COMPUTING A DIAMETER-CONSTRAINED MINIMUM SPANNING TREE

by

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ABSTRACT

In numerous practical applications, it is necessary to find the smallest possible tree with a bounded diameter. A diameter-constrained minimum spanning tree (DCMST) of a given undirected, edge-weighted graph, G, is the smallest-weight spanning tree of all spanning trees of G which contain no path with more than k edges, where k is a given positive integer. The problem of finding a DCMST is NP-complete for all values of k; $4 \leq k \leq (n - 2)$, except when all edge-weights are identical.

A DCMST is essential for the efficiency of various distributed mutual exclusion algorithms, where it can minimize the number of messages communicated among processors per critical section. It is also useful in linear lightwave networks, where it can minimize interference in the network by limiting the traffic in the network lines. Another practical application requiring a DCMST arises in data compression, where some algorithms compress a file utilizing a tree data-structure, and decompress a path in the tree to access a record. A DCMST helps such algorithms to be fast without sacrificing a lot of storage space.

We present a survey of the literature on the DCMST problem, study the expected diameter of a random labeled tree, and present five new polynomial-time algorithms for an approximate DCMST. One of our new algorithms constructs an approximate DCMST in a modified greedy fashion, employing a heuristic for selecting an edge to be added to

the tree in each stage of the construction. Three other new algorithms start with an unconstrained minimum spanning tree, and iteratively refine it into an approximate DCMST. We also present an algorithm designed for the special case when the diameter is required to be no more than 4. Such a diameter-4 tree is also used for evaluating the quality of other algorithms. All five algorithms were implemented on a PC, and four of them were also parallelized and implemented on a massively parallel machine—the MasPar MP-1. We discuss convergence, relative merits, and implementation of these heuristics. Our extensive empirical study shows that the heuristics produce good solutions for a wide variety of inputs.

To my parents

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LIST OF SYMBOLS

$\forall$	the universal quantifier
$\lvert S \rvert$	the number of nodes in tree S, (if S is a tree), or the number of elements in set S (if S is a set).
b	the bandwidth of a line in a network
D	the maximum node-degree in the tree
e	a small positive constant
r	number of leaves adjacent to a given bicenter node in a tree of diameter 3
$O(g(n))$	bounded by $g(n)$
$W(g(n))$	bounded below by $g(n)$
a, b, c	nodes
a_1, a_2, a_i, a_s	diameters from a tree-diameter set, described in Chapter 1
$Adj(v)$	nodes adjacent to v within the given set of edges
BFST	breadth-first spanning tree
C	the set of edges near the center of the spanning tree
CIR	the composite-iterative-refinement algorithm
d	the diameter of a spanning tree
$D_d(x)$	the enumerator (generating function) for all trees with diameter d
$\overline{d}_n$	the expected value of diameter for a labeled tree with n nodes
DCMST	diameter-constrained minimum spanning tree

DCMST(k)	diameter-constrained minimum spanning tree with diameter no more than k
$dist(u, v)$	the (unweighted) distance from node u to node v
$distc(l)$	the distance of edge l from the center of the tree, plus 1
$dists(i)$	the distance from the source node (or edge) to node i
e	Napier's constant ≈ 2.71828···
E, E', E^*	sets of edges
$ecc(u)$	the eccentricity of node u: the maximum distance from node u to any node in the same tree
$ecc_T(u)$	the eccentricity of node u with respect to tree T
ERM1, ERM2	the edge-replacement methods used by Algorithm IR2
G, G', G^*	graphs
$G_h(x)$	the enumerator (generating function) for all trees with height ≤ h
h	the height of a tree
$H_h(x)$	the enumerator (generating function) for all trees with height = h
h_v	a function that determines the candidate "Hubs" in the algorithm developed by Paddock and described in Chapter 1
i	[in Chapter 2] a subscript
i	[in all chapters except for Chapter 2] a node
IR1	the first general-iterative-refinement algorithm
IR2	the second general-iterative-refinement algorithm
j	a node
k	the given constant-bound on diameter
l	an edge
L	the lightest excluded-edge in the algorithm developed by Paddock and described in Chapter 1

l_c	the center edge of a tree of odd diameter
l_{ij}, l_{sj}	boolean decision variables, used in the mixed-integer-linear-programming formulation developed by Achuthan *et al*. and described in Chapter 1
$\ln n$	the natural logarithm (logarithm base e) of n
$\log n$	the logarithm of n with any constant base
m	the number of edges in the given graph
m_1, m_2, m_h	variables
$M(P)$	a problem of finding the smallest tree isomorphic to prototype P
$MAX\{\ldots\}$	the maximum of two or more values
$\underset{a \in A}{MAX}\{\ldots\}$	the maximum of values calculated for all a in set A
$\underset{a \in A}{MIN}\{\ldots\}$	the minimum of values calculated for all a in set A
MIPS	million instructions per second
MST	minimum spanning tree
n	the number of nodes in the given graph
$near(u)$	a tree node candidate to be incident to u in the approximate DCMST
NSM1, NSM2	node-selection methods used with Algorithm OTTC
NSM3	a node-selection method used with Algorithm OTTC
$O(g(n))$	bounded above by $g(n)$
OTTC	the one-time-tree-construction algorithm
p	the number of processes allowed to be executing in their critical section at the same time
P	a prototype
q	the number of selected start nodes in Algorithm OTTC
r	the radius of a graph

s	a source node in a directed graph
SIMD	single instruction-stream over multiple data-streams
t	the time required to traverse a link in a network
$t_n(h)$	the number of labeled rooted-trees with n nodes, a specified root, and height no more than h
T_v	a breadth-first spanning tree
T_g	a tree from a sequence
$\langle T_g \rangle$	a sequence of trees
u, v	nodes
V, V'	sets of nodes
v_c	the center node of a tree of even diameter
$W\langle u, v \rangle$	the weight of the directed edge from u to v
$W(u, v)$	the weight of edge (u, v)
$W(T)$	the weight of tree T
$w_1, w_2, \ldots, w_n$	weights of edges, also used in referencing edges
$w_{\max}$	the largest edge-weight in the current spanning tree
$w_{\min}$	the smallest edge-weight in the current spanning tree
X	a bitmap vector
x	[in Chapter 2] a variable (used in generating functions)
x	[in all chapters except for Chapter 2] a node
y, z, z_0	nodes

CHAPTER 1

INTRODUCTION

The Diameter-Constrained Minimum Spanning Tree (DCMST) problem can be stated as follows: given an undirected, edge-weighted graph, G, and a positive integer, k, find a spanning tree with the smallest weight among all spanning trees of G which contain no path with more than k edges. The length of the longest (unweighted) path in the tree is called the *diameter* of the tree.

This problem was shown to be NP-complete by transformation from the Exact Cover by 3-Sets problem [33]. Let n denote the number of nodes in G. It can be easily shown that the problem can be solved in polynomial time for the following four special cases: $k = 2$, $k = 3$, $k = (n - 1)$, or when all edge weights are identical. As stated in [33], the other cases are NP-complete, even when edge weights are randomly selected from the set $\{1, 2\}$. We consider G to be connected; where the edge-weights of G are non-negative numbers, randomly chosen with equal probability, following the Erdös-Rényi model [28].

1.1 Motivation

The DCMST problem has applications in several areas, such as in distributed mutual exclusion, linear lightwave networks, and bit-compression for information retrieval. In distributed systems, where message passing is used for interprocessor communication, some algorithms use a DCMST to limit the number of messages. For example, Raymond's algorithm [22, 66] imposes a logical spanning tree structure on a network of processors. Messages are passed among processors requesting entrance to a critical section and processors granting the privilege to enter. The maximum number of messages generated per critical-section execution is $2d$, where d is the diameter of the spanning tree. Therefore, a small diameter is essential for the efficiency of the algorithm. Minimizing edge weights reduces the cost of the network. A fault-tolerant protocol was introduced by Revannaswamy and Bhatt [69] as an extension to this algorithm. The protocol makes the algorithm tolerant to single node/link failure and associated network partition. This is done by utilizing non-tree edges without increasing the upper bound on the number of passed messages.

Satyanarayanan and Muthukrishnan [74] modified Raymond's original algorithm to incorporate the "least executed" fairness criterion and to prevent starvation, also using no more than $2d$ messages per process. In a subsequent paper [75], they presented a distributed algorithm for the *readers and writers* problem, where multiple nodes need to access a shared, serially reusable resource. In this distributed algorithm, the number of

messages generated by a *read* operation and a *write* operation has an upper bound of $3d$ and $2d$, respectively.

In another paper on distributed mutual exclusion, Wang and Lang [86] presented a token-based algorithm for solving the *p-entry critical-section* problem, where a maximum of p processes are allowed to be in their critical section at the same time. If a node owns one of the p tokens of the system, it may enter its critical section; otherwise, it must broadcast a request to all the nodes that own tokens. Each request passes at most $2pd$ messages.

The critical section protocol presented by Seban [76, 77] keeps a bound on the logical clock using a distributed predictive "clock squashing" mechanism. For simplicity, he assumed a constant-size message requiring a constant time t to traverse a network link, thus modeling the network by an undirected graph. For a general graph whose spanning tree has diameter d, it takes $d \cdot t$ time to transfer a message between the two most distant nodes, and each node must process messages to and from all adjacent nodes. Therefore, the protocol has control-section access-time performance $\Theta(MAX\{d, D\}) \cdot t$, where D is the maximum node-degree in the tree. The worst-case performance occurs for linear and star graphs, where a star is a tree with at most one non-leaf. It is desirable to construct the spanning tree with diameter and maximum degree in $O(\log n)$, where n is the number of nodes in the graph. This problem is related to the DCMST problem, especially if the time taken by a message to traverse an edge is variable. A solution to the DCMST problem can be a spanning tree with the smallest total edge-traversal time, and diameter $O(\log n)$. If the maximum degree is not $O(\log n)$, the tree must be refined into the desired form.

For such cases, a new algorithm needs to be developed to refine the node degrees of the DCMST without increasing its diameter.

In addition to its applications in distributed mutual-exclusion, a DCMST is useful in information retrieval, where large data structures called *bitmaps* are used in compressing large files. It is required to compress the files, so that they will occupy less memory space, while allowing reasonably fast access. Bookstein and Klein [17, 18] proposed a preprocessing stage, in which bitmaps are first clustered and the clusters are used to transform their member bitmaps into sparser ones that can be more effectively compressed. This is done by associating pairs of bitmap vectors such that the occurring of a 1-bit in a vector increases the likelihood of a 1-bit occurring in the same position in the other vector. The association is made by an XOR operation of the pair, creating a new vector with a smaller number of 1's. The number of 1's in the resulting cluster is called the *Hamming distance* between the two associated bitmap vectors. A small Hamming distance is desirable when compressing bitmap vectors because a smaller number of 1-bits allows more efficient compression. Between the two associated bitmap vectors, the vector with the higher number of 1's is discarded. The other vector may be associated again, with a new bitmap vector, in the same manner. All non-discarded vectors and clusters are compressed. An efficient clustering method starts by generating a complete edge-weighted-graph structure on the bitmaps, where the nodes represent bitmaps, and the weighted edges represent Hamming distances. Then, the method uses a spanning tree of this graph to cluster and compress vectors along the paths from a chosen node, the root, to all the leaves of the spanning tree. To recover a given bitmap vector, X,

it is required to decompress all nodes in the path from the root to X. Therefore, a spanning tree with a small diameter can provide high-speed retrieval. However, the total Hamming distance of the spanning tree must be low in order to conserve storage space. Consequently, a DCMST provides the necessary balance between access speed and storage space.

The DCMST problem also arises in linear lightwave networks, where multi-cast calls are sent from each source to multiple destinations. It is desirable to use a spanning tree with a small diameter for each transmission to minimize interference in the network. An algorithm by Bala *et al*. [14] decomposes a linear lightwave network into edge disjoint trees with at least one spanning tree. The algorithm builds trees with small diameters by computing trees whose maximum node-degree is less than a given parameter, rather than by optimizing the diameter directly. Furthermore, the lines of the network are assumed to be identical. If the linear lightwave network has lines of different bandwidths, lines of higher bandwidth should be included in the spanning trees to be used more often and with more traffic. Employing an algorithm that solves the DCMST problem can help find a better tree decomposition for this type of network. The network can be modeled by an edge-weighted graph, where an edge of weight $1/\boldsymbol{b}$ is used to represent a line of bandwidth $\boldsymbol{b}$.

1.2 Existing Algorithms for the DCMST Problem

Three exact-solution algorithms for the DCMST problem, developed by Achuthan *et al*. [5], were based on a mixed-integer-linear-programming formulation of the problem [4, 6]. Branch-and-Bound methods were used to reduce the number of subproblems. The algorithms were implemented on a SUN SPARC II workstation operating at 28.5 MIPS. The algorithms were tested on complete graphs of different orders ($n \leq 40$), using 50 cases for each order, where edge-weights were randomly generated numbers between 1 and 1000. The fastest of the three algorithms with diameter bound $k = 4$ produced an exact solution in less than one second on average when $n = 20$, but it took an average of 550 seconds when $n = 40$.

Subsequently, Achuthan *et al*. [7] developed a better mixed-integer-linear-programming formulation of the DCMST problem, thus improving the three branch-and-bound algorithms. The improved mixed-integer-linear-programming formulation distinguishes the cases of odd and even diameter-constraint, k. Let $W(i, j)$ denote the weight of undirected edge (i, j), and $W\langle i, j\rangle$ denote the weight of the directed edge from i to j. For the DCMST being constructed, let l_{ij} denote a decision variable that takes the value 1 when edge $\langle i, j\rangle$ or $\langle j, i\rangle$ is in the spanning tree, and 0 when neither edge is in the spanning tree. When k is even, the following formulation is used to solve the DCMST problem. Extend the given graph $G = (V, E)$ to a directed graph $G' = (V', E')$, where $V' = V \cup \{s\}$, and E' is obtained by replacing each edge $(i, j) \in E$ with the directed edges

$\langle i, j\rangle$ and $\langle j, i\rangle$, with $W(i, j) = W\langle i, j\rangle = W\langle j, i\rangle$, and adding a zero-weight edge $\langle s, u\rangle$ for every $u \in V$. The mixed-integer-linear-programming formulation for constructing a DCMST with diameter no more than k is: Construct a graph $G^* = (V', E^*)$, $E^* \subseteq E'$, which minimizes:

$$f = \sum_{i, j \in V'} W\langle i, j\rangle\, l_{ij} \tag{1.1}$$

subject to the constraints:

$$\sum_{j \in V} l_{sj} = 1, \tag{1.2}$$

$$\sum_{i \in V', j \in V, i \neq j} l_{ij} = 1, \tag{1.3}$$

$$dists(i) - dists(j) + (k/2 + 1)l_{ij} \leq k/2 \qquad \forall\langle i, j\rangle \in E', \tag{1.4}$$

$$l_{ij} \in \{0, 1\} \qquad \forall\langle i, j\rangle \in E', \tag{1.5}$$

and $$0 \leq dists(i) \leq k/2 + 1 \qquad \forall i \in V', \tag{1.6}$$

where $dists(i)$ represents the distance from node s to node i in G^*.

The reasoning behind this formulation is the following. Equation 1.1 optimizes the spanning tree weight. Conditions 1.2 and 1.3 ensure that node s has outdegree 1, and all other nodes have indegree 1. By limiting the distance from s to each node in the tree being constructed, Condition 1.4 prevents cycles from forming. Together, conditions 1.2, 1.3, and 1.4 lead to the creation of a directed graph G^*, consisting of directed paths from s to all other nodes. Conditions 1.4 and 1.6 ensure that the length of each of these directed paths has length no more than $(k/2 + 1)$. The solution, which is an exact DCMST with diameter no more than k, is the underlying undirected graph of $G^* - \{s\}$.

When the diameter constraint k is odd, a similar formulation can be developed. The odd formulation does not require the use of an extra source node s. Rather, it chooses one directed edge $\langle u, v\rangle$ as the source edge, where $dists(i)$ is the distance from edge $\langle u, v\rangle$ to i. The constructed graph, G^*, is a tree that can be put in a layer structure such that all directed paths have u or v as their origin. The required DCMST is the underlying undirected graph of G^*.

The improved algorithms [7] were tested with the same type of graphs and on the same machine as the original algorithms. When tested with $k = 4$, the fastest of the improved algorithms produced exact solutions in 113, 366, and 7343 seconds on average for graphs of 40, 50, and 100 nodes, respectively. Since such exact algorithms have exponential time complexity, they are not suitable for graphs with thousands of nodes.

One special-case algorithm, which computes an approximate DCMST with diameter bound $k = 6$, was developed by Paddock [59]. The algorithm can be described as follows. First, the edges of the graph $G = (V, E)$ are sorted according to weight, and the smallest 20% edges are assigned to the set E', and all other edges are deleted. In this algorithm, all missing edges are assumed to have weight 0. Then, for each node v, compute:

$$h_v = \sum_{u \in Adj(v)} \sum_{z \in Adj(u)} \left(L - W(u, z)\right),$$

where $Adj(a)$ is the set of nodes x such that edge $(a, x) \in E'$, and L is the smallest edge-weight in E larger than all edge-weights in E'. Choose five nodes v having the largest h_v values as the candidate "Hubs." Each Hub is used to generate a candidate DCMST, where the smallest-weight DCMST among them is selected as the approximate solution. Each candidate DCMST is computed using the following greedy strategy. Initialize the

tree to a Hub, then add nodes to the tree using Prim's algorithm, but at each step, use only those edges that result in a distance from the Hub of 3 or less. Once this spanning tree is obtained, refine it by replacing some of its edges, one by one, with minimum-spanning-tree edges not in the current spanning tree. Each replacement is made only if it does not cause the distance from the Hub to any node to exceed 3. The worst-case time complexity of this algorithm is $O(nm + n^2)$, where n and m are the number of nodes and edges in the input graph, respectively. This algorithm is not effective since it only considers distances from the Hub, using a small number of Hubs, rather than using the distance between each pair of nodes in the tree being constructed. Hence, it tries to optimize the diameter by optimizing the radius. If the initial choice of Hub is poor, the produced solution may be significantly heavier than the optimal. Furthermore, generalizing this algorithm is less effective for odd values of k. A better algorithm could optimize the diameter directly, discarding fewer edges by the greedy strategy, thus making the algorithm less vulnerable to the choice of Hub.

The problem of finding a radius-constrained directed MST in a directed graph was discussed by Gouveia [35]. He presented several node-oriented formulations for this problem, where it is required to find a directed MST of a given weighted directed-graph such that each path starting from a given root to any tree node has length no more than h, $h > 0$. The formulations were based on an existing formulation for the traveling-salesman problem [55], and they provided lower bounds for the radius-constrained directed MST problem. Computational results from a set of complete graphs, with up to 40 nodes, was also presented [35]. Dahl [24] studied a special case of the radius-constrained directed

MST problem in directed graphs, where the radius constraint is 2, and he presented a new formulation for this special case.

In undirected graphs, the problem of finding a radius-constrained MST, with a given root and radius 2, was shown to be NP-hard by Alfandari *et al*. [9, 10]. They devised an approximate polynomial-time algorithm for this problem. This algorithm guarantees a worst-case approximation ratio of $O(\log n)$ in Euclidian graphs. They tested their algorithm on a real-world problem of 80 nodes, and on randomly generated Euclidian graphs of different orders. The experimental results were significantly better than the theoretical $O(\log n)$ bound. The special-case algorithms for this problem, where the edge weights range over two possible values, were also shown to be approximatable within logarithmic ratio [9, 10].

Bar-Ilan *et al*. [15] presented two polynomial-time algorithms that find approximate DCMSTs with the diameter constrained to 4 or 5. The worst-case time complexity of the two algorithms is $O(n^2)$ and $O(mn^2)$ for constraints 4 and 5, respectively. The algorithms were specifically designed to provide a logarithmic ratio approximation when the edge-weights in the input graph are the elements of an integral, non-decreasing function. They are not suitable for the general DCMST problem.

