DERIVED CATEGORIES OF COHERENT SHEAVES AND TRIANGULATED CATEGORIES OF SINGULARITIES

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Abstract. In this paper we establish an equivalence between the category of graded D-branes of type B in Landau-Ginzburg models with homogeneous superpotential W and triangulated category of singularities of the fiber of W over zero. We also proved that the category of graded D-branes of type B in such LG-models is connected by a fully faithful functor with the derived category of coherent sheaves on the projective variety defined by the equation $W=0$.

Introduction

To any algebraic variety X one can attach two triangulated categories: the bounded derived category of coherent sheaves $\mathbf{D}^b(\mathrm{coh}(X))$ and the triangulated subcategory of perfect complexes $\mathfrak{Perf}(X)$. If the variety X is smooth then these categories coincide. For singular varieties this property is not fulfilled. In [18] we introduced a triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(X)$ as the quotient of the triangulated category $\mathbf{D}^b(\mathrm{coh}(X))$ by the full subcategory of perfect complexes $\mathfrak{Perf}(X)$. The category $\mathbf{D}_{\mathrm{Sg}}(X)$ reflects many properties of the singularities of X.

By the same way for any noetherian algebra A we can define a triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(A)$ as the quotient $\mathbf{D}^b(\mathrm{mod}{-}A)/\mathfrak{Perf}(A)$, where $\mathbf{D}^b(\mathrm{mod}{-}A)$ is the bounded derived category of finitely generated right A-modules and $\mathfrak{Perf}(A)$ is the its triangulated subcategory consisting of objects which are isomorphic to bounded complexes of projectives. The subcategory $\mathfrak{Perf}(A)$ will be called as the subcategories of perfect complexes, but usually we will write $\mathbf{D}^b(\mathrm{proj}{-}A)$ instead $\mathfrak{Perf}(A)$, because this category can also be considered as the derived of exact category of finitely generated right projective A-modules $\mathrm{proj}{-}A$ (see, e.g. [15]).

The investigation of triangulated categories of singularities is not only connected with a study of singularities but is mainly inspired by the Homological Mirror Symmetry Conjecture [16]. More precisely, they are directly related to D-branes of type B (B-branes) in Landau-Ginzburg models. Such models arise as a mirror to Fano varieties [10]. For Fano varieties one has the derived categories of coherent sheaves (B-branes) and given a symplectic form one can propose a suitable Fukaya category (A-branes). The mirror symmetry should interchange these two classes of D-branes. Thus, to extend the Homological Mirror Symmetry Conjecture to the Fano case, one should describe D-branes in Landau-Ginzburg models.

This work is done under partial financial support of the Weyl Fund, the grant RFFI (No 02-01-00468), grant of President of RF in support of young russian scientists MD-2731.2004.1, grant CRDF Award No RM1-2405-MO-02. It is also a pleasure for me to express my gratitude to the Russian Science Support Foundation.

General definition of a Landau-Ginzburg model involves, besides a choice of a target space, a choice of a holomorphic function W on it which is called superpotential. B-branes in Landau-Ginzburg models is defined as W-twisted $\mathbb{Z}_2$-periodic complexes of coherent sheaves, i.e the composition of differentials is no longer zero, but is equal to multiplication by W (see, e.g. [12, 18]). In the paper [18] we established a connection between categories of B-branes in Landau-Ginzburg models and triangulated categories of singularities. We considered singular fibres of the map W and showed that the product of the triangulated categories of singularities of these fibres is equivalent to the categories of B-branes.

In this paper we consider a graded case. Let $A = \bigoplus_i A_i$ be a graded noetherian algebra over a field $\mathbf{k}$. We can define a triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ as the quotient $\mathbf{D}^b(\mathrm{gr}{-}A)/\mathbf{D}^b(\mathrm{grproj}{-}A)$, where $\mathbf{D}^b(\mathrm{gr}{-}A)$ is the bounded derived category of finitely generated graded right A-modules and $\mathbf{D}^b(\mathrm{grproj}{-}A)$ is the its triangulated subcategory consisting of objects which are isomorphic to bounded complexes of projectives.

The graded version of triangulated category of singularities is closely related to B-branes in Landau-Ginzburg models (X, W) with an action of the multiplicative group $\mathbf{k}^*$ such that W is semi-invariant with respect this action. The grading on D-branes of type B were defined in the papers [11, 23]. If there is an action of the group $\mathbf{k}^*$, then one can construct a category of graded B-branes in the Landau-Ginzburg model (Definition 3.1 and subsection 3.3). Theorem 3.10 gives us equivalence between the category of graded B-branes and the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$, where A is the algebra such that $\mathbf{Spec}(A)$ is the fiber of W over 0-point.

This equivalence allows us to compare category of graded B-branes and the derived category of coherent sheaves on the projective variety which is defined by the superpotential W. For example, let X be the affine space $\mathbb{A}^N$ and let W be a homogeneous polynomial of degree d. Denote by $Y \subset \mathbb{P}^{N-1}$ the hypersurface of degree d which is given by the equation $W = 0$. If $d = N$, then the triangulated category of graded B-branes $\mathrm{DGrB}(W)$ is equivalent to the derived category of coherent sheaves on the variety Y. (In this case Y is a Calabi-Yau variety.) Moreover, if $d < N$ (i.e. Y is a Fano variety), then we get a fully faithful functor from $\mathrm{DGrB}(W)$ to $\mathbf{D}^b(\mathrm{coh}(Y))$, and, if $d > N$ (i.e. Y is a variety of general type), then there is a fully faithful functor from $\mathbf{D}^b(\mathrm{coh}(Y))$ to $\mathrm{DGrB}(W)$ (see Theorem 3.11).

This result follows from more general statement for graded Gorenstein algebras (Theorem 2.5). It gives a some relation between the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and the bounded derived category $\mathbf{D}^b(\mathrm{qgr}\, A)$, where $\mathrm{qgr}\, A$ is the quotient of the abelian category of graded finitely generated A-modules by the subcategory of torsion modules. More precisely, for Gorenstein algebras we obtain a fully faithful functor between $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and $\mathbf{D}^b(\mathrm{qgr}\, A)$, and the direction of this functor depends on the Gorenstein parameter a of A. In the case the Gorenstein parameter $a = 0$, there arises an equivalence between these categories. Further, the famous Serre theorem, which says that $\mathbf{D}^b(\mathrm{qgr}\, A)$ is equivalent to $\mathbf{D}^b(\mathrm{coh}(\mathbf{Proj}\,(A)))$ when A is generated by its first component, allows to apply this result to geometry.

I am grateful to Alexei Bondal, Anton Kapustin, Ludmil Katzarkov, Alexander Kuznetsov, Tony Pantev and Johannes Walcher for very useful discussions.

1. Triangulated categories of singularities for graded algebras.

1.1. **Localization in triangulated categories and semiorthogonal decomposition.** We remind that a triangulated category $\mathcal{D}$ is an additive category with the following data:

a) an additive autoequivalence $[1]:\mathcal{D}\longrightarrow\mathcal{D},$ which is called a translation functor,

b) a class of exact (or distinguished) triangles:

$$X\stackrel{u}{\longrightarrow}Y\stackrel{v}{\longrightarrow}Z\stackrel{w}{\longrightarrow}X[1],$$

which must satisfy a certain set of axioms (see [22], also [7, 13, 15, 17]).

A functor $F:\mathcal{D}\longrightarrow\mathcal{D}'$ between two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ is called exact if it commutes with the translation functors and transforms exact triangles into exact triangles.

With any pair $\mathcal{N}\subset\mathcal{D},$ where $\mathcal{N}$ is a full triangulated subcategory, in a triangulated category $\mathcal{D},$ we can associate the quotient category $\mathcal{D}/\mathcal{N}.$ To construct it let us denote by $\Sigma(\mathcal{N})$ a class of morphisms s in $\mathcal{D}$ embedding into an exact triangle

$$X\stackrel{s}{\longrightarrow}Y\longrightarrow N\longrightarrow X[1]$$

with $N\in\mathcal{N}$. It can be checked that $\Sigma(N)$ is a multiplicative system. Define the quotient $\mathcal{D}/\mathcal{N}$ as a localization $\mathcal{D}[\Sigma(\mathcal{N})^{-1}]$ (see [6, 7, 22]). We endow the category $\mathcal{D}/\mathcal{N}$ with a translation functor induced by the translation functor in the category $\mathcal{D}$. The category $\mathcal{D}/\mathcal{N}$ becomes a triangulated category by taking for exact triangles such that are isomorphic to the images of exact triangles in $\mathcal{D}$. The quotient functor $Q:\mathcal{D}\longrightarrow\mathcal{D}/\mathcal{N}$ annihilates $\mathcal{N}$. Moreover, any exact functor $F:\mathcal{D}\longrightarrow\mathcal{D}'$ between triangulated categories for which $F(X)\simeq 0$ when $X\in\mathcal{N}$ factors uniquely through Q. The following lemma is evident.

Lemma 1.1. *Let $\mathcal{N}$ and $\mathcal{N}'$ be full triangulated subcategories of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ respectively. Let $F:\mathcal{D}\to\mathcal{D}'$ and $G:\mathcal{D}'\to\mathcal{D}$ be adjoint pair of exact functors such that $F(\mathcal{N})\subset\mathcal{N}'$ and $G(\mathcal{N}')\subset\mathcal{N}$. Then they induce functors*

$$\overline{F}:\mathcal{D}/\mathcal{N}\longrightarrow\mathcal{D}'/\mathcal{N}',\qquad \overline{G}:\mathcal{D}'/\mathcal{N}'\longrightarrow\mathcal{D}/\mathcal{N}$$

which are adjoint too. Moreover, if the functor $F:\mathcal{D}\to\mathcal{D}'$ is fully faithful then the functor $\overline{F}:\mathcal{D}/\mathcal{N}\longrightarrow\mathcal{D}'/\mathcal{N}'$ is fully faithful too.

Now recall some definitions and facts concerning admissible subcategories and semiorthogonal decompositions (see [3, 4]). Let $\mathcal{N}\subset\mathcal{D}$ be a full triangulated subcategory. The right orthogonal to $\mathcal{N}$ is a full subcategory $\mathcal{N}^{\perp}\subset\mathcal{D}$ consisting of all objects M such that $\mathrm{Hom}(N,M)=0$ for any $N\in\mathcal{N}$. The left orthogonal ${}^{\perp}\mathcal{N}$ is defined analogously. The orthogonals are also triangulated subcategories.

Definition 1.2. *Let* $I : \mathcal{N} \hookrightarrow \mathcal{D}$ *be an embedding of a full triangulated subcategory* $\mathcal{N}$ *in a triangulated category* $\mathcal{D}$*. We say that* $\mathcal{N}$ *is* right admissible *(resp.* left admissible*) if there is a right (resp. left) adjoint functor* $Q : \mathcal{D} \to \mathcal{N}$*. The subcategory* $\mathcal{N}$ *will be called admissible if it is right and left admissible.*

Remark 1.3. The property to be right admissible for the subcategory $\mathcal{N}$ is equivalent to the following: for each $X \in \mathcal{D}$ there is an exact triangle $N \to X \to M$, where $N \in \mathcal{N}$ and $M \in \mathcal{N}^{\perp}$.

