

In Defense of the Ideal

2nd DRAFT

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This paper lies at the edge of the topic of the workshop. We can write down a Π_1^1 axiom whose models are precisely the \in -structures $\langle R_\alpha, \in \cap R_\alpha^2 \rangle$ where $\alpha > 0$ and R_α is the collection of all (pure) sets of rank $< \alpha$. From this, one can consider the introduction of new axioms concerning the size of α . The question of the grounds for doing so is perhaps the central question of the workshop. But I want to discuss another question which, as I said, arises at the periphery: How do we know that there are structures $\langle R_\alpha, \in \cap R_\alpha^2 \rangle$? How do we know that there exist such things as sets and how do we know that, given such things, the axioms we write down are true of them? These seem very primitive questions, but the skepticism implicit in them has deep (and ancient) roots. In particular, they are questions about ideal objects in general, and not just about the actual infinite. I want to explain why I think the questions (as intended) are empty and the skepticism unfounded.¹

I will be expanding the argument of the first part of my paper “Proof and truth: the Platonism of mathematics” [1986a].² The argument in question has two parts: first, that skepticism about ideal objects, such as sets, based upon the fact that we don’t perceive or otherwise interact with them in the natural world leads naturally to a venerable skepticism about even empirically observable objects; and secondly, that both skeptical arguments are based on the same misconception of meaning, of how language works.

The present paper differs from [1986a] in that I now see more clearly the extent to which my argument is really a thread, I think the dominant thread, running through Wittgenstein’s *Philosophical Investigations*, to which I will refer simply as the *Investigations*. It may seem shocking to some that a

¹I write “as intended” because one might question the existence of sets because one believes that *ZF*, say, is inconsistent; and that belief is not empty—although we certainly hope it is false.

²The second part of that paper, about the Curry-Howard theory of types, was riding a hobby horse of mine at the time and is, whether sensible or not, entirely separable from the first part.

defense of the ideal and in particular of the actual infinite could be based on the thought of the skeptic and anti-Platonist, Wittgenstein. But I would say that careful reading reveals the *Investigations* to be not skeptical at all; in fact it is anti-skeptical (as I had already argued in [1986b]). And Wittgenstein's problem with mathematics was that his conception of it belonged to the eighteenth century. For him, the language of set theory was 'language on holiday' and he reveals no awareness that it was a well-established part of mathematics by the time he began developing his views in 1929 leading up to *Investigations*.

1. The 'Augustinian' Picture of Meaning.

A natural view of language and how it works involves the idea that words name things and that the relationship among the words in an elementary ('atomic') sentence expresses a certain relationship among the corresponding named things—the sentence being true when the latter relationship obtains. Wittgenstein, who himself in his *Tractatus Logico-Philosophicus* offered a version of this static picture of language, refers it in §§1-32 of *Investigations* to Augustine's *Confessions*, Book 1, viii, which he quotes in §1. As long as we are considering language only in its descriptive function, which is indeed our primary interest, there is nothing wrong with this picture *as a program for describing the semantics of a language*. The truth definition relative to a model for a formal language in the framework of predicate logic [Tarski, 1936] shows how it works for those (relatively impoverished) fragments of language that can be 'regimented' into such formal languages. More generally, the program of compositional semantics in linguistics, as I understand it, aims at displaying the meaning of an expression as a function of the meaning of its parts. But it is a mistake to think that, even where it applies, this semantics provides an ultimate account of the meaning of words in the sense of reducing what we need to know in order to use or understand complex expressions, such as sentences, to what we need to know in order to use or understand its constituents, such as names³—it is the thesis that there is such a reduction to which I refer as the 'Augustinian picture of meaning'. According to this

³In "General semantics" David Lewis makes a similar distinction:

I distinguish two topics: first, the description of possible languages or grammars as abstract semantic systems whereby symbols are associated with aspects of the world; and, second, the description of the psychological and sociological facts whereby a particular one of these abstract semantic systems is the one used by a person or population. Only confusion comes of mixing these two topics. [1970, p. 19]

picture, what we need to know in order to use or understand names must be given to us extra-linguistically, since it is presupposed for the understanding or use of sentences. The difficulty with this picture comes into focus when we note that the definition of truth has itself to be given in some language, a ‘metalanguage’, and so the definition only tells us in the metalanguage what the expressions in the language in question, the ‘object language’, mean: it is essentially a schema for *translating* the sentences of the object language into the metalanguage. Even at the level of the elementary sentences, say Rs , expressing that the object s has the property R , this difficulty arises: How else are we to refer to s in stating the truth condition other than by using a synonym for “ s ” or a description either in the same language or in a metalanguage? The alternative, that we establish its denotation by some gesture such as pointing, is disqualified by noting that ostensive gestures convey meaning only in the context of a language already in place. (See §6 and §§27-8 of the *Investigations*.) In §32 Wittgenstein sums up what is wrong with the Augustinian picture as an account of meaning:

Augustine describes the learning of human language as if the child [learning the language] came into a foreign country and did not understand the language of the country; that is, as if he already had a language, only not this one. §32.

When we think about the relation of language to reality, we are such children perforce. Armed with my command of English, I view and can describe the natural world as having a more or less definite structure, consisting of objects of this or that sort, having this or that property and entering into this or that relationship. (And, parenthetically, if I have studied some physics, then I also know of other structures in terms of which I can view the world; i.e. I have other languages in terms of which to describe ‘it’. Nor need these different visions of the natural world even be mutually compatible or compatible with our every day one.) So faced with questions of the reference of words of some other language, I can answer by referring to kinds of objects, properties or relations in my own: I can translate.

But when instead of a foreign country we are at home and the language is our own—when in Willard V.O. Quine’s terms we ‘go domestic’—we can no longer be thinking about translation, and the questions we ask may become silly: the reasonable question about what the native speaker of an exotic language means by the word “gavagai” becomes the odd question of what *we* mean by the word “rabbit.” Of course, we are often faced with the question of the reference of words in our native language, but these we answer by recourse to synonyms or descriptions, perhaps combined with

ostension (e.g. *that* person, *that* body, *that* color, etc.). But that is because, or better, to the extent that we already have a language.

Deriving from Aristotle’s *On Interpretation*, Chapter 1, 16^a3 – 18, the idea of a language already possessed was developed in the Middle Ages in the form of ‘the natural language’ of the mind. This we all possess in common, and it is in contrast to the various conventional languages (which we now call ‘natural’)⁴ that we speak and which we learn by ‘translation’ into the natural language.⁵ It is against the background of this notion of the natural language that the passage from the *Confessions* should be read. The idea of a language of the mind has been reintroduced in recent times by Jerry Fodor in his *Language of Thought* [1975; 2008] and Steven Pinker in *The Language Instinct* [1994], for example. In a somewhat different direction, contemporary connectionist theories of cognitive processes identify a range of concepts with certain structures in the neural network.■ But it is difficult to see how a theory of meaning based on structures of the mind or brain can meet the challenge of accounting for the norms of language use, for grammatical and logical norms. If I compute $7 + 5$ to 13, I, i.e. my internal computer, is wrong. Of course, there could be internal mechanisms for error-checking and which, in particular, could catch this error. But suppose not; suppose that the computation is not an error from the point of view of my internal language? Then on what grounds, other than external ones, can we say I was wrong? For such reasons, I don’t think that we can take seriously that the source of the meaning of our linguistic expressions is an internal language of the mind (or brain). But I should emphasize that this does not imply rejection of the possibility that there are structures of the mind or brain that account for our *mastery* of our language. I for one believe that possibility to be a certainty.

2. Skepticism about Ideal Objects.

The Augustinian picture of meaning is basic to the view that propositional knowledge must be about things to which we can refer extra-linguistically, since this is required in order for the elementary sentences of the language in which the knowledge is framed to have meaning. A contemporary statement of this view is in Benacerraf’s “Mathematical truth” [1973], in which naming

⁴The term “conventional” is used today in connection with linguistics to express the thought that our common language results from a kind of negotiation among us, with our individual idiolects, in the interest of communication. See [Higgenbotham, 2006, pp.142-3].

⁵Indeed, William of Ockham’s ‘nominalism’ consists in his view that universals are nothing more than common names *in the language of thought*. See *Summa Logicae*, Part I.

comes about by a causal chain leading back to a baptism; but it has a long history, from Aristotle's doctrine that all knowledge begins with sense perception to Kant's that concepts without intuitions are blind.

A consequence of this view is skepticism about the existence of 'abstract' or, as I prefer to call them, *ideal* objects and, in particular, the objects of mathematics, such as numbers, sets and functions—objects with which, most of us believe, we have no extra-linguistic connection. For example, the Benacerraf paper just cited contains a version of just this skeptical line of thought, as does Michael Dummett's paper "Platonism" [1978a]. It has led, even in recent times, to various forms of *nominalism*, according to which there are no ideal objects. We won't dwell on the distinctions among these various versions of nominalism, since our aim ultimately is to disarm the arguments that lead to any of them.⁶

So I am taking the term "ideal" here to be the opposite of "natural" or "empirical", where the latter refers to objects 'found' in the natural world. It would in many ways make sense to refer to objects of the latter kind as 'real', but unfortunately the contemporary use of that term would suggest that I am capitulating on my ultimate aim by agreeing that ideal things are 'unreal'. (Of course I am still open to the charge that I am taking them to be unnatural.) I will not trouble myself (or you) with a discussion of exactly what the non-ideal things are, for example whether they include 'theoretical entities', etc., although I believe that the argument of this paper will challenge the grounds for at least some of the various 'realism-antirealism' debates in philosophy of science. ■ What is characteristic of natural objects in my sense is that an appeal to the natural world is essential to establish their existence and properties. The boundary between what is natural and what is ideal may still seem not to be clearly drawn by this criterion because of a no-man's land: we can, for example, speak of the set of pens on the desk in front of me. As a set, this certainly has an element of the ideal, but the question of what its elements are is an empirical question and inevitably anything nontrivial we say about it will have empirical content. But to repeat, I take as a mark of the ideal that empirical data are irrelevant to the truth of what we say about it. So, in my sense, these 'mixed' objects, along with what Quine [1960, p. 233] refers to as 'abstract particulars'—e.g. the Equator and the North Pole—are empirical. The distinction I am making has been directly challenged by Quine [1953, footnote 1] by remarking that the statement that there is no ratio between the number of centaurs and the

