

WEIGHTED NORM INEQUALITIES, OFF-DIAGONAL ESTIMATES AND ELLIPTIC OPERATORS.

PART IV: RIESZ TRANSFORMS ON MANIFOLDS AND WEIGHTS

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ABSTRACT. This is the fourth article of our series. Here, we apply the results of [AM1] to study weighted norm inequalities for the Riesz transform of the Laplace-Beltrami operator on Riemannian manifolds and of subelliptic sum of squares on Lie groups, under the doubling volume property and Poincaré inequalities.

1. INTRODUCTION AND MAIN RESULTS

On \mathbb{R}^n , it is well-known that the classical Riesz transforms R_j , $1 \leq j \leq n$, are bounded on $L^p(\mathbb{R}^n, dx)$ for $1 < p < \infty$ and are of weak-type (1,1) with respect to dx . As a consequence of the weighted theory for classical Calderón-Zygmund operators, the Riesz transforms are also bounded on $L^p(\mathbb{R}^n, w(x)dx)$ for all $w \in A_p(dx)$, $1 < p < \infty$, and are of weak-type (1,1) with respect to $w(x)dx$ for $w \in A_1(dx)$. Furthermore, it can be shown that the A_p condition on the weight is necessary for the weighted L^p boundedness of the Riesz transforms (see, for example, [Gra]).

On a manifold, there has been a number of works discussing the validity of the unweighted L^p theory depending in the geometry of the manifold. Although some progress has been done in this direction, the general picture is far from clear. A difficulty is that one has to leave the class of Calderón-Zygmund operators. In particular, the Riesz transforms on the manifold may not have Calderón-Zygmund kernels, either because one does not have regularity estimates, or worse because one does not even have size estimates. It turns out also that the range of p for which one obtains L^p boundedness may not be $(1, \infty)$. See [ACDH] for a detailed account on all this and Section 2 below.

Here, we wish to develop a weighted theory: we want to obtain weighted L^p estimates for a range of p and for Muckenhoupt weights with respect to the volume form. Of course, they must encompass the unweighted estimates so we shall restrict ourselves to situations where the unweighted theory has been developed. Nothing new will be done on the unweighted case (except the commutator result in Section 4). We assume that the volume form satisfies the doubling condition. In that case, we are able to apply a machinery developed in the first article of our series [AM1].

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Let M be a complete non-compact Riemannian manifold with d its geodesic distance. Assume that the volume form μ verifies the doubling condition,

$$\mu(B(x, 2r)) \leq C \mu(B(x, r)) < \infty,$$

for all $x \in M$ and $r > 0$ where $B(x, r) = \{y \in M : d(x, y) < r\}$. Then M equipped with the geodesic distance and the volume form is a space of homogeneous type. Non-compactness of M implies infinite diameter, which together with the doubling condition yields $\mu(M) = \infty$ (see for instance [Mar]).

Let $1 \leq p < \infty$. One says that M satisfies the L^p -Poincaré property, we write M satisfies (P_p) , if there exists $C > 0$ such that, for every ball B and every f with $f, \nabla f \in L^p_{\text{loc}}(\mu)$,

$$\int_B |f - f_B|^p d\mu \leq Cr(B)^p \int_B |\nabla f|^p d\mu. \quad (P_p)$$

Here, $r(B)$ is the radius of B , f_B is the mean value of f over B , ∇f is the Riemannian gradient of f , $|\nabla f|$ its length in the tangent space TM . It is known that (P_p) implies (P_q) when $q > p$ (see for instance [HK]). Thus the set of p 's such that (P_p) holds is, if not empty, an interval unbounded on the right. A recent deep result from [KZ] implies that this set is open in $[1, \infty)$. We define

$$q_- = \inf \{p \in [1, \infty) : (P_p) \text{ holds}\}.$$

The Riesz transform is the (vector) operator $\nabla \Delta^{-1/2}$, where Δ is the positive Laplace-Beltrami operator on M .

Theorem 1.1. *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property and (P_2) . Then, there exists $\varepsilon \in (0, \infty]$ such that*

$$\| |\nabla \Delta^{-1/2} f| \|_p \leq C_p \|f\|_p \quad (R_p)$$

holds for $1 < p < 2 + \varepsilon$ and all f bounded with compact support.