Lemma 1.4. *Let* $\mathcal{N}$ *be a full triangulated subcategory in a triangulated category* $\mathcal{D}$*. If* $\mathcal{N}$ *is right (resp. left) admissible then the quotient category* $\mathcal{D}/\mathcal{N}$ *is equivalent to* $\mathcal{N}^{\perp}$ *(resp.* ${}^{\perp}\mathcal{N}$ *). Inverse, if the quotient functor* $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{N}$ *has a left (right) adjoint then* $\mathcal{D}/\mathcal{N}$ *is equivalent to* $\mathcal{N}^{\perp}$ *(resp.* ${}^{\perp}\mathcal{N}$ *).*

If $\mathcal{N} \subset \mathcal{D}$ is a right (resp. left) admissible subcategory then we say that the category $\mathcal{D}$ has weak semiorthogonal decompositions of the form $\langle \mathcal{N}^{\perp}, \mathcal{N} \rangle$ (resp. $\langle \mathcal{N}, {}^{\perp}\mathcal{N} \rangle$).

Definition 1.5. *A sequence of full triangulated subcategories* $(\mathcal{N}_1, \dots, \mathcal{N}_n)$ *in a triangulated category* $\mathcal{D}$ *will be called a* weak semiorthogonal decomposition *of* $\mathcal{D}$ *if there is a sequence of left admissible subcategories* $\mathcal{D}_1 = \mathcal{N}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$ *such that* $\mathcal{N}_p$ *is left orthogonal to* $\mathcal{D}_{p-1}$ *in* $\mathcal{D}_p$ *. We will denote this as* $\mathcal{D} = \langle \mathcal{N}_1, \dots, \mathcal{N}_n \rangle$ *. If all* N_p *are admissible in* $\mathcal{D}$ *then this decomposition is called simply* semiorthogonal*.*

The simplest example of semiorthogonal decomposition is a case when all subcategories $\mathcal{N}_p$ are equivalent to the bounded derived category of finite dimensional vector spaces.

Definition 1.6. *An object* E *of a* $\mathbf{k}$*-linear triangulated category* $\mathcal{T}$ *is called* exceptional *if* $\mathrm{Hom}(E, E[p]) = 0$ *when* $p \neq 0$*, and* $\mathrm{Hom}(E, E) = \mathbf{k}$ *. An* exceptional collection *in* $\mathcal{T}$ *is a sequence of exceptional objects* $(E_0, \dots, E_n)$ *satisfying the semiorthogonal condition* $\mathrm{Hom}(E_i, E_j[p]) = 0$ *for all* p *when* $i > j$*.*

If a triangulated category $\mathcal{D}$ has an exceptional collection $(E_0, \dots, E_n)$ which generates the whole $\mathcal{D}$ then such collection is called full. In this case $\mathcal{D}$ has the semiorhtogonal decomposition with $\mathcal{N}_p = \langle E_p \rangle$ that are equivalent to the bounded derived category of finite dimensional vector spaces. In this case we write $\mathcal{D} = \langle E_0, \dots, E_n \rangle$.

1.2. **Triangulated categories of singularities for algebras.** Let $A = \bigoplus_{i \geq 0} A_i$ be a noetherian graded algebra over a field $\mathbf{k}$. Denote by $\mathrm{mod}{-}A$ and $\mathrm{gr}{-}A$ the category of finitely generated right modules and the category of finitely generated graded right modules respectively. Note that morphisms in $\mathrm{gr}{-}A$ are homomorphisms of degree zero. These categories are abelian. We will also use the notation $\mathrm{Mod}{-}A$ and $\mathrm{Gr}{-}A$ for the abelian categories of all right modules and all graded right modules and we will often omit the prefix "right". Left A-modules are considered as right A°-modules and $A-B$ bimodules as right $A^\circ - B$-modules, where A° is the opposite algebra.

The shift functor (p) on the category $\mathrm{gr}{-}A$ is defined as follows: it takes a graded module $M = \oplus_i M_i$ to the shifted module $M(p)$ such that $M(p)_i = M_{p+i}$ and takes a morphism $f : M \longrightarrow N$ to the same morphism between shifted modules $f(p) : M(p) \longrightarrow N(p)$.

Consider the bounded derived categories $\mathbf{D}^b(\mathrm{gr}{-}A)$ and $\mathbf{D}^b(\mathrm{mod}{-}A)$. They can be endowed with the natural structures of triangulated categories. The categories $\mathbf{D}^b(\mathrm{gr}{-}A)$ and $\mathbf{D}^b(\mathrm{mod}{-}A)$ have the triangulated subcategories consisting of objects which are isomorphic to bounded complexes of projectives. These subcategories can also be considered as the derived of exact categories of projective modules $\mathbf{D}^b(\mathrm{grproj}{-}A)$ and $\mathbf{D}^b(\mathrm{proj}{-}A)$ respectively (see, e.g. [15]). They will be named as the subcategories of perfect complexes. We should note that the category $\mathbf{D}^b(\mathrm{gr}{-}A)$ (resp. $\mathbf{D}^b(\mathrm{mod}{-}A)$) is equivalent to the category $\mathbf{D}^b_{\mathrm{gr}-A}(\mathrm{Gr}{-}A)$ (resp. $\mathbf{D}^b_{\mathrm{mod}-A}(\mathrm{Mod}{-}A)$) of complexes of all modules with finitely generated cohomologies (see [2]) and we use this equivalence as a matter of course.

Definition 1.7. *We define triangulated categories of singularities* $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *and* $\mathbf{D}_{\mathrm{Sg}}(A)$ *as the quotient* $\mathbf{D}^b(\mathrm{gr}{-}A)/\mathbf{D}^b(\mathrm{grproj}{-}A)$ *and* $\mathbf{D}^b(\mathrm{mod}{-}A)/\mathbf{D}^b(\mathrm{proj}{-}A)$ *respectively.*

Remark 1.8. As in commutative case (see [18]), if A has a finite homological dimension then the triangulated categories of singularities are trivial, because in this case any A-module has a bounded projective resolution, i.e. the subcategories of perfect complexes coincide with whole of bounded derived categories of finitely generated modules.

An important example of functors is coming from a homomorphism of (graded) algebras $f : A \to B$. If B has a finite Tor-dimension as A-module then we get the functor $\overset{\mathbf{L}}{\otimes}_A B$ between bounded derived categories of finitely generated modules and it preserves perfect complexes. Therefore, we get functors between triangulated categories of singularities

$$\overset{\mathbf{L}}{\otimes}_A B : \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(B) \quad \text{and} \quad \overset{\mathbf{L}}{\otimes}_A B : \mathbf{D}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}_{\mathrm{Sg}}(B).$$

If, in addition, B is finitely generated as A-module then there is the right adjoint functor which sends a complex of B-modules to itself considered as the complex of A-modules.

More generally, let ${}_A\underline{M}^{\cdot}_B$ be a complex of graded $A-B$ bimodules which as a complex of graded B-modules is quasi-isomorphic to a perfect complex. Suppose that ${}_A\underline{M}^{\cdot}$ has a finite Tor-amplitude as the left A-module. Then we can define the derived tensor product functor $\overset{\mathbf{L}}{\otimes}_A \underline{M}^{\cdot}_B : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}B)$. Moreover, since $\underline{M}^{\cdot}_B$ is perfect over B this functor sends perfect complexes to perfect complexes. Therefore, we get an exact functor

$$\overset{\mathbf{L}}{\otimes}_A \underline{M}^{\cdot}_B : \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(B).$$

In ungraded case we also get the functor $\overset{\mathbf{L}}{\otimes}_A \underline{M}^{\cdot}_B : \mathbf{D}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}_{\mathrm{Sg}}(B)$.

1.3. **Morphisms in categories of singularities.** In general, it is not so easy to calculate spaces of morphisms between objects in a quotient category. The following lemma and proposition gives us a some possibility to do it for triangulated categories of singularities.

Lemma 1.9. *For any object $T \in \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ (resp. $T \in \mathbf{D}_{\mathrm{Sg}}(A)$) and for sufficiently large k there is a module $M \in \mathrm{gr}{-}A$ (resp. $M \in \mathrm{mod}{-}A$) such that T is isomorphic to to the image of $M[k]$ in the triangulated category of singularities. If, in addition, the algebra A has a finite injective dimension, then for any sufficient large k we have $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$.*

Proof. The object T is represented by a bounded complex of modules $\underline{T}^{\cdot}$. Let us take a bounded above projective resolution $\underline{P}^{\cdot} \stackrel{\sim}{\to} \underline{T}^{\cdot}$. Consider a brutal truncation $\sigma^{\geq -k+1}\underline{P}^{\cdot}$ for sufficiently large $k \gg 0$. Denote by M the cohomology $H^{-k+1}(\sigma^{\geq -k+1}\underline{P}^{\cdot})$. It is clear that $T \cong M[k]$ in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ (resp. $\mathbf{D}_{\mathrm{Sg}}(A)$).

If now A has a finite injective dimension then $\mathrm{Hom}(\underline{T}^{\cdot}, A[i])$ in $\mathbf{D}^b(\mathrm{gr}{-}A)$ (resp. $\mathbf{D}^b(\mathrm{mod}{-}A)$) are trivial except for finite number of $i \in \mathbb{Z}$. This implies that for any sufficiently large k there are equalities $\mathrm{Ext}^i_A(M, A) = 0$ when $i > 0$. □

Proposition 1.10. *Let M be an A-module such that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. Then for any A-module N*

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{Sg}}(A)}(M, N) \cong \mathrm{Hom}_A(M, N)/\mathcal{R}$$

where $\mathcal{R}$ is the subspace of elements factoring through a projective, i.e. $e \in \mathcal{R}$ iff $e = \beta\alpha$ with $\alpha : M \to P$ and $\beta : P \to N$, where P is projective. If M is a graded module, then for any graded A-module N

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(M, N) \cong \mathrm{Hom}_{\mathrm{gr}-A}(M, N)/\mathcal{R}.$$

Proof. We consider the graded case. By the definition of localization any morphism from M to N in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ can be represented by a pair of morphisms in $\mathbf{D}^b(\mathrm{gr}{-}A)$ of the form

$$M \stackrel{a}{\longrightarrow} \underline{T}^{\cdot} \stackrel{s}{\longleftarrow} N \tag{1}$$

such that the cone $\underline{C}^{\cdot}(s)$ is a perfect complex. Consider a bounded above projective resolution $\underline{Q}^{\cdot} \to N$ and its brutal truncation $\sigma^{\geq -k}\underline{Q}^{\cdot}$ for sufficiently large k. There is an exact triangle

$$E[k] \longrightarrow \sigma^{\geq -k}\underline{Q}^{\cdot} \longrightarrow N \longrightarrow E[k+1],$$

where E is a notation for the module $H^{-k}(\sigma^{\geq -k}\underline{Q}^{\cdot})$. We can take k sufficiently large such that $\mathrm{Hom}(\underline{C}^{\cdot}(s), E[i]) = 0$ for all $i > k$. Using the triangle

$$\underline{C}^{\cdot}(s)[-1] \longrightarrow N \stackrel{s}{\longrightarrow} \underline{T}^{\cdot} \longrightarrow \underline{C}^{\cdot}(s),$$

we find that the map $s' : N \to E[k+1]$ can be lifted to a map $\underline{T}^{\cdot} \to E[k+1]$. The map $\underline{T}^{\cdot} \to E[k+1]$ induces a pair of the form

$$M \stackrel{a'}{\longrightarrow} E[k+1] \stackrel{s'}{\longleftarrow} N, \tag{2}$$

and this pair gives the same morphism in $\mathbf{D}_{\mathrm{Sg}}(A)$ as the pair (1). Since $\mathrm{Ext}^i(M, P) = 0$ for $i > 0$ when P is a projective module, we obtain

$$\mathrm{Hom}(M,\ (\sigma^{\geq -k}\underline{Q}^{\cdot})[1]) = 0.$$

Hence, the map $a' : M \to E[k+1]$ can be lift to a map f which completes the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ & {\scriptstyle a'}\searrow \quad \swarrow{\scriptstyle s'} & \\ & E[k+1] & \end{array}$$

Thus, the map f is equivalent to the map (2) and, as consequence, to the map (1). Hence, any morphism from M to N in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is represented by a morphism from M to N in the category $\mathbf{D}^b(\mathrm{gr}{-}A)$.