⁶The reader will find a full discussion of the varieties of nominalism in [Burgess and Rosen, 1997].

number of unicorns has empirical content. One might question this, as does Saul Kripke [1972, 157] implicitly, on the grounds that the non-existence of mythical creatures is a non-empirical truth; but then Quine’s example can be replaced by ones that do have empirical content—the number of honest politicians and the like. From my point of view, the relevant objection to Quine’s argument is that there is an ideal, i.e. non-empirical, content of the statement, namely that division by 0 is undefined: this is a mathematical fact. The statement about centaurs and unicorns, or about honest politicians, is an *application* of this fact to mixed objects, the set of unicorns, etc. The form of expression “the number of F ’s” belongs to the ideal realm only if the concept F does. But the system of (the natural) numbers, which can be *applied* as cardinal numbers, was characterized by Dedekind [1888b] completely free of reference to the natural world.⁷

A rather singular take-off from skepticism about ideal objects is Quine’s conception according to which mathematical theories gain meaning and vindication only as a part of a grand theory of everything, which itself is grounded in its empirical consequences. Quine abandoned nominalism in favor of this position because the former does not permit the development of a sufficient amount of mathematics to support empirical sciences.⁸ For example, according to him the existence of the natural numbers and the Dedekind-Peano axioms asserting what we take to be their fundamental properties are contingent truths, high up perhaps in the scale of non-revisability, but nevertheless revisable in principle like any truth in (the contemporary version of) the grand theory.

I call Quine’s approach to mathematical objects ‘singular’ because, in a characteristic way, it blurs the distinction that I want to maintain between the natural, which is subject to empirical investigation, and the ideal, which is not. Quine’s has perhaps been the most influential attack in the twentieth century against the claim that there is a sharp distinction between them. According to him, there is just one notion of existence and, when we distinguish an ideal domain, such as the natural numbers from the natural, we

⁷This is not so of Frege’s definition of a number, into which he attempts to build the notion of cardinality. His definition of “the number of F ’s”, leaving aside that it is inconsistent, is to apply to all concepts F , ideal or empirical. This feature of his treatment is preserved by the so-called *neo-logicists* treatment of number. See for example [Hale and Wright, 2001].

⁸See [Quine, 1986]. There are versions of nominalism which seek to refute Quine on this point by expanding the notion of physical object to include things like parts of space, where the term “part” is to be understood quite broadly. For example, see [Field, 1980]. But, again, we need not discuss this, since we plan to argue that nominalism in any of its forms is based on a misconception.

are only speaking of two different ‘classes’ of existing things; and existential assertions about what I am calling ideal things are theoretical assumptions, like those of a physical theory. Indeed, they are all a part of our current grand theory of everything, and are revisable as empirical evidence warrants. On his view it could turn out to be wrong that the natural numbers exist, and this, not because there is anything internally wrong with number theory (such as that it is inconsistent), but just because on holistic grounds we are led to the conclusion that the numbers don’t exist. Suffice it to say of this faulty understanding of the relation between mathematics and natural science that its motivation was certainly the skeptical argument concerning ideal objects: as we have noted, he rejected nominalism because an insufficient amount of the mathematics needed for the grand theory could be developed on a nominalist basis. But we don’t interact with mathematical objects and so the only grounds that we have for assuming them to exist is the successful role that they play in our account of the things with which we do interact. In this way they are an extreme form of ‘theoretical entity’.

3. Abstract Objects.

Let me stop to explain why I prefer to speak of *ideal* rather than ‘abstract’ objects, which is the commonly used term. Abstract objects, properly speaking, ought to be abstracted from something; they are formed by taking something away. What is left is a universal, for example a species or genus or property of members of some species, etc. Historically, the question about the existence of abstract objects concerned whether universals, abstracted from sensible substances, existed in their own right—however that is to be understood. But of course these universals are not ideal objects and, in particular, are not mathematical objects. We can replace the universal by the class of things falling under the universal (assuming, contrary to fact in at least most cases, that the class is well-defined), and it is in this way that the issue of the existence of abstract objects entered the twentieth century debate over the existence of mathematical objects. ■ Quine and Goodman ■ But the class corresponding to the universal is not an ideal object, either. In our terminology, it is a mixed object.

I hope that this brief discussion makes it clear why I resist referring to mathematical objects as ‘abstract’, in spite of the consensus use of that term. The term “abstract” should be reserved for Aristotelian universals or perhaps the corresponding classes. The term “ideal” does have the disadvantage that it is used *within* mathematics to refer to objects introduced into a structure (typically) to simplify the structure in some respect: point

and lines at infinity, ideals in ring theory, etc. But we shall not generally be speaking *in* mathematics, only sometimes *about* it. And the term accords very well with Plato's language when he (for the first time as far as we know) argued that the objects of exact science, however much it developed as science about the natural world, cannot be natural objects.

For those thinkers, such as Aristotle, who have rejected the existence of ideal objects, mathematics concerned empirical objects. In particular, geometrical objects are just sensible substances regarded only with respect to their extension. Thus, geometric concepts such as 'sphere', 'cube', etc., then are universals whose instances are the extensions of physical objects. Of course, if one adopts this position with respect to geometry, one must reject (as did Aristotle) those objects which we regard as basic in geometry, namely points, line and surfaces, which in no way can be regarded as extensions of sensible substances. The early fourteenth century nominalist Ockham adopted Aristotle's view in connection with geometry and attempted to deal with the problem of the apparent need to refer to points, lines and surfaces in geometry. His general program was to demonstrate that propositions that, on the face of it, seemed to imply the existence of universals, ideal or mixed things, can be reformulated in specific contexts in a way that no longer makes reference to them. (See for example [Ockham, 2009, p. 21f].) In particular, he attempted to do this in the case of points, lines and surfaces by showing that propositions about them could be understood as propositions about sensible objects that 'approximate' them. For example, he would rephrase the proposition that a sphere and a plane tangent to it have just one point in common as the proposition that they touch, but no part of the plane is a part of the sphere. ■ A systematic foundation for this idea was attempted in the early twentieth century by Alfred North Whitehead with his 'method of extensive abstraction' ([1919, Part III], [1920, Chapter IV], [1929, Part IV]) for constructing spacetime. According to Whitehead's construction, a geometric object such as a point, curve, etc., is an *abstraction class* of the things we can actually perceive that intuitively 'contain' it. There is a difference in that, for Whitehead, the things we can perceive are 'events' or 'regions' (depending on the version of his method one follows) in spacetime rather than sensible substances. Also, Whitehead's motivation was somewhat different from Ockham's in that his aim was to explain how the domain of the precise objects of geometry is connected to the world of things that we can perceive, the events say, with their 'ragged edges' [Whitehead, 1920, p. 50]. However, the abstraction classes—they are 'mixed objects' in the terminology we introduced above—are not themselves extensions of physical objects, and so Whitehead's construction falls short of

the demands of nominalism. Nevertheless, by taking the abstractive classes to be suitable classes of sensible substances rather than of events or regions, Whitehead's construction provides a ground for Ockham's elimination of reference to ideal geometric objects: a proposition about these objects is always to be understood as a proposition about elements of the corresponding abstraction classes.

4. Ideal Objects in Mathematics.

There is a problem with the Aristotelian conception of mathematical objects already in connection with classical Greek geometry. As Plato had already observed, things in the world of appearance both are and are not or (to repeat) as Whitehead more recently put it, nature observed has ragged edges. If truths of geometry are simply to be truths about sensible substances limited to the language of extension, then the precision demanded by Greek geometry is lost. Plato noted in the *Phaedo*, 74a9-13, that typical statements about geometric objects, namely that they are equal (or, as we say, 'equal in magnitude'), cannot be understood as statements about natural things, unless we have already *idealized* nature. For example, the statement that, at this instant (already an idealization), my pen has a definite ratio in length, given by a real number, to the standard meter bar is not one that anyone would want to defend. As we now know, by the time we get to trying to measure how many billionths of a centimeter are contained in them, both objects have dissolved into ill-defined swirls of atoms. ■ Of course, we can often establish *inequalities* in magnitude between sensible things A and B , say $A < B$ —for example, I'm taller than my pen. But when we ask for the exact *difference* in length, that makes no real sense. Yet in Greek geometry, if a magnitude A is less than a like magnitude B , it is assumed that there is a unique magnitude $C = B - A$, i.e. such that $A + C = B$.⁹ Likewise one of the great discoveries of Greek mathematics (perhaps late 5th century BCE?), namely that there are incommensurable linegments such as the side and diagonal of a square, has no meaning in terms of empirical objects. My pen and the standard meter bar are commensurable in length to within any degree of accuracy it makes sense to speak of. Sensible substances and, likewise, Whitehead's events or regions do not have precise boundaries; and that is why Whitehead's project, like Ockham's, fails: it requires the question of whether two events or regions

⁹For the Greeks, the geometric objects a of the various kinds—the bounded lines, surfaces, and solids—were the magnitudes. For simplicity I am taking them to be the ratios $A = a : u$ between the object a and a fixed 'unit' object of the same kind.

are extensively connected (i.e. whether they touch not) to be well-defined, thus betraying the very ragged edginess that he was attempting to reconcile with the exactness of geometry.

It was on just such grounds that Plato insisted on the ideal, on what Aristotle called the “separate forms”. Precisely what makes them ‘separate’—and this was the source of Aristotle’s complaint about them—is that the truths concerning them are not truths about empirical things. Of course, this is not to say that geometry cannot be applied to natural phenomena. (The sensible figures ‘participate’ in the forms.) In fact the known history certainly suggests that geometry and arithmetic grew out of empirical concerns—exchanging goods, surveying and parcelling land, building alters, etc. But reason is a subversive thing. Start with the Pythagorean theorem for isosceles triangles, that the square on the diagonal is twice the square on the side, the fact that in a numerical ratio $m : n$ m and n can be assumed relatively prime, and the fact that, when the positive integer n is odd, then $n \times n$ is odd, all of which had been long known and none of which in its own right would challenge the view that mathematics is just about natural objects and their relations. Now put them together and infer that the ratio of the side to the diagonal is irrational—a statement that has no meaning in terms of natural objects.¹⁰ However useful geometry may be in dealing with the sensible world, in the hands of reason, its truths outrun that world.