This result is a combination of [CD] for $1 < p < 2$ (in fact, a condition weaker than (P_2) suffices, namely, the on-diagonal upper bound $p_t(x, x) \leq C\mu(B(x, \sqrt{t}))^{-1}$ for all $t > 0$ and all $x \in M$) and [AC] for the existence of ε . Its value or its expression in terms of geometric quantities is not known.

We set

$$q_+ = \sup \{p \in (1, \infty) : (R_p) \text{ holds}\}.$$

Define \tilde{q}_+ as the supremum of those $p \in (1, \infty)$ such that for all $t > 0$,

$$\| |\nabla e^{-t\Delta} f| \|_p \leq C t^{-1/2} \|f\|_p. \quad (1.1)$$

By analyticity of the heat semigroup, one always have $\tilde{q}_+ \geq q_+$. Under the doubling condition and (P_2) , it is shown in [ACDH, Theorem 1.3] that $q_+ = \tilde{q}_+$ and by Theorem 1.1, $q_+ > 2$.

Let us turn to weighted estimates. Properties of Muckenhoupt weights A_p and Reverse Hölder classes RH_s are reviewed in [AM1, Section 2]. If $w \in A_\infty(\mu)$, one can define $r_w = \inf\{p > 1 : w \in A_p(\mu)\} \in [1, \infty)$ and $s_w = \sup\{s > 1 : w \in RH_s(\mu)\} \in (1, \infty]$. Given $1 \leq p_0 < q_0 \leq \infty$, we introduce the (possibly empty) set

$$\mathcal{W}_w(p_0, q_0) = \left(p_0 r_w, \frac{q_0}{(s_w)'} \right) = \{p : p_0 < p < q_0, w \in A_{\frac{p}{p_0}}(\mu) \cap RH_{(\frac{q_0}{p})'}(\mu)\}.$$

Here, $q' = \frac{q}{q-1}$ is the conjugate exponent to q . And note that RH_1 means no condition on the weight (besides A_∞).

Theorem 1.2. *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property and (P_2) . Let $w \in A_\infty(\mu)$.*

- (i) *If $\mathcal{W}_w(q_-, q_+) \neq \emptyset$, then, for $p \in \mathcal{W}_w(1, q_+)$, the Riesz transform is of strong-type (p, p) with respect to $w d\mu$, that is,*

$$\| |\nabla \Delta^{-1/2} f| \|_{L^p(M, w)} \leq C_{p, w} \|f\|_{L^p(M, w)} \quad (1.2)$$

for all f bounded with compact support.

- (ii) *If $w \in A_1(\mu) \cap RH_{(q_+/q_-)'}(\mu)$, then the Riesz transform is of weak-type $(1, 1)$ with respect to $w d\mu$, that is,*

$$\| |\nabla \Delta^{-1/2} f| \|_{L^{1, \infty}(M, w)} \leq C_{1, w} \|f\|_{L^1(M, w)} \quad (1.3)$$

for all f bounded with compact support.

If $q_+ = \infty$ then $\mathcal{W}_w(q_-, \infty) = (r_w q_-, \infty)$ is never empty and we can apply the above theorem. Thus, given $w \in A_\infty(\mu)$, the Riesz transform is bounded on $L^p(M, w)$ for $r_w < p < \infty$, that is, for $w \in A_p(\mu)$, and we obtain the same weighted theory as for the Riesz transform on \mathbb{R}^n :

Corollary 1.3. *Let M be a complete non-compact Riemannian manifold satisfying the doubling volume property and (P_2) . Assume that the Riesz transform is bounded on $L^p(M)$ for all $1 < p < \infty$. Then the Riesz transform is bounded on $L^p(M, w)$ for all $w \in A_p(\mu)$ and $1 < p < \infty$ and it is of weak-type $(1, 1)$ with respect to $w d\mu$ for all $w \in A_1(\mu)$.*

The proof of Theorem 1.2, part (i), has two steps. In the first one, we prove the $L^p(w)$ inequality for $p \in \mathcal{W}_w(q_-, q_+)$. This is where we use Poincaré inequalities. There are two reasons for introducing the number $q_- < 2$. First, we have $\mathcal{W}_w(2, q_+) \subset \mathcal{W}_w(q_-, q_+)$, so the condition $\mathcal{W}_w(q_-, q_+) \neq \emptyset$ allows more weights. Second, if $q_- = 1$, then the proof of part (ii) is completed in the first step. In a second step we extend the range to $\mathcal{W}_w(1, q_+)$ and Poincaré inequalities are not needed. Its proof reveals a worth mentioning result in the spirit of [CD]:

