wedge^i \Omega_X[i] \otimes T_X[-1]$ as an $\oplus_i \wedge^i \Omega_X[i]$ module. It follows that the top exterior power of D_1 over $\oplus_i \wedge^i \Omega_X[i]$ is isomorphic to D_n as $\oplus_i \wedge^i \Omega_X[i]$ modules.

8. We can consider the element $\mathrm{id} \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \Omega \otimes T_X)$. Identifying $\Omega \otimes T_X$ with $\Omega[1] \otimes T_X[-1]$, we can think of id as en element of $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \Omega[1] \otimes D_1)$. This is a "connection" on the algebra $\oplus_i \wedge^i \Omega_X[i]$ in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$. We can take the top exterior power of id to obtain an element of $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \wedge^n \Omega_X[n] \otimes \wedge^n T_X[-n])$. Tensoring this with $\oplus_i \wedge^i \Omega_X[i]$ (and applying the right swap) gives us an element (which we call id^n of $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\oplus_i \wedge^i \Omega_X[i], \wedge^n \Omega_X[n] \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n])$. Finally , we note that $S_X = \wedge^n \Omega_X[n]$. Thus we have an element $\mathrm{id}^n \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\oplus_i \wedge^i \Omega_X[i], S_X \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n])$. We finally recall that $\oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n]$ is just the top exterior power of D_1 over $\oplus_i \wedge^i \Omega_X[i]$.

9. Observe that we can obtain an element of $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, S_X \otimes D_n)$ simply by taking the top exterior power over $\oplus_i \wedge^i \Omega_X[i]$ of $\bar{\varphi_R} \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \Omega[1] \otimes D_1)$. We denote this top exterior power by $\bar{\varphi_R}^n$. The key computation we need to make is to "compare" $\bar{\varphi_R}^n$ with id^n. In other words, we want an explicit formula for $(\mathrm{id}^n)^{-1} \bar{\varphi_R}^n \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \oplus_i \wedge^i \Omega_X[i])$. This is where the Todd genus begins to appear.

3.3. **Comparison between id^n and $\bar{\varphi_R}^n$.** The keys to this comparison are the following observations

1. id^n just identifies $\mathcal{O}_X$ with $S_X \otimes \wedge^n T_X[-n]$. It this follows that if $f \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \oplus_i \wedge^i \Omega_X[i])$, then $f = (\mathrm{id}^n)^{-1} f.\mathrm{id}^n$. Therefore, we must find f so that $\bar{\varphi_R}^n = f.\mathrm{id}^n \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])$.

2. We therefore need to calculate the top wedge power of the map $\bar{\varphi_R}$ over $\oplus_i \wedge^i \Omega_X[i]$. It is enough for us to calculate the top wedge power over $\oplus_i \wedge^i \Omega_X[i]$ of the map $\bar{\varphi_R} \circ \iota$ in $\mathrm{D}(\mathrm{mod} - \mathcal{O}_X)$ where $\iota : \Omega[1] \to \oplus_i \wedge^i \Omega_X[i]$ is the map in

D(mod$-\mathcal{O}_X$) induced by the natural inclusion. This is because the top wedge power of $\bar{\varphi_R}$ (over $\oplus_i \wedge^i \Omega_X[i]$) is the restriction to $\wedge^n \Omega_X[n]$ of the n th tensor power of $\bar{\varphi_R}$ restricted to $(\oplus_i \wedge^i \Omega_X[i])^{\otimes n}$. Since we are interested in what $\bar{\varphi_R}^n$ does to the degree n part of this object in D(mod $-\mathcal{O}_X$), we note that the degree n part of $\oplus_i \wedge^i \Omega_X[i]^{\otimes n}$ consists of $\Omega^{\otimes n}[n]$ and parts made up by tensoring other powers of $\Omega[1]$ together with some $\mathcal{O}_X$ terms. But since $\bar{\varphi_R} \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \Omega[1] \otimes D_1)$, the composite $\mathcal{O}_X \to \oplus_i \wedge^i \Omega_X[i] \to \Omega[1] \otimes \oplus_i \wedge^i \Omega_X[i]$ where the last arrow is $\bar{\varphi_R}$ is 0.

3. The composite $\bar{\varphi_R} \circ \iota : \Omega[1] \to \Omega[1] \otimes \oplus_i \wedge^i \Omega_X[i]$ in D(mod $-\mathcal{O}_X$) is given by the formula $\frac{z}{\exp(z)-1}$ where $z = \alpha_{T_X}$ is the Atiyah class of T_X. Here, we recall that $\alpha_{T_X} \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(T_X[-1] \otimes T_X[-1], T_X[-1])$ and ${\alpha_{T_X}}^i \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(T_X[-1] \otimes T_X^{\otimes i}[-i], T_X[-1])$. By z^i we mean the restriction of ${\alpha_{T_X}}^i$ to an element of $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(T_X[-1] \otimes \wedge^i T_X[-i], T_X[-1])$. This is a direct consequence of Theorem 1.