1.3 A Generalization of the DCMST Problem

The DCMST problem was introduced by Garey and Johnson [33] as an NP-complete problem. Papadimitriou and Yannakakis [61] discussed the general problem, $M(P)$, of finding the smallest spanning-tree that is isomorphic to a given prototype, P. They showed that the complexity of such a problem, $M(P)$, depends explicitly on the rate of growth of the family of prototypes, P. In one case, they proved that if P is "isomorphic to a path," then $M(P)$ is NP-complete. Then, they proved the following theorem: Let $\langle T_g \rangle$ be an *efficiently given* sequence of trees such that the diameter of every tree, T_g, in the sequence, is in $\grave{U}(|T_{\tilde{a}}|^{\mathring{a}})$ for some $e > 0$, then the problem of finding the smallest spanning tree of minimum weight that is isomorphic to $\langle T_g \rangle$ is NP-complete, where $|T_{\tilde{a}}|$ denotes the number of nodes in tree T_g. A sequence of trees $\langle T_g \rangle$ is *efficiently given* if (i) $\langle T_g \rangle$ is infinite, (ii) for all $g \geq 1$, $|T_{\tilde{a}}| < |T_{\tilde{a}+1}| < g(|T_{\tilde{a}}|)$ for some polynomial g, and (iii) there is a polynomial-time algorithm to decide whether there exists a tree in $\langle T_g \rangle$ with a given number of nodes and to return that tree if it exists. Johnson [44] listed a few problems proven to be NP-complete by showing that they contain the DCMST problem as a special case.

1.4 Related Optimization and Decision Problems

Some of the well-known constrained minimum-spanning-tree problems require minimizing the weighted diameter of the spanning tree of a randomly-weighted graph; *i.e.*, the maximum path-weight in the spanning tree. These problems are closely related to the problems that require optimizing the weighted radius of the spanning tree—the maximum path-weight in the spanning tree from a given node to all other nodes in the tree. The main difference between these problems and the DCMST problem lies in the way they disregard the number of edges in the longest path in the tree. In these problems, it is desired to have a spanning tree with small weighted diameter, even if it is a Hamiltonian path. However, approaches to solve these problems can be sometimes modified to solve the DCMST problem, and vise versa. For example, Cong *et al*. [23, 45] modified Prim's algorithm to compute a spanning tree with a small weighted-radius. Their modification follows Prim's algorithm, adding the nearest node, u, first, as long as the weighted-radius constraint is not violated; otherwise, backtracking is performed and u is added via an edge of minimum weight that does not violate the constraint. Our modified Prim algorithm, explained in Chapter 7, adds the nearest node u first, unless it violates the diameter constraint. In this case, the violating edge to node u is discarded and processing resumes with the next lightest edges not in the tree. We can investigate backtracking as an alternative to adding the next nearest node, and study the affects on speed and solution quality of our algorithm.

The general problem of finding a minimum weighted-diameter spanning-tree when the edge weights are random (possibly negative) real numbers is NP-complete [19]. An approximate heuristic for this problem was developed by Butelle *et al*. [19]. When the given graph is Euclidean, the minimum weighted-diameter spanning-tree can be solved in $O(mn + n^2 \log n)$ time, as shown by Hassin and Tamir [39]. In a more recent paper, Cho and Breen [21] presented an approximate algorithm for the general problem, but they only tested it on Euclidean graphs. A formulation of the minimum weighted-radius spanning tree problem, in the context of message routing, was presented in [84, 85]. An approximate algorithm for the bounded weighted-radius problem, based on Prim's algorithm, was presented in [23, 45].

It was shown by Ho *et al*. [40] that the minimum-weighted-diameter minimum spanning tree problem and minimum-weighted-radius minimum spanning tree problem are NP-hard. The minimum-weighted-diameter minimum spanning tree problem requires finding the smallest spanning tree among all spanning trees with minimum weighted-diameter. The minimum-weighted-radius minimum spanning tree problem is defined similarly. In the following four corresponding decision problems: bounded-weighted-diameter minimum spanning tree problem, minimum-weighted-diameter bounded spanning tree problem, bounded-weighted-radius minimum spanning tree problem, and minimum-weighted-radius bounded spanning tree problem, it is required to find a tree minimizing one weight objective while keeping a bound on the other. All four decision-problems are NP-complete [40]. In each of the other two corresponding decision problems: bounded-weighted-diameter bounded spanning tree problem and bounded-

weighted-radius bounded spanning tree problem, two bounds are given. Both of these decision problems are NP-hard [40]. An approximate algorithm for the bounded-weighted-radius bounded spanning tree problem was presented in [23, 45].

Another version of the bounded-weighted-diameter minimum spanning tree problem was presented by Salama *et al*. [71]. In this problem, there are two different weights associated with each edge of the graph, corresponding to the delay and cost of each edge. The objective of the problem is to find a bounded-weighted-diameter minimum spanning tree where the diameter is measured in terms of delay, and the total spanning tree weight is calculated using edge-costs. Salama *et al*. [71] showed this problem is NP-complete, and they presented an approximate algorithm for solving it. An alternative statement of this problem, as indicated by Maffioli [52], would be minimizing cost-to-time ratio, which introduces a new graph with a single weight for each edge. Other related optimization problems can be found in [25, 27, 46, 48, 53, 62, 64, 65, 81].

1.5 Diameter Sets and the Dynamic DCMST

The tree-diameter set $(a_1, a_2, \ldots, a_s)$ of a connected graph G is the set of all diameters of the spanning trees of G, listed in increasing order. When designing an algorithm to obtain a spanning tree with a small diameter, we should know if such a spanning tree exists. In addition, algorithms like IR1, IR2, and CIR, explained in Chapters 4, 5, and 6, iterate through a set of spanning trees of different diameters, trying to find a low-weight

tree with the desired diameter. The behavior of diameter sets may provide an insight that helps analyze and improve such algorithms. However, properties of diameter sets are more useful to dynamic trees—spanning trees that repeatedly need to replace one or more of their edges with different edges from the graph.

It is interesting to know which type of graph has a spanning tree with diameter equal to its own diameter. A connected graph with radius r has a diameter preserving spanning tree if and only if either: (i) its diameter is equal to $2r$ or (ii) its diameter is equal to $2r - 1$ and contains a pair of adjacent vertices that have no common neighbor whose eccentricity is equal to theirs [16].

One algorithm, developed by Harary *et al*. [37], transforms any spanning tree of a 2-connected graph G into any other spanning tree of G. The transformation uses a sequence of steps; each step yields a spanning tree of G whose diameter differs from that of the previous step by at most 1. Sankaran and Krishnamoorthy [73] extended this result to all connected graphs having exactly one cut-node. For a random graph in general, Shibata and Fukue [79] show that $a_{i+1} \leq a_i + 0.5a_1$. Interpolation style properties for distance sums of spanning trees in 2-connected graphs were presented by Plantholt [63].

Interpolation properties are helpful when developing algorithms for dynamic trees where a minimum weighted-diameter needs to be maintained while we replace one edge with another. In some cases, the added edge is given, and it is required to remove an edge to break the cycle, while keeping the weighted diameter small. An algorithm by Alstrup *et al*. [13] finds the best edge to remove in $O(n)$ time. When the removed edge is given, and it is required to find the best replacement from the graph, an algorithm by

Italiano and Ramaswami [42, 43] finds the best replacement in $O(n)$ time. Nardelli *et al*. [58] developed an algorithm that finds all the best replacements for all edges in the tree in $O(n\sqrt{m})$ time and $O(m + n)$ space, where n and m are the number of nodes and edges in the graph, respectively.

1.6 The Diameter of a Random Tree

There is a significant amount of literature on the height and diameter of a random tree, including the diameter of an unrooted random-labeled-tree. The average distance between a pair of nodes in a random p-ary tree is $\Theta(\log n)$, as shown by Shen [78]. The average height of a binary plane tree and the average distance between nodes in a random tree are $\Theta(2(\pi n)^{1/2})$ and $\Theta((\pi n/2)^{1/2})$, respectively [29, 30, 54]. The expected height of a random labeled tree is $(2 \pi n)^{1/2}$, and it was derived by Rényi and Szekeres [68]. Łuczak [49, 50, 51] studied the asymptotic behavior of the height of a random rooted-tree and used the results to realize the behavior of diameter in random graphs. The problem of tree enumeration by height and diameter, for different types of random trees, was addressed in several papers, such as [38, 51, 68, 70], but the equation for the expected value of diameter in a random labeled-tree is due to Szekeres [82]. He showed that for a random labeled-tree of order n, as n approaches infinity, the expected value of diameter and the diameter of maximum probability are $3.342171n^{1/2}$ and $3.20151315n^{1/2}$, respectively.

1.7 Outline

This dissertation is organized as follows. In Chapter 2, we experimentally study the expected value of MST diameter in complete graphs with uniformly-distributed random edge-weights. Then, in Chapter 3, we discuss polynomially-solvable exact cases of the DCMST problem and present a new heuristic that computes an approximate DCMST with diameter no more than 4. Our approximate algorithms for solving the general DCMST problem employ two distinct strategies: iterative refinement and one-time tree-construction. The first general-iterative-refinement algorithm, IR1, is described in Chapter 4. It starts with an unconstrained MST, and then iteratively increases the weights of edges near the center of the MST and recomputes the MST until the diameter constraint is met or the number of iterations reaches a given number. The second general-iterative-refinement algorithm, IR2, which is described in Chapter 5, starts with an unconstrained MST, and then replaces carefully selected edges, one by one, to transform the MST into a spanning tree satisfying the diameter constraint. A composite iterative-refinement algorithm, CIR, is presented in Chapter 6. Algorithm CIR starts with an unconstrained MST, and then uses IR1 until it (IR1) terminates. Then, if necessary, it uses IR2 to transform the output of IR1 into a spanning tree satisfying the diameter constraint. The one-time tree-construction algorithm, presented in Chapter 7, grows a spanning tree within the desired diameter-constraint using our modified version of Prim's algorithm. All of our algorithms were implemented on a PC with a Pentium III / 500 MHz processor. Algorithms IR1, IR2, CIR, OTTC, and the special-case algorithm for

$k = 4$ were also parallelized and implemented on the MasPar MP-1, which is an SIMD parallel computer with 8192 processors. We analyze the empirical data, from implementation of these five algorithms, in their respective chapters. We compare their overall performance and relative merits in Chapter 8.

CHAPTER 2

EXPECTED VALUE OF MST-DIAMETER

In a randomly-weighted complete graph, every spanning tree is equally likely to be an MST. Thus, there is a one-to-one correspondence between the set of possible minimum spanning trees of a randomly-weighted complete graph with n nodes and the set of all n^{n-2} labeled trees with n nodes. Therefore, the behavior of MST diameter in a randomly-weighted complete graph can be studied using unweighted random labeled-trees.

Rényi [60, 67] proved that the probability that a node in a random labeled-tree is a leaf is $1/e$. If we repeatedly remove $1/e$ of the nodes from the tree, a single node or single edge would result after $(2.18 \ln n)$ iterations. However, this does not imply that the expected diameter is $O(\ln n)$. The leaf-deletion procedure does not actually remove $1/e$ of the nodes, except in the first iteration. After the leaves are removed, the remaining subtree does not have the same probability distribution as the original tree. For example, removing all the leaves of a tree may produce a Hamiltonian path, but removing the leaves of a Hamiltonian path will not produce anything other than a Hamiltonian path. Therefore, the probability of a Hamiltonian path in the subtree produced by removing the leaves is strictly greater than the probability of a Hamiltonian path in the original random-tree.

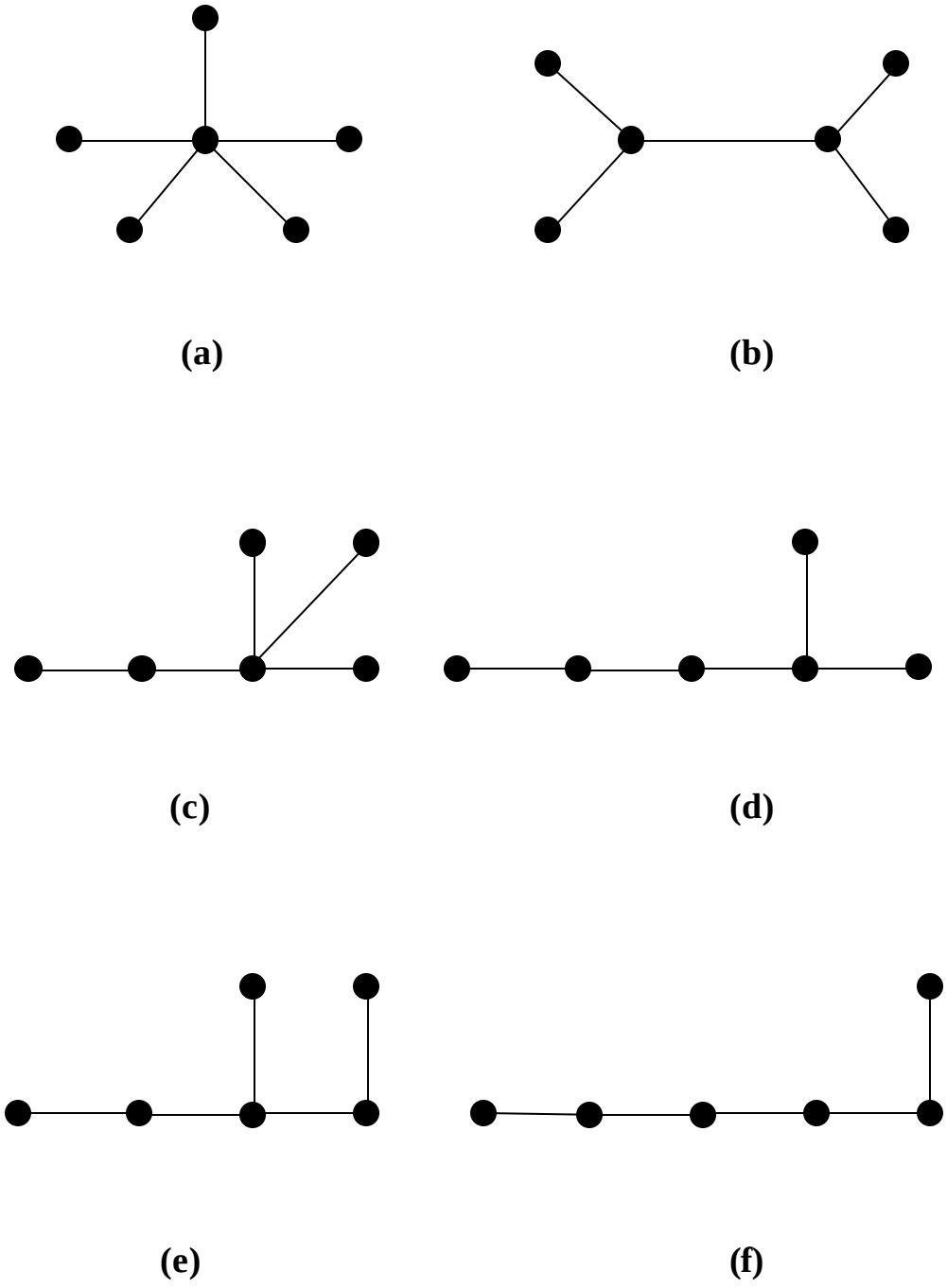


Figure 2.1 The unlabeled trees of order 6

2.1 Exact Average-Diameter

When the number of nodes is small, like n = 6, the expected diameter for labeled and unlabeled trees can be calculated utilizing a listing of all unlabeled trees of the same order. For example, there are only six unlabeled trees of order 6, shown in Figure 2.1. Since there are 1, 2, 2, and 1 unlabeled trees of diameter 2, 3, 4, and 5, respectively, the expected value of diameter for an unlabeled tree of order 6 is:

$$(1 \times 2 + 2 \times 3 + 2 \times 4 + 1 \times 5) / 6 = 3.5.$$

The number of different ways to label the trees in Figures 1(a), 1(b), 1(c), 1(d), 1(e), and 1(f) is 6, 90, 120, 360, 360, and 360, respectively. Thus, the expected value of diameter for a labeled tree of order 6 is:

$$(6 \times 2 + (90 + 120) \times 3 + (360 + 360) \times 4 + 360 \times 5) / 6^4 \approx 4.106.$$

This method of computing the expected value of diameter becomes less feasible as n grows. For example, there are 47 unlabeled trees of order 9, and 65 unlabeled trees of order 10, as shown by Harary and Prins [38].

Employing Riordan's method for enumerating labeled trees by diameter [70, 56], we can use the following method to compute the expected value of diameter for all labeled trees with n nodes. First, compute $t_n(h)$, the number of labeled rooted-trees with n nodes, a specified root, and height no more than h, using the following equation:

$$t_n(h) = \sum_{m_1+m_2+\ldots+m_h=n-1} \frac{(n-1)!}{m_1!\,m_2 \ldots !\,m_h!} m_1^{m_2} m_2^{m_3} \ldots m_{h-1}^{m_h}, \quad \text{where}$$

$$1 \le h \le (n-2),\ 0 \le m_i \le (n-1),\ 1 \le i \le h, \text{ and } t_n(1) = 0^0 = 1. \qquad (2.1)$$

Clearly, $t_n(h) = n^{n-1}$ for $h \geq n - 1$. When $h = 2$, Equation 2.1 can be simplified to:

$$t_n(2) = \sum_{m_1=0}^{n-1} \binom{n-1}{m_1} m_1^{n-m_1-1}.$$

The values for $t_n(h)$ are then substituted into the following expression for their enumerator:

$$G_h(x) = \sum_{n=1}^{\infty} \frac{t_n(h)}{(n-1)!} x^n, \quad h \geq 0. \tag{2.2}$$

Now, $H_h(x)$, the enumerator for the number of labeled trees with n nodes and height exactly h, is given by:

$$H_h(x) = G_h(x) - G_{h-1}(x) = \sum_{n=1}^{\infty} \frac{t_n(h) - t_n(h-1)}{(n-1)!} x^n, \quad h \geq 1. \tag{2.3}$$

Now, $D_d(x)$, the enumerator for the number of labeled trees with diameter d, is obtained for odd and even values of d:

$$D_{2h+1}(x) = \tfrac{1}{2} H_h^2(x), \quad h \geq 0, \tag{2.4}$$

$$D_{2h}(x) = H_h(x) - H_{h-1}(x)\, G_{h-1}(x), \quad h \geq 1. \tag{2.5}$$

After that, we obtain $\boldsymbol{d}_n(d)$, the number of labeled trees with n nodes and diameter d, by comparing terms from Equations 2.4 and 2.5 with the terms in the following enumerator:

$$D_d(x) = \sum_{n=1}^{\infty} \frac{\boldsymbol{d}_n(d)}{n!} x^n, \quad d \geq 1. \tag{2.6}$$

Unlike the enumerators in Equations 2.2 and 2.3, this enumerator has a denominator of $n!$ in each term, instead of $(n - 1)!$, to account for the different ways to choose a root in a

labeled tree of diameter d. Finally, $\bar{d}_n$, the expected value of the diameter for a labeled tree with n nodes, is given by:

$$\bar{d}_n = \left(\sum_{d=1}^{n} d \cdot \boldsymbol{d}_n(d) \right) \cdot n^{2-n}, \quad n \geq 3. \tag{2.7}$$

Explicitly, Equation 2.1 can be used to calculate:

$$t_4(2) = \sum_{m_1=0}^{3} \binom{3}{m_1} m_1^{3-m_1} = 10,$$

$$t_5(2) = \sum_{m_1=0}^{4} \binom{4}{m_1} m_1^{4-m_1} = 41, \text{ and}$$

$$t_5(3) = \sum_{m_1+m_2+m_3=4} \frac{4!}{m_1!m_2!m_3!} m_1^{m_2} m_2^{m_3} = 101.$$

Calculate more values using $t_n(h) = n^{n-2}$ for $h \geq n - 1$, then substitute for $t_n(h)$ in Equation 2.2 to obtain:

$G_0(x) = x$,

$G_1(x) = x + x^2 + x^3/2 + x^4/6 + x^5/24 + x^6/120 + \ldots$,

$G_2(x) = x + x^2 + 3x^3/2 + 5x^4/3 + 41x^5/24 + 49x^6/30 + \ldots$,

$G_3(x) = x + x^2 + 3x^3/2 + 8x^4/3 + 101x^5/24 + \ldots$, and

$G_4(x) = x + x^2 + 3x^3/2 + 8x^4/3 + 125x^5/24 + \ldots$

Then, use Equation 2.3 to obtain:

$H_0(x) = x$,

$H_1(x) = x^2 + x^3/2 + x^4/6 + x^5/24 + x^6/120 + \ldots$, and

$H_2(x) = x^3 + 3x^4/2 + 5x^5/3 + 13x^6/8 + \ldots$

After that, use Equations 2.4 and 2.5 to obtain:

$D_1(x) = x^2/2$,

$D_2(x) = x^3/2 + x^4/6 + x^5/24 + x^6/120 + \ldots$,

$D_3(x) = x^4/2 + x^5/2 + 7x^6/24 + \ldots$,

$D_4(x) = x^5/2 + x^6 + \ldots$, and

$D_5(x) = x^6/2 + \ldots$

Finally, solve for the values of $\boldsymbol{d}_n(d)$ using Equation 2.6, and compute the average diameter for a labeled tree with n nodes, $\overline{d}_n$, using Equation 2.7.