And when we turn to the mathematics as it has developed through the last two centuries, Aristotelian abstraction is obviously inadequate as a foundation: from what do we abstract an infinite set? So the ancient skepticism rubs off on the actual infinite, not because it is infinite, but because it is ideal.

5. Nominalism and Revisionist Conceptions of Mathematics.

Nominalism is certainly a very radical position and so one must believe that, rare on the ground though the nominalist may be, undercutting the skeptical argument on which he stands would be of some value. But it isn’t only nominalism that is at issue: there have been, from at least the beginning of the twentieth century, more moderate, but still quite restrictive, positions concerning the nature of mathematics, such as predicativism and constructivism, that draw their rational in part from the skeptical argument. I am not referring to the development of predicative mathematics or constructive

¹⁰One might speculate that, contrary to Aristotle’s suggestion, it was the existence of incommensurable line segments rather than simply Heraclitus’s influence that led Plato to the view that the objects of exact science are ideal.

mathematics in its own right: each can be understood as a part of mathematics (in the unfettered sense) that is, each in its own way, of intrinsic interest. Rather I refer to the view, supported by Poincaré and (at one time) Weyl, that mathematical objects (other than the natural numbers) *must be* definable without circles (see [Poincaré, 1906; Weyl, 1918; Weyl, 1919] and the view, supported by Brouwer, Weyl (at a later time) and Bishop, that mathematical objects *must be* ‘constructible’ by us (see [Brouwer, 1908; Brouwer, 1913; Brouwer, 1927; Weyl, 1921; Bishop, 1967; Bishop and Bridges, 1985]). In both cases there is an attempt to appease the specter of skepticism by drawing the domain of mathematical objects in some sense closer to us.¹¹

To be sure, both predicativity and constructivism drew support in the early twentieth century from the so-called ‘paradoxes of set theory’, which were regarded as object lessons against overstepping the bounds of what we can really know. But, with the development of a better understanding of these ‘paradoxes’, with the clarification of the notion of ‘set’ involved in terms of ϵ -structures (see [Gödel, 1947]), and the understanding they arise from the assumption that *essentially potential* infinities, such as the domain of all ordinal numbers or of all sets, are actual infinities, this particular support for revisionist versions of mathematics has considerably weakened. But for many the support from skepticism remains. For both Brouwer and Weyl, in particular, the assurance of consistency of classical (i.e. non-constructive) mathematical theories would in any case not be enough: it would still remain to give real meaning to the terms of the theory that justify its existence claims.

6. A Deeper Skepticism.

Those who subscribe to the skeptical argument mentioned above in support of nominalism or revisionist mathematics, namely that we do not causally interact with ideal objects, tend to ignore a venerable and more radical skepticism that cuts deeper: it challenges even our knowledge about natural objects. For it notices that our talk about interaction with natural objects already *presupposes* that there are such things and that, absent that presupposition, all we really have to talk about are at our end of the supposed interactions, namely sensory inputs. The structure that is imposed on these inputs in order to give meaning to our statements about the natural world

¹¹It should be noted, too, that a more radical resistance to the nineteenth century transformation of mathematics and its acceptance of the actual infinite began even earlier with Kronecker’s finitism. Kronecker’s reaction to the breakdown of geometric foundations was to abandon geometry and any part of function theory that could not be represented by computation with the natural numbers.

does not derive from the inputs themselves (cf. *Theaetetus* 184b4-185e2), and one can reasonably ask why one should be less skeptical about this structure than about the ‘pure’ structure posited in mathematics.¹²

This deeper skepticism, which has led to various forms of idealism and solipsism, and can easily lead to further, more radical, views according to which even the self dissolves into isolated instants of sensations, memories, and feelings, certainly demands a response from us, an indication of where the skeptical argument goes wrong, of why the premise that our supposed interactions with the natural world can be challenged on the grounds that all we really have are sensory inputs does not yield the conclusion that we can have no knowledge of the natural world.

I have already argued in outline in [1986a] that the skeptical argument leading to idealism and solipsism is parallel to that in support of nominalism and revisionism in mathematics and that the same path that leads to the collapse of skepticism about natural objects and other minds leads to the collapse of skepticism about ideal objects, too. The two cases are, to be sure, not completely parallel. In the case of knowledge concerning sensible objects, we may speak of two kinds of knowledge: knowledge *of* or *acquaintance with* objects on the one hand, and propositional knowledge—knowledge *that* or knowledge *about* objects—on the other. We won’t discuss the concept of knowledge *of* further; but the thing to notice is that it is restricted to objects that we can perceive (and so, incidentally, will exclude many things to which both some present day nominalists and advocates of causal theories of meaning would ascribe existence) and, notwithstanding a dubious reading of a passage in [Gödel, 1964, p.271], the view that ideal objects can be known by something like acquaintance, is surely neither widespread nor plausible. Indeed, it is in part this difference that is the starting point of skepticism about ideal objects. Bernays, in his paper “Sur le platonisme dans les mathématiques” [1935], speaks in this respect of the ‘remoteness’, of the objects of classical mathematics. But the possibility of ‘presence’, although a feature *sometimes* of the physical objects, living things, people,

¹²This is very likely part of what Gödel meant when he wrote

That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the ideas of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the “given” underlying mathematics is closely related to the abstract elements contained in our empirical ideas. [1964, pp. 271-2]

etc., that we speak about, cannot be the *sine qua non* of meaning. That is what the deeper skepticism tells us. The Augustinian picture of meaning, which lies behind the belief that presence is in some way essential to meaning, is a picture that we can draw only after we have language—at least *some* language—already in place. So when we ask about ultimate meanings, we must look elsewhere than to this picture. For it is not an object *simpliciter* that is present, but an object embodied or, in a sense I want eventually to make clear, ‘*constituted*’ in language.

7. The *Investigations*.

What is needed is a conception of meaning on the basis of which the descriptive or representational use of language, highlighted by the Augustinian picture, becomes intelligible without the presupposition of a pre-linguistic relation of reference connecting the word to the thing. And, on my reading, that is just what Wittgenstein develops throughout the *Investigations*. Wittgenstein’s idea is that we should look at language, not primarily from the point of view of semantics, but as a particular form of social interaction, and understand the meaning of linguistic expressions in terms of how they are used in these interactions. This is the point of view he introduces in §§1-32 in the midst of his critique of the Augustinian picture and that he continues to develop and defend throughout the *Investigations*.

Quine was quite right to point out in his paper “Ontological relativity” [1969, pp. 26-68] that roughly the conception of language in the *Investigations* was anticipated in a sense by Dewey, for example in Chapter V of *Experience and Nature* [1929], entitled “Nature and Communication”. But Quine’s easy dismissal of Wittgenstein in his essay is unjustified. A very important part of the contribution of the *Investigations* is the argument that ostension cannot be the source of the sought-after prelinguistic relationship between name and thing named, because ostensive gestures can be acts of naming only in the context of an established language—that the full meanings of the words first introduced by gestures of this kind are not exhausted by being the target of the gestures, since the latter are ambiguous in many respects as to exactly what their targets are:

...an ostensive definition can be variously interpreted in *every* case. (The *Investigations*, §28)

When Quine writes “I am not worrying, as Wittgenstein did, about simple cases of ostension,” and then refers to Wittgenstein’s discussion of the color word “sepia” in §30 as an example, he is ignoring the profound discussion of

ostension as a means of fixing reference in the early sections of the *Investigations* and, in particular, in §§28-35. It is clear to anyone who has seriously read these passages that Wittgenstein fully understood the difficulty with ostensive definition. Quine contends that Wittgenstein was dealing only with simple cases of ostension because “sepia” is a mass term and Quine believes that a deeper difficulty with ostensive definition concerns sortal terms (terms with ‘divided reference’, as Quine calls them.) But in §28 Wittgenstein also includes, for example, the ostensive definition of the number 2, as the common name (i.e. sortal) of couples of distinct things. Quine’s point about sortals can be illustrated with his own example, “gavagai”, which is the term (sortal) for ‘rabbit’ in an exotic language. The point is that, when pointing to a rabbit, I am always pointing also to a time-slice of a rabbit and to an undetached rabbit part. So ostension in itself can never determine which of these are really the denotation of the term “gavagai”. When we point to a color, on the other hand, although it is true that we are at the same time pointing to a shape, a surface, etc., we can always vary these while still pointing to the color. Quine writes

The color word “sepia”, to take one of his examples, can certainly be learned by an ordinary conditioning process, or induction. One need not even be told that sepia is a color and not a shape or a material or an article. (p. 31)

But, contrary to Quine’s contention, “sapia” in fact presents the same difficulties as “rabbit”. When I point to a color, I am always pointing to a colored surface—vary its shape as you will—and to the thing with that surface and to an undetached part of that surface and to an undetached part of that thing with that surface and to a time-slice of one of these, etc.. So does “sepia” mean a particular color, sepia, or a sepia colored surface, or a thing with a sepia colored surface or ... ?

Concerning Dewey’s priority, Quine was also wrong when he wrote (p. 27) “When Dewey was writing in this naturalistic vein, Wittgenstein still held his copy theory of language.” We are speaking of the year 1929, more or less, and Wittgenstein no longer held the views expressed in the *emph-Tractatus*.

But more importantly, Quine’s evaluation is unjustified because it was Wittgenstein and not Dewey—nor Quine—whose understanding of language frees us from skepticism about ideal things. Indeed, Dewey was, throughout his book, doing battle with the idea of just such things, and we know Quine’s view on this. When one thinks that the only practice in town is navigating and understanding the natural world around us, as both Dewey

and Quine did, then there is no room for the study of mathematical structures independently of their empirical applications or, indeed, of whether or not they have empirical applications. The autonomy of reason (which is the essence of true Platonism) is lost. We see this in Quine's view that all truths are revisable, even those of logic and mathematics, in the face of empirical evidence.