Now if f is the 0-morphism in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ then we can suppose that the map a is zero map, i.e. $a' = 0$ too. This implies that f factors through a morphism $M \to \sigma^{\geq -k}\underline{Q}^{\cdot}$. By the assumption on M any such morphism can be lifted to a morphism $M \to Q^0$. Hence, if f is the 0-morphism in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ then it factors through Q^0. The same proof works in ungraded case (see [18]). □

Now we describe a construction. Let $\underline{M}^{\cdot}$ and $\underline{N}^{\cdot}$ be two bounded complexes of (graded) A-modules. Assume that $\mathrm{Hom}(\underline{M}^{\cdot}, A[i])$ in the bounded derived categories of A-modules are trivial except for finite number of $i \in \mathbb{Z}$. By Lemma 1.9 for sufficiently large k there are modules $M, N \in \mathrm{gr}{-}A$ (resp. $M, N \in \mathrm{mod}{-}A$) such that $\underline{M}^{\cdot}$ and $\underline{N}^{\cdot}$ are isomorphic to to the images of $M[k]$ and $N[k]$ respectively in the triangulated category of singularities. Moreover, it follows immediately from the assumption on $\underline{M}^{\cdot}$ and the construction of M that for any sufficiently large k there are equalities $\mathrm{Ext}^i_A(M, A) = 0$ when $i > 0$. Hence, by Proposition 1.10 we get

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(\underline{M}^{\cdot}, \underline{N}^{\cdot}) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(M, N) \cong \mathrm{Hom}_A(M, N)/\mathcal{R}$$

where $\mathcal{R}$ is the subspace of elements factoring through a projective module. This procedure work in ungraded situation by the same way.

2. Categories of coherent sheaves and categories of singularities.

2.1. **Quotient categories of graded modules.** Let $A = \bigoplus_{i\geq 0} A_i$ as above be a noetherian graded algebra. We suppose that A is connected, i.e. $A_0 = \mathbf{k}$. Denote by $\mathrm{tors}{-}A$ the full subcategory of $\mathrm{gr}{-}A$ which consists of all finite dimensional graded A–modules.

An important role will be played by the quotient abelian category $\mathrm{qgr}\, A = \mathrm{gr}{-}A/\mathrm{tors}{-}A$. It has the following explicit description. The objects of $\mathrm{qgr}\, A$ are the objects of $\mathrm{gr}{-}A$ (we denote by πM the object in $\mathrm{qgr}\, A$ which corresponds to a module M). The morphisms in $\mathrm{qgr}\, A$ are given by

$$\mathrm{Hom}_{\mathrm{qgr}}(\pi M, \pi N) = \varinjlim_{M'} \mathrm{Hom}_{\mathrm{gr}}(M', N) \tag{3}$$

where M' runs over submodules of M such that M/M' is finite dimensional.

Given a graded A-module M and an integer p, the graded A-submodule $\bigoplus_{i\ge p} M_i$ of M is denoted by $M_{\ge p}$ and it is called a tail of M. Since A is noetherian, we have

$$\mathrm{Hom}_{\mathrm{qgr}}(\pi M, \pi N) = \lim_{p\to\infty} \mathrm{Hom}_{\mathrm{gr}}(M_{\ge p}, N).$$

We will also identify M_p with the quotient $M_{\ge p}/M_{\ge p+1}$. By the same way we can define a tail $\underline{M}^{\cdot}_{\ge p}$ for any complex of modules $\underline{M}^{\cdot}$.

Similarly, we can consider the subcategory $\mathrm{Tors}{-}A \subset \mathrm{Gr}{-}A$ of torsion modules. Recall that a module M is called torsion if for any element $x \in M$ one has $xA_{\ge p} = 0$ for some p. Denote by $\mathrm{QGr}\, A$ the quotient category $\mathrm{Gr}{-}A/\mathrm{Tors}{-}A$. The category $\mathrm{QGr}\, A$ contains $\mathrm{qgr}\, A$ as a full subcategory. Sometimes it is convenient to work in $\mathrm{QGr}\, A$ instead of $\mathrm{qgr}\, A$.

Denote by Π and π the canonical projections of $\mathrm{Gr}{-}A$ to $\mathrm{QGr}\, A$ and of $\mathrm{gr}{-}A$ to $\mathrm{qgr}\, A$ respectively. The functor Π has a right adjoint Ω and, moreover, for any $N \in \mathrm{Gr}{-}A$

$$\Omega\Pi N \cong \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{QGr}}(\Pi A, \Pi N(n)). \tag{4}$$

For any $i \in \mathbb{Z}$ we can consider full abelian subcategories $\mathrm{Gr}{-}A_{\ge i}$ and $\mathrm{gr}{-}A_{\ge i}$ which consist of all modules M such that $M_p = 0$ when $p < i$. The natural projection functor $\Pi_i : \mathrm{Gr}{-}A_{\ge i} \longrightarrow \mathrm{QGr}-A$ has a right adjoint Ω_i which has the form

$$\Omega_i\Pi_i N \cong \bigoplus_{n=i}^{\infty} \mathrm{Hom}_{\mathrm{QGr}}(\Pi A, \Pi_i N(n)).$$

Since the category $\mathrm{QGr}\, A$ is an Ab5-category with enough injectives there is the right derived functor

$$\mathbf{R}\Omega_i : \mathbf{D}^{+}(\mathrm{QGr}\, A) \longrightarrow \mathbf{D}^{+}(\mathrm{Gr}{-}A_{\ge i}).$$

Assume now that the algebra A satisfies condition "χ" [1]. We recall that a connected noetherian graded algebra A is said to satisfy condition "χ" if for every $M \in \mathrm{gr}{-}A$ the space $\mathrm{Ext}^i_A(\mathbf{k}, M)$ has right limited grading for all i. In this case it was proved in [1] (Prop. 3.14) that the restriction of the functors Ω_i on subcategory $\mathrm{qgr}\, A$ gives a functor $\omega_i : \mathrm{qgr}\, A \longrightarrow \mathrm{gr}{-}A_{\ge i}$ which is right adjoint to π_i. Moreover, it follows from [1] (Th. 7.4) that the functor ω_i has the right derived

$$\mathbf{R}\omega_i : \mathbf{D}^{+}(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^{+}(\mathrm{gr}{-}A_{\ge i})$$

and all $\mathbf{R}^j\omega_i \in \mathrm{tors}{-}A$ for $j > 0$.

If, in addition, the algebra A has a finite injective dimension as module over itself we obtain the right derived functor

$$\mathbf{R}\omega_i : \mathbf{D}^{b}(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^{b}(\mathrm{gr}{-}A_{\ge i})$$

between bounded derived categories (see [1], Prop. 7.10(4)). It is important to note that the functor $\mathbf{R}\omega_i$ is fully faithful because $\pi_i\mathbf{R}\omega_i$ is isomorphic to the identity functor ([1], Prop. 7.2).

2.2. **Triangulated categories of singularities for Gorenstein algebras.** The main aim of this section is to establish a connection between the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and the derived category $\mathbf{D}^b(\mathrm{qgr}\, A)$ in the case of algebra A to be Gorenstein.

When the algebra A has finite injective dimension as right and as left module over itself (i.e. A is a dualizing complex for itself) we get two functors

$$D \ := \mathbf{R}\operatorname{Hom}_A\ (-, A) : \mathbf{D}^b(\mathrm{gr}{-}A\ )^\circ \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}A^\circ),$$
$$D^\circ := \mathbf{R}\operatorname{Hom}_{A^\circ}(-, A) : \mathbf{D}^b(\mathrm{gr}{-}A^\circ)^\circ \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}A\ ),$$

which are quasi-inverse triangulated equivalences (see [24]).

Definition 2.1. *We say that the algebra A is Gorenstein if it has a finite injective dimension n and $D(\mathbf{k}) = \mathbf{R}\operatorname{Hom}_A(\mathbf{k}, A)$ is isomorphic to $\mathbf{k}(a)[-n]$ for some interger a which is called the Gorenstein parameter of A. (Such algebra is also called AS-Gorenstein, where "AS" stands for "Artin-Schelter".)*

Remark 2.2. It is known that any Gorenstein algebra satisfies condition "χ" (see [25]). Hence, for any Gorenstein algebra A and for any $i \in \mathbb{Z}$ we have derived functors

$$\mathbf{R}\omega_i : \mathbf{D}^b(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$$

which are fully faithful.

Now we describe the images of the functors $\mathbf{R}\omega_i$. Denote by $\mathcal{D}_i$ the subcategories of $\mathbf{D}^b(\mathrm{gr}{-}A)$ which are the images of the composition of $\mathbf{R}\omega_i$ and the natural inclusion of $\mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$ to $\mathbf{D}^b(\mathrm{gr}{-}A)$. All $\mathcal{D}_i$ are equivalent to $\mathbf{D}^b(\mathrm{qgr}\,{-}A)$. Further, for any integer i denote by $\mathcal{S}_{<i}(A)$ (or simple $\mathcal{S}_{<i}$) the full triangulated subcategory of $\mathbf{D}^b(\mathrm{gr}{-}A)$ generated by the modules $\mathbf{k}(e)$ with $e > -i$. In other words, objects of $\mathcal{S}_{<i}$ are complexes $\underline{M}^{\cdot}$ for which the tail $\underline{M}^{\cdot}_{\geq i}$ is isomorphic to zero. Analogously, we define $\mathcal{S}_{\geq i}$ as a triangulated subcategory which is generated by the objects $\mathbf{k}(e)$ with $e \leq -i$. In other words, objects of $\mathcal{S}_{\geq i}$ are complexes of torsion modules from $\mathrm{gr}{-}A_{\geq i}$. We can see that $\mathcal{S}_{<i} \cong \mathcal{S}_{<0}(-i)$ and $\mathcal{S}_{\geq i} \cong \mathcal{S}_{\geq 0}(-i)$.

Furthermore, denote by $\mathcal{P}_{<i}$ the full triangulated subcategory of $\mathbf{D}^b(\mathrm{gr}{-}A)$ generated by the free modules $A(e)$ with $e > -i$ and denote by $\mathcal{P}_{\geq i}$ the triangulated subcategory which is generated by the free modules $A(e)$ with $e \leq -i$. We can see that $\mathcal{P}_{<i} \cong \mathcal{P}_{<0}(-i)$ and $\mathcal{P}_{\geq i} \cong \mathcal{P}_{\geq 0}(-i)$.