4. The following lemma follows from the above observations

Lemma 2. *$\bar{\varphi_R} = S.id^n$ where $S = det(\frac{z}{exp(z)-1})$. Here, z^i is as in the previous paragraph.*

4. Some exterior algebra

4.1. **Some observations and calculations.** This section is a necessary digression we need to undertake. We make a suitable definition of the adjoint map from $\mathrm{R}\mathcal{H}om_X(\oplus_i \wedge^i \Omega_X[i], \oplus_i \wedge^i \Omega_X[i])$ to itself. For some time we will make all our definitions and prove all our lemmas for differential graded vector spaces over a field of characteristic 0. What we do in this case will be seen to extend easily to the category DG(mod $-\mathcal{O}_X$) and hence, will work for D(mod $-\mathcal{O}_X$) as well.

Let V be a vector space over a field of characteristic 0. Let V^* denote its dual. Let $x_1, .., x_n$ be a basis of V and let $y_1, ..., y_n$ be the basis of V^* dual to $x_1, ..., x_n$. All DG algebras we work with in this section will have 0 differential. We make the following observations

1. The composite
$\oplus_i \wedge^i V[i] \otimes \oplus_i \wedge^i V^*[-i] \xrightarrow{\iota \otimes \iota} \mathrm{End}(\oplus_i \wedge^i V[i])^{\otimes 2} \xrightarrow{\circ} \mathrm{End}(\oplus_i \wedge^i V[i])$ is an isomorphism, where $\iota : \oplus_i \wedge^i V[i] \to \mathrm{End}(\oplus_i \wedge^i V[i])$ is the map taking an element of $\oplus_i \wedge^i V[i]$ to wedge product on the left by that element and $\iota : \oplus_i \wedge^i V^*[-i] \to \mathrm{End}(\oplus_i \wedge^i V[i])$ is the map taking an element of $\oplus_i \wedge^i V^*[-i]$ to contraction by that element. Here, we clarify that if $w_1, w_2 \in \oplus_i \wedge^i V^*[-i]$, then $\iota_{w_1} \circ \iota_{w_2} = \iota_{w_1 \wedge w_2}$. We call the inverse of this isomorphism isomorphism f_l.

2. There is a non-degenerate pairing on $\oplus_i \wedge^i V[i]$ which we denote by $<,>$, such that if $w_1, w_2 \in \oplus_i \wedge^i V[i]$, then $< w_1, w_2 >= p_n(w_1 \wedge w_2)$ where $p_n : \oplus_i \wedge^i V[i] \to \wedge^n V[n]$ is the natural projection.

For an operator $L \in \mathrm{End}(\oplus_i \wedge^i V[i])$ we define its adjoint $A(L)$, which we shall also denote by L^+ (following Markarian [2]) to be the operator in $\mathrm{End}(\oplus_i \wedge^i V[i])$

satisfying $< L(v), w >= (-1)^{|L||v|} < v, L^+(w) >$ for all $v, w \in \oplus_i \wedge^i V[i]$. Here, $|L|$ and $|v|$ denote the degrees of L and v respectively.

We make the following remarks
(a) If $Y \in \oplus_i \wedge^i V^*[-i]$ is a homogenous element, then $A(\iota_Y) = (-1)^{|Y|}\iota_Y$.
(b) If $X \in \oplus_i \wedge^i V[i]$ is a homogenous element, then $A(\iota_X) = \iota_X$.
(c) If $L_1, L_2 \in \text{End}(\oplus_i \wedge^i V[i])$ are homogenous operators, then $A(L_1 \circ L_2) = (-1)^{|L_1||L_2|}A(L_2) \circ A(L_1)$.
(d) It follows that if $X \in \oplus_i \wedge^i V[i]$ and $Y \in \oplus_i \wedge^i V^*[-i]$ are homogenous elements, then $(\iota_X \circ \iota_Y)^+ = (-1)^{|X||Y|}(-1)^{|Y|}\iota_Y \circ \iota_X$.

3. We can treat $\text{End}(\oplus_i \wedge^i V[i])$ as a Lie super-algebra. If D_1 is the image of $\oplus_i \wedge^i V[i] \otimes V^*[-1]$ in $\text{End}(\oplus_i \wedge^i V[i])$ under the isomorphism $(f_l)^{-1}$, then D_1 can be thought of as the space of "first order differential operators" on $\oplus_i \wedge^i V[i]$. We can consider the top exterior power of D_1 over $\oplus_i \wedge^i V[i]$, which is a 1 dimensional $\oplus_i \wedge^i V[i]$ module that can be identified with $\oplus_i \wedge^i V[i] \otimes \wedge^n V^*[-n]$. We have a map $\text{id}^n : \oplus_i \wedge^i V[i] \to \oplus_i \wedge^i V[i] \otimes \wedge^n V^*[-n]$ taking an element $Z \in \oplus_i \wedge^i V[i]$ to $Z \otimes y_n \wedge .... \wedge y_1$. We have the following lemma

Lemma 3. *If $L \in D_1 \in End(\oplus_i \wedge^i V[i])$, the following diagram commutes*

$$\begin{array}{ccc}
\oplus_i \wedge^i V[i] & \xrightarrow{id^n} & \oplus_i \wedge^i V[i] \otimes \wedge^n V^*[-n] \\
{\scriptstyle -L^+}\big\downarrow & & \big\downarrow{\scriptstyle L} \\
\oplus_i \wedge^i V[i] & \xrightarrow{id^n} & \oplus_i \wedge^i V[i] \otimes \wedge^n V^*[-n]
\end{array}$$

Before beginning the proof, we need to explain what the arrow on the right means. We identify $\oplus_i \wedge^i V[i] \otimes \wedge^n V^*[-n]$ with the top exterior power of D_1. As $D_1 \subset \text{End}(\oplus_i \wedge^i V[i])$, is a Lie subalgebra (this can be checked by direct computation). We can treat the top exterior power (over $\oplus_i \wedge^i V[i]$) of D_1 as a module over the Lie super-algebra D_1. By the map $L : \oplus_i \wedge^i V[i] \otimes \wedge^n V^*[-n] \to \oplus_i \wedge^i V[i] \otimes \wedge^n V^*[-n]$ we actually mean $ad(L)$ in this sense. Having said this, we are ready to write down the proof, which consists of straightforward computations.