Table 1 Exact average-diameter for n = 4 to 19

Nodes	Mean Diameter
4	2 + 3 / 4
5	3 + 11 / 25
6	4 + 23 / 216
7	4 + 244 / 343
8	5 + 8803 / 32768
9	5 + 423973 / 531441
10	6 + 3042239 / 10000000
11	6 + 168968246 / 214358881
12	7 + 1296756731 / 5159780352
13	7 + 95955348055 / 137858491849
14	8 + 505795179247 / 4049565169664
15	8 + 70033243048672 / 129746337890625
16	8 + 4242384791970699 / 4503599627370496
17	9 + 56007173241669065 / 168377826559400929
18	9 + 4807321332780086399 / 6746640616477458432
19	10 + 23817830747172067306 / 28844141356762116768 1

Riordan [70] listed the number of labeled trees with n nodes and diameter d, for $3 \le n \le 10$ and $2 \le d \le (n - 1)$. We present the average diameter, produced by our

implementation of the method described above, for labeled trees with 4 to 19 nodes in Table 1.

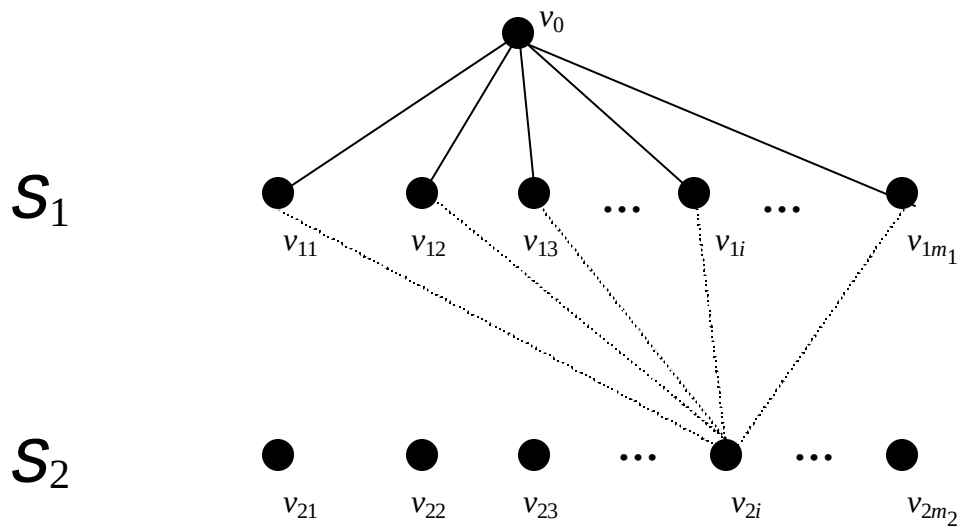


Figure 2.2 Different ways to connect a node from S_2 to a node in S_1

To calculate Equation 2.3, we must use all solutions to the linear equation:

$$m_1 + m_2 + \ldots + m_h = n - 1 \ . \tag{2.8}$$

Each solution to Equation 2.8 represents one way to distribute $n - 1$ nodes into h classes S_j, where $|S_j| = m_j$, $1 \le j \le h$. Each class, S_j, contains the nodes at distance j from the root. Corresponding to each solution to Equation 2.4, there are:

$$\frac{(n-1)!}{m_1!\, m_2!\, \Lambda \quad m_h!}$$

different ways to distribute the $(n - 1)$ nodes into the h classes. As seen in Figure 2.2, node v_{2i} can be connected to a node in S_1 in m_1 different ways. Since there are m_2 nodes in S_2, there are $m_1^{m_2}$ ways to connect the nodes in S_2 to the nodes in S_1, and similarly,

$m_2^{m_3}$ ways to connect the nodes in S_3 to the nodes in S_2, and so on. By allowing the classes to be empty and defining $0^0 = 1$, Equation 2.3 includes all the possibilities where the height of the rooted labeled-tree is less than h. Connecting the nodes of a non-empty class S_j to an empty class S_{j-1} will produce $m_{j-1}^{m_j} = 0$, which prevents such impossible cases from being counted. The product $m_1^{m_2} m_2^{m_3} \ldots m_{h-1}^{m_h}$ gives all the different ways to connect all the nodes in the h different classes.

Equation 2.1 requires computing and adding

$$\frac{(n+h-2)!}{((n-1)!)^2}$$

terms for specified values of n and h, corresponding to the number of nonnegative integer solutions to Equation 2.8. We implemented this method of computing the exact value of average diameter on a PC with a Pentium III / 500 MHz processor. Employing the dynamic programming strategy to reduce the time taken by this method, our implementation required $O(n^2)$ space and approximately $(4.2)^{n-10}$ seconds to compute the average diameter for all sets of trees of orders 4 to n. This high time-complexity makes this method too slow for graphs with thousands of nodes. Therefore, we conducted an empirical study of the expected value of diameter.

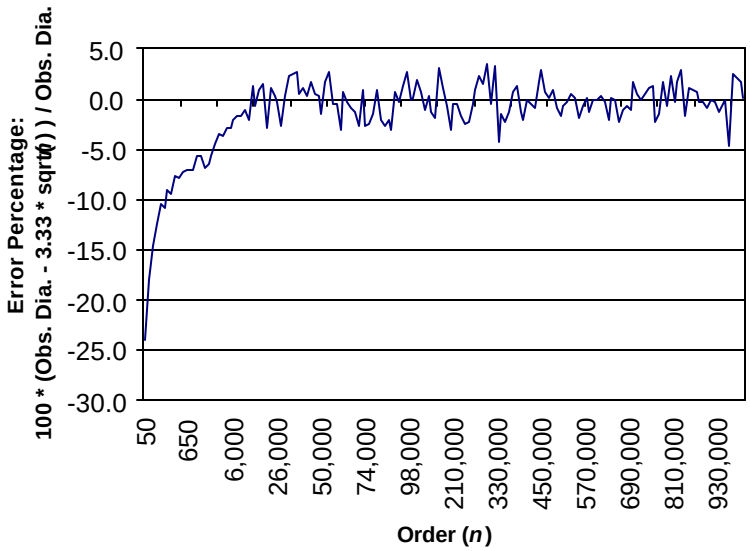


Figure 2.3 Percentage error in approximate average-diameter

2.2 Approximate Average-Diameter

We compared the average diameter in computer-generated random labeled-trees to the expected value computed using Szekeres' formula. The mean diameter was computed for randomly generated trees with up to one million nodes, and averaged for 100 different trees of each order. The curve fitting result for the diameter means, obtained using a least-square-fit program, was $3.33125n^{½}$, showing a difference of $0.010921n^{½}$ from

Szekeres' formula. The curve fitting error, illustrated in Figure 2.3, stabilizes for $n \geq 1100$.

To generate a random labeled-tree in linear time, we used a randomly generated Prüfer code [47]. Then, we examined three different approaches for calculating the diameters of the trees. The first diameter-calculation algorithm takes a naïve approach by examining the paths between all pairs of leaves in the tree. Employing the Warshall-Floyd algorithm, this approach takes $O(n^3)$ time. The second approach repeatedly removes all the leaves in the tree, until a single path remains. The diameter is equal to twice the number of deletions, plus the length of the remaining path. Using an efficient data structure, such as a queue, to maintain the order by which leaves are deleted, this approach takes linear time. The third approach, proposed by Handler [36], performs a depth-first search from an arbitrary node, u, in the tree to find the farthest node v from u. Node v will always be at one end of a longest path in the tree [84, 85]. Then, the algorithm performs a second depth-first search to find the farthest node z from v. The length of this path from v to z gives the diameter of the tree [36]. This method computes the diameter in linear time.

To distinguish the speed of the two linear-time methods of computing diameter, we compared their execution times on a PC with a Pentium III / 500 MHz processor, using trees with 50 to one million nodes. The time taken by each algorithm was less than 3 milliseconds for $n \leq 1000$. As the number of nodes increased, the leaf-deletion algorithm became clearly faster than Handler's algorithm. Using a polynomial-fit program, we

determined that the execution time, in microseconds, taken by Handler's algorithm and the leaf-deletion algorithm was ($2.79n - 43265$) and ($0.67n - 11716$), respectively.

The above three methods for computing tree-diameter may be used with unlabeled trees, where an arbitrary labeling of the nodes can be assigned before applying the algorithms. A random unlabeled-tree can be generated in polynomial time using an algorithm devised by Alonso *et al*. [11, 12].

CHAPTER 3

QUALITY OF AN APPROXIMATE DCMST

3.1 Polynomially-Solvable Cases

Four cases of the DCMST problem can be exactly solved in polynomial time. When the diameter constraint $k = n - 1$, an MST is the solution. When $k = 2$, the solution is a smallest-weight star, where a star is a tree with at most one non-leaf. Let DCMST(k) denote (an optimal) DCMST with diameter no more than k. A DCMST(2) can be computed in $O(n^2)$ time by comparing the weight of every n-node star in G. A DCMST(3) can be computed in $O(n^3)$ time by computing all spanning trees with diameter no more than 3 and choosing a spanning tree having the smallest weight as follows: Clearly, in a DCMST(3) of graph G, every node must be of degree 1 except at most two nodes, call them u and v. The edge (u, v) is the *central edge* of DCMST(3). To construct a spanning tree with diameter no more than 3, select an edge to be the central edge, (u, v). Then, for every node x in G, $x \notin \{u, v\}$, include in the spanning tree the smaller of the two edges (x, u) and (x, v). To obtain a DCMST(3), compute all such spanning trees — with every edge in G as the central edge of one spanning tree — and select the one with

the smallest weight. In a graph of m edges, we have to compute m different spanning trees. Each of these trees requires $(n - 2)$ comparisons to select (x, u) or (x, v). Therefore, the total number of comparisons required to obtain a DCMST(3) is $(n - 2)m$.

For the case when all edge-weights in G are equal, we can consider G unweighted, and the spanning tree with the smallest diameter is an optimal solution for any $k \geq 2$, if and only if a solution exists. Finding such a solution is trivial for all unweighted graphs that contain an n-node star. For graphs not containing a spanning star, a minimum-diameter spanning tree can be constructed as follows [8, 41]: For every node v in G, construct a breadth-first spanning tree (BFST), T_v. The radius of T_v is the maximum path length from v to any node in T_v, and it can be computed by keeping track of the distance of every node u from v when u is added to T_v during the BFST construction, without increasing the time complexity. All minimum-radius spanning trees will have diameter $2r$ or $2r - 1$. A minimum-diameter spanning tree will be a BFST of minimum radius r that contains exactly one node with distance r from the root, if such a tree exists, or any minimum-radius spanning tree if such a tree does not exist. Since each BFST is computed in $O(m)$ time, and there are n possible BFSTs, the time complexity of finding a minimum-diameter spanning tree is $O(mn)$.

3.2 The Special-Case Algorithm for DCMST(4)

We developed a special-case algorithm to compute an approximate DCMST(4). The algorithm starts with an exact DCMST(3), then it replaces higher-weight edges with smaller-weight edges, allowing the diameter to increase to 4. The refinement process first arbitrarily selects one end-node, u, of the central edge, (u, v), of DCMST(3), to be the center of DCMST(4). Let $W(a, b)$ denote the weight of an edge (a, b). For every node x adjacent to v, the algorithm attempts to obtain another tree of smaller weight by replacing edge (x, v) with an edge (x, y), where y is adjacent to u, and $W(x, y) < W(x, v)$. Furthermore, for all nodes z adjacent to u, $W(x, y) \leq W(x, z)$. Figure 3.1 illustrates an example of this possible replacement. If no such edge exists, we keep edge (v, x) in the tree. We use the same method to compute a second approximate DCMST(4), with v as its center. Finally, the algorithm certifies the DCMST(4) having the smaller weight as the approximate solution.

Suppose there are r leaves adjacent to u in DCMST(3). Then, there are $(n - r - 2)$ leaves adjacent to v. Therefore, it is required to make

$$2r(n - r - 2) \tag{3.1}$$

comparisons to get an approximate DCMST(4). The probability that u is connected to r leaves is:

$$\frac{2\tilde{n}(n - \tilde{n} - 2)}{\sum_{r=1}^{n=3} 2\tilde{n}(n - \tilde{n} - 2)} = \frac{6\tilde{n}(n - \tilde{n} - 2)}{(n-1)(n-2)(n-3)}. \tag{3.2}$$

To find the expected value for $\boldsymbol{r}$, treat Equation 3.2 as a continuous function, take its first derivative with respect to $\boldsymbol{r}$, and then set the derivative equal to zero and solve for $\boldsymbol{r}$. This gives the value: $\boldsymbol{r} = (n - 2)/2$. Substituting this value of $\boldsymbol{r}$ into Equation 3.1, it shows that, employing our special-case algorithm, the expected number of comparisons required to obtain an approximate DCMST(4) from a DCMST(3) is $(n^2 - 8n - 12)/2$.

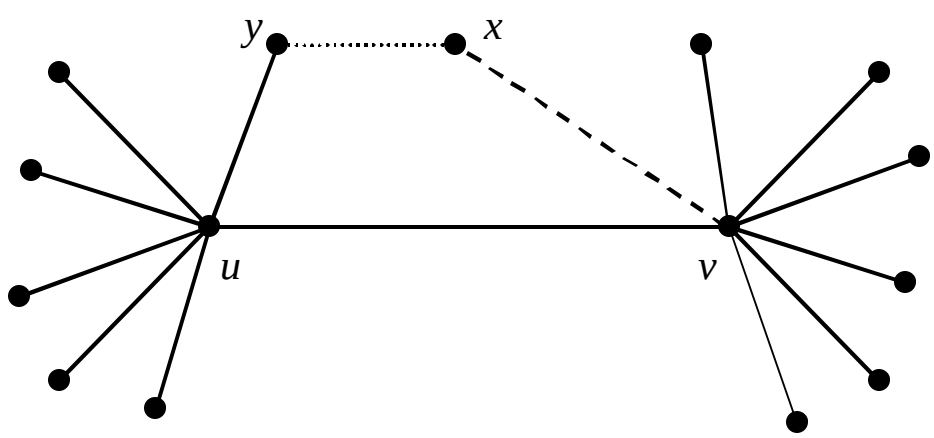


Figure 3.1 One step in constructing an approximate DCMST(4) from DCMST(3)

To obtain a crude upper bound on the approximate DCMST(k) weight (where k is the diameter constraint), observe that DCMST(3) and DCMST(4) are feasible (but often grossly suboptimal) solutions of DCMST(k) for all $k > 4$. Using the special-case heuristic, we compute an approximate DCMST(4) and compare its weight with that of DCMST(3) to verify that the heuristic provides a tighter upper-bound for approximate DCMST(k). Let $W(T)$ denote the weight of tree T. Then, clearly for any $k \geq 5$,

$$W(\text{MST}) \leq W(\text{DCMST}(k)) \leq W(\text{DCMST}(4)) \leq W(\text{DCMST}(3)).$$

Since the exact DCMST for large graphs cannot be determined in a reasonable time, we use the upper bounds, along with the ratio of the weight of the approximate DCMST to the weight of the unconstrained MST, as a rough measure of the quality of the solution.

We implemented the special-case heuristic for DCMST(4) sequentially on a PC with a Pentium III / 500 MHz processor. We also parallelized it and implemented it on the MasPar MP-1. We used complete graphs of orders between 50 and 3000, with randomly generated weights either ranging between 1 and 1000 or ranging between 1 and 10000. We also used complete graphs forced to have Hamiltonian-path MSTs with the same ranges of edge weights. The sequential and parallel implementations produced similar results for both random graphs and randomly generated graphs forced to have Hamiltonian-path MSTs. The change of the upper bound on edge weight did not have any noticeable effect, either. The special-case heuristic for DCMST(4) produced approximate solutions with weight roughly half that of exact DCMST(3), independent of n, as will be shown in Figures 8.1 and 8.2. The time to refine a DCMST(3) into an approximate DCMST(4) was about 1% of the time needed to calculate a DCMST(3), independent of n. This heuristic is not suited for incomplete graphs since they are unlikely to contain a spanning tree with diameter 3.

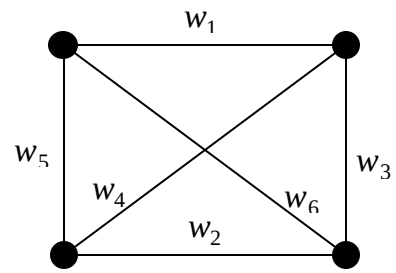
CHAPTER 4

THE IR1 ITERATIVE-REFINEMENT ALGORITHM

Our three general iterative-refinement-strategy-algorithms first compute an unconstrained MST, and then iteratively refine this MST by edge-replacement until the diameter constraint is satisfied. General iterative-refinement-algorithm IR1, which we present in this chapter, iteratively penalizes the edges near the center of the MST by increasing their weight and then recomputes the MST. This attempts to lower the diameter by breaking up long paths from the middle, replacing them by shorter ones.

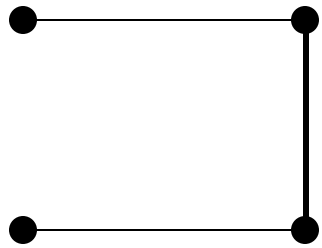
4.1 The Algorithm

The heart of Algorithm IR1 is a problem-specific *penalty function*. A penalty function succinctly encodes how many edges to penalize, which edges to penalize, and what the penalty amount must be, where the penalty is an increase in edge weight. In each iteration of IR1, as described in Algorithm 1, an MST of the graph with the current weights is computed, and then a subset of tree edges are penalized (using the penalty function), so that they are discouraged from appearing in the MST in the next iteration.