Lemma 2.3. *Let $A = \bigoplus_{i\geq 0} A_i$ be a noetherian connected graded algebra. Then the subcategories $\mathcal{S}_{<i}$ and $\mathcal{P}_{<i}$ are left and respectively right admissible for any $i \in \mathbb{Z}$. Moreover, there are weak semiorthogonal decompositions*

$$\mathbf{D}^b(\mathrm{gr}{-}A) = \langle \mathcal{S}_{<i}, \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})\rangle, \qquad \mathbf{D}^b(\mathrm{tors}{-}A) = \langle \mathcal{S}_{<i}, \mathcal{S}_{\geq i}\rangle, \tag{5}$$

$$\mathbf{D}^b(\mathrm{gr}{-}A) = \langle \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i}), \mathcal{P}_{<i}\rangle, \qquad \mathbf{D}^b(\mathrm{grproj}{-}A) = \langle \mathcal{P}_{\geq i}, \mathcal{P}_{<i}\rangle \tag{6}$$

Proof. For any complex $\underline{M}^{\cdot} \in \mathbf{D}^b(\text{mod}-A)$ there is an exact triangle of the form

$$\underline{M}^{\cdot}_{\geq i} \longrightarrow \underline{M}^{\cdot} \longrightarrow \underline{M}^{\cdot}/\underline{M}^{\cdot}_{\geq i}.$$

We see that the object $\underline{M}^{\cdot}/\underline{M}^{\cdot}_{\geq i}$ belongs to $\mathcal{S}_{<i}$ and, in addition, the object $\underline{M}^{\cdot}_{\geq i}$ is in the left orthogonal ${}^{\perp}\mathcal{S}_{<i}$. Hence, by Remark 1.3, $\mathcal{S}_{<i}$ is left admissible. Moreover, $\underline{M}^{\cdot}_{\geq i}$ also belongs to $\mathbf{D}^b(\text{gr}-A_{\geq i})$, i.e. $\mathbf{D}^b(\text{gr}-A_{\geq i}) \cong {}^{\perp}\mathcal{S}_{<i}$ in the category $\mathbf{D}^b(\text{gr}-A)$. If $\underline{M}^{\cdot}$ is a complex of torsion modules then $\underline{M}^{\cdot}_{\geq i}$ belongs to $\mathcal{S}_{\geq i}$. Thus, we obtain both decompositions of (5).

To prove the existence of decompositions (6), first, note that any finitely generated graded projective A-module is free, because A is connected. Second, any finitely generated free module P has the canonical split decomposition of the form

$$0 \longrightarrow P_{<i} \longrightarrow P \longrightarrow P_{\geq i} \longrightarrow 0,$$

where $P_{<i} \in \mathcal{P}_{<i}$ and $P_{\geq i} \in \mathcal{P}_{\geq i}$. Third, any bounded complex of finitely generated A-modules $\underline{M}^{\cdot}$ has a bounded above free resolution $\underline{P}^{\cdot} \to \underline{M}^{\cdot}$ such that $P^{-k} \in \mathcal{P}_{\geq i}$ for all $k \gg 0$. This implies that the object $\underline{P}^{\cdot}_{<i} \in \mathcal{P}_{<i}$ from the exact sequence of complexes

$$0 \longrightarrow \underline{P}^{\cdot}_{<i} \longrightarrow \underline{P}^{\cdot} \longrightarrow \underline{P}^{\cdot}_{\geq i} \longrightarrow 0,$$

is a bounded complex. Further, the complex $\underline{P}^{\cdot}_{\geq i}$ as the complex $\underline{P}^{\cdot}$ is quasi-isomorphic to a bounded complex $K^{\cdot}$ from $\mathbf{D}^b(\text{gr}-A_{\geq i})$. Thus, any object $\underline{M}^{\cdot}$ has a decomposition

$$\underline{P}^{\cdot}_{<i} \longrightarrow \underline{M}^{\cdot} \longrightarrow \underline{K}^{\cdot},$$

where $\underline{P}^{\cdot}_{<i} \in \mathcal{P}_{<i}$ and $\underline{K}^{\cdot} \in \mathbf{D}^b(\text{gr}-A_{\geq i})$. This proves decompositions (6). □

Lemma 2.4. *Let* $A = \bigoplus_{i\geq 0} A_i$ *be a noetherian connected graded algebra which is Gorenstein. Then the subcategories* $\mathcal{S}_{\geq i}$ *and* $\mathcal{P}_{\geq i}$ *are right and respectively left admissible. Moreover, for any* $i \in \mathbb{Z}$ *there are weak semiorthogonal decompositions*

$$\mathbf{D}^b(\text{gr}-A_{\geq i}) = \langle \mathcal{D}_i, \mathcal{S}_{\geq i}\rangle, \quad \mathbf{D}^b(\text{gr}-A_{\geq i}) = \langle \mathcal{P}_{\geq i}, \mathcal{T}_i\rangle, \tag{7}$$

where the subcategory $\mathcal{D}_i$ *is equivalent to* $\mathbf{D}^b(\text{qgr}\, A)$ *under the functor* $\mathbf{R}\omega_i$, *and* $\mathcal{T}_i$ *is equivalent to* $\mathbf{D}^{\text{gr}}_{\text{Sg}}(A)$.

Proof. The functor $\mathbf{R}\omega_i$ is fully faithful and has the left adjoint π_i. Thus, we obtain a semiorthogonal decomposition

$$\mathbf{D}^b(\text{gr}-A_{\geq i}) = \langle \mathcal{D}_i, {}^{\perp}\mathcal{D}_i\rangle,$$

where $\mathcal{D}_i \cong \mathbf{D}^b(\text{qgr}\, A)$. Furthermore, the orthogonal ${}^{\perp}\mathcal{D}_i$ consists of all objects $\underline{M}^{\cdot}$ such that $\pi_i(\underline{M}^{\cdot}) = 0$. Thus, ${}^{\perp}\mathcal{D}_i$ coincides with $\mathcal{S}_{\geq i}$. Hence, $\mathcal{S}_{\geq i}$ is right admissible in $\mathbf{D}^b(\text{gr}-A_{\geq i})$ which is right admissible in whole $\mathbf{D}^b(\text{gr}-A)$. This implies that $\mathcal{S}_{\geq i}$ is right admissible in $\mathbf{D}^b(\text{gr}-A)$ too.

The equivalence D establish an equivalence of the subcategory $\mathcal{P}_{\geq i}(A)^{\circ}$ with the subcategory $\mathcal{P}_{<-i+1}(A^{\circ})$ which is right admissible by Lemma 2.3. Hence, $\mathcal{P}_{\geq i}(A)$ is left admissible and there is a decomposition of the form

$$\mathbf{D}^b(\mathrm{gr}{-}A_{\geq i}) = \langle \mathcal{P}_{\geq i}, \mathcal{T}_i \rangle$$

with some $\mathcal{T}_i$.

Now applying Lemma 1.1 to the full embedding of $\mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$ to $\mathbf{D}^b(\mathrm{gr}{-}A)$ and using Lemma 1.4 we get the fully faithful functor from $\mathcal{T}_i \cong \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})/\mathcal{P}_{\geq i}$ to $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) = \mathbf{D}^b(\mathrm{gr}{-}A)/\mathbf{D}^b(\mathrm{grproj}{-}A)$. Finally, since this functor is essentially surjective on the objects it is actually an equivalence. □

Theorem 2.5. *Let* $A = \bigoplus_{i\geq 0} A_i$ *be a noetherian connected graded algebra which is Gorenstein with the parameter* a. *Then, there is the following relation between the triangulated categories* $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *and* $\mathbf{D}^b(\mathrm{qgr}\, A)$ *:*

(i) *if* $a > 0$, *there are fully faithful functors* $\Phi_i : \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \longrightarrow \mathbf{D}^b(\mathrm{qgr}\, A)$ *and semiorthogonal decompositions*

$$\mathbf{D}^b(\mathrm{qgr}\, A) = \langle \pi A(-i-a+1), \dots, \pi A(-i), \Phi_i \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \rangle,$$

where $\pi : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^b(\mathrm{qgr}\, A)$ *is the natural projection;*

(ii) *if* $a < 0$, *there are fully faithful functors* $\Psi_i : \mathbf{D}^b(\mathrm{qgr}\, A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *and semiorthogonal decompositions*

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) = \langle q\mathbf{k}(-i), \dots, q\mathbf{k}(-i+a+1), \Psi_i \mathbf{D}^b(\mathrm{qgr}\, A) \rangle,$$

where $q : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *is the natural projection;*

(iii) *if* $a = 0$, *there is an equivalence* $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{qgr}\, A)$.

Proof. Lemmas 2.3 and 2.4 gives us that the subcategory $\mathcal{T}_i$ is admissible in $\mathbf{D}^b(\mathrm{gr}{-}A)$ and the right orthogonal $\mathcal{T}_i^{\perp}$ has the weak semiorthogonal decomposition of the form

$$\mathcal{T}_i^{\perp} = \langle \mathcal{S}_{<i}, \mathcal{P}_{\geq i} \rangle. \tag{8}$$

Now let us describe the right orthogonal to the subcategory $\mathcal{D}_i$. First, since A is Gorenstein the functor D takes the subcategory $\mathcal{S}_{\geq i}(A)$ to the subcategory $\mathcal{S}_{<-i-a+1}(A^{\circ})$. Hence, D sends the right orthogonal $\mathcal{S}_{\geq i}^{\perp}(A)$ to the left orthogonal ${}^{\perp}\mathcal{S}_{<-i-a+1}(A^{\circ})$ which coincides with the right orthogonal $\mathcal{P}^{\perp}_{<-i-a+1}(A^{\circ})$ by Lemma 2.3. Therefore, the subcategory $\mathcal{S}^{\perp}_{\geq i}$ coincides with ${}^{\perp}\mathcal{P}_{\geq i+a}$. On the other hand, by Lemmas 2.3 and 2.4 we have that

$${}^{\perp}\mathcal{P}_{\geq i+a} = \mathcal{S}^{\perp}_{\geq i} \cong \langle \mathcal{S}_{<i}, \mathcal{D}_i \rangle.$$

This implies that the right orthogonal $\mathcal{D}_i^{\perp}$ has the following decomposition

$$\mathcal{D}_i^{\perp} = \langle \mathcal{P}_{\geq i+a}, \mathcal{S}_{<i} \rangle. \tag{9}$$

Assume that $a \geq 0$. In this case, decomposition (9) is not only semiorthogonal but it is mutually orthogonal, because $\mathcal{P}_{\geq i+a} \subset \mathbf{D}^b(\mathrm{gr}{-}A_{\geq i})$. Hence, we can interchange $\mathcal{P}_{\geq i+a}$ and $\mathcal{S}_{<i}$, i.e.

$$\mathcal{D}_i^{\perp} = \langle \mathcal{S}_{<i}, \mathcal{P}_{\geq i+a} \rangle.$$

Thus, we obtain that $\mathcal{D}_i^{\perp} \subset \mathcal{T}_i^{\perp}$ and, consequently, $\mathcal{T}_i$ is the full subcategory of $\mathcal{D}_i$. Moreover, we can describe the right orthogonal to $\mathcal{T}_i$ in $\mathcal{D}_i$. Actually, there is a decomposition

$$\mathcal{P}_{\geq i} = \langle \mathcal{P}_{\geq i+a}, \mathcal{P}_i^a \rangle,$$

where $\mathcal{P}_i^a$ is the subcategory generated by the modules $A(-i-a+1), \dots, A(-i)$. Moreover, these modules forms an exceptional collection. Thus, the category $\mathcal{D}_i$ has the following semiorthogonal decomposition

$$\mathcal{D}_i = \langle A(-i-a+1), \dots, A(-i), \mathcal{T}_i \rangle,$$

Since $\mathcal{D}_i \cong \mathbf{D}^b(\mathrm{qgr}\, A)$ and $\mathcal{T}_i \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ we obtain the decomposition

$$\mathbf{D}^b(\mathrm{qgr}\, A) \cong \langle \pi A(-i-a+1), \dots, \pi A(-i), \Phi_i \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \rangle,$$

where the fully faithful functor Φ_i is the composition $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \xrightarrow{\sim} \mathcal{T}_i \hookrightarrow \mathbf{D}^b(\mathrm{gr}{-}A) \xrightarrow{\pi} \mathbf{D}^b(\mathrm{qgr}\, A)$.