Proof. Without loss of generality, $L = \iota_Z \circ \iota_{y_1}$ where Z is a homogenous element of $\oplus_i \wedge^i V[i]$. Let H be any homogenous element of $\oplus_i \wedge^i V[i]$. Then $\text{id}^n(-L^+(H)) = \text{id}^n((-1)^{|Z|}\iota_{y_1}\iota_Z H)$ by Observation 2 (d) of this section. This is equal to $\text{id}^n((-1)^{|Z|}\iota_{y_1}[Z \wedge H]) = \text{id}^n((-1)^{|Z|}(\iota_{y_1}Z) \wedge H + Z \wedge \iota_{y_1}H) = [Z \wedge \iota_{y_1}H + (-1)^{|Z|}(\iota_{y_1}Z) \wedge H] \otimes y_n \wedge ..... \wedge y_1$.

On the other hand, $L(\text{id}^n H) = L(H \otimes y_n \wedge ..... \wedge y_1)$. We make the following calculations (we keep in mind that the product in $\oplus_i \wedge^i V[i]$ is the wedge product).

$[L, H]Y = Z.\iota_{y_1}(H.Y) - (-1)^{|H|(|Z|+1)}H.Z.\iota_{y_1}Y = Z.\iota_{y_1}H.Y - (Z.H)(-1)^{|H|}\iota_{y_1}Y - (-1)^{|H|(|Z|+1)}H.Z.\iota_{y_1}Y = Z.\iota_{y_1}H.Y$. Thus, $[L, H] = \iota_{(Z.\iota_{y_1}H)}$.

$[L,\iota_{y_i}]Y = Z.\iota_{y_1}\iota_{y_i}Y - (-1)^{|Z|+1}\iota_{y_i}(Z.\iota_{y_1}Y) = Z.\iota_{y_1}\iota_{y_i}Y - (-1)^{|Z|+1}[(\iota_{y_i}Z).(\iota_{y_1}Y) + (-1)^{|Z|}Z.\iota_{y_i}\iota_{y_1}Y] = (-1)^{|Z|}(\iota_{y_i}Z).(\iota_{y_1}Y)$. Thus $[L,\iota_{y_i}] = (-1)^{|Z|}\iota_{\iota_{y_i}Z}\circ\iota_{y_1}$.

It follows that if $i \neq 1$, then $y_n \wedge .... \wedge [L,y_i] \wedge ... \wedge y_1 = 0$ where we identify y_i with ι_{y_i}. The only surviving term therefore corresponds to $i = 1$. It follows that $L(H \otimes y_1 \wedge .... \wedge y_n) = [Z \wedge \iota_{y_1}H + (-1)^{|Z|}(\iota_{y_1}Z) \wedge H] \otimes y_1 \wedge ..... \wedge y_n$. Since $y_n \wedge ... \wedge y_1 = (-1)^{\frac{n(n-1)}{2}} y_1 \wedge .... \wedge y_n$ the desired result follows.

□

We note that these observations apply in the category $\mathrm{DG}(\mathrm{mod}-\mathcal{O}_X)$ if we replace $\oplus_i \wedge^i V[i]$ by $\oplus_i \wedge^i \Omega_X[i]$ and $\oplus_i \wedge^i V^*[-i]$ by $\oplus_i \wedge^i T_X[-i]$. They therefore also carry over to $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$. We note that $\varphi_L \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\Omega[1] \otimes D_1)$. We can therefore, ask for the adjoint of φ_L, by which we mean $(A \otimes \mathrm{id})_*\varphi_L$ where A is the adjoint map on $\mathrm{End}(\oplus_i \wedge^i \Omega_X[i]$. This will be calculated in an explicit fashion in the following subsection.

4.2. **The adjoint of φ_L.** We make the following observations
1. $\varphi_L, \bar{\varphi_R} \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\oplus_i \wedge^i \Omega_X[i], \Omega[1] \otimes \oplus_i \wedge^i \Omega_X[i])$ "commute" with each other. We have to be more precise here. It follows from (Observation 3, Section 3.2) that $\varphi_L, \bar{\varphi_R} \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \Omega[1] \otimes D_1)$. By saying that they "commute" $\varphi_L \circ \bar{\varphi_R} - \tau \circ \bar{\varphi_R} \circ \varphi_L = 0$ where $\tau : \Omega[1]^{\otimes 2} \to \Omega[1]^{\otimes 2}$ swaps factors with the appropriate sign, (-1) in this case . To see this, it is better to go to $\widehat{C^{\cdot}}(X)$ where these operators are more explicit. Here, we need to see that α_L and α_R commute in the above sense. But α_L and α_R are of total degree 0 operators . Their (super) commutator is therefore $\alpha_L \circ \alpha_R - \tau \circ \alpha_R \circ \alpha_L$. On $r_0 \otimes .... \otimes r_n$ this gives $(-1)^{n-1} r_0 dr_n \otimes dr_1 \otimes dr_2 \otimes .... \otimes r_{n-1} - \tau \circ (-1)^{n-2} r_0 dr_1 \otimes dr_n \otimes r_2 \otimes .... \otimes r_{n-1} = 0$.