Theorem 1.4. *Assume that M (complete, non-compact) satisfies the doubling volume property and $p_t(x, x) \leq C\mu(B(x, \sqrt{t}))^{-1}$ for all $t > 0$ and all $x \in M$. Let w be a weight with $(\tilde{q}_+)' < s_w$. If there is some $q \in (r_w, \infty)$, for which the Riesz transform is of weak-type (q, q) with respect to $w d\mu$, then it is of strong-type (p, p) with respect to $w d\mu$ for all $p \in (r_w, q)$. In addition, if $w \in A_1(\mu)$, then the Riesz transform is of weak-type $(1, 1)$ with respect to $w d\mu$.*

When $w = 1$, then one can take $q = 2$ and this is the result in [CD]. For other weights, we do not know how to obtain the existence of q unless assuming further the Poincaré inequalities (P_2) .

In [AM1, Lemma 4.6] examples of weights in $A_p(\mu) \cap RH_q(\mu)$ are given. The computations are done in the Euclidean setting, but most of them can be carried out in spaces of homogeneous type. In particular, given $f, g \in L^1(M, \mu)$ (or Dirac masses) $1 \leq r < \infty$ and $1 < s \leq \infty$, we have that $w(x) = M_\mu f(x)^{-(r-1)} + M_\mu g(x)^{1/s} \in$

$A_p(\mu) \cap RH_q(\mu)$ (M_μ is the Hardy-Littlewood maximal function) for all $p > r$ and $q < s$ (and $p = r$ if $r = 1$ and $q = s$ if $s = \infty$). Thus, $r_w \leq r$ and $s_w \geq s$.

We next provide some applications, then proofs of the above two theorems and eventually we add a word on estimates for commutators with bounded mean oscillation functions.

2. APPLICATIONS

Unweighted L^p bounds for Riesz transforms in different specific situations were reobtained in a unified manner in [ACDH] and the methods used there are precisely those which allowed us to start the weighted theory. Therefore, it is natural to apply this theory in return to Riesz transforms on manifolds. Let us concentrate on four situations (more is done in [ACDH]).

2.1. Manifolds with non-negative Ricci curvature. In this case, the Riesz transform is bounded on (unweighted) L^p for $1 < p < \infty$ ([Ba1],[Ba2]). Thus $q_+ = \infty$. Such manifolds are known to satisfy the doubling condition (see [Cha, Theorem 3.10] and (P_2) and even (P_1) [Bus] (see, for instance, [HK] or [SC2] for other references). By Corollary 1.3, we obtain strong-type (p, p) for $A_p(\mu)$ weights and weak-type $(1, 1)$ for $A_1(\mu)$ weights.

2.2. Co-compact covering manifolds with polynomial growth deck transformation group. In this case, one has the doubling condition and (P_2) (see [SC2])*. That the Riesz transform is of unweighted strong type (p, p) for $1 \leq p \leq 2$ is due to [CD]. For $2 < p < \infty$ this is first done in [Dun] and hence $q_+ = \infty$. By Corollary 1.3, we obtain strong-type (p, p) for $A_p(\mu)$ weights and weak-type $(1, 1)$ for $A_1(\mu)$ weights.

2.3. Conical manifolds with compact basis without boundary. As mentioned in [ACDH], this is not strictly speaking a smooth manifold but it is stochastically complete and this is what is needed to develop the unweighted theory for the Riesz transform: it is shown in [Li] that q_+ is a finite value related to the bottom of the spectrum on the Laplace operator on the compact basis. Also, one has doubling and (P_2) (see [Li] and [CL]) and even (P_1) by using the methods in [GS], so $q_- = 1$. Hence, $\mathcal{W}_w(1, q_+) = (r_w, q_+ / (s_w)')$ is (contained in) the range of L^p boundedness for a given weight provided this is not empty. In other words, if $1 < p < q_+$ and $w \in A_p(\mu) \cap RH_{(q_+/p)'(\mu)}$ then one has strong type (p, p) with respect to $wd\mu$. For $p = 1$, one has weak-type $(1, 1)$ with respect to $wd\mu$ if $w \in A_1(\mu) \cap RH_{(q_+)'}(\mu)$.