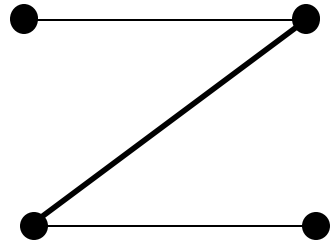


$w_1 < w_2 < w_3 < w_4 < w_5 < w_6$

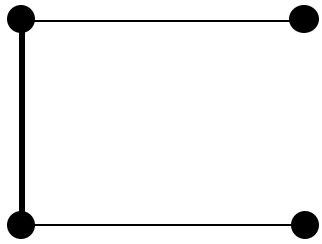
(a)



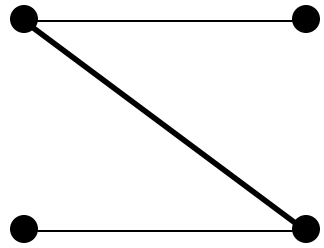
(b)



(c)



(d)



(e)

Figure 4.1 An example of cycling in IR1

Obviously, an edge at the center of a long path is a good candidate to be penalized, since

it would split each of the longest paths in the current tree into two subpaths of equal length. However, penalizing only one edge per iteration may not be sufficient, as illustrated by the example of Figure 4.1.

```
ALGORITHM 1 (IR1(G, k)).
begin
    fails := 0;
    G' := G;
    T_min := MST of G;
    T := T_min;
    while (((diameter of T) > k) and (fails ≤ 15)) do
        G' := G' with edges closest to the center of T_min penalized;
        T_min := MST of G' ;                /* computed using the new edge-weights */
        if (((diameter of T_min) < (diameter of T))
            or (((diameter of T_min) = (diameter of T)) and (W(T_min) < W(T))))
        then  begin
                T := T_min;
                fails := 0;
              end
        else
            fails := fails + 1;
    end while
    return T
end.
```

For this complete graph and a specified diameter bound of 2, the MST is the path (w_1, w_3, w_2), shown in Figure 4.1(b). After penalizing the center edge, w_3, and recomputing the MST, we get the path (w_1, w_4, w_2), shown in Figure 4.1(c). The center edge w_4 on this path is penalized next, producing the path in Figure 4.1(d). The algorithm fails to reduce the diameter of this tree as well, producing the tree in Figure 4.1(e), which, in the next iteration, regenerates the original MST. The iterative refinement cycles

among these paths of diameter 3, and never finds any of the four spanning trees of diameter 2.

However, if two edges are penalized in every iteration, there is no cycling for this example. The solution is found in three iterations, as shown in Figure 4.2. Such is the case for every edge-weighted graph with $n = 4$. But for $n = 5$, penalizing two edges per iteration may not be sufficient.

To reduce the possibility of cycling, the number of edges to be penalized per iteration should increase with n. However, it must be kept in mind that penalizing too many edges may result in the solution being too far from optimal. This is because in the space of all n^{n-2} labeled spanning trees, the iterative refinement in such a case would jump (in a single iteration) from one tree to another, which is many edges different, thereby missing a number of feasible solutions with perhaps smaller weight. Therefore, the number of edges penalized must be a slow-growing function of n, say $\log_2 n$. All the edges penalized should be close to the center of the current spanning tree where the center of a tree consists either of one node or one edge, depending on whether its diameter is even or odd. The edges to be penalized should be the ones incident to the center. If more edges are required to be penalized (when the degree of the center node is less than $\log_2 n$), then the edges at distance two from the center node should be chosen, and so on. A tie can be broken by choosing the higher-weight edge to penalize.

Another issue to consider in designing a penalty function is the penalty amount. To be effective without causing overflow, the penalty value must relate to the range of the weights in the spanning tree. Let $W(l)$ denote the current weight of an edge l being

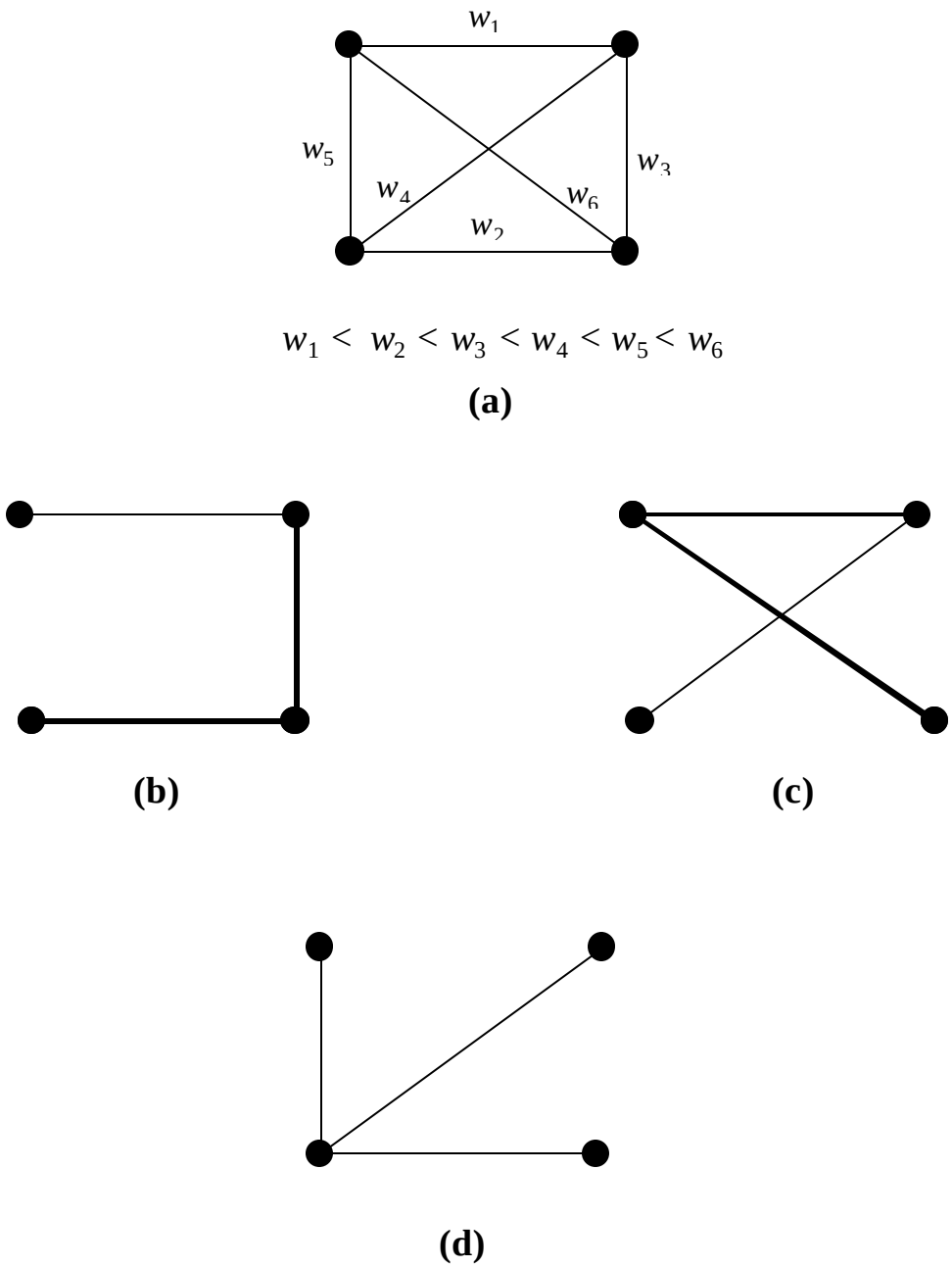


Figure 4.2 Finding an approximate DCMST(2) by penalizing 2 edges per iteration

penalized, and $w_{\max}$ and $w_{\min}$ denote the largest and the smallest edge-weight,

respectively, in the current MST. Also, let $distc(l)$ denote the distance of an edge l from the center node, plus one. When the center is a unique node, v_c, all the edges l incident to v_c have $distc(l) = 1$, the ones at distance one from v_c have $distc(l) = 2$, and so on. When the center is an edge l_c, it has $distc(l_c) = 1$, an edge l incident to only one end-point of the center edge has $distc(l) = 2$, and so on. Therefore, the penalty amount imposed on the tree edge l is given by:

$$MAX\left\{\frac{(W(l) - w_{\min})w_{\max}}{distc(l)\,(w_{\max} - w_{\min})}, \boldsymbol{e}\right\},$$

where $\boldsymbol{e} > 0$ is a minimum mandatory penalty imposed on an edge, chosen to be penalized. This minimum penalty ensures that the iterative refinement makes progress in every iteration, and does not stay at the same spanning tree by imposing zero penalties to all the edges (in situations, for example, when all the penalized edges have weights equal to $w_{\min}$). In a typical implementation, in which weights are stored as integer values, the value of $\boldsymbol{e}$ may be set to 1.

Clearly, the penalty amount is proportional to the weight of the penalized edge and inverse-proportional to its distance from the center of the current MST. The penalty amount can be as high as $w_{\max}/distc(l)$, and it decreases as the penalized edge becomes farther away from the center of the tree. This was done because replacing an edge with a small $distc(l)$ in the current tree can break a long path into two significantly shorter subpaths, rather than a short subpath and a long one. Also, an edge with a smaller weight is penalized by a smaller amount than one with a larger weight if they have the same

value of *distc*(.) to makes it less likely for the larger-weight edge to appear in the next MST.

4.2 Implementation

We parallelized Algorithm IR1 and implemented it on the MasPar MP-1. We ran the code for IR1 on random graphs with up to 3000 nodes, whose minimum spanning trees are forced to be Hamiltonian paths, and whose edge weights were randomly selected numbers between 1 and 1000. The tree weights resulting from IR1 are reported as factors of the unconstrained MST weight. The average constrained spanning-tree weights with diameter $n/10$ were 1.068, 1.036, and 1.024 for n = 1000, 2000, and 3000, respectively. This indicates remarkable performance of this iterative-refinement algorithm when the diameter constraint is a large fraction of the number of nodes. The algorithm was also fast, as it reduced the diameter of a 3000-node complete graph from 2999 to 103 in about 15 minutes. Nonetheless, this iterative-refinement algorithm was not able to obtain approximate DCMST(k) when k is a small fraction of the number of nodes, such as $n/20$. Thus, it should be used only for large values of k.

4.3 Convergence

One problem with the approach of Algorithm IR1 is that it recomputes the MST in every iteration, which sometimes reproduces trees that were already examined, even when the replacement increases the diameter. Algorithm IR1 terminates when the current MST diameter is no more than k, or when it cannot improve the current MST further. The latter case is identified by 15 consecutive iterations that reduce neither the diameter nor the weight of the current MST. Our empirical study showed that allowing IR1 to continue past 15 consecutive non-improving iterations did not result in better solutions when the edge weights ranged from 1 to 10000. When it was allowed to run for 500 iterations (regardless of non-improving iterations), Algorithm IR1 succeeded in finding a solution when the diameter constraint $k \geq n/10$, but failed to find a DCMST when k was a small constant. We present a different iterative-refinement algorithm in the next chapter that avoids the cycling problem, and produces solutions with smaller values of k.

CHAPTER 5

THE IR2 ITERATIVE-REFINEMENT ALGORITHM

The next iterative-refinement algorithm, IR2, does not recompute the MST in every iteration; rather, a new spanning tree is computed by modifying the previously computed one. The modification performed does not regenerate previously generated trees and it guarantees the algorithm will terminate. Unlike IR1, this algorithm removes one edge at a time and prevents cycling by moving away from the center of the spanning tree whenever cycling becomes imminent. Figure 5.1 illustrates how this technique prevents cycling for the original graph of Figure 4.1. After computing the MST, the algorithm considers the middle edge (shown in bold) as the candidate for removal, as in Figure 5.1(b). But this edge does not have a replacement that can reduce the diameter, so the algorithm considers edges a little farther away from the center of the tree. The edge shown in bold in Figure 5.1(c) is the highest-weight such edge. As seen in Figure 5.1(d), the algorithm is able to replace it by another edge, and that reduces the diameter. This algorithm guarantees that the diameter does not increase in any iteration and in fact can reduce the diameter to a small constant (less than 1% of the number of nodes in the graph).

IR2 starts by computing the unconstrained MST for the input graph $G = (V, E)$. Then, in each iteration, it removes one edge that breaks a longest path in the spanning tree and

replaces it by a non-tree edge without increasing the diameter. The algorithm requires computing eccentricity values for all nodes in the spanning tree in every iteration.

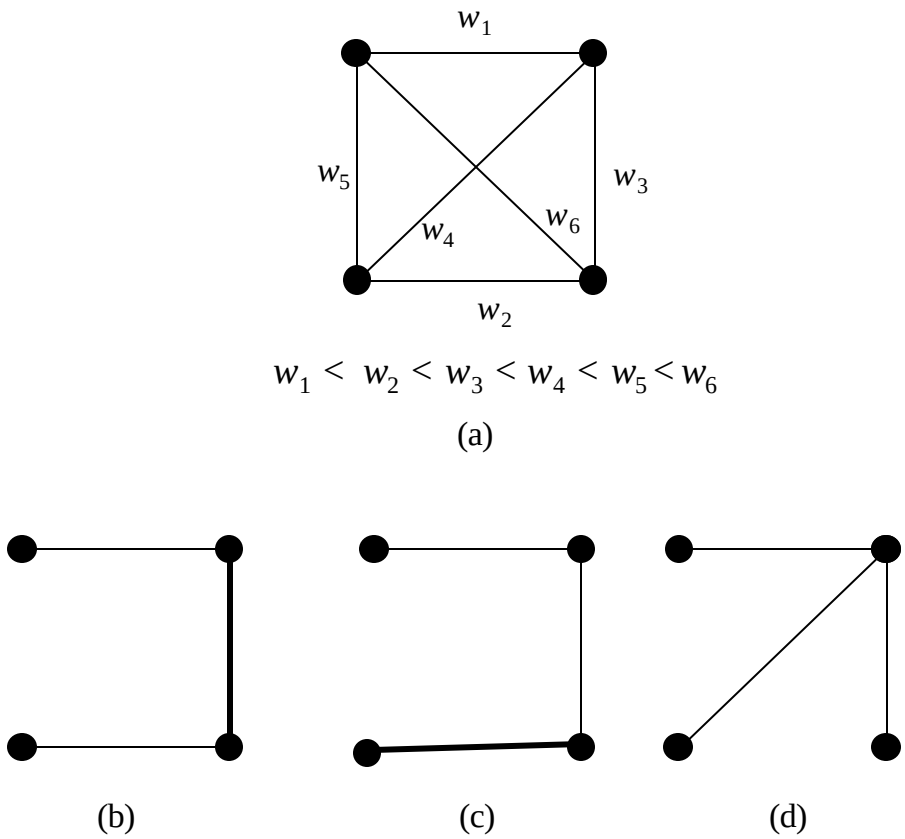


Figure 5.1 An example of IR2

The initial MST can be computed using Prim's algorithm. The initial eccentricity values for all nodes in the MST can be computed using a preorder tree-traversal where each node visit consists of computing the distances from that node to all other nodes in the spanning tree. This requires a total of $O(n^2)$ computations. As the spanning tree changes, we only recompute the eccentricity values that change. After computing the

MST and the initial eccentricity values, the algorithm identifies one edge to remove from the tree and replaces it by another edge from G until the diameter constraint is met or the algorithm fails. When implemented and executed on a variety of inputs, we found that this process required no more than $3n$ iterations. Each iteration consists of two parts. In the first part, described in Section 5.1, we find an edge whose removal can contribute to reducing the diameter, and in the second part, described in Section 5.2, we find a good replacement edge. The IR2 algorithm is shown in Algorithm 2, and its two different edge-replacement subprocedures are shown in Algorithms 3 and 4. We use $ecc_T(u)$ to denote the eccentricity of node u with respect to spanning tree T; the maximum distance from u to any other node in T. The diameter of a spanning tree T is given by $MAX\{ecc_T(u)\}$ over all nodes u in T.

```
ALGORITHM 2 (IR2(G, T, k)).
begin
    if (T is undefined)
        then
            T := MST of G;
    compute ecc_T (z) for all z in V;                    /* G = (V, E) */
    C := ∅;
    move := false;
    repeat
        diameter := MAX_{z∈V} {ecc_T (z)} ;
        if (C = ∅)
            then
                if (move = true)
                    then   begin
                                move := false;
                                C := edges (u, z) that are one edge farther from the
                                        center of T than in the previous iteration;
                           end
                    else
                        C := edges (u, z) at the center of T;
        repeat
            (x, y) := highest weight edge in C;
            /* This splits T into two trees: subtree1 and subtree2 */
        until ((C = ∅) or ( MAX_{u∈subtree1} {ecc_T (u)} = MAX_{z∈subtree2} {ecc_T (z)} ));
        if (C = ∅)
            then                                 /* no good edge to remove was found */
                move := true;
            else   begin
                        remove (x, y) from T;
                        get a replacement edge and add it to T;
                        recompute ecc_T (z) for all z in V;
                   end
    until ((diameter ≤ k) or (edges to be removed are farthest from center of T));
    return T
end.
```

5.1 Selecting Edges for Removal

To reduce the diameter, the edge removed must break a longest path in the tree and should be near the center of the tree. The center of spanning tree T can be found by identifying the nodes u in T with $ecc_T(u) = \lceil diameter/2 \rceil$, the node (or two nodes) with minimum eccentricity.

Since we may have more than one edge candidate for removal, we keep a sorted list of candidate edges. This list, which we call C, is implemented as a max-heap sorted according to edge weights, so that the highest-weight candidate edge is at the root.

Removing an edge from a tree does not guarantee breaking all longest paths in the tree. The end nodes of a longest path in T have maximum eccentricity, which is equal to the diameter of T. Therefore, we must verify that removing an edge splits the tree T into two subtrees, $subtree1$ and $subtree2$, such that each of the two subtrees contains a node v with $ecc_T(v)$ equal to the diameter of the tree T. If the highest-weight edge from heap C does not satisfy this condition, the algorithm removes it from C and considers the next highest. This process continues until the algorithm either finds an edge that breaks a longest path in T or the heap, C, becomes empty.

If the algorithm goes through the entire heap, C, without finding an edge to remove, it must consider edges farther from the center. This is done by identifying the nodes u with $ecc_T(u) = \lceil diameter/2 \rceil + bias$, where $bias$ is initialized to zero, and incremented by 1 every time the algorithm goes through C without finding an edge to remove. Then, the

algorithm recomputes C as all the edges incident to set of nodes u. Every time the algorithm succeeds in finding an edge to remove, *bias* is reset to zero.

This method of examining edges helps prevent cycling since we consider a different edge every time until an edge that can be removed is found. But to guarantee the prevention of cycling, always select a replacement edge that reduces the length of a path in T. This will ensure that the refinement process will terminate, since it will either reduce the diameter below the bound, k, or *bias* will become so large that the algorithm tries to remove the edges incident to the end-points of the longest paths in the tree.

In the worst case, computing heap C examines many edges in T, thereby requiring $O(n)$ comparisons. In addition, sorting C will take $O(n \log n)$ time. A replacement edge is found in $O(n^2)$ time since the algorithm must recompute eccentricity values for all nodes to find the replacement that helps reduce the diameter. Therefore, the iterative process, which removes and replaces edges for n iterations, will take $O(n^3)$ time in the worst case. Since heap C has to be sorted every time it is computed, the execution time can be reduced by a constant factor if we prevent C from becoming too large. This is achieved by an edge-replacement method that keeps the tree T fairly uniform so that it has a small number of edges near the center, as we will show in the next section. Since C is constructed from edges near the center of T, this will keep C small.

5.2 Selecting a Replacement Edge

When an edge is removed from a tree T, the tree T is split into two subtrees: *subtree*1 and *subtree*2. Then, we select a non-tree edge to connect the two subtrees in a way that reduces the length of at least one longest path in T without increasing the diameter. The diameter of T will be reduced when all longest paths have been so broken. We develop two methods, ERM1 and ERM2, to find such replacement edges.