Assume now that $a \leq 0$. In this case, the decomposition (8) is not only semiorthogonal but it is mutually orthogonal, because the algebra A is Gorenstein and $\mathbf{R}\mathrm{Hom}_A(\mathbf{k}, A) = \mathbf{k}(a)[-n]$ with $a \leq 0$. Hence, we can interchange $\mathcal{P}_{\geq i}$ and $\mathcal{S}_{<i}$, i.e.

$$\mathcal{T}_i^{\perp} = \langle \mathcal{P}_{\geq i}, \mathcal{S}_{<i} \rangle.$$

Now we see that $\mathcal{T}_i^{\perp} \subset \mathcal{D}_{i-a}^{\perp}$ and, consequently, $\mathcal{D}_{i-a}$ is the full subcategory of $\mathcal{T}_i$. Moreover, we can describe the right orthogonal to $\mathcal{D}_{i-a}$ in $\mathcal{T}_i$. Actually, there is a decomposition of the form

$$\mathcal{S}_{<i-a} = \langle \mathcal{S}_{<i}, \mathbf{k}(-i), \dots, \mathbf{k}(-i+a+1) \rangle.$$

Therefore, the category $\mathcal{T}_i \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has a semiorthogonal decomposition of the form

$$\mathcal{T}_i = \langle \mathbf{k}(-i), \dots, \mathbf{k}(-i+a+1), \mathcal{D}_{i-a} \rangle,$$

Since $\mathcal{D}_{i-a} \cong \mathbf{D}^b(\mathrm{qgr}\, A)$ and $\mathcal{T}_i \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ we obtain the decomposition

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \cong \langle q\mathbf{k}(-i), \dots, q\mathbf{k}(-i+a+1), \Psi_i \mathbf{D}^b(\mathrm{qgr}\, A) \rangle,$$

where the fully faithful functor Ψ_i can be defined as the composition $\mathbf{D}^b(\mathrm{qgr}\, A) \xrightarrow{\sim} \mathcal{D}_{i-a} \hookrightarrow \mathbf{D}^b(\mathrm{gr}{-}A) \xrightarrow{q} \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. If $a = 0$, then we get equivalence. □

2.3. **Categories of coherent sheaves for Gorenstein varieties.** Let X be a projective Gorenstein variety of dimension n and let $\mathcal{L}$ be a very ample line bundle such that the dualizing sheaf ω_X is isomorphic to $\mathcal{L}^{-r}$ for some $r \in \mathbb{Z}$. Denote by A the graded coordinate algebra $\bigoplus_{i\geq 0} H^0(X, \mathcal{L}^i)$. The famous Serre theorem [20] asserts that the abelian category of coherent sheaves $\mathrm{coh}(X)$ is equivalent to the quotient category $\mathrm{qgr}\, A$.

Assume also that $H^j(X, \mathcal{L}^k) = 0$ for all $k \in \mathbb{Z}$ when $j \neq 0, n$. For example, if X is a complete intersection in $\mathbb{P}^N$ then it satisfies these conditions. In this case, Theorem 2.5 allows us to compare the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ with the bounded derived category of coherent sheaves $\mathbf{D}^b(\mathrm{coh}(X))$. To apply that theorem we need the following lemma.

Lemma 2.6. *Let X be a projective Gorenstein variety of dimension n and let $\mathcal{L}$ be a very ample line bundle such that $\omega_X \cong \mathcal{L}^{-r}$ for some $r \in \mathbb{Z}$ and $H^j(X, \mathcal{L}^k) = 0$ for all $k \in \mathbb{Z}$ when $j \neq 0, n$. Then the graded algebra $A = \bigoplus_{i\geq 0} H^0(X, \mathcal{L}^i)$ is Gorenstein with the parameter $a = r$.*

Proof. Consider the projection functor $\Pi : \mathrm{Gr}{-}A \to \mathrm{QGr}\, A$ and its right adjoint $\Omega : \mathrm{QGr}\, A \to \mathrm{Gr}{-}A$ which is given by the formula (4)

$$\Omega\Pi N \cong \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{QGr}}(\Pi A, \Pi N(n)).$$

The functor Ω has the derived functor $\mathbf{R}\Omega$ and it is given by the formula

$$\mathbf{R}^j\Omega(\Pi N) \cong \bigoplus_{n=-\infty}^{\infty} \mathrm{Ext}^j_{\mathrm{QGr}}(\Pi A, \Pi N(n))$$

(see, e.g. [1], Prop. 7.2.).

The assumptions on X and $\mathcal{L}$ imply that $\mathbf{R}^j\Omega(\Pi A) \cong 0$ for all $j \neq 0, n$. Moreover, since X is Gorenstein and $\omega_X \cong \mathcal{L}^{-r}$ the Serre duality for X yields that

$$\mathbf{R}^0\Omega(\Pi A) \cong \bigoplus_{i=-\infty}^{\infty} H^0(X, \mathcal{L}^i) \cong A \qquad \text{and} \qquad \mathbf{R}^n\Omega(\Pi A) \cong \bigoplus_{i=-\infty}^{\infty} H^n(X, \mathcal{L}^i) \cong A^*(r),$$

where $A^* = \mathrm{Hom}_{\mathbf{k}}(A, \mathbf{k})$. By adjunction of functors Π and $\mathbf{R}\Omega$ we have

$$\mathbf{R}\,\mathrm{Hom}_{\mathrm{Gr}}(\mathbf{k}(s), \mathbf{R}\Omega(\Pi A)) \cong \mathbf{R}\,\mathrm{Hom}_{\mathrm{QGr}}(\Pi\mathbf{k}(s), \Pi A) = 0$$

for all s. Further, we know that $\mathbf{R}\,\mathrm{Hom}_A(\mathbf{k}, A^*) \cong \mathbf{R}\,\mathrm{Hom}_A(A, \mathbf{k}) \cong \mathbf{k}$. This implies that $\mathbf{R}\,\mathrm{Hom}_A(\mathbf{k}, A) \cong \mathbf{k}(r)[-n-1]$. This isomorphism gives us that the affine cone $\mathbf{Spec}A$ is Gorenstein in the vertex and assumption on X implies that $\mathbf{Spec}A$ is Gorenstein scheme ([9], §9). Since $\mathbf{Spec}A$ has a finite Krull dimension, the algebra A is a dualizing complex for itself, i.e. it has a finite injective dimension. Thus, the algebra A is Gorenstein with the parameter r. □

Theorem 2.7. *Let X be a projective Gorenstein variety of dimension n and let $\mathcal{L}$ be a very ample line bundle such that $\omega_X \cong \mathcal{L}^{-r}$ for some $r \in \mathbb{Z}$. Suppose that $H^j(X, \mathcal{L}^k) = 0$ for all $k \in \mathbb{Z}$ when $j \neq 0, n$. Then, there is the following relation between derived category of coherent sheaves $\mathbf{D}^b(\mathrm{coh}(X))$ and triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$, where $A = \bigoplus_{i\geq 0} H^0(X, \mathcal{L}^i)$:*

(i) *if* $r > 0$ *(i.e.* X *is a Fano variety), there is a semiorthogonal decomposition*

$$\mathbf{D}^b(\mathrm{coh}(X)) = \langle \mathcal{L}^{-r+1}, \dots, \mathcal{O}_X, \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)\rangle,$$

(ii) *if* $r < 0$ *(i.e.* X *is a variety of general type), there is a semiorthogonal decomposition*

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) = \langle q\mathbf{k}(r+1), \dots, q\mathbf{k}, \mathbf{D}^b(\mathrm{coh}(X))\rangle,$$

where $q : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *is the natural projection,*

(iii) *if* $r = 0$ *(i.e.* X *is a Calabi-Yau variety), there is an equivalence*

$$\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{coh}(X)).$$

Proof. By the Serre theorem, since $\mathcal{L}$ is very ample the bounded derived category $\mathbf{D}^b(\mathrm{coh}(X))$ is equivalent to the category $\mathbf{D}^b(\mathrm{qgr}\, A)$, where $A = \bigoplus_{i \ge 0} H^0(X, \mathcal{L}^i)$. Since $H^j(X, \mathcal{L}^k) = 0$ for $j \ne 0, n$ and all $k \in \mathbb{Z}$, Lemma 2.6 implies that A is Gorenstein. Now, the theorem immediately follows from Theorem 2.5. □

Corollary 2.8. *Let* X *be a projective Gorenstein Fano variety of dimension* n *with rational singularities. Let* $\mathcal{L}$ *be a very ample line bundle such that* $\omega_X^{-1} \cong \mathcal{L}^r$ *for some* $r \in \mathbb{N}$. *Then, there is a semiorthogonal decomposition of the form*

$$\mathbf{D}^b(\mathrm{coh}(X)) = \langle \mathcal{L}^{-r+1}, \dots, \mathcal{O}_X, \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)\rangle,$$

where $A = \bigoplus_{i \ge 0} H^0(X, \mathcal{L}^i)$.

Proof. The Kawamata-Viehweg vanishing theorem ([14], Th.1.2.5) gives us that $H^j(X, \mathcal{L}^k) = 0$ for $j \ne 0, n$ and all k. Hence, we can apply Theorem 2.7(i). □

Corollary 2.9. *Let* X *be a projective variety with rational singularities and with trivial canonical sheaf* $\omega_X \cong \mathcal{O}_X$ *such that* $H^j(X, \mathcal{O}_X) = 0$ *for* $j \ne 0, n$ *(i.e.* X *is a Calabi-Yau variety). Let* $\mathcal{L}$ *be a some very ample line bundle on* X. *Then there is an equivalence*

$$\mathbf{D}^b(\mathrm{coh}(X)) \cong \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$$

where $A = \bigoplus_{i \ge 0} H^0(X, \mathcal{L}^i)$.

Proof. The variety X has rational singularities hence it is Cohen-Macaulay. Moreover, X is Gorenstein, because $\omega_X \cong \mathcal{O}_X$. The Kawamata-Viehweg vanishing theorem ([14], Th.1.2.5) gives us that $H^j(X, \mathcal{L}^k) = 0$ for $j \ne 0, n$ and all $k \ne 0$. Further, by assumption $H^j(X, \mathcal{O}_X) = 0$ for $j \ne 0, n$. Therefore, we can apply Theorem 2.7 (iii). □

Proposition 2.10. *Let* $X \subset \mathbb{P}^N$ *be a complete intersection of* m *hypersurfaces* $D_1, \dots, D_m$. *Then* X *and* $\mathcal{L} = \mathcal{O}_X(1)$ *satisfy the conditions of Theorem 2.7 with Gorenstein parameter* $r = N + 1 - \sum_{i=1}^m d_i$, *where* d_i *are degrees of* D_i.