For both Dewey and Quine, language is anchored in the natural world, and, by failing to see the sense in which even natural objects are constituted in our language about them, they failed to see that the ideal can also be constituted in language.

8. The Objectivity of Meaning and Understanding.

The account of the meaning of linguistic expressions and understanding (grasping) the meaning of expressions developed in the *Investigations* takes both of them out of the mind and locates them, each in its own way, in the public domain. In this respect it challenges the views of many contemporary writers in philosophy of language and of mind, including both philosophers who count themselves to be supporters of what they take to be its views on this topic and some who count themselves to be in opposition. The account in the *Investigations* in particular rejects the view that meaning and understanding fall outside nature, the domain of natural science, as we usually understand that term. Wittgenstein himself may have been skeptical of the possibility of a cognitive science,¹³ but despite some decades of questionable or premature claims made for this science by philosophers, computer scientists and linguists, it would be quite foolish to question its importance, much less its legitimacy, and I would quite definitely not want to follow Wittgenstein there. Rather, I think that a consequence of his analysis (as opposed, possibly, to his prejudices) is the separation of two questions: *What are meaning and understanding?* on the one hand, and *What mechanisms account for them?* on the other. (Cf. especially *Investigations* §149.) The former question asks for conceptual clarification, the latter is in the domain of natural science: it asks for causal explanations—for example of how languages develop and evolve and how we, individually, develop linguis-

¹³The accusation that he was skeptical about cognitive science has its source, as far as I know, in remarks in *Zettle* [1970, §§608 ff]. But in those passages he is primarily questioning whether all psychological phenomena can be investigated physiologically. When I speak of cognitive science here, I am happy to leave open the question of whether or not it can be pursued entirely in physiological terms. Wittgenstein's doubts have in any case certainly turned out to be largely ill-founded, but then they were part-and-parcel with a more general rash and negative conservatism.

tic competence. It was the former question that concerned Wittgenstein in the *Investigations*. There has been some *explicit* confusion in contemporary philosophy of mind between concept and cause, but I believe that the confusion also *implicitly* underlies many of the objections to the the *Investigations* account of understanding as well as many of the misunderstandings of it, without having been brought out into the open.¹⁴

Although a consequence of Wittgenstein's analysis is that meaning and understanding are objective phenomena, there for all of us to inspect, this by no means is intended to exclude the possibility that the terms in which we might account causally for our understanding might involve the psychological in a way that is not reducible or at least not known to be reducible to reference to the brain. If we take physicalism to be the doctrine (roughly) that all mental phenomena reduce to physical phenomena, I am not arguing for or against physicalism. But I do want to endorse what I take to be Wittgenstein's message, that the phenomena *to be explained* do *not* involve the mental. One may think that, on the contrary, the phenomenon of linguistic behavior involves an essential intentional element; but although an intention might be part of the cause of a linguistic act, it is not an intrinsic part of it. For I thin we want my intention to explain why I acted as I did; but the intention must be prior to the act.

I should stop here to say here that, when I use the term "act" without qualification, I am using it in the extensional sense: throwing the switch and turning on the lamp are the same act. This contrasts with the intensional sense, according to which a person's acts are individuated also in terms of his or her intentions. (Please don't be confused by the juxtaposition of "intensional" and "intentional". They are related here only because the intensional element in the intensional notion of 'act' is an intention.) And I use the term "disposition" to refer simply to a propensity and not to some cause of the propensity. The only criterion for whether a coin is disposed to

¹⁴There is one version of this confusion which, although wrong, makes a certain sense. Secondary qualities such as colors of physical objects have a physical explanation and the meaning of color terms have shifted from referring to the secondary quality to referring to the cause, to the physical source, to wave lengths: when we want to test for color, we now test for wave length. But when Fodor writes in opposition to Ryle that "intelligent behavior *is* intelligent because it has the etiology that it has" [1975, p. 3], the situation is somewhat different. Leaving aside the difficulties with his own theory of what the causes are, however optimistic one might feel about the possibility of finding a mental or physiological explanation of understanding or of intelligent behavior in general, there is no such explanation now: there is nothing to shift the meaning of "understanding" to! And so we still have to keep in mind what it is that needs explaining while we are looking for the explanations.

turn up heads 50% of the time is how it behaves, more or less, in tests. We may account for what the coin does in terms of its symmetry, but that is a causal explanation of its disposition, not the disposition itself. This is an important terminological point, since Wittgenstein in the passage §149 just cited uses the term “disposition” for the cause and that usage has persisted in the literature on Wittgenstein. (Cf. [Kripke, 1982] for example.) On my use of the term “disposition,” when I speak as I did just now of a disposition to act, it refers only to the propensity to act in the extensional sense of “act,” where the only criterion for someone having that disposition is that he or she acts in accordance, more or less, with it.

There are of course two ways to understand the role of intention in acts. One is causal: “I do such-and-such because I intended to”; but where there is no claim that the act itself has an intentional component. In *Intentions* [1957, p. 1] Elizabeth Anscombe makes the distinction between *intention to do* and *intention in doing*. The position taken here, as well as on behalf of Wittgenstein, is that the relation of the intention *to do* to the act can only be that of a cause. The response to the possible objection that the relation may rather be that of a *reason* and that this is *not* a cause is, in brief, essentially Gilbert Ryle’s [1946; 1949] and, ultimately, Lewis Carroll’s¹⁵ question: What mediates the relation between the reason, which is presumably the belief in some principle—a proposition, and the act?

One may or may not believe of our cognitive behavior that, in contrast perhaps with other forms of life, explanations in terms of the mind (e.g. in terms of intentions to do) are ultimate and will not, in the end, be reduced to explanations that make no reference to mental phenomena. A preliminary consideration against this view is that, after all, we witness the behavior of members of other species and are inclined in the same way to describe it in terms of their intentions. But I conjecture that few of us would think that, in describing the behavior of a mosquito landing on the back of one’s neck, the reference to its intentions is ineliminable. But my target at the moment is the other way in which the role of the intentional in linguistic behavior has been understood, namely that the linguistic act of understanding or of saying or writing something involves *in itself* an intentional component and that it is that component that makes it a linguistic act. So the linguistic act in this sense is not an act in the extensional sense I want to use the term; it is an act *armed with* an intention. This roughly corresponds to Anscombe’s notion of *intention in doing*. These two conceptions of the role of the intentional in linguistic acts, the intention as (be)cause and intention

¹⁵In “What the Tortoise said to Achilles”.

as intrinsic to the act, sometimes get confused in the contemporary literature on philosophy of mind ■; and so here we again have an example of the confusion of the causal with the conceptual. Also, I think in this case that the confusion is most often only implicit: the obvious way in which intentions often very well serve to *explain* our acts becomes grounds for the idea that a linguistic act is itself an intensional act—a kind of action at a distance: the action resulting from my intention is simultaneous, indeed intrinsic to, the action itself. It is this latter way in which linguistic acts are thought to involve the intentional that I want, following Wittgenstein, to contest—the view that the acts themselves are intrinsically intentional and so the question of whether someone understands becomes a question at least in part about mental states. This latter would not conflict with the thesis that understanding is a public phenomena providing that the mental states in question simply amounted to dispositions to act in certain ways; for in that case, the only criteria for the states would be how the subject acts, and that is there for all to observe. But of course that minimalist conception of the role of the mind in linguistic action is not at all what is involved in the thesis that linguistic acts are intentional. Rather that thesis is motivated by the perceived need to make a distinction between linguistic acts and, more generally, meaningful acts and rational acts, on the one hand, and those acts that are just conditioned response, on the other, where the distinction in question is to be made in terms of what is intrinsic to the act.

It follows from the argument of the *Investigations* that this distinction between meaningful acts and conditioned responses is *not* intrinsic to the act itself but rather arises from the nature of the surrounding circumstances, circumstances that are open to all to observe. I speak English correctly (when I do) or compute or reason correctly (ditto) because I am, under suitable conditions, disposed to do it that way. Among those conditions may be my own prior intentions. These are prior and count as (be)causes and not something intrinsic to the act. However, to defend the claim that the question of whether or not I understand is open for all to evaluate, we must see that my intentions, too, *in the relevant respect*, are open for all to evaluate.

9. Wittgenstein and Scepticisms.

Whether or not my interpretation of Wittgenstein's position is in every respect correct or not, there is no doubt that he, himself, felt that his view dissolved skeptical arguments about ontology. Thus, he wrote

For *this* is what disputes between idealists, solipsists and realists look like. The one party attacks the normal form of expression as if they were attacking a statement; the others defend it, as if they were stating facts recognized by every reasonable human being. §402

To avoid misunderstanding, though, I should probably mention at the outset that I *am not* following Wittgenstein's lead on the nature of mathematics. Suffice it to say that his conception of mathematics did not match the state of the subject in his time, and so his criticism of what was part of the warp and woof of mathematics by the 1920's (see §426, for example) is in conflict with his statement that philosophy "leaves mathematics as it is" (§124).¹⁶ In particular, the nominalist should be listed with the idealist and the solipsist in the camp of the transcendental "they don't exist" in §402 and the correlative realist about mathematical objects should be included along with the realists about physical objects and about other minds in the camp of the transcendental "they do exist". There is some irony here, though, in that the ultimate aim of this paper, expanding the argument in (Tait 1986), is to defend ordinary mathematical practice as it developed through the nineteenth and twentieth centuries against skepticism about mathematical objects. The argument leads us somewhat far a field to a position on the nature of meaningful discourse, before leading finally to §402. But there we find conspicuously absent from the catalogue of empty skepticisms the one it is my concern to dissolve.

But notice that it is not just skepticism that Wittgenstein is addressing in §402; he is complaining, too, about the realist, who believes that there is a transcendental notion of existence according to which physical objects, other minds or, *pace* Wittgenstein, ideal objects, exist—just as the skeptic believes that there is a transcendental notion of existence according to which they don't. His argument in the large is that, if we look at our language as it is, an instrument of communal interaction, we will understand notions such as that of object, existence and truth as notions that take their meaning *in* the language, not as a part of a scaffolding on which the language is built.