2.4. Lie groups with polynomial volume growth endowed with a sublaplacian. One starts with left-invariant vector fields X_j satisfying the Hörmander condition and μ is the left (and right) invariant Haar measure. The sublaplacian is $\Delta = -\sum_{j=1}^n X_j^2$. One has the doubling condition and (P_1) , hence $q_- = 1$ (see [Var] or [HK, p. 70] for a statement and references). The statement of Theorem 1.2 applies with no change to the Riesz transforms $X_j \Delta^{-1/2}$. In this case, $q_+ = \infty$ from [Ale]. By Corollary 1.3, the weighted theory for these Riesz transforms is the same as the ones in \mathbb{R}^n for $1 \leq p < \infty$: $L^p(w)$ boundedness for $w \in A_p$, $1 < p < \infty$, and weak-type $(1, 1)$ with respect to $w d\mu$ for $w \in A_1$.

* (P_1) also holds by a discretization method [CS, Théorème 7.2] and Poincaré inequalities for discrete groups (see [HK, p.76]).

3. PROOF OF THE MAIN RESULTS

We advice the reader to have [AM1] and [ACDH] handy.

Proof of Theorem 1.2, (i). The first step consists in obtaining the desired estimate for $p \in \mathcal{W}_w(q_-, q_+)$. Let us fix such a p . Then (see, for instance, [AM1, Proposition 2.1]) there exist p_0, q_0 such that

$$q_- < p_0 < p < q_0 < q_+ \quad \text{and} \quad w \in A_{\frac{p}{p_0}}(\mu) \cap RH_{\left(\frac{q_0}{p}\right)'(\mu)}.$$

The desired estimate follows from [AM1, Theorem 3.7] (extended to spaces of homogeneous types in [AM1, Section 5]) applied to $Tf = |\nabla \Delta^{-1/2} f|$, $S = I$ and $\mathcal{A}_r = I - (I - e^{-r^2 \Delta})^m$ with m large enough. We need to show that for any ball B with radius $r = r(B)$ and all $x \in B$,

$$\left(\int_B |\nabla \Delta^{-1/2} (I - e^{-r^2 \Delta})^m f|^{p_0} d\mu \right)^{\frac{1}{p_0}} \leq C M_\mu(|f|^{p_0})^{\frac{1}{p_0}}(x), \quad (3.1)$$

for all $f \in L_c^\infty(M)$ (i.e., f bounded with compact support), and for $k \in \mathbb{N}$, $1 \leq k \leq m$,

$$\left(\int_B |\nabla e^{-k r^2 \Delta} f|^{q_0} d\mu \right)^{\frac{1}{q_0}} \leq C M_\mu(|\nabla f|^{p_0})^{\frac{1}{p_0}}(x), \quad (3.2)$$

for all $f \in W^{1,p_0}(M)$ (the Sobolev space) where M_μ is the Hardy-Littlewood maximal operator. We used the notation

$$\int_B h d\mu = \frac{1}{\mu(B)} \int_B h d\mu.$$

It is shown in [ACDH, Section 3.1] that the conjunction of (1.1) for all $2 < q < q_+$ (which holds as recalled above), the doubling property, the (trivial) boundedness of the Riesz transform on $L^2(M)$ and (P_2) imply (3.1) and (3.2) with exponent 2 in lieu of p_0 . As the Riesz transform is already bounded on $L^{p_0}(M)$ and (P_{p_0}) holds since $q_- < p_0 < q_+$, we can therefore reproduce *mutatis mutandis* the same argument with 2 replaced by p_0 to obtain (3.1) and (3.2) (the L^{p_0} -Poincaré inequality (P_{p_0}) is used when proving (3.2)).