2. We consider the top wedge power (over $\oplus_i \wedge^i \Omega_X[i]$) of $\bar{\varphi_R}$. This gives us an element in $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \wedge^n\Omega_X[n] \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n])$. We identify $\oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n]$ with the top wedge power of D_1 over $\oplus_i \wedge^i \Omega_X[i]$. $\bar{\varphi_R} \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \Omega[1] \otimes D_1)$ then gives us an element in $\mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n] \otimes \wedge^n \Omega_X[n], \wedge^n \Omega_X[n] \otimes \Omega[1] \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n])$ where D_1 acts on its top wedge power (over $\oplus_i \wedge^i \Omega_X[i]$ as described in Lemma 3. It follows from this description and Observation 1 in this section and that $\varphi_L \circ \bar{\varphi_R}^n = 0$.

3. By Lemma 3 the following diagram commutes in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$

$$\begin{array}{ccc}
\oplus_i \wedge^i \Omega_X[i] & \xrightarrow{\mathrm{id}^n} & \wedge^n \Omega_X[n] \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n] \\
{\scriptstyle -\varphi_L{}^+}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_L} \\
\oplus_i \wedge^i \Omega_X[i] \otimes \Omega[1] & \xrightarrow{\mathrm{id}^n} & \wedge^n \Omega_X[n] \otimes \Omega[1] \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n T_X[-n]
\end{array}$$

From this diagram and the fact that $\varphi_L \circ \bar{\varphi_R}^n = 0$, it follows that $-\mathrm{id}^n \circ \varphi_L{}^+ \circ (\mathrm{id}^{n-1}\bar{\varphi_R}^n) = 0$. But if $f \in \mathrm{Hom}_{\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)}(\mathcal{O}_X, \oplus_i \wedge^i \Omega_X[i])$ then $\mathrm{id}^n \circ f = 0 \iff f = 0$. Thus $\varphi_L{}^+ \circ (\mathrm{id}^{n-1}\bar{\varphi_R}^n) = 0$. But Lemma 2 says that $(\mathrm{id}^{n-1}\bar{\varphi_R}^n) = S$ where $S = \det(\frac{z}{\exp(z)-1})$ where $z = \alpha_{T_X}$, the Atiyah class of the tangent bundle of

X. It follows that $\varphi_L^+(S) = 0$.

4. We can now make the final calculation that explicitly gives us φ_L^+. Given any two sections a, b of $\oplus_i \wedge^i \Omega_X[i]$, we have $<\varphi_L(a), b> = <\varphi_L(a), b.S.S^{-1}> = <\varphi_L(a), S^{-1}.b.S>$. The second equality is because $S \in \oplus_i \mathrm{H}^i(X, \Omega^i)$ which commutes with everything. But, φ_L is a differential operator on $\oplus_i \wedge^i \Omega_X[i]$ which means that $\varphi_L(a.S^{-1}.b) = \varphi_L(a).S^{-1}b + a.\varphi_L(S^{-1}.b)$. By the definition of the pairing $<,>$, we therefore have $<\varphi_L(a), S^{-1}.b.S> = <\varphi_L(a.S^{-1}b), S> - <a, \varphi_L(S^{-1}.b).S>$. But $<\varphi_L(a.S^{-1}b), S> = C <(a.S^{-1}b), \varphi_L^+ S>$ where C is a constant and $\varphi_L^+ S = 0$. Thus ,$<\varphi_L(a), S^{-1}.b.S> = -<a, \varphi_L(S^{-1}.b).S> = <a, -S.\varphi_L(S^{-1}.b)>$. The following lemma has therefore been proven

Lemma 4. *Following the conventions mentioned in this section, we have* $\varphi_L^+ = -S. \circ \varphi_L \circ S^{-1}$. *where* $S = det(\frac{z}{exp(z)-1})$ *where* $z = \alpha_{T_X}$ *, the Atiyah class of the tangent bundle of* X *(by* $S.$ *we actually mean* ι_S*).*

5. Relating the HKR and the twisted HKR maps

Recall that $\Delta : X \to X \times X$ is the diagonal embedding. We now make the following observations

1. ${S_X}^{-1}\otimes$ is an isomorphism between $\mathrm{RHom}_X(\Delta^*\Delta_*\mathcal{O}_X, S_X)$ and $\mathrm{RHom}_X({S_X}^{-1}\otimes \Delta^*\Delta_*\mathcal{O}_X, \mathcal{O}_X)$.

2. $S_X^{-1} \otimes \Delta^*\Delta_*\mathcal{O}_X$ is isomorphic to $\Delta^*\Delta_!\mathcal{O}_X$ where $\Delta_!$ is the left adjoint of Δ^*. (Recall that $\Delta_! = S_{X\times X}^{-1}\Delta_* S_X$). Thus, $S_X^{-1}\otimes$ identifies $\mathrm{RHom}_X(\Delta^*\Delta_*\mathcal{O}_X, S_X)$ with $\mathrm{RHom}_X(\Delta^*\Delta_!\mathcal{O}_X, \mathcal{O}_X)$.