5.2.1 Edge-Replacement Method ERM1

The first edge-replacement-method, shown in Algorithm 3, selects a minimum-weight edge (a, b) in G connecting a central node a in *subtree*1 to a central node b in *subtree*2. Among all edges that can connect *subtree*1 to *subtree*2, no other edge (c, z) will produce a tree such that the diameter of $(subtree1 \cup subtree2 \cup \{(c, z)\})$ is smaller than the diameter of $(subtree1 \cup subtree2 \cup \{(a, b)\})$. However, such an edge (a, b) is not guaranteed to exist in incomplete graphs.

ALGORITHM 3 (ERM1(G, T, *subtree*1, *subtree*2, *move*)).

```
begin
    recompute ecc_subtree1(.) and ecc_subtree2(.) for all nodes in each subtree;
    m1 := MIN_{u∈subtree1} {ecc_subtree1(u)} ;
    m2 := MIN_{u∈subtree2} {ecc_subtree2(u)} ;
    (a, b) := minimum-weight edge in G that has:
        (a ∈ subtree1) and (b ∈ subtree2) and (ecc_subtree1(a) = m1)
        and (ecc_subtree2(b) = m2);
    if (such an edge (a, b) is found)
        then
            add edge (a, b) to T;
        else  begin
                add the removed edge (x, y) back to T;
                move := true;
              end
    if ((C = ∅) or (bias = 0))
        then  begin
                move = true;
                C = ∅;
              end
    return edge (a, b)
end.
```

Since there can be at most two central nodes in each subtree, there are at most four edges to select from. The central nodes in the subtrees can be found by computing $ecc_{subtree1}(.)$ and $ecc_{subtree2}(.)$ in each subtree, then taking the nodes v with $ecc_{subtree}(v) = MIN\{ecc_{subtree}(u)\}$ over all nodes u in the subtree that contains v. This selection can be done in $O(n^2)$ time.

Finally, the boolean variable *move* is set to *true* every time an edge incident to the center of the tree is removed. This causes the removal of edges farther from the center of the tree in the next iteration of the algorithm, which prevents removing the recently added edge, (a, b).

This edge-replacement method seems fast at the first look, because it selects one out of four edges. However, in the early iterations of the algorithm, this method creates nodes of high degree near the center of the tree, which causes C to be very large. This, as we have shown in the previous section, causes the time complexity of the algorithm to increase by a constant factor. Furthermore, having at most four edges from which to select a replacement often causes the tree weight to increase significantly.

5.2.2 Edge-Replacement Method ERM2

The second edge-replacement-method, shown in Algorithm 4, computes $ecc_{subtree1}(.)$ and $ecc_{subtree2}(.)$ values for each subtree individually, as in ERM1. Then, the two subtrees are joined as follows. Let the removed edge (x, y) have $x \in subtree1$ and $y \in subtree2$. The replacement edge will be the smallest-weight edge (a, b) which (i) guarantees that the new edge does not increase the diameter, and (ii) guarantees reducing the length of a longest path in the tree at least by one. We enforce condition (i) by:

$$ecc_{subtree1}(a) \leq ecc_{subtree1}(x) \text{ AND } ecc_{subtree2}(b) \leq ecc_{subtree2}(y) ,$$

and condition (ii) by:

$$ecc_{subtree1}(a) < ecc_{subtree1}(x) \text{ OR } ecc_{subtree2}(b) < ecc_{subtree2}(y) .$$

If no such edge (a, b) is found, we must remove an edge farther from the center of the tree, instead.

ALGORITHM 4 (ERM2(G, T, *subtree*1, *subtree*2, x, y, *move*)).

```
begin
    recompute ecc_subtree1(.) and ecc_subtree2(.) for all nodes in each subtree;
    m1 := ecc_subtree1(x);
    m2 := ecc_subtree2(y);
    (a, b) := minimum-weight edge in G that has:
        (a ∈ subtree1) and (b ∈ subtree2) and (ecc_subtree1(a) ≤ m1)
        and (ecc_subtree2(b) ≤ m2) and ((ecc_subtree1(a) < m1 or (ecc_subtree2(b) < m2));
    if (such an edge (a, b) is found)
        then
            add edge (a, b) to T;
        else  begin
                add the removed edge (x, y) back to T;
                move := true;
              end
    return edge (a, b)
end.
```

Since ERM2 is not restricted to the centers of the two subtrees, it works better than ERM1 on incomplete graphs. In addition, it can produce DCMSTs with smaller weights because it selects a replacement from a large set of edges, instead of 4 or fewer edges as in ERM1. The larger number of edges increases the total time complexity of IR2 with ERM2 by a constant factor over IR2 with ERM1. However, this method does not create nodes of high degree near the center of the tree as in ERM1. This helps keep the size of heap C small in the early iterations, reducing the time complexity of IR2 by a constant factor.

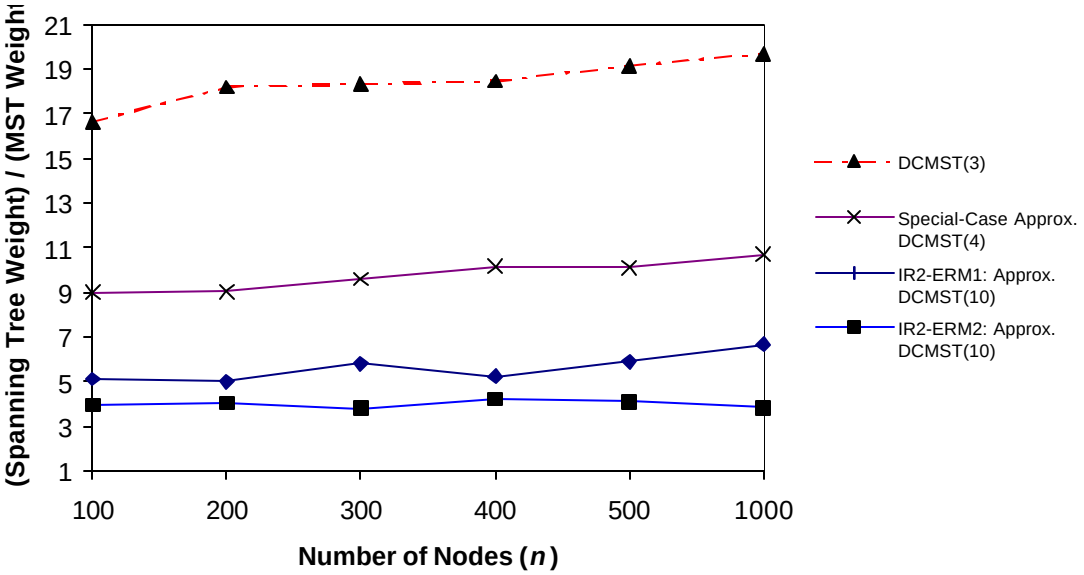


Figure 5.2 Weight quality of approximate solution, in randomly weighted complete-graphs with Hamiltonian-path MSTs, produced by IR2 using two different edge-replacement methods

5.3 Implementation

First, we parallelized Algorithm IR2 and implemented it on the MasPar MP-1, using complete random-graphs and complete graphs forced to have Hamiltonian-path MSTs, where edge weights were randomly selected integers between 1 and 1000. We also implemented IR2 sequentially on a PC with a Pentium III / 500 MHz processor using random-graphs and graphs forced to have Hamiltonian-path MSTs, where edge weights

were randomly selected integers between 1 and 10000, and the graph densities ranged from 20% to 100%. All input graphs had orders ranging from 50 to 2000, where 20 different graphs were generated for each order, density, and type of graphs. As expected, IR2 did not enter an infinite loop, and it always terminated within 3*n* iterations.

The weight quality of approximate DCMST(10) successfully obtained by this iterative-refinement algorithm using the two different edge replacement methods, ERM1 and ERM2, for graphs with Hamiltonian-path MSTs is shown in Figure 5.2. The diagram shows the weight of the computed approximate DCMST as a multiple of the weight of the unconstrained MST. It is clear that IR2 produced approximate solutions lower than the upper bounds, and IR2 using ERM2 produced lower weight solutions than IR2 using ERM1. As expected, the time required by IR2 using ERM1 to obtain approximate DCMSTs was greater than the time required by IR2 using ERM2. In addition, ERM1 required more memory space than ERM2, because the size of *C* when we use ERM1 is significantly larger than its size when ERM2 is used. This is caused by the creation of high-degree nodes by ERM1, as explained in Section 5.2. For the remainder of this dissertation, we will discuss the behavior of Algorithm IR2 only using ERM2 as the edge-replacement method.

When IR2 (using ERM2) was tested on random complete-graphs, the weight quality of approximate DCMST(10) produced by IR2 exceeded the weight of approximate DCMST(4) produced by the special-case algorithm when the edge weights were randomly selected integers between 1 and 1000, but not when the range of edge weights was 1 to 10000. In the latter case, IR2 also produced approximate DCMST(5) with

weight lower than the approximate DCMST(4) produced by the special-case algorithm. No spanning tree of diameter 3 was found in our samples of sparse graphs, and therefore, the special-case heuristic did not obtain any spanning trees of diameter 4 in those graphs. The average time required to produce approximate solutions with n = 2000 for DCMST(5) and DCMST(10), respectively, was 1924 and 1296 seconds in random complete-graphs, and 1231 and 538 seconds in random graphs with 20% density. The average weight of solutions with n = 2000 for DCMST(5) and DCMST(10), respectively, as a factor of the unconstrained MST weight, was 159 and 48 in random complete-graphs and 29 and 10.8 in random graphs with 20% density. In random graphs of all tested densities, the weight of solutions, as a factor of the unconstrained-MST weight, increased with n.

In graphs with Hamiltonian-path MSTs, the weight of approximate DCMST(10) produced by IR2 (using ERM2) was lower than the weight of approximate DCMST(4) produced by the special-case algorithm, regardless of the range of edge weights. The upper bounds (trees of diameter 3 and 4) were not available for sparse graphs of this type, either. The average time required by IR2 to produce approximate solutions, in graphs with Hamiltonian MSTs, with n = 2000 for DCMST(5) and DCMST(10), respectively, was 1488 and 1038 seconds in random complete-graphs, and 3038 and 1053 seconds in random graphs with 20% density. The average weight of solutions as a factor of the unconstrained MST weight, was approximately 26 and 9 for DCMST(5) and DCMST(10), respectively, in random complete-graphs, independent of n. In random graphs with 20% density, the weight of solution, as a factor of the unconstrained MST

weight, decreased with n. The weights of DCMST(5) and DCMST(10), respectively, as a factor of MST weight, was 44.6 and 18.9 for n = 50 and 21.5 and 11.1 for n = 2000.

The weight of solutions, as a factor of MST weight, in our samples of graphs with Hamiltonian-path MSTs did not increase with n because of the way these graphs were generated. To force a randomly generated graph to have a Hamiltonian-path MST, we randomly selected edges to include in the Hamiltonian path and randomly assigned them integer weights between 1 and 100. The rest of the edges were randomly generated integer-weights between 101 and 10000. Therefore, the average weight of an MST-edge is 50, and the average weight of a non-MST edge is 5050. However, there are only (n – 1) edges in the MST and there are $O(n^2)$ non-tree edges in the rest of the graph. Thus, as n increases, the ratio:

(average weight of a non-MST edge) / (average weight of an MST edge)

decreases. This effect becomes clearer as the number of edges exceeds 10000. Consequently, we evaluate the solutions' weights in this type of graphs based on the upper and lower bounds (whenever available) calculated for the same set of graphs. However, the time taken by the algorithm can be compared with other types of graphs, where it can be seen that IR2 requires a longer time to obtain a solution when the diameter of the unconstrained MST is larger.

With all input graphs used for IR2, the weights of solutions and time required to obtain them increased whenever the diameter bound, k, was decreased. The quality of IR2 will be discussed further, in Chapter 8, when it is compared to the other algorithms we developed for the DCMST problem.

5.4 Convergence

As was shown in Sections 5.2 and 5.3, Algorithm IR2 is guaranteed to terminate, but it is not guaranteed to produce a solution. The failure rate of IR2 does not depend on what fraction of n the value of the bound on diameter, k, is. Rather, it depends on how small the constant, k, is. To see this, we must take a close look at the way we move away from the center of the tree while selecting edges for removal. Note that the algorithm will fail only when it tries to remove edges incident to the end-points of the longest paths in the spanning tree. Also note that the algorithm moves away from the center of the spanning tree every time it goes through the entire set C without finding a good replacement edge, and it returns to the center of the spanning tree every time it succeeds. Thus, the only way the algorithm fails is when it is unable to find a good replacement edge in $\lceil diameter/2 \rceil$ *consecutive* attempts, each of which includes going through a different set of C. Empirical results show that it is unlikely that the algorithm will fail for 8 consecutive times, which makes it suitable for finding an approximate DCMST when the value of k is a constant greater than or equal to 15.

When the input graphs were forced to have Hamiltonian-path MSTs, Algorithm IR2 was unable to find a spanning tree with diameter no more than 10 in some cases. In graphs with $100 \leq n \leq 2500$, our empirical results show a failure rate of 10% for $k = 10$ and 15% for $k = 5$. The success rate of IR2 (using ERM2) with (unrestricted) random complete-graphs was 90% for $n \geq 200$. In all graphs, the times required by IR2 to obtain a solution increased when the value of k was decreased.

When tested on incomplete graphs, Algorithm IR2 (using ERM2) was more than 65% successful in obtaining an approximate DCMST(5) for random graphs and graphs with Hamiltonian-path MSTs, where the density was at least 20% and $n \geq 500$. The success rate dropped slowly as the density of the input graph was decreased. For the same types of graphs and the same densities, the success rate also dropped when n was reduced below 500, where Algorithm IR2 becomes only 30% successful in finding an approximate DCMST(5) in graphs with $n = 50$ and density = 20%. This is understandable since the number of edges grows faster than the number of nodes. For example, when density is 20%, there are 24950 edges in a graph of 500 nodes, but only 245 edges in a graph of 50 nodes.

We measured and analyzed the time taken by IR2 to terminate. We measured the time taken by IR2 (using ERM2) to terminate successfully on the Pentium III / 500 MHz machine, and we obtained the following equations using a polynomial-fit program. When using complete random-graphs, IR2 required $(0.111n^3 + 62.7n^2 - 29583.7n + 2170981)$ and $(0.0736n^3 - 21.5n^2 + 10100n - 1230000)$ microseconds for $k = 5$ and $k = 10$, respectively. For random graphs with 20% density, IR2 required $(0.191n^3 + 77.2n^2 - 42250.8n + 3152147)$ and $(0.0639n^3 + 9.55\ n^2 - 5573.5n - 342626)$ microseconds for $k = 5$ and $k = 10$, respectively. This shows that the time required by IR2 for this type of graphs is almost unaffected by the change in graph density. When using complete graphs with Hamiltonian-path MSTs, IR2 required $(0.187n^3 + 50.2n^2 - 28288.1n + 2119730)$ and $(0.121n^3 + 22n^2 - 11709.6n + 74699121)$ microseconds for $k = 5$ and $k = 10$, respectively. For graphs with Hamiltonian-path MSTs and 20%

density, IR2 required ($0.248n^3 + 38.4n^2 - 26746.2n + 2120446$) and ($0.181n^3 - 133.8n^2 + 71780.63n - 7707461$) microseconds for $k = 5$ and $k = 10$, respectively. This shows that, in this type of graphs, the time required by IR2 increases slightly when the graph density is reduced.

CHAPTER 6

THE CIR ITERATIVE-REFINEMENT ALGORITHM

The composite-iterative-refinement algorithm, CIR, first computes an unconstrained MST, then if the MST does not satisfy the constraint, iteratively refines this MST by edge replacements until the diameter constraint is satisfied or the algorithm fails. Algorithm CIR consists of two stages: Algorithms IR1 and IR2, described in Chapters 4 and 5. The first stage, Algorithm IR1, is fast, but it normally cannot reduce the diameter to the given bound, k. It terminates when it is unable to improve the current spanning tree for 15 consecutive iterations. The second stage, Algorithm IR2, refines the spanning tree further, until a solution is found or the algorithm fails. The only difference in our use of IR2 in this chapter is that it takes the spanning tree produced by IR1 as input, instead of starting with an unconstrained MST as in Chapter 5.

6.1 Implementation

Algorithm CIR was implemented sequentially on a PC with a Pentium III /500 MHz processor using graphs of orders 50 to 3000 and densities from 20% to 100%. Twenty different random graphs and twenty different graphs with Hamiltonian-path MSTs were

randomly generated for each order and density. CIR always terminated after no more than $3n$ iterations of IR2. It was successful in more than 80% obtaining an approximate solution in complete graphs, including the cases when n = 3000 and k = 5, where the weight ratio (approximate DCMST to MST) was 251 for random graphs and 27 for graphs with Hamiltonian-path MSTs. The weights of solutions produced by CIR were lower than the upper bounds in complete graphs. IR1 terminated after reducing the diameter by about 20%, producing a spanning tree with weight approximately 1.05 times MST weight. When the density of the graphs was gradually reduced to 20%, the success rate of CIR remained above 60%, but the weights of solutions deteriorated quickly. Overall, the time taken by IR1 was about 1% of the total CIR time. The time taken by CIR, in seconds, to obtain an approximate DCMST(5) in randomly-generated complete-graphs was 245, 815, and 1924 for n = 1000, 1500, and 2000, respectively. Using graphs with Hamiltonian-path MSTs as input, the weights of solutions, as compared to the upper and lower bounds, had the same quality as in random graphs, with the same success rate. However, using graphs with Hamiltonian-path MSTs, CIR obtained an approximate DCMST(5) in 292, 1130, and 3300 seconds for n = 1000, 1500, and 2000, respectively. For all graphs, the weights of solutions, and the time required by CIR to obtain them, increased whenever k was reduced or whenever n was increased.

6.2 Convergence

The first phase of Algorithm CIR, which is IR1, is guaranteed to terminate because it will either reduce the diameter below the given bound, or it will stop after failing to improve the solution for 15 consecutive iterations. Empirical evidence shows that allowing IR1 to continue past 15 consecutive unsuccessful-iterations does not result in noticeably lower-weight solutions for the DCMST problem by CIR. The second phase of CIR, which is IR2, is guaranteed to terminate (successfully or unsuccessfully), as shown in Chapter 5. Consequently, CIR will always terminate.

Empirical results show that the success rate of CIR is slightly lower than the success rate of IR2 (without IR1). Algorithm IR1 is faster than Algorithm IR2. However, the time required by CIR is longer than the time required when IR2 is run using an unconstrained MST as input. In the first stage of CIR, the MST is recomputed by IR1, producing a spanning tree with a smaller diameter. This spanning tree, however, contains $O(\log n)$ edges that are not in the unconstrained MST. Furthermore, these edges are close to the center of the spanning tree produced by IR1. In the second stage of CIR, some of these $O(\log n)$ edges are replaced by IR2, which also replaces other edges and reduces the diameter further. If the edge replacements in the first stage of CIR are not very carefully selected, they may cause the second stage to require more iterations, and possibly decrease the weight-quality of the final solution.

The time taken by CIR was measured on a PC with a Pentium III / 500 MHz processor. The input graphs had 50 to 3000 nodes, and densities ranging from 20% to

100%. The edge weights were randomly-selected integers between 1 and 10000. A polynomial-fit program was used to obtain the following equations for the time taken by CIR to obtain an approximate solution. When using random graphs as input, the time required by CIR to compute approximate DCMST(5) and DCMST(10), respectively, was $(0.236n^3 + 9.92n^2 - 2430.3n - 1050113)$ and $(0.169n^3 - 16.2n^2 + 4941.7n - 402038)$ microseconds for complete graphs, and $(0.278n^3 + 74.1n^2 - 42266.8n + 3030118)$ and $(0.151n^3 + 23.5n^2 - 16146.5n - 1194858)$ microseconds for graphs with 20% density. This shows that the time required by CIR, for this type of graphs, is almost unaffected by the change in graph density. Using graphs with Hamiltonian-path MSTs, the time required by CIR to compute approximate DCMST(5) and DCMST(10), respectively, was $(0.461n^3 - 239.4n^2 + 75896.98n - 3266577)$ and $(0.233n^3 - 30.5n^2 + 13153.98n - 1039968)$ microseconds for complete graphs, and $(0.394n^3 - 1.11n^2 + 203.51n + 704228)$ and $(0.171n^3 + 103.12n^2 - 50536.3n + 4291839)$ microseconds for graphs with 20% density. This shows that an increase in graph density in this type of graphs causes an increase the time required by CIR to obtain a solution.