Leaving aside Wittgenstein's own conservatism and, in particular, his peculiar time-warp in connection with mathematics, there is a wonderful liberation implicit in §402. For those of us who have found distasteful the constraint and demand for humility bound up with Kant's response to skepticism—reason restricted to its 'proper' object, the natural world given to us in intuition, and the basic laws of this world hard-wired into its

¹⁶We discuss this further in Section ??.

givenness—the vision is most welcome that the objects of a domain of discourse and the basic laws governing them are constituted in the discourse. We are indeed in this respect, as Dedekind wrote [1888a], a divine race: we created and can create the discourse.

There is another form of skepticism that has been attributed to Wittgenstein, notably by Kripke [1982] and less explicitly by earlier writers, namely skepticism about meaning. To give an example, the skeptical problem as Kripke sees it is to explain how it is that I can mean *plus* (i.e. addition of natural numbers) by the symbol “+” and not, say, some other function *quus* on the natural numbers when the two functions agree when applied to any numbers $> k$, where k is the greatest number to which I have ever applied +.

I have discussed this in [1986b], where I argue both that Kripke’s reading of Wittgenstein is mistaken and that the skeptical problem itself is based on a confusion. Kripke’s problem is about what *I* mean by “plus” and his argument is that nothing in me can count as warrant for new cases of addition. But what I mean is parasitic off what the word “plus” means in English; and it is in its English usage that warrant abides. Kripke ascribes this problem to Wittgenstein, but this is again a mistake. In the passage to which Kripke refers as stating the problem, namely §201, Wittgenstein is speaking about understanding, and he refers to a paradox that has already been resolved. The paradox lay in thinking that, behind our understanding of or grasping meaning, say of a rule, must lie something else that we understand—so that there is an infinite regress. The paradox has already been dissolved for him, however, with the observation that our understanding is nothing more than the disposition to act in accordance with the rule. (We will come back to this point.) There is no ‘skeptical paradox’ requiring skeptical solutions or otherwise, either in *Investigations* or in fact.

10. Meaning.

There are three formulas involved in Wittgenstein’s philosophy that are well-known. The first and central one

Meaning is use

was not quite stated by him in the *Investigations* §43, but suitably understood it is accurate enough. The central thesis is that, when we speak of the meaning of an expression in a language, say English, usually what we are saying will ultimately cash out as a statement about how the expression is correctly used in English.

Of course, English usage of an expression may change over time and may

vary from one region of the English speaking world to another. Even at a given time and place, usage can differ because there are different linguistic communities of, broadly speaking, English speakers at the same time and place. So to avoid this complication, we should understand the term “language” to be used always relative to a suitably circumscribed community of users. Naturally, what counts as ‘suitable’ here will depend upon the expression in question, and even then we cannot think of the circumscribed language as a precisely defined object with respect to which all questions of correct use have determinate answers. So I am using the term “language” in the same sense that many writers on Wittgenstein use the term “practice” or more specifically, “linguistic practice”.

So understood then, the question of linguistic meaning reduces ultimately to questions of correct usage. Of course, the qualification ‘ultimately’ is essential here: We often ask for the meaning of an expression and are content with a synonym or, if the expression is not in our language, with a translation. But finally such dictionary definitions come to an end, and the end consists of a list of examples of core uses of the expression.

Notice that, in speaking of meaning, I am referring to what the *expression* means (e.g. as an expression in English), not what I or any other particular person mean by it. I take this as the basic sense of the term. We can speak of *my meaning* of an expression, but I can only tell you (or myself) about it in some language common to us, perhaps using synonyms or examples—or perhaps using gestures. But all of these are meaningful only in the context of an established language. There are writers, such as Donald Davidson [1974], Jonathan Bennett [1976] and John Searle [1983], for whom explicitly the notion of personal meaning is the primitive one: we all have our own languages or *idiolects*, and the possibility of communication rests upon a handshake across idiolects—we *interpret* one another, according to Davidson.¹⁷ Let me be clear that the issue here is not about so-called ‘private languages’. There is no argument that in principle a person could not have a private language, nor do I think that Wittgenstein intended to give one: *His* argument was directed at the idea of a meaning-giving act—that I, at one time and in one nonlinguistic act, give meaning to a sign for myself. (Of course, *having a language*, I can make up my own private definitions of expressions.) Rather the issue is whether, for example, *English* can be understood in terms of a community of idiolects. I won’t dwell on this further

¹⁷There are also writers who, without explicit statement, implicitly hold the view that language is essentially idiolect, since when they speak of the meaning of an expression they are speaking of its personal meaning rather than its public meaning.

here (the reader may want also to consult [Dummett, 1991b, pp.86-88]); but an important part of the argument against such a view starts with a conception of meaning that is Fregean to the extent that one takes the meaning of a sentence to be a function of the meaning of its constitutive expressions and the truth-value of a sentence to be a function of its meaning. Then, for example, if the only non-trivial belief that I had about Groucho Marx was that he wrote *Das Kapital*, it cannot be *my* meaning of the term “Groucho Marx” that determines the truth-value of the sentence “Groucho Marx wrote *Das Kapital*.”

Anyway, in normal cases of communication there is *no evidence* of the kind of interpretation that Davidson postulates. For example, when we read or listen to someone speaking on the radio, there is no sense of entering into a handshake with the author—of interpreting him or her: it will often be someone we don’t know and are in no way focused on. Rather the situation is that he or she writes or speaks in English and what we read or hear is *the English*. Providing that a speaker’s English is reasonable, we understand the spoken words, *not* their author. It is only in the relatively rare cases, such as those in which the speaker’s English is in some way sufficiently idiosyncratic, that we pay attention to him or her and have the sense of ‘interpreting’. But even this is not entirely like the case of interpreting what is expressed in one language into another: it is rather the case of accepting a particular sequence of sounds *as* English, but having difficulty recognizing exactly *what* English is intended. Davidson’s introduction of a mental process of ‘interpreting’ to account for (public) meaning is typical of the bootleg cognitive science that frequently occurs in philosophy of mind.

11. Existence.

One natural companion to the central formula, and the one most pertinent to the ulterior aims of this paper, is a paraphrase of part of §10 of the *Investigations*:

What a word signifies is shown by the kind of use it has.

From the context in §10, it is clear that “signifies” here has or at least includes the sense of “refers to”, in which case what is signified is an object. This passage is itself in two respects an enrichment of Frege’s so-called ‘context principle’

... we ought always to keep before our eyes a complete sentence. Only in a sentence have the words really a meaning. It is enough if the sentence taken as a whole has a sense; it is this that confers

on its parts also their content. [Frege, 1884, §60]

In one sense §10 is an enrichment because, as Michael Dummett has pointed out, it fills in one of “two lacunas in Frege’s account, at opposite ends: one at the end of sense, and the other at that of a speaker’s grasp of sense.” Dummett writes about the first of these:

[On one account of how to fill these lacunas], to be found in its clearest and most explicit form in Wittgenstein, the sense of an expression consists in its role within the complex social practice constituting the communal use of the language, a practice open to view and not itself involving any hidden mental operations. [1991, p. 16]

The notion of sense is analyzed or, at least in its role in determining reference, is replaced by the notion of use, and its source is located in the communal practice, the language. I will have occasion later to mention the other sense in which the passage in §10 is an enrichment of the context principle.

As we have already noticed, one *prima facie* objection to our central formula “meaning is use” is precisely that it *does* lead to this new formula and that, therefore, it fails to account for the semantical/descriptive or *representational* use of language. The argument is that at least one use of language is to describe reality; and in this context, the norms for correct usage must derive from that reality and not from our linguistic practice. Wittgenstein concentrates on this objection in the early part of the *Investigations*, especially §§1-32. In part his response is that language has many diverse uses, not all of which can be reduced to this representational use; and it is reasonable to believe that there is a unitary conception of meaning, not one conception that applies to the representational use of language and others that apply to these other uses. But the major part of his response to the objection is to argue that it is based upon an erroneous picture of the relation between descriptive language and reality. According to this picture, which we have referred to as the ‘Augustinian picture’ reality is given to us as a well-defined structure independent of our language, that learning language is just a matter of learning the proper names of elements of that structure, and that what is grammatical or true is merely read off what is given to us. Contrary to this picture, Kant had already convincingly argued that not even our ideas of object, space, time, causation and logic (including what Kant called “*demonstration*”), which inform what we take to be ‘reality’, are read off in this way. For him, the objects of our experience are

constituted in that experience. Kant's view presumed far more hard-wiring of the human mind than turned out to be plausible and furthermore implied a restriction on the admissible ontology that is inadequate for mathematics and science as it developed in the nineteenth century. The message of the *Investigations*, summarized in the passage from §10 that is paraphrased above, is in this respect a correction of Kant: Objects are constituted, not not in our minds, but *in our language*.

Note that the modular role of the sentence in Frege's account of meaning disappears in Wittgenstein's. This is inevitable, given the latter's more general concern for language as opposed to simply its representational use. But it follows ultimately even for Frege; for the context principle implies that the reference (truth or falsity) of a sentence (and so its sense, as well) also depend upon the sense of *its* sentential contexts, and so Frege's account of the meaning of an expression finally depends upon *a whole system of sentences*, with their logical relationships and external criteria for truth and falsity. Thus, it is this system in its entirety that ultimately confers meaning on the individual expressions.¹⁸ But, and this was Dummett's point, Frege nowhere discusses the provenance of this system and it was left to Wittgenstein to do so. And, when he does, lost is the Fregean sense of one monolithic language in which Reality is described; it is replaced with the comparison of our language with an

ancient city: a maze of little streets and squares, of old and new houses, of houses with extensions from various periods, and all this surrounded by a multitude of new suburbs with straight and regular streets and uniform houses. §18

What should be concluded from Wittgenstein's discussion of the 'Augustinian picture' is that no adequate account of meaning can be derived from a prior notion of reference, from the view that meaning derives simply from the correlation of the word and the thing for which it stands.

Here the word, there the meaning. The money and the cow one can buy with it §120.

Taking off from Frege's lead, he argues that what our words stand for is determined by how they are used.