3. $\mathrm{RHom}_X(\Delta^*\Delta_!\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\gamma} \mathrm{RHom}_{X\times X}(\Delta_!\mathcal{O}_X, \Delta_*\mathcal{O}_X)$
is am isomorphism where for $\alpha \in \mathrm{RHom}_X(\Delta^*\Delta_!\mathcal{O}_X, \mathcal{O}_X)$, $\gamma(\alpha) = \Delta_*\alpha \circ \eta$ where $\eta : \Delta_!\mathcal{O}_X \to \Delta_*\Delta^*\Delta_!\mathcal{O}_X$ is the adjunction map.

4. $\mathrm{RHom}_{X\times X}(\Delta_!\mathcal{O}_X, \Delta_*\mathcal{O}_X)$ is isomorphic to $\mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_X)$. This isomorphism takes $f \in \mathrm{RHom}_{X\times X}(\Delta_!\mathcal{O}_X, \Delta_*\mathcal{O}_X)$ to the composite
$\mathcal{O}_X \xrightarrow{\eta} \Delta^*\Delta_!\mathcal{O}_X \xrightarrow{\Delta^* f} \Delta^*\Delta_*\mathcal{O}_X$
Here, $\eta : \mathcal{O}_X \to \Delta^*\Delta_!\mathcal{O}_X$ is the adjunction map.

5. The inverse of the duality map $D_\Delta : \mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\Delta_*\mathcal{O}_X) \to \mathrm{RHom}_X(\Delta^*\Delta_*\mathcal{O}_X, S_X)$ therefore, takes an element θ of $\mathrm{RHom}_X(\Delta^*\Delta_*\mathcal{O}_X, \mathcal{O}_X)$ to $\Delta^*(\gamma(S_X^{-1}\otimes\theta)) \circ \eta$

6. $\Delta^*\gamma(\alpha) = \Delta^*\Delta_*\alpha \circ \Delta^*\eta$ by Observation 3, where α is an element of $\mathrm{RHom}_X(\Delta^*\Delta_!\mathcal{O}_X, \mathcal{O}_X)$. We note that $\Delta^*\eta : \Delta^*\Delta_!\mathcal{O}_X \to \Delta^*\Delta_*\Delta^*\Delta_!\mathcal{O}_X$ is an adjunction map. Identifying $\Delta^*\Delta_!\mathcal{O}_X$ with $S_X^{-1} \otimes \Delta^*\Delta_*\mathcal{O}_X$ we use Lemma 6 of Markarian [2] to say that $\eta = \mathrm{id}_{S_X^{-1}} \otimes \Delta$ where Δ denotes the co-multiplication on $\Delta^*\Delta_*\mathcal{O}_X$ in this context. Similarly, we can check that $\Delta^*\Delta_*\alpha = \alpha \otimes \mathrm{id}_{\Delta^*\Delta_*\mathcal{O}_X}$.

7. It follows that $D_{\Delta}^{-1}(\theta)$ is given by the following composition of morphisms in $\mathrm{D}(\mathrm{mod}-\mathcal{O}_X)$

$$\mathcal{O}_X \xrightarrow{\eta} S_X^{-1}\otimes\Delta^*\Delta_*\mathcal{O}_X \xrightarrow{(\mathrm{id}\otimes\Delta)} S_X^{-1}\otimes\Delta^*\Delta_*\mathcal{O}_X\otimes\Delta^*\Delta_*\mathcal{O}_X$$

$$S_X^{-1}\otimes\Delta^*\Delta_*\mathcal{O}_X\otimes\Delta^*\Delta_*\mathcal{O}_X \xrightarrow{\mathrm{id}\otimes\theta\otimes\mathrm{id}} \Delta^*\Delta_*\mathcal{O}_X$$

8. We now use the HKR quasi-isomorphism to identify $\Delta^*\Delta_*\mathcal{O}_X$ with $\oplus_i\wedge^i\Omega_X[i]$. In this picture, the adjunction $\eta:\mathcal{O}_X\to\Delta^*\Delta_!\mathcal{O}_X$ just amounts to id^n. Assume that $\theta=\widehat{\mathrm{I}_{HKR}}(x)$, where $x\in\mathrm{RHom}_X(\mathcal{O}_X,\oplus_I\wedge^i\Omega_X[i])$. We identify $\oplus_i\wedge^i\Omega_X[i]$ with $S_X\otimes\oplus_i\wedge^i T_X[-i]$.

9. The map $(\theta\otimes\mathrm{id})\circ\Delta:\Delta^*\Delta_*\mathcal{O}_X\to\Delta^*\Delta_*\mathcal{O}_X\otimes\Delta^*\Delta_*\mathcal{O}_X\to S_X\otimes\Delta^*\Delta_*\mathcal{O}_X$ coincides with $((x*)\otimes\mathrm{id})\circ(\mathrm{I}_{HKR}\otimes\mathrm{id})\circ\Delta$. Here, $*:\oplus_i\wedge^i\Omega_X[i]\otimes\oplus_j\wedge^jT_X[-j]\to\mathcal{O}_X$ is the standard contraction. By our definition of $\varphi_L\in\mathrm{RHom}_X(\oplus_i\wedge^i\Omega_X[i],\Omega[1]\otimes\oplus_i\wedge^i\Omega_X[i])$, this amounts to $x*\exp(\varphi_L)$.