Proof. Since the variety X is a complete intersection it is Gorenstein. The canonical class ω_X is isomorphic to $\mathcal{O}(\sum d_i - N - 1)$. It can be easily proved by induction that $H^j(X, \mathcal{O}_X(k)) = 0$ for all k and $j \neq 0, n$, where $n = N - m$ is the dimension of X. Actually, assume that for $Y = D_1 \cap \cdots \cap D_{m-1}$ these conditions hold. Then, consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(k - d_m) \longrightarrow \mathcal{O}_Y(k) \longrightarrow \mathcal{O}_X(k) \longrightarrow 0.$$

Since the cohomologies $H^j(Y, \mathcal{O}_Y(k)) = 0$ for all k and $j \neq 0, n+1$ we obtain that $H^j(X, \mathcal{O}_X(k)) = 0$ for all k and $j \neq 0, n$. The base of the induction is evident. □

Theorem 2.7 can be extended on the case of quotient stacks. To do it we note that the famous Serre theorem [20] has a some generalization. The Serre theorem says that if a commutative connected graded algebra $A = \bigoplus_{i\geq 0} A_i$ is generated by the first component the category $\operatorname{qgr} A$ is equivalent to the category of coherent sheaves $\operatorname{coh}(X)$ on the projective variety $X = \mathbf{Proj}\, A$. (Such equivalence holds for the categories of quasicoherent sheaves $\operatorname{Qcoh}(X)$ and $\operatorname{QGr} A$ too.)

Consider now a commutative connected graded $\mathbf{k}$-algebra $A = \bigoplus_{i\geq 0} A_i$ which is not necessary generated by the first component. The grading on A induces an action of the group $\mathbf{k}^*$ on the affine scheme $\mathbf{Spec} A$. Let $\mathbf{0}$ be the closed point of $\mathbf{Spec} A$ that corresponds to the ideal $A_+ = A_{\geq 1} \subset A$. This point is invariant under the action.

Denote by $\mathbb{P}\mathrm{roj}\, A$ the quotient stack $\left[(\mathbf{Spec} A \backslash \mathbf{0})/\mathbf{k}^*\right]$. (Note that there is a natural map $\mathbb{P}\mathrm{roj}\, A \to \mathbf{Proj}\, A$, which is an isomorphism if the algebra A is generated by A_1.)

Proposition 2.11. *Let $A = \bigoplus_{i\geq 0} A_i$ be a graded connected finitely generated algebra. Then the category of (quasi)coherent sheaves on the quotient stack $\mathbb{P}\mathrm{roj}\,(A)$ is equivalent to the category $\operatorname{qgr} A$ (resp. $\operatorname{QGr} A$).*

Proof. Let $\mathbf{0}$ be the closed point on the affine scheme $\mathbf{Spec} A$ which corresponds to the maximal ideal $A_+ \subset A$. Denote by U the complement $\mathbf{Spec} A \backslash \mathbf{0}$. We know that the category of (quasi)coherent sheaves on the stack $\mathbb{P}\mathrm{roj}\, A$ is equivalent to the category of $\mathbf{k}^*$-equivariant (quasi)coherent sheaves on U. The category of (quasi)coherent sheaves on U is equivalent to the quotient of the category of (quasi)coherent sheaves on $\mathbf{Spec} A$ by the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$ (see [5]). This is also true for the categories of $\mathbf{k}^*$-equivariant sheaves. But the category of (quasi)coherent $\mathbf{k}^*$-equivariant sheaves on $\mathbf{Spec} A$ is just the category $\mathrm{gr}{-}A$ (resp. $\mathrm{Gr}{-}A$) of graded modules over A, and the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$ coincides with the subcategory $\mathrm{tors}{-}A$ (resp. $\mathrm{Tors}{-}A$). Thus, we obtain that $\operatorname{coh}(\mathbb{P}\mathrm{roj}\, A)$ is equivalent to the quotient category $\operatorname{qgr} A = \mathrm{gr}{-}A/\mathrm{tors}{-}A$ (and $\operatorname{Qcoh}(\mathbb{P}\mathrm{roj}\, A)$ is equivalent to $\operatorname{QGr} A = \mathrm{Gr}{-}A/\mathrm{Tors}{-}A$). □

Corollary 2.12. *Assume that a noetherian Gorenstein connected graded algebra A from Theorem 2.5 is finitely generated and commutative. Then instead the bounded derived category $\mathbf{D}^b(\operatorname{qgr} A)$ in Theorem 2.5 we can substitute the category $\mathbf{D}^b(\operatorname{coh}(\mathbb{P}\mathrm{roj}\, A))$, where $\mathbb{P}\mathrm{roj}\, A$ the quotient stack $\left[(\mathbf{Spec} A \backslash \mathbf{0})/\mathbf{k}^*\right]$.*

3. Categories of graded D-branes of type B in Landau-Ginzburg models.

3.1. **Categories of graded pairs.** Let $B=\bigoplus_{i\ge 0} B_i$ be a finitely generated connected graded algebra over a field $\mathbf{k}$. Let $W\in B_n$ be a central element of degree n which is non-zero-divisor, i.e. $Wb=bW$ for any $b\in B$ and $bW=0$ only for $b=0$. Denote by J the two-sided ideal $WB=BW$ and denote by A the quotient graded algebra B/J.

With any such element $W\in B_n$ we associated two categories: an exact category $\mathrm{GrPair}(W)$ and a triangulated category $\mathrm{DGrB}(W)$.[1] Objects of these categories are ordered pairs

$$\overline{P} := \Big(P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \Big)$$

where $P_0, P_1\in \mathrm{gr}{-}B$ are finitely generated free graded right B-modules, p_1 is a map of degree 0 and p_0 is a map of degree n (i.e a map from P_0 to $P_1(n)$) such that the compositions p_0p_1 and $p_1(n)p_0$ are the left multiplications by the element W. A morphism $f:\overline{P}\to\overline{Q}$ in the category $\mathrm{GrPair}(W)$ is a pair of morphisms $f_1: P_1\to Q_1$ and $f_0:P_0\to Q_0$ of degree 0 such that $f_1(n)p_0=q_0f_0$ and $q_1f_1=f_0p_1$. The morphism $f=(f_1,f_0)$ is null-homotopic if there are two morphisms $s:P_0\to Q_1$ and $t:P_1\to Q_0(-n)$ such that $f_1=q_0(n)t+sp_1$ and $f_0=t(n)p_0+q_1s$. Morphisms in the category $\mathrm{DGrB}(W)$ are the classes of morphisms in $\mathrm{GrPair}(W)$ modulo null-homotopic morphisms.

In other words, objects of both categories are infinite sequence

$$\underline{K}^{\cdot} := \{\cdots \longrightarrow K^i \stackrel{k^i}{\longrightarrow} K^{i+1} \stackrel{k^{i+1}}{\longrightarrow} K^{i+2} \longrightarrow \cdots\},$$

of morphisms in $\mathrm{gr}{-}B$ between *free* graded right B-modules such that it is quasi-periodic, i.e. $\underline{K}^{\cdot}[2]=\underline{K}^{\cdot}(n)$, and the composition of two near-by morphisms $k^{i+1}k^i$ is equal to multiplication by W. Thereby,

$$K^{2i-1}\cong P_1(i),\ K^{2i}\cong P_0(i),\ k^{2i-1}=p_1(i),\ k^{2i}=p_0(i).$$

A morphism $f:\underline{K}^{\cdot}\longrightarrow \underline{L}^{\cdot}$ in the category $\mathrm{GrPair}(W)$ is a family of morphisms $f^i: K^i\longrightarrow L^i$ in $\mathrm{gr}{-}B$ which is quasi-periodic, i.e $f^{i+2}=f^i(n)$, and which commutes with k^i and l^i, i.e. $f^{i+1}k^i=l^if^i$.

Morphisms in the category $\mathrm{DGrB}(W)$ are morphisms in $\mathrm{GrPair}(W)$ modulo null-homotopic morphisms, and we consider only quasi-periodic homotopies, i.e. such families $s^i: K^i\longrightarrow L^{i-1}$ that $s^{i+2}=s^i(n)$.

Definition 3.1. *The category* $\mathrm{DGrB}(W)$ *constructed above will be called* the category of graded D-branes of type B *for the pair* $(B=\bigoplus_{i\ge 0}B_i, W)$.

Remark 3.2. If B is commutative, then we can consider the affine scheme $\mathbf{Spec}B$. The grading on B gives the action of the algebraic group $\mathbf{k}^*$ on $\mathbf{Spec}B$. The element W is considered as a regular function on $\mathbf{Spec}B$ which is semi-invariant with respect to this action. This way, we get a

[1]One can also construct a differential graded category the homotopy category of which is equivalent to DGrB.

(non-smooth) Landau-Ginzburg model $(\mathbf{Spec}B, W)$ with an action of torus $\mathbf{k}^*$. Thus, Definition 3.1 is a definition of the category of *graded* D-branes of type B for this model. (see also [11, 23]).

It is clear that the category $\mathrm{GrPair}(W)$ is an exact category with respect to componentwise monomorphisms and epimorphisms (see definition in [19]). While the category $\mathrm{DGrB}(W)$ can be endowed with a natural structure of a triangulated category. To determine it we have to define a translation functor $[1]$ and a class of exact triangles.

The translation functor as usually is defined as a functor that takes an object $\underline{K}^{\cdot}$ to the object $\underline{K}^{\cdot}[1]$, where $K[1]^i = K^{i+1}$ and $d[1]^i = -d^{i+1}$, and takes a morphism f to the morphism $f[1]$ which coincides componentwise with f.

For any morphism $f : \underline{K}^{\cdot} \to \underline{L}^{\cdot}$ from the category $\mathrm{GrPair}(W)$ we define a mapping cone $\underline{C}^{\cdot}(f)$ as an object

$$\underline{C}^{\cdot}(f) = \{\cdots \longrightarrow L^i \oplus K^{i+1} \stackrel{c^i}{\longrightarrow} L^{i+1} \oplus K^{i+2} \stackrel{c^{i+1}}{\longrightarrow} L^{i+2} \oplus K^{i+3} \longrightarrow \cdots\}$$

such that

$$c^i = \begin{pmatrix} l^i & f^{i+1} \\ 0 & -k^{i+1} \end{pmatrix}.$$

There are maps $g : \underline{L}^{\cdot} \to \underline{C}^{\cdot}(f)$, $g = (\mathrm{id}, 0)$ and $h : \underline{C}^{\cdot}(f) \to \underline{K}^{\cdot}[1]$, $h = (0, -\mathrm{id})$.

Now we define a standard triangle in the category $\mathrm{DGrB}(W)$ as a triangle of the form

$$\underline{K}^{\cdot} \stackrel{f}{\longrightarrow} \underline{L}^{\cdot} \stackrel{g}{\longrightarrow} \underline{C}^{\cdot}(f) \stackrel{h}{\longrightarrow} \underline{K}^{\cdot}[1].$$

for some $f \in \mathrm{GrPair}(W)$.