CHAPTER 7

THE ONE-TIME-TREE-CONSTRUCTION ALGORITHM

In the one-time tree-construction strategy, a modification of Prim's algorithm is used to compute an approximate DCMST in one pass. Prim's original algorithm is chosen since it has been experimentally shown to be the fastest algorithm for computing an MST for large dense-graphs [57]. In addition, Prim's algorithm keeps the partially-formed tree connected in every iteration of the MST-construction process. This makes it easier to keep track of the diameter, as opposed to keeping track of the diameters of the trees of a forest in Kruskal's algorithm. Furthermore, using Kruskal's algorithm, a greedy strategy will not be able to construct a spanning tree of diameter k in one pass if an intermediate step creates two non-spanning trees of diameter $k/2$. That makes it necessary to keep more information about the forest while the spanning tree is being constructed, and it may be necessary to do backtracking in some cases.

7.1 The Algorithm

The one-time-tree-construction algorithm, OTTC, starts with an arbitrary node, and grows a spanning tree by connecting the nearest neighbor that does not violate the

diameter constraint. Algorithm OTTC is described in Algorithm 4, where we maintain the following information for each node, u:

$near(u)$ is a tree node candidate to be incident to u in the approximate DCMST.

$W(u, v)$ is the weight of edge (u, v).

$dist(u, v)$ is the (unweighted) distance from u to v if u and v are in the current tree, and is set to -1 if u or v is not in the tree.

$ecc(u)$ is the eccentricity of node u (the distance in the tree from u to the farthest node) if u is in the current tree, and is set to -1 if u is not in the tree.

To select $near(u)$, we determine the edges that connect u to partially-formed tree, T, without increasing the diameter (as the first criterion) and among all such edges we want the one with minimum weight.

In Segment 1 of Algorithm OTTC (described in Algorithm 4), the $dist(.)$ and $ecc(.)$ values for node z are set by copying from its parent node $near(z)$. In Segment 2, the values of $dist(.)$ and $ecc(.)$ for the parent node are updated in n steps. In Segment 3, the values of $dist(.)$ and $ecc(.)$ are update for other nodes. We make use of the $dist(.)$ and $ecc(.)$ arrays, as described above, to simplify the OTTC computation.

ALGORITHM 4 (OTTC(G, k, z_0)).

```
begin
    V_T := {z_0};                                          /* T = (V_T, E_T) */
    E_T := ∅;
    near(u) := z_0 ,  for every u ∉ V_T ;
    find a node z such that: W(z, near(z)) = MIN_{u ∈ V_T} {W(u, near(u))} ;
    while (|E_T| < (n – 1)) do
        select the node z with the smallest value of W(z, near(z));
        V_T := V_T ∪ {z};
        E_T := E_T ∪ {(z, near(z))};

        /* 1. set dist(z, u) and ecc(z) */
        for u = 1 to n do
            if (dist(near(z), u) > –1)
                then
                    dist(z, u) := dist(near(z), u) + 1;
        end for
        dist(z, z) := 0;
        ecc(z) := ecc(near(z)) + 1;

        /* 2. update dist(near(z), u) and ecc(near(z)) */
        dist(near(z), z) := 1;
        if (ecc(near(z)) < 1)
            then
                ecc(near(z)) := 1;

        /* 3. update other nodes' values of dist(.) and ecc(.) */
        for each tree node u other than z or near(z) do
            dist(u, z) := dist(u, near(z)) + 1;
            ecc(u) := MAX{ecc(u), dist(u, z)};
        end for

        /* 4. update the near(.) values for other nodes in G */
        for each node u not in the tree do
            if ((ecc(near(u)) + 1) > k)
                then
                    examine all nodes in T to determine near(u);
                else
                    compare W(u, near(u)) to W(u, z);
        end for
    end while
    return T
end.
```

In Segment 4, we update the *near*(.) values for every node not yet in the tree. If adding node z to the tree would increase the diameter beyond the constraint, we must reexamine all nodes of the tree to find a new value for *near*(z). This can be achieved by examining *ecc*(u) for nodes u in the tree; *i.e.*, we need not recompute the tree diameter. This computation includes adding a new node, z, to the tree, where $W(z, near(z))$ is minimum, and the addition does not increase the tree diameter beyond the constraint. The time complexity of Segment 4 is $O(n^2)$ since it requires examining each node in the tree once for each non-tree node. The *while* loop requires $(n - 1)$ iterations. Each iteration requires at most $O(n^2)$ steps, which makes the worst-case time-complexity of the algorithm $O(n^3)$.

When G is incomplete, this algorithm does not *always* find an existing solution. Furthermore, the algorithm is sensitive to the node chosen for starting the spanning tree. Therefore, we compute n such trees, one for each starting node. Then, we select the spanning tree with the smallest weight.

To reduce the time needed to compute a DCMST further, we select q starting nodes (q is independent of n) for OTTC at random, or we select the q nodes with the smallest sum of weights of the edges incident to each node. Now, producing q spanning trees, instead of n, reduces the overall time complexity by a factor $O(n)$ when q is a constant. For incomplete graphs, we can choose the q nodes with the highest degrees, and break a tie by choosing the node with the smaller sum of weights of edges incident to it.

7.2 Implementation

We implemented Algorithm OTTC on a PC with a Pentium III / 500 MHz processor for graphs of up to 2000 nodes and densities ranging from 20% to 100%, where 20 random graphs and 20 graphs with Hamiltoniam-path MSTs were generated for each order and density. The edge weights were randomly-selected integers between 1 and 10000. We also parallelized OTTC and implemented it on the MasPar MP-1 for complete random-graphs and graphs with Hamiltonian-path MSTs, where edge weights were randomly-selected integers between 1 and 1000. The parallel and sequential implementations produced similar results in terms of the performance of OTTC.

The time required by OTTC to obtain an approximate DCMST using n start nodes was significantly larger than the time required by the other four algorithms presented in this dissertation. We addressed this issue by running the algorithm for a carefully selected small set of start nodes. We examined three different methods for choosing the set of start nodes. One node-selection-method, NSM1, selects the center nodes of the q smallest stars in G as start nodes. The second method, NSM2, selects q nodes from G at random. The third method, NSM3, is specific to incomplete graphs, and it selects the q nodes with the highest degrees in G, and breaks a tie by choosing the node with the smaller sum of weights of edges incident to it. We report the results using $q = 20$, where other small values for q (between 1 and 50) produced similar results. The quality of solution improved slowly with q, and the success rate of OTTC in obtaining a solution in incomplete graphs increased slowly with q.

Algorithm OTTC was always successful in obtaining a solution for complete graphs, including when $k = 4$. However, the success rate of OTTC declined as the density of input graphs was reduced, especially when using less than n start nodes. Here, we report the results (approximate solutions) for complete random-graphs. When using n start nodes, Algorithm OTTC required 600, 2154, and 5130 seconds to compute an approximate DCMST(5) with $n = 1000$, 1500, and 2000. OTTC using NSM1 produced approximate solutions for DCMST(5) in 29, 91, and 206 seconds with $n = 1000$, 1500, and 2000, respectively. OTTC using NSM2 produced approximate solutions with weight almost identical to those obtained using NSM1, but it took 6, 14, and 26 seconds to compute an approximate DCMST(5) with $n = 1000$, 1500, and 2000, respectively. The weights of approximate solutions obtained using n start nodes was better than using 20 start nodes by more than two times the weight of MST, but it took about $n/20$ times longer to run to completion. This confirms the theoretical analysis that predicted an improvement of $O(n)$ in execution time over the n-iteration algorithm. The time required by OTTC grew slowly with k. For example, using OTTC with NSM2 to compute an approximate DCMST(150) required only ½ second longer than the time required to compute an approximate DCMST(4) for the same set of graphs with $n = 1500$. There was no difference in the weight quality (compared to the upper and lower bounds), or in the actual running time, when graphs with Hamiltonian-path MSTs were used as input for OTTC.

In incomplete graphs, OTTC with NSM2 required the same time it did with complete graphs. Using OTTC with NSM3 instead of NSM1 required a small amount of extra

time, but it had no noticeable effect on the weights of solutions or the success rate of the algorithm. Overall, the time required by OTTC was not affected by the change in graph density and whether or not the graph had a Hamiltonian-path MST.

7.3 Convergence

Algorithm OTTC starts with an arbitrary node, and grows a spanning tree by adding a node via the smallest-weight edge that does not violate the diameter constraint. When adding an edge would violate the diameter constraint, an alternative edge, which does not violate the diameter constraint, must be found. Since such an edge is guaranteed to exist in a complete graph, OTTC will always converge in complete graphs. However, in incomplete graphs, the algorithm is likely to fail as the density of the graph is reduced.

As expected, empirical results show OTTC was always successful in obtaining an approximate solution in all types of complete graphs. When the density was reduced to 50%, it was necessary to run OTTC an average of 1.25 times for n = 100 and an average of 7.3 times for n = 1500, each time with a different arbitrary node, before an approximate DCMST(10) was found. In graphs with 20% density, it was necessary to run OTTC approximately 29 times for n = 100 and approximately 67 times for n = 1500, each time with a different arbitrary node, before an approximate DCMST(10) was found. The number of attempts required to obtain a solution increased with n, but it was always possible for OTTC to obtain an approximate solution for DCMST(10) within n attempts

in graphs with density of at least 20%. However, OTTC, using n source nodes, was unable to find a spanning tree with diameter 5 in sparse graphs when at least one existed, and its success rate in dense graphs was less than 100% for $k = 5$. The time required by OTTC was not affected by the change in graph density, where NSM2 required the same time as in complete graphs, and NSM3 required approximately the same time as NSM1. The success rate of NSM2 and NSM3, in obtaining a solution, was approximately the same.

We used a polynomial-fit program to determine the equation for the time taken by OTTC on a PC with a Pentium III / 500 MHz processor. The time taken by OTTC using one source node was approximately $(0.00299n^3 + 0.33n^2 - 28.4n + 1667)$ microseconds.

CHAPTER 8

PERFORMANCE COMPARISONS

In this chapter, we present empirical results by implementing the iterative refinement and one-time-tree-construction algorithms on a Pentium III / 500 MHz processor, and we compare the performance of these algorithms based on the weight of approximate solutions obtained, required time, convergence and success rate, and the effects of different types of input.

Graphs of different orders and densities, represented by their $(n \times n)$ weight matrices, were used as input. The input graphs had $50 \leq n \leq 3000$, where edge weights were randomly generated integers, 1 to 10000. Twenty different graphs were generated for each order and density, and the mean values were calculated. In addition, we used randomly generated graphs with Hamiltonian-path MSTs, *i.e.*, graphs whose MSTs are forced to have diameter $(n - 1)$.

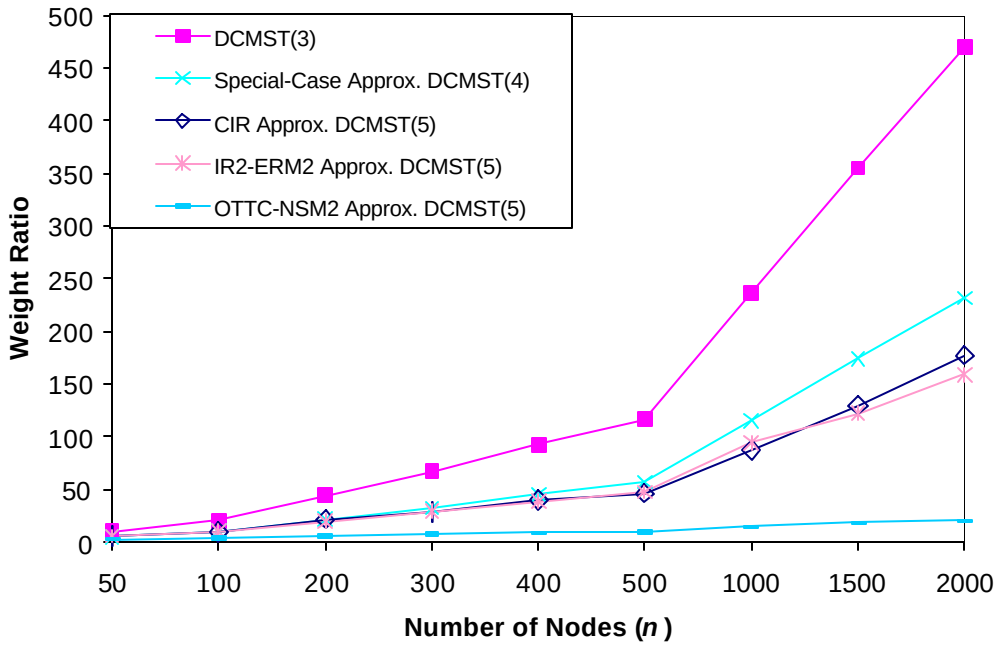


Figure 8.1 The ratio ((spanning-tree weight) / (MST weight)) in randomly weighted complete-graphs

Among all four general-algorithms we presented, only OTTC was *always* successful in obtaining an approximate solution for the DCMST problem in complete graphs regardless of the value of the diameter bound, k. However, the success rate of OTTC (using one source node) diminished quickly when the density of the graphs was reduced, especially with small values of the diameter bound, such as $k = 5$. Algorithms IR2 and CIR were not always successful in obtaining an approximate solution, but their success rate remained above 60%, even for graphs with 20% density and diameter constraint $k = 5$. When OTTC was run using n start nodes, it was successful in obtaining an approximate DCMST(10) in all graphs we tested, regardless of their density. However, OTTC using n start nodes was unable to obtain an approximate DCMST(5) in any graph

with 20% density, and it was unsuccessful in obtaining an approximate DCMST(5) in graphs with 50% density and $n \geq 1000$, where algorithms IR2 and CIR verified that such solutions existed. Algorithm IR1 was always successful in obtaining an approximate solution in complete graphs when the diameter bound, k, was not less than $n/10$, but it was unsuccessful in obtaining a solution when k was a small constant, or when the graph was not dense.

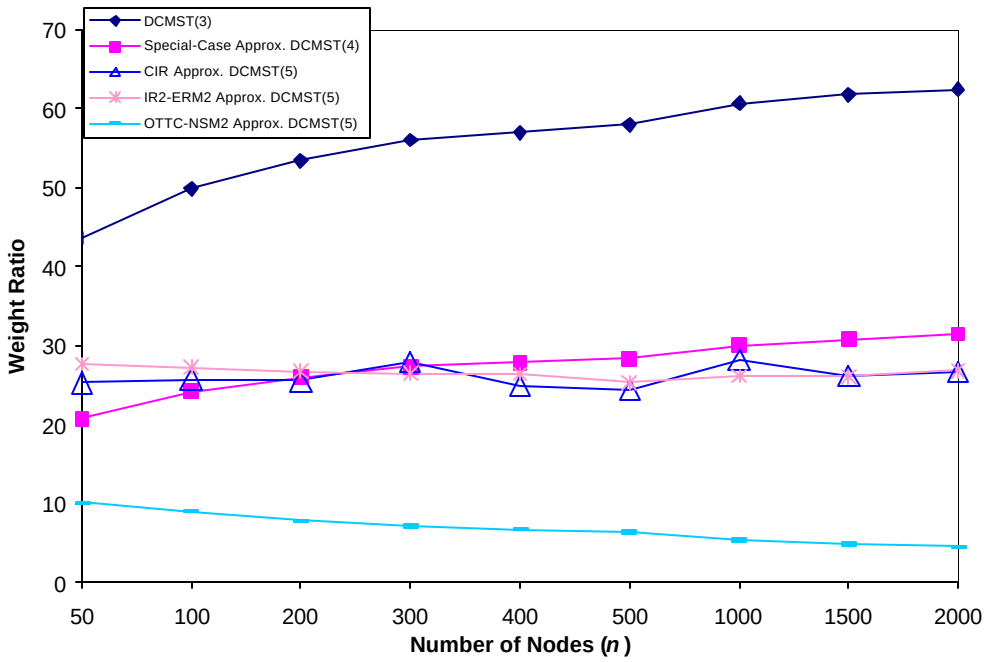


Figure 8.2 The ratio ((spanning-tree weight) / (MST weight)) in randomly weighted complete-graphs with Hamiltonian-path MSTs

The special-case algorithm for DCMST(4) was always successful in finding an approximate solution in a graph that contains a spanning tree of diameter 3, and it was unsuccessful otherwise. This special-case algorithm provided an upper bound on the weights of approximate solutions produced by the general algorithms.

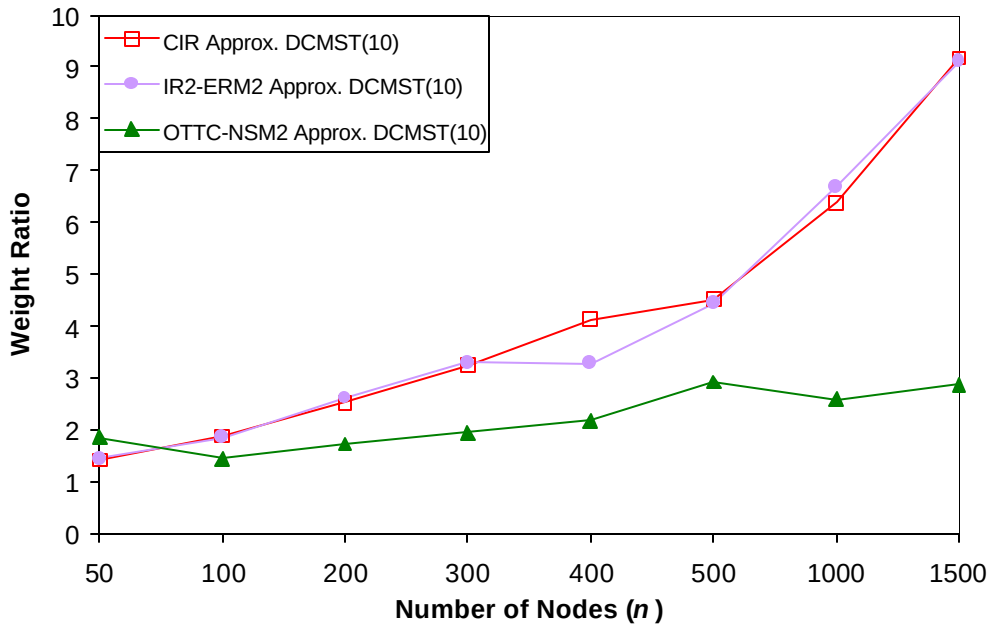


Figure 8.3 The ratio((spanning-tree weight) / (MST weight)) in randomly weighted graphs with 20% density

As seen in Figures 8.1, 8.2, and 8.3, Algorithm OTTC provided the lowest-weight solutions, regardless of the density of the graph, and whether or not it has a Hamiltonian-path MST. Algorithm CIR produced approximate solutions with weight similar to the weight of approximate solutions produced by IR2, and lower than the weight of the approximate DCMST(4) produced by the special-case algorithm. Among all five algorithms we presented for the DCMST problem, only IR2 and CIR could produce approximate solutions for values of k as small as 5 in sparse graphs.