10. We can think of $\varphi_L\in\mathrm{RHom}_X(\mathcal{O}_X,\Omega[1]\otimes\mathrm{R}\mathcal{H}om_X(\oplus_i\wedge^i\Omega_X[i],\oplus_i\wedge^i\Omega_X[i])$. Thus, $x*\exp(\varphi_L)\in\mathrm{RHom}_X(\mathcal{O}_X,S_X\otimes\mathrm{R}\mathcal{H}om_X(\oplus_i\wedge^i\Omega_X[i],\oplus_i\wedge^i\Omega_X[i]))$. We recall from Section 4 that $f_l:\mathrm{R}\mathcal{H}om_X(\oplus_i\wedge^i\Omega_X[i],\oplus_i\wedge^i\Omega_X[i])\to\oplus_i\wedge^i\Omega_X[i]\otimes\oplus_j\wedge^jT_X[-j]$ is an isomorphism (Observation 1 Section 4.1). We introduce a new isomorphism $f_r:\mathrm{R}\mathcal{H}om_X(\oplus_i\wedge^i\Omega_X[i],\oplus_i\wedge^i\Omega_X[i])\to\oplus_i\wedge^i\Omega_X[i]\otimes\oplus_j\wedge^jT_X[-j]$. Recalling the notation of Section 4.1 , f_r^{-1} is the composite

$$\oplus_i\wedge^iV[i]\otimes\oplus_i\wedge^iV^*[-i]\xrightarrow{\tau}\oplus_i\wedge^iV^*[-i]\otimes\oplus_i\wedge^iV[i]\xrightarrow{\iota\otimes\iota}\mathrm{End}(\oplus_i\wedge^iV[i])^{\otimes 2}$$

$$\mathrm{End}(\oplus_i\wedge^iV[i])^{\otimes 2}\xrightarrow{\circ}\mathrm{End}(\oplus_i\wedge^iV[i])$$

Here, τ is a swap map that takes degrees into account. We revert back to the notation of this section. Let $\pi_l:\oplus_i\wedge^i\Omega_X[i]\otimes\oplus_j\wedge^jT_X[-j]\to\wedge^l\Omega_X[l]\otimes\oplus_i\wedge^iT_X[-i]$ be the natural projection.

If $y\in\mathrm{RHom}_X(\mathcal{O}_X,S_X\otimes\mathrm{R}\mathcal{H}om_X(\oplus_i\wedge^i\Omega_X[i],\oplus_i\wedge^i\Omega_X[i]))$, then , if $l>0$ the composite

$$\mathcal{O}_X\xrightarrow{\mathrm{id}^n}S_X^{-1}\otimes\oplus_i\wedge^i\Omega_X[i]\xrightarrow{\mathrm{id}\otimes\pi_l(f_r(y))}\oplus_i\wedge^i\Omega_X[i]$$

is 0 simply because if $l>0$, then multiplying by $\wedge^l\Omega_X[l]$ kills $\wedge^n\Omega_X[n]$. It follows that for our purpose of finding $D_{\Delta}^{-1}(\theta)$ we are interested in the composition

$$\mathcal{O}_X\xrightarrow{\mathrm{id}^n}S_X^{-1}\otimes\oplus_i\wedge^i\Omega_X[i]\xrightarrow{\mathrm{id}\otimes\pi_0(f_r((x*\exp(\varphi_L))))}\oplus_iS_X^{-1}\otimes S_X\wedge^i\Omega_X[i]$$

11. $\pi_0(f_r(y))=I\pi_0(f_l(y^+))$. This is a consequence of Observation 2(d) of Section 4.1.

12. We therefore need to calculate $(x*\exp(\varphi_L))^+$. This is $x*\exp(\varphi_L^+)$ by definition, since φ_L has total degree 0 . By Lemma 4, this is equal to $x*\exp(-S\varphi_LS^{-1})$ where $S\in\mathrm{RHom}_X(\mathcal{O}_X,\oplus_i\wedge^i\Omega_X[i])$ is as described in Lemma 4. Let I be the operator multiplying $\wedge^iT_X[-i]$ by -1^i. Then $x*\exp(-S\varphi_LS^{-1})=Ix*\exp(S\varphi_LS^{-1})=$

$S(Ix * \exp(\varphi_L))S^{-1}$.

13. $\pi_0(f_l(\alpha \circ \beta)) = \pi_0(f_l(\pi_0(f_l(\alpha)) \circ \beta))$. This follows from the following calculation (which is done in the notation of Section 4.1).

Assume that $Z \in \oplus_i \wedge^i V[i]$ and $y \in \oplus_i \wedge^i V^*[-i]$, where Z is non-constant and homogenous. We denote by Z the endomorphism ι_Z and by Y the endomorphism ι_Y. Then, $f_l(Z \circ Y \circ \beta) = Zf_l(Y \circ \beta)$. Since Z is nonconstant, it follows in this case that $\pi_0(f_l(Z \circ Y \circ \beta)) = 0$.

14. From Observations 12 and 13 it follows that $I\pi_0(f_l(S(Ix * \exp(\varphi_L))S^{-1})) = I\pi_0(f_l((Ix * \exp(\varphi_L))S^{-1})$, since $f_l(S) = S$ and $\pi_0(f_l(S)) = 1$.