Definition 3.3. *A triangle* $\underline{K}^{\cdot} \to \underline{L}^{\cdot} \to \underline{M}^{\cdot} \to \underline{K}^{\cdot}[1]$ *in* $\mathrm{DGrB}(W)$ *will be called an exact (distinguished) triangle if it is isomorphic to a standard triangle.*

Proposition 3.4. *The category* $\mathrm{DGrB}(W)$ *endowed with the translation functor* $[1]$ *and the above class of exact triangles becomes a triangulated category.*

We omit the proof of this proposition which is moreless the same as the proof of the analogous result for a usual homotopic category (see, e.g. [7, 13]).

3.2. **Categories of graded pairs and categories of singularities.** With any object $\underline{K}^{\cdot}$ one associates a short exact sequence

$$0 \longrightarrow K^{-1} \stackrel{k^{-1}}{\longrightarrow} K^0 \longrightarrow \mathrm{Coker}\ k^{-1} \longrightarrow 0. \tag{10}$$

We can attach to an object $\underline{K}^{\cdot}$ the right B-module $\mathrm{Coker}\ k^{-1}$. It can be easily checked that the multiplication by W annihilates it. Hence, the module $\mathrm{Coker}\ k^{-1}$ can be considered as the right A-module. Any morphism $f : \underline{K}^{\cdot} \to \underline{L}^{\cdot}$ in $\mathrm{GrPair}(W)$ induces a morphism between cokernels. This construction defines a functor $\mathrm{Cok} : \mathrm{GrPair}(W) \longrightarrow \mathrm{gr}{-}A$. Using the functor Cok we can construct an exact functor between triangulated categories $\mathrm{DGrB}(W)$ and $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$.

Proposition 3.5. *There is a functor* F *which completes the following commutative diagram*

$$\begin{array}{ccc} \mathrm{GrPair}(W) & \xrightarrow{\mathrm{Cok}} & \mathrm{gr}{-}A \\ \downarrow & & \downarrow \\ \mathrm{DGrB}(W) & \xrightarrow{F} & \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A). \end{array}$$

Moreover, the functor F *is an exact functor between triangulated categories.*

Proof. We have the functor $\mathrm{GrPair}(W) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ which is the composition of Cok and the natural functor from $\mathrm{gr}{-}A$ to $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. To prove the existence of a functor F we need to show that any morphism $f : \underline{K}^{\cdot} \to \underline{L}^{\cdot}$ which is null-homotopic goes to 0-morphism in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. Fix a homotopy $s = (s^i)$ with $s^i : K^i \to L^{i-1}$. Consider the following decomposition of f :

$$\begin{array}{ccccc} K^{-1} & \xrightarrow{k^{-1}} & K^0 & \longrightarrow & \mathrm{Coker}\ k^{-1} \\ \downarrow{\scriptstyle (s^{-1},f^{-1})} & & \downarrow{\scriptstyle (s^0,f^0)} & & \downarrow \\ L^{-2} \oplus L^{-1} & \xrightarrow{u^{-1}} & L^{-1} \oplus L^0 & \longrightarrow & L^0 \otimes_B A \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} & & \downarrow \\ L^{-1} & \xrightarrow{l^{-1}} & L^0 & \longrightarrow & \mathrm{Coker}\ l^{-1} \end{array} \qquad \text{where} \qquad u^{-1} = \begin{pmatrix} -l^{-2} & \mathrm{id} \\ 0 & l^{-1} \end{pmatrix},$$

This yields the decomposition of $F(f)$ through a locally free object $L^0 \otimes_B A$. Hence, $F(f) = 0$ in the category $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. By Lemma 3.7 proved below the tensor product $\underline{K}^{\cdot} \otimes_B A$ is an acyclic complex. Hence, there is an exact sequence $0 \to \mathrm{Coker}\ k^{-1} \to K^1 \otimes_B A \to \mathrm{Coker}\ k^0 \to 0$. Since $K^1 \otimes_B A$ is free, we have $\mathrm{Coker}\ k^0 \cong \mathrm{Coker}\ k^{-1}[1]$ in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. But, $\mathrm{Coker}\ k^0 = F(\underline{K}^{\cdot}[1])$. Hence, the functor F commutes with translation functors. It is easy to see that F takes a standard triangle in $\mathrm{DGrB}(W)$ to an exact triangle in $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. Thus, F is exact. □

Lemma 3.6. *The functor* Cok *is full.*

Proof. Any map $g : \mathrm{Coker}\ k^{-1} \to \mathrm{Coker}\ l^{-1}$ between A-modules can be considered as the map of B-modules and can be extended to a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{-1} & \xrightarrow{k^{-1}} & K^0 & \longrightarrow & \mathrm{Coker}\ k^{-1} & \longrightarrow & 0 \\ & & {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L^{-1} & \xrightarrow{l^{-1}} & L^0 & \longrightarrow & \mathrm{Coker}\ l^{-1} & \longrightarrow & 0, \end{array}$$

because K^0 is free. This gives us a sequence of morphisms $f = (f^i), i \in \mathbb{Z}$, where $f^{2i} = f^0(i)$ and $f^{2i-1} = f^{-1}(i)$. To prove the lemma it is sufficient to show that the family f is a map from $\underline{K}^{\cdot}$ to $\underline{L}^{\cdot}$, i.e $f^1 k^0 = l^0 f^0$. Consider the sequence of equalities

$$l^1(f^1k^0 - l^0f^0) = f^2k^1k^0 - Wf^0 = f^2W - Wf^0 = f^0(2)W - Wf^0 = 0.$$

Since l^1 is an embedding, we obtain that $f^1k^0 = l^0f^0$. □

Lemma 3.7. *For any sequence* $\underline{K}^{\cdot} \in \mathrm{GrPair}(W)$ *the tensor product* $\underline{K}^{\cdot} \otimes_B A$ *is an acyclic complex of* A*-modules and the* A*-module* $\mathrm{Coker}\ k^{-1}$ *satisfies the condition*

$$\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-1}, A) = 0 \qquad \text{for all} \quad i > 0.$$

Proof. It is clear that $\underline{K}^{\cdot} \otimes_B A$ is a complex. Applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^{i-2} & \xrightarrow{k^{i-2}} & K^{i-1} & \longrightarrow & \mathrm{Coker}\ k^{i-2} & \longrightarrow & 0 \\
 & & {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & K^{i} & \xrightarrow{k^{i}} & K^{i+1} & \longrightarrow & \mathrm{Coker}\ k^{i} & \longrightarrow & 0,
\end{array}$$

we obtain an exact sequence

$$0 \to \mathrm{Coker}\ k^{i-2} \longrightarrow K^i \otimes_B A \xrightarrow{k_i|_W} K^{i+1} \otimes_B A \longrightarrow \mathrm{Coker}\ k^i \to 0.$$

This implies that $\underline{K}^{\cdot} \otimes_B A$ is an acyclic complex.

Further, consider the dual sequence of left B-modules $\underline{K}^{\cdot\vee}$, where $\underline{K}^{\cdot\vee} \cong \mathrm{Hom}_B(\underline{K}^{\cdot}, B)$. By the same reasons as above $A \otimes_B \underline{K}^{\cdot\vee}$ is an acyclic complex. On the other hand, the cohomologies of the complex $\{(K^0)^\vee \longrightarrow (K^{-1})^\vee \longrightarrow (K^{-2})^\vee \longrightarrow \cdots\}$ are isomorphic to $\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-1}, A)$. And, as we see, they are equal to 0 for all $i > 0$. □

Lemma 3.8. *If* $F\underline{K}^{\cdot} \cong 0$, *then* $\underline{K}^{\cdot} \cong 0$ *in* $\mathrm{DGrB}(W)$.

Proof. If $F\underline{K}^{\cdot} \cong 0$, then the A-module $\mathrm{Coker}\ k^{-1}$ is a perfect as complex of A-modules. Let us show that $\mathrm{Coker}\ k^{-1}$ is projective in this case. Indeed, there is a natural number m such that $\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-1}, N) = 0$ for any A-module N and any $i \geq m$. Considering the exact sequence

$$0 \longrightarrow \mathrm{Coker}\ k^{-2m-1} \longrightarrow K^{-2m} \otimes_B A \longrightarrow \cdots \longrightarrow K^{-1} \otimes_B A \longrightarrow K^0 \otimes_B A \longrightarrow \mathrm{Coker}\ k^{-1} \longrightarrow 0$$

and taking into account that all A-modules $K^i \otimes_B A$ are free, we find that for all modules N $\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-2m-1}, N) = 0$ when $i > 0$. Hence, $\mathrm{Coker}\ k^{-2m-1}$ is projective A-module. This implies that $\mathrm{Coker}\ k^{-1}$ is also projective, because it is isomorphic to $\mathrm{Coker}\ k^{-2m-1}(-m)$.

Since $\mathrm{Coker}\ k^{-1}$ is projective there is a map $f : \mathrm{Coker}\ k^{-1} \to K^0 \otimes_B A$ which splits the epimorphism $\mathrm{pr} : K^0 \otimes_B A \to \mathrm{Coker}\ k^{-1}$. It can be lifted to a map from the complex $\{K^{-1} \xrightarrow{k^{-1}} K^0\}$ to the complex $\{K^{-2} \xrightarrow{W} K^0\}$. Denote it by (s^{-1}, u). Consider the following diagram

$$\begin{array}{ccccc}
K^{-1} & \xrightarrow{k^{-1}} & K^0 & \longrightarrow & \mathrm{Coker}\ k^{-1} \\
{\scriptstyle s^{-1}}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f} \\
K^{-2} & \xrightarrow{W} & K^0 & \longrightarrow & K^0 \otimes_B A \\
{\scriptstyle k^{-2}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle pr} \\
K^{-1} & \xrightarrow{k^{-1}} & K^0 & \longrightarrow & \mathrm{Coker}\ k^{-1}.
\end{array}$$

Since the composition $\mathrm{pr}\, f$ is identical, the map $(k^{-2}s^{-1}, u)$ from the pair $\{K^{-1} \xrightarrow{k^{-1}} K^0\}$ to itself is homotopic to the identity map. Hence, there is a map $s^0 : K^0 \to K^{-1}$ such that

$$\mathrm{id}_{K^{-1}} - k^{-2}s^{-1} = s^0 k^{-1} \qquad \text{and} \qquad k^{-1}s^0 = \mathrm{id}_{K^0} - u.$$

Moreover, we have the following equalities

$$0 = (uk^{-1} - Ws^{-1}) = (uk^{-1} - s^{-1}(n)W) = (u - s^{-1}(n)k^0)k^{-1}.$$

This gives us that $u = s^{-1}(n)k^0$, because there are no maps from $\mathrm{Coker}\ k^{-1}$ to K^0. Finally, we get the sequence of morphisms $s^i : K^i \longrightarrow K^{i-1}$, where $s^{2i-1} = s^{-1}(i), s^{2i} = s^0(i)$, such that $k^{i-1}s^i + k^i s^{i+1} = \mathrm{id}$. Thus the identity morphism of the object $\underline{K}^{\cdot}$ is null-homotopic. Hence, the object $\underline{K}^{\cdot}$ is isomorphic to the zero object in the category $\mathrm{DGrB}_0(W)$. □

Theorem 3.9. *The exact functor* $F : \mathrm{DGrB}(W) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *is fully faithful.*

Proof. By Lemma 3.7 we have $\mathrm{Ext}^i_A(\mathrm{Coker}\ k^{-1}, A) = 0$ for $i > 0$. Now, Proposition 1.10 gives an isomorphism

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(\mathrm{Coker}\ k^{-1}, \mathrm{Coker}\ l^{-1}) \cong \mathrm{Hom}_{\mathrm{gr}-A}(\mathrm{Coker}\ k^{-1}, \mathrm{Coker}\ l^{-1})/\mathcal{R},$$

where $\mathcal{R}$ is the subspace of morphisms factoring through projective modules. Since the functor Cok is full we get that the functor F is also full.