In the second step we extend the range to $\mathcal{W}_w(1, q_+)$. To do so, we apply [AM1, Theorem 8.7] (extended to spaces of homogeneous type in [AM1, Section 8.4]) to $Tf = |\nabla \Delta^{-1/2} f|$ with $\mathcal{A}_r = I - (I - e^{-r^2 \Delta})^m$ for some large integer m , $p_0 = 1$ and $q_0 = q_+$ (do not confuse with p_0 and q_0 in the previous argument). It suffices to check the following list of items:

- (a) There exists $q \in \mathcal{W}_w(1, q_+)$ such that $\nabla \Delta^{-1/2}$ is bounded from $L^q(w)$ to $L^{q,\infty}(w)$.
- (b) For all $j \geq 1$, there exist constants α_j such that for any ball B with $r(B)$ its radius and for any $f \in L_c^\infty(M)$ supported in B ,

$$\left(\int_{C_j(B)} |\mathcal{A}_{r(B)} f|^{q_+} d\mu \right)^{\frac{1}{q_+}} \leq \alpha_j \int_B |f| d\mu. \quad (3.3)$$

- (c) There exists $\beta > (s_w)'$, i.e., $w \in RH_{\beta'}(\mu)$, with the following property: for all $j \geq 2$, there exist constants α_j such that for any ball B with $r(B)$ its radius and

for any $f \in L_c^\infty(M)$ supported in B ,

$$\left(\int_{C_j(B)} |\nabla \Delta^{-1/2}(I - \mathcal{A}_{r(B)})f|^\beta d\mu \right)^{1/\beta} \leq \alpha_j \int_B |f| d\mu. \quad (3.4)$$

(d) $\sum_j \alpha_j 2^{D_w j} < \infty$ where D_w is the doubling order of $w d\mu$.

Here, $C_j(B) = 4B$ for $j = 1$ and $C_j(B) = 2^{j+1}B \setminus 2^j B$ for $j \geq 2$, where λB is the ball co-centered with B and radius $\lambda r(B)$. Also, we have used the notation

$$\int_{C_j(B)} h d\mu = \frac{1}{\mu(2^{j+1}B)} \int_{C_j(B)} h d\mu.$$

The conclusion is that $\nabla \Delta^{-1/2}$ is bounded on $L^p(M, w)$ for all $p \in \mathcal{W}_w(1, q_+)$ with $p < q$. By the first step one can choose any $q \in \mathcal{W}_w(q_-, q_+) \subset \mathcal{W}_w(1, q_+)$ hence $\nabla \Delta^{-1/2}$ is bounded on $L^p(M, w)$ for all $p \in \mathcal{W}_w(1, q_+)$.

Now, we check the items in the list. As just explained, (a) holds by the first step for $q \in \mathcal{W}_w(q_-, q_+)$. To obtain (b), we deduce from the Gaussian upper bound on the kernel $p_t(x, y)$ of $e^{-t\Delta}$ —which holds under doubling and on-diagonal upper bound, see [ACDH]—that for any fixed integer m there exist $c, C > 0$ such that for all $j \geq 1$, all ball B , all $f \in L_c^\infty(M)$ supported in B and all $1 \leq k \leq m$,

$$\sup_{C_j(B)} |e^{-kr(B)^2 \Delta} f| \leq C e^{-c4^j} \int_B |f| d\mu. \quad (3.5)$$

This easily implies that $\mathcal{A}_r = I - (I - e^{-r^2 \Delta})^m$ satisfies (3.3) with $\alpha_j = C e^{-c4^j}$.

The proof of (c) is based on the following result.

Lemma 3.1. *Assume that M (complete, non-compact) satisfies the doubling volume property and the on-diagonal upper bound $p_t(x, x) \leq C\mu(B(x, \sqrt{t}))^{-1}$ for all $t > 0$ and all $x \in M$. Then for all $\beta \in [1, \tilde{q}_+) \cup [1, 2]$, one has the following estimate: for all $m \geq 1$, there exists $C > 0$ such that for all $j \geq 2$, all ball B , all $f \in L_c^\infty(M)$ with support in B ,*

$$\left(\int_{C_j(B)} |\nabla \Delta^{-1/2}(I - e^{-r(B)^2 \Delta})^m f|^\beta d\mu \right)^{\frac{1}{\beta}} \leq C 4^{-jm} \int_B |f| d\mu. \quad (3.6)$$

To finish the proof of (c), notice that $\mathcal{W}_w(q_-, q_+) \neq \emptyset$ implies that $q_+/(s_w)' > q_- r_w \geq 1$. In particular, $q_+ > (s_w)'$. We also know that $\tilde{q}_+ = q_+ > 2$. We select β with $\max\{(s_w)', 2\} < \beta < \tilde{q}_+$. Hence, (3.4) holds with $\alpha_j = C 4^{-jm}$

Eventually (d) holds if we choose $m > D_w/2$. \square

Proof of Lemma 3.1. First, this estimate is known for $\beta = 2$ (see [ACDH]). Also, the inequality for a fixed β_0 implies the same one for all β with $1 \leq \beta \leq \beta_0$. It suffices to assume $\beta > 2$, which happens only if $\tilde{q}_+ > 2$.