The time required by IR1, IR2, and CIR increased when k was reduced. The time required by OTTC increased slightly when k was increased. However, OTTC (using NSM2) was the fastest of the four general algorithms in obtaining an approximate solution. Furthermore, it was the only general algorithm that did not require more time

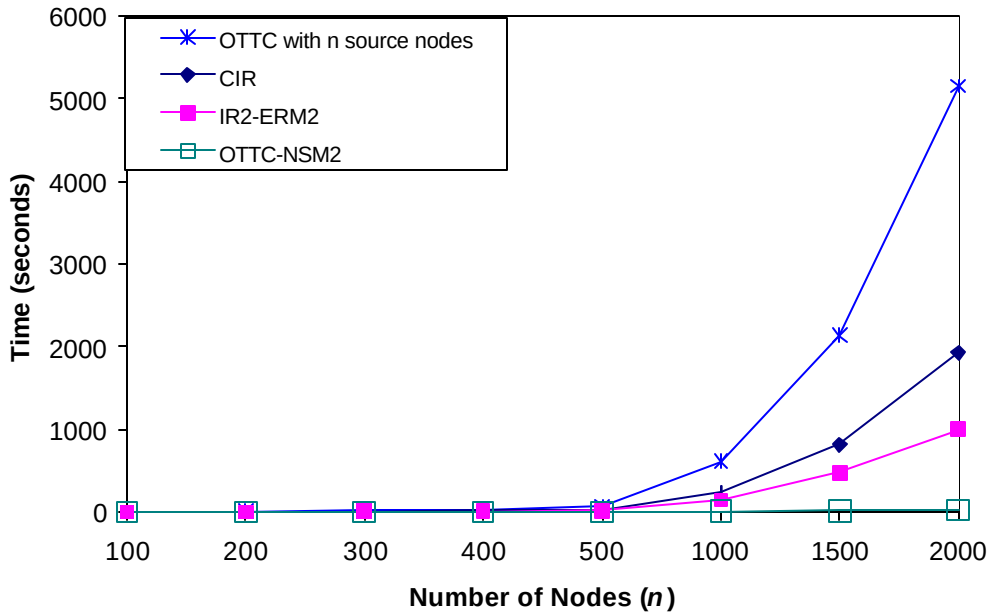


Figure 8.4 The time required by different algorithms to obtain an approximate DCMST(5) in randomly weighted complete-graphs

when the diameter of the unrestricted MST was increased. Figure 8.4 illustrates the time required by IR2, CIR, and OTTC to compute an approximate DCMST(5) in the same set of complete graphs. As the number of nodes increased, OTTC using NSM2 required less time than the other algorithms illustrated in the figure, but OTTC using n start nodes required more time than the other algorithms illustrated in this figure. These results are reflected by the coefficient of n^3 in the equation of time required by OTTC, given in Chapter 7, which is lower than the coefficients of n^3 in the equations of time required by other algorithms, given in earlier chapters. When OTTC was run with n source nodes, its required time was multiplied by n. The equations for the time required by OTTC and IR2, to obtain a DCMST(5) in randomly weighted complete-graphs, can be used to show that OTTC run with n source nodes requires more time than IR2 for graphs with $n \geq 81$.

CHAPTER 9

CONCLUSION

The DCMST problem has various practical applications, such as in distributed mutual-exclusion, bit-compression for information retrieval, and linear lightwave networks. We presented a survey of the literature on the DCMST problem and presented new efficient approximate-algorithms that solve it. We presented a study of the behavior of MST-diameter in randomly generated graphs. Then, we developed five approximate algorithms for the DCMST problem. The special-case algorithm for DCMST(4) constructs an approximate DCMST(4) from an exact DCMST(3), and it is used in providing an upper bound on the weight of approximate solutions in dense graphs. When the diameter constraint is a large fraction of n, Algorithm IR1 uses the iterative-refinement strategy to provide high-quality solutions in a short amount of time. It starts with an unconstrained MST, and then it iteratively increases the weight of (log n) edges near the center of the MST and recomputes the MST. Algorithm IR2 uses the iterative-refinement strategy with more careful edge-replacements, and without recomputing the MST, to provide feasible solutions when k is a small constant. It serves as a tighter upper bound for other general algorithms for the DCMST problem, and it is the algorithm of choice for sparse graphs. The composite-iterative-refinement algorithm, CIR, combines IR1 and IR2, providing insight into their behavior. Algorithm OTTC, which grows a

spanning tree of the desired diameter, was always successful in obtaining an approximate solution in complete graphs, and the solution was always lower than all upper bounds, even when using a small number of start nodes. When OTTC was used with a small number of start nodes, it was also the fastest of the four general algorithms we presented for solving the DCMST problem. Consequently, OTTC is an excellent algorithm for providing solutions for the DCMST problem, except in sparse graphs, where its success rate was significantly lower than that of IR2.

APPENDIX A

PROGRAM CODE FOR COMPUTING EXACT AVERAGE-DIAMETER

```
/*******************************************************************************************
*  Calculation of exact average-diameter for a tree of order n using Riordan's
*  equations for computing the number of trees with a given diameter.
*
*******************************************************************************************/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <sys/time.h>
#include <sys/resource.h>
#include "gmp.h"  /* this multi-precision library is available free from GNU  */

#define maxn 52

#define MY_SEC (((after.ru_utime.tv_sec*1000000+after.ru_utime.tv_usec) \
-(before.ru_utime.tv_sec*1000000+before.ru_utime.tv_usec))/1000000.0)

struct rusage before, after;
long hsec, husec;
long int who=RUSAGE_SELF;

int S[maxn/2];
mpz_t power[maxn-2][maxn-4];  /* mpz_t is an arbitrarily-long integer */
mpz_t Factorial[maxn-1];
mpq_t G[maxn/2][maxn-1];      /* mpq_t is an arbitrarily-long rational number */
mpq_t D[maxn-1][maxn-1];

void Exponent(mpz_t *ans, int base, int expo)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   if (expo < 0)
        mpz_set_ui(*ans,0);  /* unacceptable exponents */
   else if (expo == 1)
        mpz_set_ui(*ans,(unsigned long)base);
   else if (expo == 0 || base == 1)
        mpz_set_ui(*ans,1); /* includes 0^0: The Curious Identity */
   else if (base == 0)
        mpz_set_ui(*ans,0);
   else mpz_set(*ans,power[base-2][expo-2]);
}

void AllPowers(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* power[i][j] contains (i+2)^(j+2) which means power[0][0] contains 2^2          */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int base, expo;

   for (base = 2; base <= n_hi; base++) {
     mpz_init_set_ui(power[base-2][0],(unsigned long)base);
     mpz_mul_ui(power[base-2][0],power[base-2][0],(unsigned long)base);
     for (expo = 3; expo <= n_hi - 2; expo++)
```

```
         mpz_mul_ui(power[base-2][expo-2], power[base-2][expo-3],(unsigned long)base);
   }
}

void AllFactorials(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i;

   mpz_init_set_ui(Factorial[0], 1);
   for (i=1; i <= n_hi; i++) {
      mpz_init_set(Factorial[i], Factorial[i-1]);
      mpz_mul_ui(Factorial[i], Factorial[i], (unsigned long)i);
   }
}

void PolyMult(mpq_t *poly_prod, mpq_t *poly1, mpq_t *poly2,  int len1, int len2, int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute the product of two polynomials. Assume all terms of each polynomial are      */
/* present, where poly[i] is the coefficient of x^(i+1)                                  */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, j;
   mpq_t temp;

   mpq_init(temp);
   for (i = 0; i < len1+len2-1 && i < n_hi-1; i++)
      mpq_set_ui(poly_prod[i], 0, 1);

   for (i = 1; i < len1; i++)
      for (j = 1; j < len2 && i+j < n_hi; j++) {
         mpq_mul(temp, poly1[i-1], poly2[j-1]);
         mpq_add(poly_prod[i+j-1], poly_prod[i+j-1], temp);
      }
   mpq_clear(temp);
}

void Initialize_G(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, j;

   for (i=0; i < n_hi/2; i++)
      for (j=0; j < n_hi-1; j++) {
         mpq_init(G[i][j]);
         mpq_set_ui(G[i][j], 0, 1);
      }
}

void Initialize_D(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, j;
```

```
   for (i=0; i < n_hi-1; i++)
      for (j=0; j < n_hi-1; j++)
         mpq_init(D[i][j]);
}

void Compute_G2n(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Note: G[i][j] contains G_i+1_j+1 which means G[0][0] contains G_1_1                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, n;
   mpz_t tempz1, tempz2;
   mpq_t tempq;

   mpz_init(tempz1);
   mpz_init(tempz2);
   mpq_init(tempq);

   for (n = 3; n < n_hi; n++) {
      mpq_set_ui(G[1][n-1], 0, 1);
      for (i = 1; i < n; i++) {
         mpz_mul(tempz1, Factorial[i], Factorial[n-i-1]);
         Exponent(&tempz2,i,n-i-1);
         mpz_set(mpq_numref(tempq), tempz2);
         mpz_set(mpq_denref(tempq), tempz1);
         mpq_canonicalize(tempq);
         mpq_add(G[1][n-1], G[1][n-1], tempq);
      }
   }

   mpz_clear(tempz1);
   mpz_clear(tempz2);
   mpq_clear(tempq);
}

void Compute_Gkn(int k, int n, int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Note: G[i][j] contains G_i+1_j+1 which means G[0][0] contains G_1_1                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i;
   mpq_t tempq;
   mpz_t tempz;
   mpq_t result;

   mpq_init(tempq);
   mpz_init(tempz);
   mpq_init(result);

   mpz_set_ui(mpq_numref(result), 1);
   mpz_set(mpq_denref(result), Factorial[S[0]]);
   for (i = 0; i <= k-2; i++) {
```

```
       Exponent(&tempz,S[i],S[i+1]);
       mpz_set(mpq_numref(tempq), tempz);
       mpz_set(mpq_denref(tempq), Factorial[S[i+1]]);
       mpq_canonicalize(tempq);
       mpq_mul(result, result, tempq);
    }

    for (i = k-1; i < n && i < n_hi/2; i++) {
       Exponent(&tempz,S[i],S[i+1]);
       mpz_set(mpq_numref(tempq), tempz);
       mpz_set(mpq_denref(tempq), Factorial[S[i+1]]);
       mpq_canonicalize(tempq);
       mpq_mul(result, result, tempq);
       mpq_add(G[i][n-1], G[i][n-1], result);
    }

    mpq_clear(tempq);
    mpz_clear(tempz);
    mpq_clear(result);
}

void Compute_Gn(int n, int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    int diameter_value;
    int i, k, subtotal;

    for (i = 0; i < n-2; i++)
       S[i] = 0;

    S[0] = n - 1;
    subtotal = 0;  /* total of S[1] through S[n-2]  */
    k = 2;

    while (S[n-2] < n-1 /* && k < n-1 && k < n_hi/2 */) {
         for (i = 1; i <= n-2; i++)
              if (subtotal < n - 1) {  /* no carry */
                S[i]++;
                subtotal++;
                if (S[i] == n-1)
                  k++;
                i = n;   /* exit for loop */
              }
              else{
                  subtotal -= S[i];
                  S[i] = 0;   /* carry 1 */
              }
         S[0] = n - subtotal - 1;

         Compute_Gkn(k,n, n_hi);
    }   /* end while */
}
```

```
void Compute_G(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, j;
   mpz_t tempz;
   mpq_t rest;

   Initialize_G(n_hi);
   mpz_init(tempz);
   mpq_init(rest);

   /* case n == 2: */
   for (i = 0; i < n_hi-1; i++) {
      mpq_set_ui(G[0][i],1,1);
      mpz_set(mpq_denref(G[0][i]), Factorial[i]);
   }
   for (i = 0; i < n_hi/2; i++) {
      mpq_set_ui(G[i][0], 1, 1);
      mpq_set_ui(G[i][1], 1, 1);
   }

   /* case n > 2: */
   for (i = 3; i < n_hi-1; i++) {
      Compute_Gn(i, n_hi);
      for (j = 2; j < i-1 && j < n_hi/2; j++)
         mpq_add(G[j][i], G[j][i], G[0][i]);
      Exponent(&tempz,i,i-2);
      mpz_set(mpq_numref(rest), tempz);
      mpz_set(mpq_denref(rest), Factorial[i-1]);
      mpq_canonicalize(rest);
      for (j = i-1; j < n_hi/2; j++)
         mpq_set(G[j-1][i-1], rest);
   }

   mpz_clear(tempz);
   mpq_clear(rest);
   Compute_G2n(n_hi);
}

void Compute_D(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Assume G has already been computed                                              */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, j;
   mpq_t *temp1, *temp2, *temp;
   mpq_t h1[maxn-1], h2[maxn-1], prod[maxn-1];
   unsigned long ul_i;

   mpq_init(h1[0]);
   mpq_set_ui(h1[0], 1, 1);
   mpq_init(h2[0]);
   mpq_set_ui(h2[0], 0, 1);
```

```
    for (i = 1; i < n_hi-1; i++) {
       mpq_init(h1[i]);
       mpq_set_ui(h1[i], 0, 1);
       mpq_init(h2[i]);
       mpq_set(h2[i], G[0][i]);
    }

    Initialize_D(n_hi);

    for (i = 0; i < n_hi-1; i++) {
       ul_i = (unsigned long)i + 1;
       mpq_set_ui(D[0][i], ul_i, 1);
       mpz_set(mpq_denref(D[0][i]), Factorial[i+1]);
       mpq_canonicalize(D[0][i]);
    }

    temp1 = h1;
    temp2 = h2;

    for (i = 1; i < n_hi/2 ; i++) {
       temp = temp1;
       temp1 = temp2;
       temp2 = temp;

       for (j = 0; j < n_hi-1; j++)
           mpq_sub(temp2[j], G[i][j], G[i-1][j]);

       /* Even diameter */
       PolyMult(D[2*i],temp1,G[i-1],n_hi-1,n_hi-1,n_hi);
       for (j = 0; j < n_hi-1; j++)
           mpq_sub(D[2*i][j], temp2[j], D[2*i][j]);

       /* Odd diameter */
       PolyMult(D[2*i-1],temp1,temp1,n_hi-1,n_hi-1,n_hi);
       for (j=0; j < n_hi-1; j++) {
          mpz_mul_ui(mpq_denref(D[2*i-1][j]), mpq_denref(D[2*i-1][j]), 2);
          mpq_canonicalize(D[2*i-1][j]);
       }
    }
/* May free h1 and h2 here */
}

void ExpectedDiameter(int n_hi)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Calculate the expected diameter of a labeled tree as the average over all possible trees of     */
/* trees of the same order                                                                          */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    int i, n;
    mpq_t mean_diameter;
    mpq_t tempq;
    mpz_t powerz;
```

```
   mpz_t mean_quotient;
   mpz_t mean_remainder;

   printf("\n   Order      Mean\n");
   printf("~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~\n");

   mpq_init(mean_diameter);
   mpq_init(tempq);
   mpz_init(powerz);
   mpz_init(mean_quotient);
   mpz_init(mean_remainder);

   for (n = 4; n < n_hi-1; n++)
   {
      mpq_set_ui(mean_diameter, 0, 1);
      for (i = 2; i <= n-1; i++) {
         mpq_set(tempq, D[i-2][n-1]);
         mpz_mul(mpq_numref(tempq), mpq_numref(tempq), Factorial[n]);
         mpq_canonicalize(tempq);
         mpz_mul_ui(mpq_numref(tempq), mpq_numref(tempq), (unsigned long)i);
         mpq_canonicalize(tempq);
         mpq_add(mean_diameter, mean_diameter, tempq);
      }
      Exponent(&powerz, n, n-2);
      mpz_mul(mpq_denref(mean_diameter), mpq_denref(mean_diameter), powerz);
      mpq_canonicalize(mean_diameter);
      mpz_tdiv_qr(mean_quotient, mean_remainder,
                  mpq_numref(mean_diameter), mpq_denref(mean_diameter));

      printf(" %5d     ", n);
      mpz_out_str (stdout, 10, mean_quotient);
      printf("  +  ");
      mpz_out_str (stdout, 10, mean_remainder);
      printf(" / ");
      mpz_out_str (stdout, 10, mpq_denref(mean_diameter));
      printf("\n");
   }

   mpq_clear(mean_diameter);
   mpq_clear(tempq);
   mpz_clear(powerz);
   mpz_clear(mean_quotient);
   mpz_clear(mean_remainder);
}

/*======================================================================*/
int main(int argc, char *argv[])
{
   int n_hi;
   double total_time;

   n_hi = atoi(argv[1]);
   printf("\n\nNumber of nodes: from 4 to %d \n",n_hi);
```

```
    n_hi += 2;
    if (n_hi > maxn) {
      printf("\nMaximum number of nodes allowed is %d\n\n",maxn-2);
      exit(1);
    }

/*  total_time = 0.0;  */
/*  getrusage(who,&before);   */

    AllFactorials(n_hi);
    AllPowers(n_hi);
    Compute_G(n_hi);
    Compute_D(n_hi);
    ExpectedDiameter(n_hi);

/*  getrusage(who,&after);    */
/*  total_time += MY_SEC;   */
/*  printf("\n\nTotal time is %2.2lf \n\n", total_time);   */

    return 0;
}

/************************************   End of File  ****************************************/
```