Let show that F is faithful. It is a standard consideration. Let $f : \underline{K}^{\cdot} \to \underline{L}^{\cdot}$ be a morphism for which $F(f) = 0$. Include f in an exact triangle $\underline{K}^{\cdot} \xrightarrow{f} \underline{L}^{\cdot} \xrightarrow{g} \underline{M}^{\cdot}$. Then the identity map of $F\underline{L}^{\cdot}$ factors through the map $F\underline{L}^{\cdot} \xrightarrow{Fg} F\underline{M}^{\cdot}$. Since F is full, there is a map $h : \underline{L}^{\cdot} \to \underline{L}^{\cdot}$ factoring through $g : \underline{L}^{\cdot} \to \underline{M}^{\cdot}$ such that $Fh = \mathrm{id}$. Hence, the cone $\underline{C}^{\cdot}(h)$ of map h goes to zero under the functor F. By Lemma 3.8 the object $\underline{C}^{\cdot}(h)$ is the zero object as well, so h is an isomorphism. Thus $g : \underline{L}^{\cdot} \to \underline{M}^{\cdot}$ is a split monomorphism and $f = 0$. □

Theorem 3.10. *Suppose that the algebra* B *has a finite homological dimension. Then the functor* $F : \mathrm{DGrB}(W) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ *is an equivalence.*

Proof. We know that F is fully faithful. To prove the theorem we need to show that each object $T \in \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is isomorphic to $F\underline{K}^{\cdot}$ for some $\underline{K}^{\cdot} \in \mathrm{DGrB}(W)$.

The algebra B has a finite homological dimension and, as consequence, it has a finite injective dimension. This implies that $A = B/J$ has a finite injective dimension too. By Lemma 1.9 any object $T \in \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is isomorphic to the image of an A-module M such that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. This means that the object $D(M) = \mathbf{R}\mathrm{Hom}_A(M, A)$ is a left A-module. We can consider a projective resolution $\underline{Q}^{\cdot} \to D(M)$. The dual to it gives us a right projective A-resolution

$$0 \longrightarrow M \longrightarrow \{P^0 \longrightarrow P^1 \longrightarrow \cdots\}.$$

Consider M as B-module and take some epimorphism $K^0 \twoheadrightarrow M$ from free B-module K^0. Denote by $k^{-1} : K^{-1} \to K^0$ the kernel of this map.

The short exact sequence $0 \to B \xrightarrow{W} B \to A \to 0$ gives us that for a projective A-module P and any B-module N we have equalities $\mathrm{Ext}^i_B(P,N) = 0$ when $i > 1$. This also yields that $\mathrm{Ext}^i_B(M,N) = 0$ for $i > 1$ and any B-module N, because M has a right projective A-resolution and the algebra B has finite homology dimension. Therefore, $\mathrm{Ext}^i_B(K^{-1},N) = 0$ for $i > 0$ and any B-module N, i.e. B-module K^{-1} is projective. Since A is connected and finitely generated, any graded projective module is free. Hence, K^{-1} is free.

Since the multiplication on W gives the zero map on M, there is a map $k^0 : K^0 \to K^{-1}(n)$ such that $k^0k^{-1} = W$ and $k^{-1}(n)k^0 = W$. This way, we get a sequence $\underline{K}^{\cdot}$ with

$$K^{2i} \cong K^0(i),\ K^{2i-1} = K^{-1}(i),\ k^{2i} = k^0(i),\ k^{2i-1} = k^{-1}(i).$$

and this sequence is an object of $\mathrm{DGrB}(W)$ for which $F\underline{K}^{\cdot} \cong T$. □

3.3. **Graded D-branes type B and coherent sheaves.** By a Landau-Ginzburg model we mean the following data: a smooth variety X with a symplectic Kähler form ω and a regular nonconstant function W on X which is considered as a flat map $W : X \longrightarrow \mathbb{A}^1$ and which should be a symplectic fibration. The function W is called superpotential. Since for the definition of D-branes of type B a symplectic form is not needed we do not fix it.

With any point $\lambda \in \mathbb{A}^1$ we can associate a triangulated category $DB_\lambda(W)$. We give constructions of these categories under the condition that $X = \mathbf{Spec}(B)$ is affine (see [12, 18]). The category of coherent sheaves on an affine scheme $X = \mathbf{Spec}(B)$ is the same as the category of finitely generated B-modules.

Objects of the category $DB_\lambda(W)$ are ordered pairs $\overline{P} := \Big(P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \Big)$, where P_0, P_1 are finitely generated projective B-modules and the compositions p_0p_1 and p_1p_0 are the multiplications by the element $(W - \lambda) \in B$. Morphisms in the category $DB(W)$ are the classes of morphisms between pairs modulo null-homotopic morphisms, where a morphism $f : \overline{P} \to \overline{Q}$ between pairs is a pair of morphisms $f_1 : P_1 \to Q_1$ and $f_0 : P_0 \to Q_0$ such that $f_1p_0 = q_0f_0$ and $q_1f_1 = f_0p_1$. The morphism f is null-homotopic if there are two morphisms $s : P_0 \to Q_1$ and $t : P_1 \to Q_0$ such that $f_1 = q_0t + sp_1$ and $f_0 = tp_0 + q_1s$.

We define a category of D-branes of type B (B-branes) on $X = \mathbf{Spec}(B)$ with the superpotential W as the product $DB(W) = \prod_{\lambda\in\mathbb{A}^1} DB_\lambda(W)$.

It was proved in the paper [18] (Cor. 3.10) that the category $DB_\lambda(W)$ for smooth affine X is equivalent to the triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(X_\lambda)$ where X_λ is the fiber over $\lambda \in \mathbb{A}^1$. Therefore, the category of B-branes $DB(W)$ is equivalent to the product $\prod_{\lambda\in\mathbb{A}^1} \mathbf{D}_{\mathrm{Sg}}(X_\lambda)$. For non-affine X the category $\prod_{\lambda\in\mathbb{A}^1} \mathbf{D}_{\mathrm{Sg}}(X_\lambda)$ can be considered as a definition of the category of D-branes of type B. Note that X_λ is $\mathbf{Spec}(A_\lambda)$, where $A_\lambda = B/(W - \lambda)B$ and, hence, the triangulated categories of singularities $\mathbf{D}_{\mathrm{Sg}}(X_\lambda)$ is the same that the category $\mathbf{D}_{\mathrm{Sg}}(A_\lambda)$.

Assume now that there is an action of the group $\mathbf{k}^*$ on the Landau-Ginzburg model (X, W) such that the superpotential W is semi-invariant of the weight d. Thus, $X = \mathbf{Spec}(B)$ and

$B = \bigoplus_i B_i$ is a graded algebra. The superpotential W is an element of B_d. Let us assume that B is positively graded and connected. In this case, we can consider the triangulated category of graded B-branes $\mathrm{DGrB}(W)$, which was constructed in subsection 3.1 (see Definition 3.1).

Denote by A the quotient graded algebra B/WB. We see that the affine variety $\mathbf{Spec}(A)$ is the fiber X_0 of W over the point 0. Denote by Y the quotient stack $[(\mathbf{Spec}(A) \setminus \mathbf{0})/\mathbf{k}^*]$, where $\mathbf{0}$ is the point on $\mathbf{Spec}(A)$ corresponding to the ideal A_+. Theorems 2.5, 3.10 and Proposition 2.11 allow us to establish a relation between triangulated category of graded B-branes $\mathrm{DGrB}(W)$ and the bounded derived category of coherent sheaves on the stack Y.

First, Theorem 3.10 gives us the equivalence F between the triangulated category of graded B-branes $\mathrm{DGrB}(W)$ and the triangulated category of singularities $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$. Second, Theorem 2.5 describes relations between the category $\mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ and the bounded derived category $\mathbf{D}^b(\mathrm{qgr}\, A)$. Third, the category $\mathbf{D}^b(\mathrm{qgr}\, A)$ is equivalent to the derived category $\mathbf{D}^b(\mathrm{coh}(Y))$ by Proposition 2.11. In particular case, when X is the affine space $\mathbb{A}^N$ with the standard action of the group $\mathbf{k}^*$, we get the following result.

Theorem 3.11. *Let X be the affine space $\mathbb{A}^N$ and let W be a homogeneous polynomial of degree d. Let $Y \subset \mathbb{P}^{N-1}$ be the hypersurface of degree d which is given by the equation $W = 0$. Then, there is the following relation between the triangulated category of graded B-branes* $\mathrm{DGrB}(W)$ *and the derived category of coherent sheaves* $\mathbf{D}^b(\mathrm{coh}(Y))$:

(i) *if $d < N$ (i.e. Y is a Fano variety), there is a semiorthogonal decomposition*

$$\mathbf{D}^b(\mathrm{coh}(Y)) = \langle \mathcal{O}_Y(d-N+1), \ldots, \mathcal{O}_Y, \mathrm{DGrB}(W)\rangle,$$

(ii) *if $d > N$ (i.e. X is a variety of general type), there is a semiorthogonal decomposition*

$$\mathrm{DGrB}(W) = \langle F^{-1}q(\mathbf{k}(r+1)), \ldots, F^{-1}q(\mathbf{k}), \mathbf{D}^b(\mathrm{coh}(Y))\rangle,$$

where $q : \mathbf{D}^b(\mathrm{gr}{-}A) \longrightarrow \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the natural projection, and $F : \mathrm{DGrB} \stackrel{\sim}{\longrightarrow} \mathbf{D}^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ is the equivalence constructed in Proposition 3.5.

(iii) *if $d = N$ (i.e. Y is a Calabi-Yau variety), there is an equivalence*

$$\mathrm{DGrB}(W) \stackrel{\sim}{\longrightarrow} \mathbf{D}^b(\mathrm{coh}(Y)).$$

Remark 3.12. We can also consider a weighted action of the torus $\mathbf{k}^*$ on the affine space $\mathbb{A}^N$ with weights $(a_1, \ldots, a_N)$ and $a_i > 0$ for all i. If the superpotential W is quasi-homogeneous then we have the category of graded B-branes $\mathrm{DGrB}(W)$. The polynomial W defines an orbifold (quotient stack) $Y \subset \mathbb{P}^{N-1}(a_1, \ldots, a_N)$. The orbifold Y is the quotient of $\mathbf{Spec}(A)\backslash\mathbf{0}$ by the action of $\mathbf{k}^*$, where $A = \mathbf{k}[x_1, \ldots, x_N]/W$. Proposition 2.11 gives the equivalence between $\mathbf{D}^b(\mathrm{coh}(Y))$ and $\mathbf{D}^b(\mathrm{qgr}\, A)$. And Theorem 2.5 shows that we get an analogue of Theorem 3.11 for the weight case as well.

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