But there is an important distinction to be made here concerning the word "determined": one may say that it is the structure of the fishnet,

¹⁸This ultimate holism does not, of course, preclude the possibility of 'local' rules, such as the rules of grammar and logic which delimit correct use.

how fine or coarse the weave, that determines the kinds of fish that will be caught. But it should be clear that that is not what Wittgenstein meant; nor, I would argue, was it what Frege meant by saying that sense determines reference. Rather, the sense of “determines” here is more like the sense in which the sculptor determines the statue or the cookie-cutter the shape of the cookie. So to speak, objects or, better, kinds of objects *are constituted* in language. On this basis, we can begin to see how the conception of language understood as a social interaction meets the demand that we be able to describe reality in language. Objects and kinds of objects *are constituted* in language, not simply picked out by means of it. But this has often been completely misunderstood in discussions of the context principle. Thus Tyler Burge writes in “Frege on knowing the third realm” [1992]

Some philosophers have suggested that Frege’s use of the context principle somehow suggests a qualification on his Platonism. Issues surrounding Frege’s context principle(s) are, of course, extremely subtle and complex. But it seems to me that the suggestion must involve some confusion. The context principles govern relations between linguistic expressions and their senses or referents. They do not bear directly on the nature of the senses or referents themselves at all. (p. 640 n. 11)

Burge’s use of the plural seems to be at least partially motivated by a possible ambiguity concerning whether content in propositions determines the sense of a word or simply its reference. Anticipating our discussion of this below, I think it clear that, although Frege had not yet stated his principle that sense determines reference, he had to be referring to sense—and *therefore* reference. But the point I want to make now is that we can see from the last two sentences of the quoted passage that, in terms of our metaphor of the net and the cookie-cutter, Burge understands the context principle as postulating that the senses of the sentential contexts of a name constitute a *net*, which ‘catches’ the sense or reference of the name (and no other). It is no wonder then that he has so little sympathy for that principle. Properly understood, it contains, not a qualification of Frege’s ‘Platonism’, but (as should have been made explicit by Wittgenstein in §402) a *disqualification* of the Platonism/antiPlatonism issue as meaningful.

From our point of view, we can also see the proper content of Quine’s formula “To be is to be the value of a bound variable”. For Quine, it was a criterion for the ontological commitment of a theory, in the sense that the theory could be wrong on *external* grounds simply because it is committed to objects that are just not there. Of course an empirical theory can be wrong

on the internal grounds that its predictions about what we should observe are wrong; but Quine intends his criterion to apply also in the case of ideal objects. (Although he speaks of ‘theories,’ this notion of theory is intended to encompass mathematical English in which we talk about numbers, sets, functions and the like.) For us, the proper understanding of Quine’s formula in that context—the understanding that derives from Frege’s and Wittgenstein’s insight—is that, on the contrary, there is *no* external, transcendental notion of existence upon which to judge the theory or language. What “existence” means is defined by the use of the existential quantifier *within* the language. To reject the objects is to reject the language which gives meaning to sentences referring to them: it is to reject the natural form of expression. There can certainly be grounds, such as inconsistency, for rejecting the language, but they are not the external and essentially meaningless grounds that the objects in question just don’t happen to exist.

I mentioned above that there is a second sense in which Wittgenstein’s §10 is an enrichment of Frege’s context principle. For Frege as for Quine, there was, it seems, just one language, one meaning of existence, one truth. (That is why it wasn’t for him just *silly* to question whether Julius Caesar is the number 0. (See [Tait, 2005, p. 9].) For him, the problem was that Kant had not provided a rich enough universe of existents. Namely, for Kant objects must be representable in sensible intuition, whereas Frege needed also the existence of courses-of-values of functions in order to found arithmetic. The context principle was employed precisely for this in [Frege, 1893, §§29–31]. His argument for it in [Frege, 1884] attempts to bridge the gap between Kant’s ontology and his own requirements by pointing out that even some empirical terms, e.g. “Earth”, are not understood by us entirely in terms of a representation of them in intuition. So, for him, the context principle simply does repair work; it is a tool employed in his argument that courses-of-values are already there in the ontology that is already there, implicit in our judgements.

There is in Frege’s ken no trace of the creative spirit of Plato, Cantor, Dedekind, and Hilbert, according to which we pick out a domain of discourse by setting out the language, the discourse, concerning it. Indeed, Frege insistently opposed Hilbert on just this issue in the case of the latter’s foundations of geometry. And he never understood that Dedekind had provided a completely adequate foundation for number theory by writing down (essentially) the axioms which uniquely characterize the system of whole numbers. On the other hand, in spite of Wittgenstein’s mistaken attitude towards mathematics, his conception of language and existence is amenable to just this conception. Indeed, he almost gets it right when he writes

... new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten, (We can get a *rough picture* of this from the changes in mathematics.) §23

This creative conception of language, which seems antithetical in spirit to Frege, is precisely what is needed to accommodate the axiomatic conception of mathematics, as we understand it. One has to wonder whether the Platonism (now in the real sense) of mathematics, developing through the nineteenth and twentieth centuries, somehow got through to him.

12. Understanding.

The other companion thesis to the formula, ‘meaning is use’, fills Dummett’s other lacuna:

Understanding or ‘grasping’ meaning is simply a competence.

This is what Wittgenstein meant when he wrote in §201 of the *Investigations* that

For what we thereby show is that there is a way of grasping a rule which is *not* an interpretation, but which, from case to case of application, is exhibited in what we call “following the rule” and “going against it.” §201

Dummett formulated the thesis in a continuation of the passage quoted above from [Dummett, 1991, p.16]:

An individual speaker’s grasp of that sense then becomes one ingredient in his ability, acquired by training, to engage in that practice. On this approach, if any explanation were needed of a possession of this ability, it would not belong to the philosophical order, but would properly pertain to psychology; such an explanation would be altogether irrelevant to a philosophical account of linguistic understanding, and hence of thought.

Parenthetically, this passage, in its entirety, is a wonderfully clear synopsis of Wittgenstein’s conception of meaning and understanding—and reflects a quite different understanding of both Frege and Wittgenstein on these topics than do some of Dummett’s earlier writings. But, while he agrees with Wittgenstein on the notion of meaning or sense, Dummett rejects his analysis of understanding or grasping the meaning. McDowell (e.g.

in [McDowell, 1984]), while also holding that the first formula (roughly) expresses Wittgenstein's view and agreeing with Dummett in rejecting this last formula concerning understanding, also denies that it is what Wittgenstein meant.

When I speak of a *competence*, I am referring to a disposition to act and react in suitable ways on suitable occasions. Recall that on my usage, the term "disposition" refers only to the way in which the agent does or would act under various conditions—i.e. to a *propensity*—and not to any underlying mental or physical state or mechanism which might account for why the agent behaves in that way. In §149 Wittgenstein unfortunately uses the term "disposition" for the underlying state in terms of which we might explain the behavior. But it is precisely in that passage that he makes the distinction that I am drawing here, between understanding as a disposition (in the proper sense) and the mental or brain state in terms of which we might account for it. His argument is that, if we don't make this distinction, then we have two criteria for understanding: what the subject does in actual cases on the one hand (which is the only criterion for it *qua* disposition), and his or her mental or brain state on the other.

It follows that the agent's understanding is a mental state only in whatever sense a propensity to act is such a state: The only criterion for an agent being in that state is what he or she does or would do under suitable circumstances. In particular, understanding is not an intentional state. That there are mental or brain states and mechanisms that figure into an explanation of the agent's understanding is to me a reasonable—indeed, an overwhelmingly reasonable—conjecture. But we want to separate that question, of giving a causal explanation of the agent's understanding, from the question of what that understanding is in itself.

Understanding then is a species of what Ryle in [1946] and subsequently in *The Concept of Mind* [1949, Chapter 2] termed "knowing *how*" (as opposed to knowing *that*, i.e. propositional knowledge). The most telling passage of Wittgenstein's account of understanding, namely §201 immediately concerns only the narrow notion of grasping a rule, of being able to follow it; but it is clear that Wittgenstein, too, intends his analysis to extend to understanding language generally and, beyond that, to knowing how. Thus he writes

The grammar of the word "know" is obviously closely related to the grammar of the words "can," "is able to." But also closely related to that of the word "understand." (To have mastered a technique.) §150

Interestingly, there is an obvious logical connection between Wittgenstein's observation

[t]hat there is a misunderstanding here is shown by the mere fact that in this chain of reasoning we place one interpretation behind another . . . §201

and Ryle's earlier argument in [1946] that the assumption that competent action (i.e. in accordance with the knowledge how) involves a 'practical inference from proposition (e.g. statement of the rule) to action (obeying it) leads to an infinite regress. Ryle's argument (following Lewis Carroll) is that a new proposition stating the connection between the given proposition and the action must intervene.

I believe with Ryle that in the most important and primary sense, knowing is knowing how. Knowing in this sense is not a state of mind or brain—again, except in the sense that dispositions are. Certainly, when we test students to find out what they know, we don't do brain scans or psychological analysis; we pose to them problems or questions and judge their knowledgeability by their responses. Notice, too, that this point of view has the very reasonable consequence that there is generally no *yes* or *no* answer to the question of whether one understands or knows: it is a matter of degree. This accords very well with how we generally think about knowing and understanding: I understand addition of whole numbers quite well. I can add two numbers if the notation for them is simple enough, and will do so correctly a reasonable proportion of times. But there is a bound on the complexity of the notations beyond which I would not be able to process them, and even well below that bound, I would likely make at least occasional errors. The Cartesian conception of knowing as a state of mind which one is either in or not in at best oversimplifies this concept as we encounter it in ordinary life, when we are not making up stories about the mind. Not only do we not do brain scans to test for the students' knowledge: we also do not generally give pass/fail grades: We give grades on a scale to indicate a level of knowledge. Nor is the student's result on an examination to be understood merely as a symptom of the student's level of knowledge, in the way that body temperature can be a symptom of a disease. Knowledge *is* the ability to perform, and that *is* what is being tested.