We use a trick from [ACDH, Proof of Lemma 3.1]. Fix a ball B , with radius r , and $f \in L_c^\infty(M)$ supported in B . We have

$$\nabla \Delta^{-1/2}(I - e^{-r^2 \Delta})^m f = \int_0^\infty g_r(t) \nabla e^{-t\Delta} f dt$$

where $g_r: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a function such that

$$\int_0^\infty |g_r(t)| e^{-\frac{c4^j r^2}{t}} \frac{dt}{\sqrt{t}} \leq C_m 4^{-jm}. \quad (3.7)$$

By definition of \tilde{q}_+ and the argument of [ACDH, p. 944] we have

$$\left(\int_M |\nabla_x p_t(x, y)|^\beta e^{\gamma \frac{d^2(x, y)}{t}} d\mu(x) \right)^{1/\beta} \leq \frac{C}{\sqrt{t} [\mu(B(y, \sqrt{t}))]^{1-1/\beta}},$$

for all $t > 0$ and $y \in M$, with $\gamma > 0$ depending on β . This implies that for all $j \geq 2$, $y \in B$ and all $t > 0$,

$$\left(\int_{C_j(B)} |\nabla_x p_t(x, y)|^\beta d\mu(x) \right)^{1/\beta} \lesssim \frac{1}{\sqrt{t}} e^{-\frac{c4^j r^2}{t}} \frac{1}{\mu(B(y, \sqrt{t}))^{1-1/\beta} \mu(2^{j+1} B)^{1/\beta}}.$$

Using the doubling condition, $\mu(2^{j+1} B) \sim \mu(B(y, 2^{j+1} r))$ uniformly in $y \in B$ and

$$\frac{\mu(B(y, 2^{j+1} r))}{\mu(B(y, \sqrt{t}))} \lesssim \max \left\{ 1, \frac{2^j r}{\sqrt{t}} \right\}^D$$

where D is the doubling order of μ . Hence, with another $c > 0$,

$$\left(\int_{C_j(B)} |\nabla_x p_t(x, y)|^\beta d\mu(x) \right)^{1/\beta} \lesssim \frac{1}{\sqrt{t}} e^{-\frac{c4^j r^2}{t}} \frac{1}{\mu(2^{j+1} B)} \leq \frac{1}{\sqrt{t}} e^{-\frac{c4^j r^2}{t}} \frac{1}{\mu(B)}.$$

We conclude using Minkowski's integral inequality and (3.7) that the left hand side of (3.6) is bounded by

$$\begin{aligned} & \int_0^\infty |g_r(t)| \int_B |f(y)| \left(\int_{C_j(B)} |\nabla_x p_t(x, y)|^\beta d\mu(x) \right)^{1/\beta} d\mu(y) dt \\ & \lesssim \int_0^\infty |g_r(t)| \frac{1}{\sqrt{t}} e^{-\frac{c4^j r^2}{t}} dt \int_B |f| d\mu \lesssim 4^{-jm} \int_B |f| d\mu. \end{aligned}$$

□

Proof of Theorem 1.2, (ii). Since $w \in A_1(\mu) \cap RH_{(q_+/q_-)' }(\mu)$, $r_w = 1$ and $s_w > (q_+/q_-)'$. Thus $q_- r_w = q_- < q_+/(s_w)'$ and $\mathcal{W}_w(q_-, q_+) \neq \emptyset$. Therefore, the first step of the proof of part (i) applies. Next, the four items checked already in the second step suffice to conclude the weak-type (1, 1) when $w \in A_1(\mu)$ by [AM1, Remark 8.9]. □

Proof of Theorem 1.4. Invoke [AM1, Remark 8.10] and inspect the way we proved (b) and (c) above. □

4. COMMUTATORS

Let us write $T = \nabla \Delta^{-1/2}$ and take $b \in \text{BMO}(\mu)$ (the space of bounded mean oscillation functions). We define the first order commutator $T_b^1 f = [b, T]f = bTf - T(bf)$, and for $k \geq 2$ the k -th order commutator is $T_b^k f = [b, T_b^{k-1}]f$.