15. If $y \in \mathrm{RHom}_X(\mathcal{O}_X, S_X \otimes \oplus_i \wedge^i T_X[-i])$, we claim that $\pi_0(f_l(y * \exp(\varphi_L))) = y$. This is because $\exp(\varphi_L) = Z \circ \Delta$ where Δ denotes the co-multiplication and Z equals the identity plus some operators on $\oplus_i \wedge^i \Omega_X[i]$ arising out of cohomology classes with coefficients in positive powers of $\Omega[1]$ which are thus killed by π_0. It follows that $y * \exp(\varphi_L) = Z \circ (y * \Delta) \implies \pi_0(f_l(y * \exp(\varphi_L))) = \pi_0(y * \Delta) = y$.

16. Thus $I\pi_0(f_l((Ix * \exp(\varphi_L))S^{-1}) = I\pi_0(f_l(Ix \circ S^{-1}))$ (by Observation 13). We claim that this is equal to $I(Ix|S^{-1})$ where $| : \oplus_j \wedge^j T_X[-j] \otimes \oplus_i \wedge^i \Omega_X[i] \to \oplus_j \wedge^j T_X[-j]$ treats a section of $\oplus_i \wedge^i \Omega_X[i]$ as a differential operator on $\oplus_j \wedge^j T_X[-j]$. This can be seen through the following computation, which again uses the notation of Section 4.1

$$\iota_{(y_1 \wedge .....\wedge y_l)}(x_l \wedge .... \wedge x_{k+1}.X_S) = \iota_{(y_1 \wedge .....\wedge y_{l-1})} \circ \iota_{y_l}(x_l \wedge .... \wedge x_{k+1}.X_S) =$$
$$\iota_{(y_1 \wedge .....\wedge y_{l-1})}(x_{l-1} \wedge .....x_{k+1}.X_S) + C.\iota_{(y_1 \wedge .....\wedge y_{l-1})}(x_l \wedge .... \wedge x_{k+1}.\iota_{y_l} X_S)$$
$$= \iota_{(y_1 \wedge .....\wedge y_{l-1})}(x_{l-1} \wedge .....x_{k+1}.X_S) + C.Dx_l \iota_{(y_1 \wedge .....\wedge y_{l-1})}(x_l \wedge .... \wedge x_{k+1}.\iota_{y_l} X_S)$$

where C and D are signs. It follows that $\pi_0(\iota_{(y_1 \wedge .....\wedge y_l)} \circ \iota_{x_l \wedge .... \wedge x_{k+1}})$
$= \pi_0(\iota_{(y_1 \wedge .....\wedge y_{l-1})} \circ \iota_{x_{l-1} \wedge .... \wedge x_{k+1}})$

17. $I(Ix|S^{-1}) = (x|JS^{-1})$ where J multiplies $\wedge^k \Omega_X[k]$ by -1^k. We note that $JS^{-1} = \mathrm{td}^{-1}$ where td is the Todd genus. It follows that $\mathrm{I}_{HKR}(D_\Delta^{-1}(\widehat{\mathrm{I}_{HKR}}(x)))$ is given by the following composition

$$\mathcal{O}_X \xrightarrow{\mathrm{id}^n} S_X^{-1} \otimes \mathcal{O}_X \otimes \wedge^n \Omega_X[n] \xrightarrow{\mathrm{id} \otimes x \otimes id} S_X^{-1} \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n \Omega_X[n]$$
$$S_X^{-1} \otimes \oplus_i \wedge^i \Omega_X[i] \otimes \wedge^n \Omega_X[n] \longrightarrow S_X^{-1} \otimes S_X \otimes \oplus_i \wedge^i T_X[-i] \otimes \wedge^n \Omega_X[n]$$
$$S_X^{-1} \otimes S_X \otimes \oplus_i \wedge^i T_X[-i] \otimes \wedge^n \Omega_X[n] \longrightarrow \wedge^i T_X[-i] \otimes \wedge^n \Omega_X[n] \xrightarrow{* \circ (|\mathrm{td}^{-1})} \oplus_i \wedge^i \Omega_X[i]$$

Here, unnamed arrows are natural identifications.

18. We claim that the above composition yields $J(x \wedge \mathrm{td}^{-1})$. Note that had y been at the extreme right, in the composition in Observation 17, and $\wedge^n \Omega_X[n]$ been identified with S_X etc, we would have obtained $x \wedge \mathrm{td}^{-1}$. While putting x where it is we have not taken into account the sign of $(-1)^{l.n}$ for the degree $\wedge^l \Omega_X[l]$

part of x that occurs if we use the signed swap, which would preserve $x \wedge \mathrm{td}^{-1}$. Further, we are finally doing a left contraction by $\wedge^{(n-l)} T_X[-(n-l)]$ instead of right contraction in the calculation prescribed in this observation. This yields an additional sign on $(-1)^{l(n-l)}$. The total sign difference is therefore, $(-1)^{l^2} = (-1)^l$. Thus, $\mathrm{I}_{HKR}(D_\Delta^{-1}(\widehat{\mathrm{I}_{HKR}}(x))) = J(x \wedge \mathrm{td}^{-1})$. The following theorem follows as a consequence