It is certainly true that we often have the *feeling* of understanding or knowing, the feeling that the light is on, that 'I understand', 'I can go on,' and there is a tendency to identify that feeling with understanding or at least to think of it as an intrinsic ingredient of understanding. But often that feeling is wrong. My own experience is that, alas, there is no difference

in the quality or strength of conviction in the cases that subsequently turn out to be totally mistaken and, in some cases, totally incoherent, from those in which it turns out that I indeed did understand. Certainly, no matter how strong my momentary conviction that I understand may be, if it turns out that I cannot make the appropriate responses, you and I are going to agree that I was wrong—that I *didn't* after all understand. So there is no *logical* connection between the sense or feeling of understanding and understanding. (Cf. the *Investigations*, §§151-5.) But there is every reason to suppose that there is a *causal* one. As we would expect, the inner signals of understanding seem to improve the more experienced we are and the better we are innately at the activity in question, but they are not infallible. On the other hand, in most cases in our intellectual life, there is no such feeling of understanding at all. As far as I can recall, what was going on in my head while writing this last sentence had little to do with the content of the sentence and more to do with searching for the appropriate keys on my keyboard. (Yes, I have to look.)

The term “understanding” or “grasping meaning” has a narrow sense, as when we ask whether someone understands the word or the rule and we agree that the person understands to the extent that (s)he can actually apply it. In the widest sense of “understanding”, we may ask whether someone understands English or number theory. But expressions are expressions only in a language and inherit their meaning from the role they play in the language. So understanding in the narrowest sense implies understanding in a wider sense of more or less understanding a ‘neighborhood’ in the language. (Cf. the *Investigations* §199) Our thesis is that when we are speaking of understanding in either the narrow or wider sense, we are speaking of a disposition to act and react more or less appropriately in a narrower or wider range of linguistic contexts; in other words, we are speaking of linguistic *competence*. Of course, the term “competence” carries with it more than the term “disposition”. I may have a disposition to twitch my nose under suitable stimulus, but there is no question (other than an aesthetic one) of whether I am right or wrong in doing so. The notion of a competence, on the other hand, involves an idea of correctness and incorrectness; there are grammatical-logical norms (in the extended sense in which I am using that expression) for linguistic acts and linguistic competence is disposition in accordance with these norms. But, on our view, there is nothing intrinsic to the disposition that makes it a competence: that is entirely a matter of the *extrinsic* fact that the disposition is in accordance with the norms. Likewise, there is nothing intrinsic to an action in accordance with the disposition that makes it a linguistic act. There is nothing intrinsic to the act of

speaking or writing a word that makes it a word in English, it is the external circumstances in which the (extensional) act is performed that makes it as linguistic act. In other words, for us to be in Sellar's *logical space of reasons* [Sellars, 1963], which in one version or another has been an anchor for many who reject this naturalistic account of understanding, is not to be outside nature; it is to be in a particular kind of natural setting involving particular kinds of interactions. I have suggested in "The myth of the mind" [2002] that a better name for this space would be the "*space of meanings*," i.e. the space of linguistic practice. These interactions are observable *as linguistic interactions* not only by us, the members of the linguistic community, but also by others who have the concept of a language. (In just this way have linguists deciphered exotic languages and languages from the remote past.)

13. Rights and Wrongs.

As I have already mentioned, the view I am presenting has been rejected both as a correct account in its own right and as Wittgenstein's account of the notion of understanding. One reason, perhaps the most compelling one, is that it seems to reduce the roll of the competent language user to that of an automaton, who makes all the right linguistic moves, but in the way in which a particle might make all the right moves through a force field, not *obeying* the grammatical-logical rules but rather being merely *described by* them. bhyBut I think that a complete answer to this objection is to repeat the distinction between the conceptual and the causal: although understanding in itself is to be understood in terms of dispositions to act, there may be *reasons* why those dispositions come into play, why we act as we do; and contrary to those who wish to assign reasons to a different category than that of causes, it is the latter category to which they belong. Logic concerns the relation between propositions or perhaps judgements; there is no logical relation between the proposition or judgement and the act in accordance with it. (Perhaps it is the belief that there is such a logical relation that provides a significant part of the motivation for the intensional concept of an intentional act: if there is a logical connection between my judgement that I should turn on the light and what I do, then what I do has to be understood as 'turning on the light' and not as 'throwing the switch.')

So on the view we are developing, norms may function as *causes* of our linguistic acts; they are not somehow intrinsic to the act. The fact is, that in much of our intellectual life norms play no role in what we do, other than the role that they played in our initial linguistic or logical training. What I am likely to be thinking about as I balance my checkbook is not the rule

for adding numbers, but how much money I spent and how little I have left. As Wittgenstein wrote :

When I follow a rule, I do not choose. I follow the rule *blindly*.
§219

Indeed, although there is no clear dividing line between when an infant is merely copying and practicing sounds or reacting to them merely as sounds and when it is really behaving linguistically,¹⁹ clearly we learn to speak and understand our mother tongue long before we are explicitly aware of any rules governing it (and in this respect I think it is safe to assume ontogeny recapitulates phylogeny). To pretend that speaking and understanding at that stage is somehow not language-using is to seriously beg the question. In general, it may be true to say that we are guided in our linguistic usage by norms, but usually that is true only in the sense that we have been trained in accordance with them and we act more or less in accordance with them, not that our each linguistic act is performed in their light. The sense in which that training provides us with a ‘second nature’ just *is* that we have been so trained; it is not some extra-natural mantel of “mindedness” that the training adds to our (‘first’) nature.

There is of course a sense that our understanding of language is not merely a matter of being able to use language, but also involves the capacity to *evaluate* what we or others are doing as to its correctness. Or a better way to put it is that normative terms are a part of our language and a more or less fully competent language user will have mastered their use, too. But this is quite different from the causal role that norms sometimes immediately play in our linguistic acts and the imagined role as reasons-not-causes that have been ascribed to them. And, although the two abilities are obviously related, being able to understand or, say, add numbers is distinguishable from being able to evaluate the attempts of oneself or others to do so. For example, a simple adding machine can do the first (better than we can) but hasn’t even the capacity to do the second.

One may suspect that part of the impetus to deny a genuinely naturalistic account of grammatical-logical norms is a reaction to the sheer complexity of such a multi-layered system of competences. But this seems to parallel the impetus from the sheer complexity of some organic structures to deny a naturalistic account of the evolution of species and accept some form of Intelligent Design, or the impetus from the truly remarkable internal coherence of mathematics or its “unreasonable effectiveness in the natural

¹⁹In this respect, compare the discussion of reading at §156 f.

sciences,” to deny a naturalistic account of the development of mathematics and ascribe it to some Platonic or Leibnizian *nous*.

Of course, grammatical and logical laws must on Wittgenstein’s account be put in the same box as other expressions: if we accept that a sentence is true in virtue of the meanings of the constituent terms and that to speak of the meaning of a term is (roughly) to speak of how it is used, then we must say that, even in the case of laws of grammar or logic, they are true in virtue of how we use the constituent terms—and *that is surely a contingent matter*: we might have used the constitutive terms differently. Indeed, we might have done so even with much of the actual background of their usage in place. For example, it is a consequence of Wittgenstein’s critique of rule following that we might, while accepting the standard rules of grammar and logic, have nevertheless rejected judgments that, as a matter of fact, for us follow from those rules. It seems that it is this circumstance that has led many writers to ascribe to Wittgenstein an empiricist account of normative statements and so a skeptical account of grammatical-logical norms.

But there is a confusion here: what we may conclude from Wittgenstein’s argument to be contingently true is not this or that sentence but rather *the fact that we use the constitutive terms in the sentence in the way we do*. But, indeed, the fact that we use *any* expression in the way that we do is *always* a contingent matter. This in no way implies that, using the constituent expressions in the way that we do, we are not expressing a necessary truth. Of course, this implies that there has in fact to be some constancy in usage in order for there to be grammatical and logical norms; indeed, without some such constancy, there would not be a viable notion of truth. In this sense, grammatical and logical necessity are founded on what is contingent; but this does not destroy their necessity. It is an empirical contingent fact that “ $2 + 2 = 4$ ” is a sentence and expresses a truth, but it is not an empirical or contingent fact that $2 + 2 = 4$. This is at least part of what Wittgenstein meant when he wrote

So you are saying that human agreement decides what is true and what is false?”—It is what human beings say that it true and false; and they agree in the language they use. That is not an agreement in opinions but in a form of life. §241

In Wittgenstein’s writings in the 1940’s and 50’s, there is no hint of skepticism about logical norms. Aside from the passage just quoted, I don’t know of passages in the *Investigations* explicitly addressing this issue, but there is extensive discussion of it in Part VI of the second edition of *Remarks on the Foundations of Mathematics* [Wittgenstein, 1978]. (Part VI was written

in the period 1941-44, right up to the time Part I of the *Investigations* was composed and is entirely consonant with the sections of the *Investigations* on rule following.) Thus:

The reason why “If you follow the rule, this is where you’ll get to” is not a prediction is that this proposition simply says: “The result of this calculation is —” and that is a true or false mathematical proposition. The allusion to the future and to yourself is mere clothing. §15.

And this series is defined by this rule. Or again by the training in proceeding according to the rule. And the inexorable proposition is that according to this rule this number is the successor of this one. And this proposition is not an empirical one. §16

The “Or again by the training in proceeding according to the rule” is perhaps troublesome; but I think that he is just taking account of the fact that the rule is a rule only in the context of a practice and that one is initiated into the practice by suitable training. But, anyway, the final sentence seems conclusive.

Should we acknowledge the rule $25^2 = 625$, if we did not all arrive at this result? Well, why then should we not be able to make use of the empirical proposition instead of the rule?—Is the answer to that: Because the contrary of the empirical proposition does not correspond to the contrary of the rule? When I write down a bit of a series for you, that you then see this regularity in it may be called an empirical fact, a psychological fact. But if you have seen this law in it, that you then continue the series in this way—that is no longer an empirical fact. §26

We see that his answer to the question posed is “Yes”:

“ $25^2 = 625$ ” cannot be the empirical proposition that people calculate like that, because $25^2 \neq 625$ would in that case not be the proposition that people get not this but another result; and also it would be true if people did not calculate at all. §30

We are told by Wittgenstein scholars that he had intended the *Investigations* to contain more on the philosophy of mathematics, and one would assume that, had he been able to include more on this subject, this picture of mathematical necessity would have been developed and made explicit. It remains to be pointed out that, although grammatical norms are both more flexible

and more changeable than logical and mathematical norms, the same considerations apply to them. When we call a linguistic construction grammatical or ungrammatical, we are generally not making an empirical assertion: in fact, we are usually making a statement that, based upon what the majority of speakers of the language actually do, is likely to be utterly false.