Theorem 4.1. *Under the assumptions of Theorem 1.2, T_b^k satisfies (1.2) for each $k \geq 1$, that is, it is bounded on $L^p(M, w)$ under the same conditions on w, p .*

The proof is almost identical to that of Theorem 1.2 and we point out the main changes. In the first step we use [AM1, Theorem 3.15]. One proves estimates slightly stronger than (3.1) and (3.2), where, following the ideas in [ACDH, Section 3.1], the right hand sides involve, instead of maximal functions, series of dyadic averages with some coefficients that decay fast enough. For the second step, we use [AM1, Remark 8.11]. Items (b) and (c) remain the same. In (a), by the first step, we have that T and T_b^ℓ , $1 \leq \ell \leq k$, are of weak-type (q, q) with respect to $w d\mu$. Finally, the series in (d) is replaced by $\sum_j \alpha_j 2^{D_w j} j^k$ and it is easily seen to be finite by taking m large enough (the same m for all k). Further details are left to the interested reader.

This theorem applies to the four situations described in Section 2. Note that even the unweighted L^p estimates for the commutators are new.

REFERENCES

- [Ale] G. Alexopoulos, *An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth*, Can. J. Math. **44** (1992), no. 4, 691–727.
- [AC] P. Auscher & T. Coulhon, *Riesz transforms on manifolds and Poincaré inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 1–25.
- [ACDH] P. Auscher, T. Coulhon, X.T. Duong & S. Hofmann, *Riesz transforms on manifolds and heat kernel regularity*, Ann. Scient. ENS Paris **37** (2004), no. 6, 911–957.
- [AM1] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights*, Preprint 2006. Available at <http://www.uam.es/chema.martell>
- [Ba1] D. Bakry, *Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*, Séminaire de Probabilités, XXI, 137–172, Lecture Notes in Math., 1247, Springer, Berlin, 1987.
- [Ba2] D. Bakry, *The Riesz transforms associated with second order differential operators*, Seminar on Stochastic Processes, 1988 (Gainesville, FL, 1988), 1–43, Progr. Probab., 17, Birkhäuser Boston, Boston, MA, 1989.
- [Bus] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. cole Norm. Sup. **15** (1982), no. 2, 213–230.
- [Cha] I. Chavel, *Riemannian geometry: a modern introduction*, Cambridge Tracts in Mathematics, 108, Cambridge University Press, 1993.
- [CD] T. Coulhon & X.T. Duong, *Riesz transforms for $1 \leq p \leq 2$* , Trans. Amer. Math. Soc. **351** (1999), 1151–1169.
- [CL] T. Coulhon & H.Q. Li, *Estimations inférieures du noyau de la chaleur sur les variétés coniques et transformée de Riesz*, Arch. Math. (Basel) **83** (2004), no. 3, 229–242.
- [CS] T. Coulhon & L. Saloff-Coste, *Varités riemanniennes isométriques l’infini. (French) [Isometric Riemannian manifolds at infinity]* Rev. Mat. Iberoamericana **11** (1995), no. 3, 687–726.
- [Dun] N. Dungey, *Heat kernel estimates and Riesz transforms on some Riemannian covering manifolds*, Math. Z. **247** (2004), no. 4, 765–794.
- [Gra] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.
- [GS] A. Grigor’yan & L. Saloff-Coste, *Stability results for Harnack inequalities*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 3, 825–890.
- [HK] P. Hajlasz & P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (200), no. 688.
- [KZ] S. Keith & X. Zhong, *The Poincaré inequality is an open ended condition*, Preprint 2003.

- [Li] H.Q. Li, *La transformation de Riesz sur les variétés coniques*, J. Funct. Anal. **168** (1999), no. 1, 145–238.
- [Mar] J.M. Martell, *Desigualdades con pesos en el Análisis de Fourier: de los espacios de tipo homogéneo a las medidas no doblantes*, Ph.D. Thesis, Universidad Autónoma de Madrid, 2001.
- [SC1] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices **1992**, no. 2, 27–38.
- [SC2] L. Saloff-Coste, *Parabolic Harnack inequality for divergence-form second-order differential operators*, Potential theory and degenerate partial differential operators (Parma). Potential Anal. **4** (1995), no. 4, 429–467.
- [Var] N. Varopoulos, *Fonctions harmoniques sur les groupes de Lie*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 17, 519–521.

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