Theorem 2. *The following diagram commutes, relating I_{HKR}, $\widehat{I_{HKR}}$ and D_Δ*

$$\begin{array}{ccc} RHom_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta) & \xrightarrow{D_\Delta} & RHom_X(\Delta^*\mathcal{O}_\Delta, S_X) \\ \Big\downarrow I_{HKR} & & \widehat{I_{HKR}} \Big\uparrow \\ \oplus_{p,q} H^p(X, \Omega^q) & \xrightarrow{(\wedge td)\circ J} & \oplus_{p,q} H^p(X, \Omega^q) \end{array}$$

This is a corrected version of Theorem 8 in Markarian [2]. This is the key theorem in this write-up. We also observe that this theorem together with Lemma 1 implies the following corollary

Consider the pairing $<,>$ on $\mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\Delta_*\mathcal{O}_X)$, given by $< v, w >= tr_X(D_\Delta(v) \circ w)$. Then,

Corollary 1. *For $v, w \in RHom_X(\mathcal{O}_X, \Delta^*\Delta_*\mathcal{O}_X)$,*

$$< v, w >= \int_X Jv \wedge w \wedge td_X$$

where J multiplies $\wedge^i \Omega_X[i]$ by $(-1)^i$.

From now on we assume that X is a complex smooth scheme. We observe that the pairing $<,>$ on the Hochschild homology of X coincides with the Mukai pairing defined by Caldararu [4]. What it descends to on the usual cohomology after applying the HKR map twisted by $\sqrt{(}\mathrm{td}_X)$ is, by this corollary equal to the Mukai pairing Caldararu [3] defines on $\oplus_{p,q}\mathrm{H}^p(X, \Omega^q)$ applied to the elements $\bar{\tau}(\mathrm{I}_{HKR}(v)\sqrt{(}\mathrm{td}_X))$ and $\mathrm{I}_{HKR}(w)\sqrt{\mathrm{td}_X}$ of $\oplus_{p,q}\mathrm{H}^p(X, \Omega^q)$. Here, $\bar{\tau}$ is the endomorphism of $\oplus_{p,q}\mathrm{H}^p(X, \Omega^q)$ multiplying $\mathrm{H}^p(X, \Omega^q)$ by $(\sqrt{-1})^{q-p}$.

We can modify Caldararu's generalization [3] of the Mukai pairing by specifying our pairing on $\oplus_{p,q}\mathrm{H}^p(X, \Omega^q)$ to be given by $< x, y >= (\bar{\tau}(x), y)$ where $(,)$ is Caldararu's Mukai pairing [3] . We note that our modification in this manner of Caldararu's Mukai pairing [3] also satisfies the properties one would want of the Mukai pairing - most importantly, it satisfies the following adjointness property

If $\Theta : \mathrm{D}(\mathrm{mod} - \mathcal{O}_X) \to \mathrm{D}(\mathcal{O}_Y - mod)$ is an integral transform left adjoint to the integral transform Ψ, then $< \Theta_* v, w >_X = < v, \Psi_* w >_Y$ for $v \in \oplus_{p,q}\mathrm{H}^p(Y, \Omega^q)$ and $w \in \oplus_{p,q}\mathrm{H}^p(X, \Omega^q)$.

This follows from the same property for Caldararu's Mukai pairing [3] together with the fact that if Θ is an integral transform, then Θ_* preserves the columns of the Hodge diamond (Caldararu [3]).

We have thus, also shown that the map $x \to \mathrm{I}_{HKR}(x).\sqrt{\mathrm{td}_X}$ from the Hochschild homology of X to $\oplus_{p,q}\mathrm{H}^p(X,\Omega^q)$ preserves the Mukai pairing provided that we interpret the Mukai pairing on $\oplus_{p,q}\mathrm{H}^p(X,\Omega^q)$ suitably. This proves a part of Caldararu's conjecture [3] regarding the equivalence between the Hochschild and Hodge structures of a complex smooth scheme X.

Finally, we observe that if $f : Y \to X$ is a proper morphism between smooth complex schemes, then f^* is left adjoint to f_*, and both are integral transforms. It follows from the adjointness property of the Mukai pairing that

$$< f^*\mathrm{I}_{HKR}^{-1}(l), \mathrm{ch}(E) >_Y = < \mathrm{I}_{HKR}^{-1}l, f_*\mathrm{ch}(E) >_X$$

if $E \in \mathrm{D}(\mathcal{O}_Y - mod)$. Here, ch denotes the Chern character to Hochcshild homology as defined by Caldararu [4] and implicitly in [1]. It has been proven in [3] that the Chern character to Hochschild homology commutes with integral transforms. It follows that

$$< l, f_*\mathrm{ch}(E) >_X = < l, \mathrm{ch}(f_*E) >_X$$

Combining this with Corollary 1 we see that

$$\int_Y J(f^*l)\mathrm{ch}(E)\mathrm{td}_Y = \int_X J(l)\mathrm{ch}(f_*E)\mathrm{td}_X$$

Noting that J is an isomorphism and that $J \circ f^* = f^* \circ J$ we see that for any $l \in \oplus_{p,q}\mathrm{H}^p(Y,\Omega^q)$ and $E \in E \in \mathrm{D}(\mathcal{O}_Y - mod)$

Theorem 3.
$$\int_Y (f^*l)\ ch(E)\ td_Y = \int_X l\ \ ch(f_*E)\ td_X$$

This is the relative Riemann-Roch theorem.

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