References

- Anscombe, G. [1957]. *Intentions*, Ithica: Cornell University Press.
- Benacerraf, P. [1973]. Mathematical truth, *The Journal of Philosophy* **70**: 661–680. Reprinted in [Benacerraf and Putnam, 1964, 403–420].
- Benacerraf, P. and Putnam, H. (eds) [1964]. *Philosophy of Mathematics: Selected Readings*, first edn, Cambridge University Press. First edition 1983.
- Bennett, J. [1976]. *Linguistic Behavior*, Cambridge: Cambridge University Press.
- Bernays, P. [1935]. Sur le platonisme dans les mathématiques, *L'enseignement mathématique* **34**: 52–69. English translation by Charles Parsons in [Benacerraf and Putnam, 1964, 274–286].
- Bishop, E. [1967]. *Foundations of Constructive Analysis*, New York: McGraw-Hill.
- Bishop, E. and Bridges, D. [1985]. *Constructive Mathematics*, Berlin: Springer-Verlag.
- Brouwer, L. [1908]. De onbetrouwbaarheid der logische principes, *Tijdschrift voor Wijsbegeerte* **2**: 152–158.
- Brouwer, L. [1913]. Intuitionism and formalism, *Bulletin of the American Mathematical Society* **20**: 81–96.
- Brouwer, L. [1927]. Über Definitionsbereiche von Funktionen, *Mathematische Annalen* **97**: 60–75.
- Burge, T. [1992]. Frege on knowing the third realm, *Mind* **101**: 634–650. Reprinted in [Tait, 1997].

- Burgess, J. and Rosen, G. [1997]. *A Subject with no Object: Strategies for Nominalistic Interpretation of Mathematics*, Oxford: Oxford University Press.
- Davidson, D. [1974]. On the very idea of a conceptual framework, *Proceedings and Addresses of the American Philosophical Association* 47. Reprinted in [Davidson, 1984].
- Davidson, D. [1984]. *Inquiries into Truth and Interpretation*, Oxford: Clarendon Press.
- Dedekind, R. [1872]. *Stetigkeit und irrationale Zahlen*, Braunschweig: Vieweg. in [Dedekind, 1932]. Republished in 1969 by Vieweg and translated in [Dedekind, 1963].
- Dedekind, R. [1888a]. Letter to Weber, *In [Dedekind, 1932]* .
- Dedekind, R. [1888b]. *Was sind und was sollen die Zahlen?*, Braunschweig: Vieweg. In [Dedekind, 1932]. Republished in 1969 by Vieweg and translated in [Dedekind, 1963].
- Dedekind, R. [1932]. *Gesammelte Werke, vol. 3*, Braunschweig: Vieweg. Edited by R. Fricke, E. Noether, and O. Ore.
- Dedekind, R. [1963]. *Essays on the Theory of Numbers*, New York: Dover. English translation by W.W. Berman of [Dedekind, 1872] and [Dedekind, 1888b].
- Dewey, J. [1929]. *Experience and Nature*, New York: Dover Publications.
- Dummett, M. [1978a]. Platonism, *Truth and Other Enigmas*, Cambridge: Harvard University Press, pp. 202–214.
- Dummett, M. [1991]. *Frege: Philosophy of Mathematics*, Cambridge: Harvard University Press.
- Dummett, M. [1991b]. *The Logical Basis of Metaphysics*, Cambridge: Harvard University Press.
- Ewald, W. (ed.) [1996]. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Oxford: Oxford University Press. Two volumes.
- Field, H. [1980]. *Science without Numbers*, Princeton: Princeton University Press.

- Fodor, J. [1975]. *The Language of Thought*, Cambridge: Harvard University Press.
- Fodor, J. [2008]. *Lot 2: The Language of Thought Revisited*, Oxford: Oxford University Press.
- Frege, G. [1884]. *Grundlagen der Arithmetik*, Breslau: Verlag von Wilhelm Koebner. German/English edition entitled “The Foundations of Arithmeti”, Translation into English by J.L. Austin, Oxford: Basil Blackwell (1950).
- Frege, G. [1893]. *Grundgesetze der Arithmetik: Begriffsschriftlich abgeleitet, Band I*, Jena: H. Pohle. Reprinted in 1962 along with [Frege, 1903] by Hildesheim:Georg Olms.
- Frege, G. [1903]. *Grundgesetze der Arithmetik: Begriffsschriftlich abgeleitet, Band II*, Jena: H. Pohle.
- Gödel, K. [1947]. What is Cantor’s continuum problem?, *American Mathematical Monthly* **54**: 515–525. Reprinted in [Gödel, 1990]. [Gödel, 1964] is a revised and expanded version.
- Gödel, K. [1964]. What is Cantor’s continuum problem?, *in* Benacerraf and Putnam [1964], pp. 258–273. Revised and expanded version of [Gödel, 1947]. Reprinted in [Gödel, 1990]. In the second edition of [Benacerraf and Putnam, 1964], the pages are 470–485.
- Gödel, K. [1990]. *Collected Works, Vol. II*, Oxford: Oxford University Press.
- Hale, B. and Wright, C. [2001]. *The Reason’s Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics*, Oxford: Oxford University Press.
- Higgenbotham, J. [2006]. *Languages and idiolects: their language and ours*, Oxford: Oxford University Press, pp. 140–148.
- Kripke, S. [1972]. *Naming and Necessity*, Cambridge: Harvard University Press.
- Kripke, S. [1982]. *Wittgenstein on Rules and Private Language: An Elementary Exposition*, Cambridge: Harvard University Press.
- Lewis, D. [1970]. General semantics, *Synthese* pp. 18–67. Reprinted in [Lewis, 1983].

- Lewis, D. [1983]. *Philosophical Papers*, Vol. 1, Oxford: Oxford University Press.
- Mancosu, P. (ed.) [1998]. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's*, Oxford: Oxford University Press.
- McDowell, J. [1984]. Wittgenstein on following a rule, *Synthese* **58**: 325–363. Reprinted in [McDowell, 1998a].
- McDowell, J. [1998a]. *Mind, Value & Reality*, Cambridge: Harvard University Press.
- Ockham, W. o. [2009]. *De Sacramento Altaris*, Eugene, Oregon: WIPF and STOCK. Edited, translated, with an Introduction, by T. Bruce Birch.
- Pinker, S. [1994]. *The Language Instinct*, New York: William Morrow and Company, Inc.
- Poincaré, H. [1906]. Les mathématiques et la logique, *Revue de métaphysique et de morale* **14**: 17–34. Translation by George Bruce Halsted in [Ewald, 1996], Vol. 2, pp 1038-52.
- Quine, W. [1953]. *From a Logical Point of View*, Cambridge: Harvard University Press.
- Quine, W. [1960]. *Word and Object*, Cambridge: The M.I.T. Press.
- Quine, W. [1969]. *Ontological Relativity and Other Essays*, New York: Columbia University Press.
- Quine, W. [1986]. Reply to Charles Parsons, pp. 396–401.
- Ryle, G. [1946]. Knowing how and knowing that, *Proceedings of the Aristotelian Society, XLVI*.
- Ryle, G. [1949]. *Concept of Mind*, London: Hutchinson and Company, Ltd.
- Schirn, M. (ed.) [1996]. *Frege: Importance and Legacy*, Berlin: de Gruyter.
- Searle, J. [1983]. *Intentionality: An Essay in the Philosophy of Mind*, Cambridge: Cambridge University Press.
- Sellars, W. [1963]. Empiricism and the philosophy of mind, pp. 127–196.

- Tait, W. [1986a]. Truth and proof: the Platonism of mathematics, *Synthese* **69**: 341–370.
- Tait, W. [1986b]. Wittgenstein and the ‘skeptical paradoxes’, *The Journal of Philosophy* **83**: 475–488. Reprinted in [Tait, 2005], pp. 198–211.
- Tait, W. [2002]. The myth of the mind, *Topoi: An International Review of Philosophy* **21**(1-2): 65–74.
- Tait, W. [2005]. *The Provenance of Pure Reason: Essays in the Philosophy of Mathematics and Its History*, Oxford: Oxford University Press.
- Tait, W. (ed.) [1997]. *Early Analytic Philosophy: Frege, Russell, Wittgenstein. Essays in honor of Leonard Linsky*, Chicago: Open Court. Reprinted in [Schirn, 1996, pp. 70–113].
- Tarski, A. [1936]. Der Wahrheitsbegriff in den formalisierten Sprachen, *Studia Philosophica* **1**: 261–405.
- Weyl, H. [1918]. *Das Kontinuum: Kritische Untersuchungen über die Grundlagen der Analysis*, Leipzig: Veit. English translation in [Weyl, 1987].
- Weyl, H. [1919]. Der cicculus vitiosus in der heutigen Begründung der Analysis, *Jahresbericht der Deutschen Mathematiker-Vereinigung* **28**: 43–50. English translation in [Weyl, 1987].
- Weyl, H. [1921]. Über die neue Grundlagenkrise der Mathematik, *Mathematische Zeitschrift* **10**: 39–79. Translated by P. Mancosu in [Mancosu, 1998].
- Weyl, H. [1987]. *The Continuum: A Critical Examination of the Foundation of Analysis*, The Thomas Jefferson Press (and in a Dover edition 1994). Translation by Stephen Pollard and Thomas Bole.
- Whitehead, A. [1919]. *The Principles of Natural Knowledge*, Cambridge University Press.
- Whitehead, A. [1920]. *The Concept of Nature*, Cambridge University Press.
- Whitehead, A. [1929]. *Process and Reality*, New York: MacMillan.
- Wittgenstein, L. [1970]. *Zettel*, Berkeley: University of California Press.

Wittgenstein, L. [1978]. *Remarks on the Foundations of Mathematics*, 2 edn, Cambridge: MIT Press. ed. G. H. von Wright, et. al., tr. G. E. M. Anscombe.