APPENDIX B

PROGRAM CODE FOR THE ITERATIVE-REFINEMENT ALGORITHMS

```
/*****************************************************************************************
*  Iterative-refinement code (written in C) for the composite algorithm
*  CIR, to find approximate solutions for Diameter-Constrained MST problem.
*  This program can be easily modified to run any combination of IR1, IR2,
*  and the special-case algorithm for DCMST(4).
*
*****************************************************************************************/

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <sys/time.h>
#include <sys/resource.h>

#define MaxWt 10000
#define MaxN  10000
#define MaxMSTWt MaxWt/100

#define BIG_NUMBER 0x7fffffff
#define TRUE  1
#define FALSE 0

#define MY_SEC (((after.ru_utime.tv_sec*1000000+after.ru_utime.tv_usec) \
-(before.ru_utime.tv_sec*1000000+before.ru_utime.tv_usec))/1000000.0)

#define StartTiming() getrusage(who,&before)
#define StopTiming(time) getrusage(who,&after); time = MY_SEC;

struct rusage before, after;
long hsec, husec;
int who=RUSAGE_SELF;

short n;            /* actual number of nodes */
short k;            /* the given diameter constraint */
short diameter;      /* current tree diameter */
short dens;          /* density of incomplete graph */
short root;

short *p;           /* parent of each node in current tree */
short *ecc;          /* eccentricity of each node in the tree */
short *deg;          /* degree of each node in the tree */
int **G;            /* the original graph */

int **dist;         /* in IR1: the graph weights, including penalty */
                   /* in IR2: distance between any pair of nodes in tree */

short *leafness;     /* indication of how far each node is from the leaves */
short *L;           /* the heap containing nodes farthest from the leaves */

typedef struct heap_type {
      short u1, u2;   /* (u1, u2) is an edge "close" to the center */
      } heap_type;
heap_type *C;         /* the heap containing edges "close" to tree center */
```

```
short sizeC;            /* Current number of elements in heap */

struct QNode_type {
            short node;
            struct QNode_type *next;
            };
typedef struct QNode_type QNode;

QNode **child_h; /* Q head and tail for children of each node in current tree */
QNode **child_t;
double original_mstwt;
short maxIterations;


void DisplayGraph(int **graph)
/*~~~
```

```
   printf("\n This is the spanning tree: (root is parent of itself)");
   printf("\n========================================================\n");
   printf("Root is: %d\n",root);
   for (i=0; i < n; i++)
      printf(" Node: %2d  Parent: %2d  Weight: %4d  Dist: %4d\n",
            i, p[i], G[i][p[i]], dist[i][p[i]]);
      printf(" Node: %2d  Degree: %2d Parent: %2d  Weight: %4d  Dist: %4d\n",
            i, deg[i], p[i], G[i][p[i]], dist[i][p[i]]);
   printf("\n\n");
}

void Span(int **graph)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Generate a random spanning tree to make the graph connected.                   */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, u, v;
   short *tree;
   short *rest;

   if (NULL == (tree = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in Span(): tree[].\n");
      exit(1);
   }
   if (NULL == (rest = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in Span(): rest[].\n");
      exit(1);
   }

   for (i=0; i < n; i++)
      tree[i] = i;

   /* Put the nodes of the graph in a random order. */
   for (i=0; i < n-1; i++)   {
      u = rand() % (n - i);
      rest[i] = tree[u];
      for (v=u; v < n-i-1; v++)
         tree[v] = tree[v+1];
   }
   /* tree[0] contains the node leftover */

   tree[1] = rest[n-2];
   graph[tree[0]][tree[1]] = random() % MaxWt + 1;
   graph[tree[1]][tree[0]] = random() % MaxWt + 1;

   for (i=2; i < n; i++) {
      u = tree[rand()%i];  /* select an arbitrary node from the tree */
      v = rest[n-i-1];     /* select an "arbitrary" node not in the tree */
      tree[i] = v;         /* add new node to the tree */
      graph[u][v] = random() % MaxWt + 1;
      graph[v][u] = random() % MaxWt + 1;
   }
```

```
    free(tree);
    free(rest);
}

void GenerateRandomGraph(int **graph, int *m)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Generate a random graph of n nodes, using the given density.            */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j;
    short d;

    /* Initialize the diagonal of the graph to zero (no self loops). */
    for (i=0; i < n; i++)
       graph[i][i] = 0;
    for (i=0; i < n; i++)
       for (j = i+1; j < n; j++)
          graph[i][j] = graph[j][i] = 0;

    *m = 0;
    Span(graph);
    for (i=0; i < n; i++)
       for (j=0; j < i; j++) {
          d = rand() % 100;
          if (d < dens && graph[i][j] == 0)
            graph[i][j] = random() % MaxWt + 1;
          if (graph[i][j] > 0) {  /* if there is an edge between i and j */
            graph[j][i] = graph[i][j];
            *m = *m + 1;
          }
          else
              graph[i][j] = graph[j][i] = 0;
       }

    for (i=0;i<n;i++)
       for (j = i+1; j < n; j++)
          dist[i][j] = dist[j][i] = graph[i][j];
}

void AllocateMemory(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Allocate memory for the 2-dimentional arrays G and dist.  This Memory    */
/* will not be freed until the program exits.                                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i;

    if (NULL == (G = (int **)calloc(n, sizeof(int)))) {
      printf("Out of memory -- function AllocateMemory() -- G\n");
      exit(1);
    }
    for (i=0; i < n; i++)
```

```
       if (NULL == (G[i] = (int *)calloc(n, sizeof(int)))) {
         printf("Out of memory -- function AllocateMemory() -- G[%d]\n", i);
         exit(1);
       }

    if (NULL == (dist = (int **)calloc(n, sizeof(int)))) {
      printf("Out of memory -- function AllocateMemory() -- dist\n");
      exit(1);
    }
    for (i=0; i < n; i++)
       if (NULL == (dist[i] = (int *)calloc(n, sizeof(int)))) {
         printf("Out of memory -- function AllocateMemory() -- dist[%d]\n", i);
         exit(1);
       }
}

void GenerateRandomHamGraph(int *m)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Generate a complete graph with random edge weights, where the MST is                              */
/* forced to be a Hamiltonian path                                                                    */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j, u;
    short *node, d;

    if (NULL == (node = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in GenerateHamRandomGraph().\n");
      exit(1);
    }

    /* Put the nodes of the graph in a random order. */
    for (i=0; i < n; i++)
       node[i] = i;
    for (i=0; i < n-1; i++) {
       u = rand() % (n - i);
       p[i] = node[u];
       for (j=u; j < n-i-1; j++)
          node[j] = node[j+1];
    }
    p[n-1] = node[0];

    for (i=0; i < n; i++)
       for (j = i; j < n; j++)
          G[i][j] = 0;

    /* Generate the Hamiltonian path MST */
    for (i=0; i < n-1; i++)
               G[p[i]][p[i+1]] = G[p[i+1]][p[i]] = random() % MaxMSTWt + 1;

    *m = 0;
    for (i=0; i < n; i++)
       for (j = i+1; j < n; j++) {
          d = rand() % 100;
```

```
         if (d < dens && G[i][j] == 0)
           G[i][j] = random() % (MaxWt-MaxMSTWt) + MaxMSTWt + 1;
         if (G[i][j] > 0) {  /* if there is an edge between i and j */
           G[j][i] = G[i][j];
           *m = *m + 1;
         }
         else
             G[i][j] = G[j][i] = 0;
      }

   for (i=0; i < n; i++)
      dist[i][i] = 0;
   for (i=0; i < n; i++)
      for (j=i+1; j < n; j++)
         dist[i][j] = dist[j][i] = G[i][j];

   free(node);
}

short FindMax(short *list, short nn)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Find the largest magnitude in the first nn elements of array list[]                    */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   short i, max;

   max = 0;
   for (i=0; i < nn; i++)
      if (abs(list[i]) >= max)
        max = abs(list[i]);
   return(max);
}

void FindMaxEccInSub(short *max1, short *max2)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Find the largest eccentricity, with respect to the entire tree, in each subtree.          */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   short i;

   *max1 = *max2 = 0;
   for (i=0; i < n; i++)
      if (ecc[i] < 0 && abs(ecc[i]) >= *max1)
        *max1 = abs(ecc[i]);
      else if (ecc[i] > 0 && abs(ecc[i]) >= *max2)
             *max2 = abs(ecc[i]);
}

double FindMST(int **graph)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute the minimum spanning tree using Prim's algorithm.                                  */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
```

```
double wt=0;
short i, j;
short *cheap, *p1;
short *inTree;
short nextEdge, nextNode;

if (NULL == (p1 = (short *)calloc(n, sizeof(short)))) {
   printf("Out of memory in FindMST(): p1[].\n");
   exit(1);
}
if (NULL == (cheap = (short *)calloc(n, sizeof(short)))) {
   printf("Out of memory in FindMST(): cheap[].\n");
   exit(1);
}
if (NULL == (inTree = (short *)calloc(n, sizeof(short)))) {
   printf("Out of memory in FindMST(): inTree[].\n");
   exit(1);
}

for (i=0; i < n; i++) {
   inTree[i] = FALSE;
   if (graph[0][i] > 0)
     cheap[i] = graph[0][i];
   else
       cheap[i] = MaxWt + 1;
   p1[i] = 0;
}

inTree[0] = TRUE;
p[0] = 0;
root = 0;

for (i=1; i < n; i++) {
   nextEdge = MaxWt;
   nextNode = -1;

   for (j=1; j < n; j++)
      if (!(inTree[j]) && (nextEdge > cheap[j])) {
        nextEdge = cheap[j];
        nextNode = j;
      }

      inTree[nextNode] = TRUE;
      wt += G[nextNode][p1[nextNode]];
      p[nextNode] = p1[nextNode];

      for (j=0; j < n; j++)
         if (cheap[j] > graph[j][nextNode] && graph[j][nextNode] > 0) {
           cheap[j] = graph[j][nextNode];
           p1[j] = nextNode;
         }
}
```

```
    free(p1);
    free(cheap);
    free(inTree);
    return wt;
}

void Enqueue(short u, QNode **head, QNode **tail)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Insert node u at the tail of the queue.                                              */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    QNode *oldtail;

    oldtail = *tail;
    if(NULL==(*tail=(struct QNode_type *)malloc(sizeof(struct QNode_type)))){
      printf ("Out of memory -- function Enqueue()\n");
      exit(1);
    }
    (*tail)->node = u;

    if (oldtail != NULL)
      oldtail->next = *tail;
    else *head = *tail;

    (*tail)->next = NULL;
}

int Dequeue(QNode **head, QNode **tail)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Delete and return the node at the head of the queue.                                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    QNode *oldhead;
    int ret;

    if (*head == NULL)
      return(-1);

    ret = (*head)->node;
    oldhead = *head;
    *head = (*head)->next;
    if (*head == NULL)
      *tail = NULL;
    oldhead->next = NULL;
    free(oldhead);

    return(ret);
}

short RemoveFromQ(QNode **prev, QNode **child)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Remove a node from the middle of the queue.                                          */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
```

```
{
    QNode *temp;

    if (*prev != *child) {
      temp = *child;
      (*prev)->next = (*child)->next;
      *child = (*child)->next;
      temp->next = NULL;
      free(temp);
      return 1;
   }
   else {
     temp = *child;
     *prev = *child = (*child)->next;
     free(temp);
     return 0;
   }
}

void ComputeDegrees(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute the degree of each node and store its children                            */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    int i;

    for (i=0; i < n; i++)
               deg[i] = 0;
    for (i=0; i < n; i++)
       child_h[i] = child_t[i] = NULL;

    for (i=0; i < n; i++) {
               deg[i]++;
               deg[p[i]]++;
       if (i != root)
         Enqueue(i, &child_h[p[i]], &child_t[p[i]]);
    }
    deg[root] -= 2;
}

short GetFromHeapL(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Remove and return the node farthest from the leaves                          */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    int i, j;
    short done;
    short ret, temp;

    if (sizeC == 0)  /* heap is empty */
      ret = -1;

    else {
```

```
       done = FALSE;
       ret = L[0];
       temp = L[sizeC-1];
       sizeC--;
       i = 0;
       j = 1;  /* j is left child of i */
       while (j < sizeC && !done) {
            if (j < sizeC-1)
              if (leafness[L[j]] < leafness[L[j+1]])
                j++;  /* j points to larger child */
            if (leafness[L[temp]] >= leafness[L[j]])
              done = TRUE;
            else {
              L[i] = L[j];                  /* move child up */
              i = j; j = 2*j + 1;           /* move i and j down */
            }
       }
       L[i] = temp;
    }
    return ret;
}

void AddToHeapL(short u)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Add node u to heap L                                                     */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, done;

    if (sizeC >= MaxN) {
      printf("Error: Heap L overflow\n");
      exit(1);
    }

    else {
       i = sizeC++;
       done = FALSE;
       while (!done)
            if (i == 0)
              done = TRUE;  /* at root */
            else if (leafness[u] < leafness[L[(i-1)/2]])
                   done = TRUE;
                 else if (leafness[u] == leafness[L[(i-1)/2]]
                          && G[u][p[u]] <= G[L[u]][L[(i-1)/2]])
                            done = TRUE;
                      else {  /* move from parent to child */
                        L[i] = L[(i-1)/2];
                        i = (i-1)/2;
                      }
       L[i] = u;
    }
}
```

```
void CreateHeapL(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i;

    sizeC = 0;
    for (i = 0; i < n; i++)
       AddToHeapL(i);
}

short Diameter(short *deg, short *cross_pointer)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute diameter using Leaf Deletion Method                                                                    */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
  short i, dia;
  short leaf_count = 0, que_size = 0;
  short num_cross_point = 0; /* tree level, starting from the leaves */
  short leaf_node;
  short front, rear, crossed = 0;
  QNode *head, *tail, *child, *prev;

  if (n==1) return 0;
  else if (n==2) return 1;

  front =  rear = 0;
  head = tail = NULL;

  for (i=0; i < n; i++)
     if (deg[i] == 1) {
                Enqueue(i, &head, &tail);
       rear++;
       leaf_count++;
       que_size++;
     }

  cross_pointer[num_cross_point] = que_size - 1;
  crossed = FALSE;
  p[root] = -1;

  if (que_size != n-1)        /* not STAR, i.e., n-1 nodes are leaf nodes. */
       while (que_size <= n)  {
            leaf_node  = Dequeue(&head, &tail);
            deg[leaf_node] = 0;
            front++;
            leaf_count--;
            leafness[leaf_node] = num_cross_point + 1;

            if (leaf_node != root) {
              deg[p[leaf_node]]--;
              if (deg[p[leaf_node]] == 1)  {
                Enqueue(p[leaf_node], &head, &tail);
               rear++;
```

```
            leaf_count++;
            que_size++;
          }
        }
        /* decrement children's deg */
        prev = child = child_h[leaf_node];
        while (child != NULL) {
              deg[child->node]--;
              if (deg[child->node] == 1) {
                Enqueue(child->node, &head, &tail);
                rear++;
                leaf_count++;
                que_size++;
              }

              if (deg[child->node] <= 1) {
                if (RemoveFromQ(&prev, &child) == 0)
                /* return 0 for child is at beginning of queue */
                  child_h[leaf_node] = child;
              }
              else {
                prev = child;
                child = child->next;
              }
        }

        if (front > cross_pointer[num_cross_point]) {
          num_cross_point++;
          cross_pointer[num_cross_point] = rear - 1;
          crossed = TRUE;
        }

        if (crossed == TRUE && rear-front <= 2)  break;
        /* Detection of Path after deleting all leaf nodes */

        crossed = FALSE;
    }

/* leafness is not needed in the star case.  IR will terminate. */
if (num_cross_point == 0 && que_size == n-1)  /* STAR case */
  dia = 2;

if (num_cross_point > 0 && leaf_count == 2) { /* a path remains */
  dia = 2*(num_cross_point) + (n - (que_size - 2)) - 1;
  i = 2*num_cross_point + 1; // + (diameter&1);
  while (que_size <= n)  {
        leaf_node = Dequeue(&head, &tail);
        i++;
        if (leaf_node == -1) break;
        leafness[leaf_node] = i/2;
        if (leaf_node != root) {
          deg[p[leaf_node]]--;
          if (deg[p[leaf_node]] == 1)  {
```

```
                Enqueue(p[leaf_node], &head, &tail);
                que_size++;
             }
           }
               prev = child = child_h[leaf_node];
               while (child != NULL) {
                    deg[child->node]--;
                    if (deg[child->node] == 1) {
                      Enqueue(child->node, &head, &tail);
                     que_size++;
                    }

                    if (deg[child->node] <= 1) {
                      if (RemoveFromQ(&prev, &child) == 0)
                        child_h[leaf_node] = child;
                    }
                    else {
                      prev = child;
                      child = child->next;
                    }
              }
      }
    }

    if ( (num_cross_point > 0) && (que_size - front) == 1) { /* node remains */
      dia = 2 * num_cross_point;
     i = Dequeue(&head, &tail);
      leafness[i] = num_cross_point + 1;
    }
    tail = head;
    while (NULL != head) {
         tail = tail->next;
         free(head);
         head = tail;
    }

    for (i = 0; i < n; i++) {
       child_t[i] = child_h[i];
       while (NULL != child_h[i]) {
            child_t[i]= (child_t[i])->next;
            free(child_h[i]);
            child_h[i] = child_t[i];
       }
    }
    p[root] = root;
    return dia;
}

void Penalize(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Increase the weight of edges "near" the center                                      */
/* Assume root == 0                                                                                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
```

```
{
   short i, near;
   short penalty, limit;
   int Wmin, Wmax;

   Wmax = 0;
   Wmin = BIG_NUMBER;
   for (i = 1; i < n; i++) {
     if (Wmin < dist[i][p[i]])
       Wmin = dist[i][p[i]];
     if (Wmax > dist[i][p[i]])
       Wmax = dist[i][p[i]];
   }

   limit = floor(log(n)/log(2));
   for (i = 0; i < limit; i++) {
     if ((near = GetFromHeapL()) == root)
       near = GetFromHeapL();
     penalty = (dist[near][p[near]] - Wmin) * Wmax / (Wmax - Wmin);
     if (penalty > 0)
       if (leafness[near] <= leafness[p[near]])
         penalty /= ((diameter + 1)/2 - leafness[near] + 1);
       else penalty /= ((diameter + 1)/2 - leafness[p[near]] + 1);
     if (penalty < 1)
       penalty = 1;

     if (dist[near][p[near]] + penalty > BIG_NUMBER) {
       printf("Overflow error in function Penalize() \n");
       exit(1);
     }
     else {
       dist[near][p[near]] += penalty;
       dist[p[near]][near] = dist[near][p[near]];
     }
   }
}

double IR1(double *IR1time)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Iterative Refinement method IR1                                                   */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   double time;
   double dcmst_wt, prev_weight;
   short iter, *cross_pointer, *prev_p, *temp;
   short MSTdiameter, prev_diameter, worse;

   if (NULL == (L = (short *)calloc(n, sizeof(short)))) {
     printf("Out of memory -- function IR2(): L[]\n");
     exit(1);
   }
   if (NULL == (leafness = (short *)calloc(n, sizeof(short)))) {
     printf("Out of memory -- function IR2(): leafness[]\n");
```

```
  exit(1);
}
if (NULL == (deg = (short *)calloc(n, sizeof(short)))) {
  printf("Out of memory -- function IR2(): deg[].\n");
  exit(1);
}
if (NULL == (cross_pointer = (short *)calloc(n-1, sizeof(short)))) {
  printf("Out of memory in IR2(): cross_pointer[].\n");
  exit(1);
}
if (NULL == (prev_p = (short *)calloc(n, sizeof(short)))) {
  printf("Out of memory -- function IR2(): prev_p[]\n");
  exit(1);
}
if (NULL == (child_h = (QNode **)calloc(n, sizeof(QNode*)))) {
   printf("Out of memory in IR2(): child_h[].\n");
   exit(1);
}
if (NULL == (child_t = (QNode **)calloc(n, sizeof(QNode*)))) {
   printf("Out of memory in IR2(): child_t[].\n");
   exit(1);
}

StartTiming();
dcmst_wt = original_mstwt = FindMST(dist);
ComputeDegrees();
MSTdiameter = diameter = Diameter(deg,cross_pointer);
if (diameter > k)
  CreateHeapL();
for (iter = 0; iter < n; iter++)
   prev_p[iter] = p[iter];

StopTiming(time);
iter = 0;
printf("\nIn IR1:\n");
printf("Iteration: %d,  Diameter: %d,  MST Weight: %0.0lf,  Time: %0.4lf \n",
        iter,diameter,dcmst_wt,time);

prev_diameter = diameter;
prev_weight = dcmst_wt;
worse = 0;  /* number of iterations without improvement to solution */
iter = 0;
while (iter < maxIterations && diameter > k && worse <= 15) {
     iter++;
     CreateHeapL();
     Penalize();
     dcmst_wt = FindMST(dist);
     ComputeDegrees();
     diameter = Diameter(deg,cross_pointer);
     if (prev_diameter > diameter ||
         (prev_diameter == diameter && prev_weight > dcmst_wt)){
       prev_diameter = diameter;
       prev_weight = dcmst_wt;
```

```
          worse = 0;
          temp = prev_p;
          prev_p = p;
          p = temp;
        }
        else
          worse++;
   }

  printf("Final Iteration: %d,  Diameter: %d,  DCMST Weight: %0.0lf,  Time: %0.4lf \n",
          iter,diameter,dcmst_wt,time);

  if (diameter >= prev_diameter) {
      temp = p;
      p = prev_p;
      prev_p = temp;
      diameter = prev_diameter;
      dcmst_wt = prev_weight;
   }

   StopTiming(time);
   *IR1time = time;
   printf("Revert to better tree: Diameter: %d,  DCMST Weight: %0.0lf,  Time: %0.4lf \n",
          diameter,dcmst_wt,time);

   free(L);
   free(leafness);
   free(deg);
   free(cross_pointer);
   free(prev_p);
   free(child_h);
   free(child_t);

   return(dcmst_wt);
}

void ComputeDistEcc(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute the distance and eccentricity values for all nodes.               */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int u, w, z;
   QNode *head, *tail;

   /* initialize dist and ecc */
   for (u=0; u < n; u++) {
      ecc[u] = 0;
      dist[u][u] = 0;
      for (w=0; w < u; w++)
         dist[w][u] = dist[u][w] = -1;
   }

   head = tail = NULL;
```

```
    Enqueue(root, &head, &tail);

    while ((u = Dequeue(&head, &tail)) != -1)
       for (w=0; w < n; w++)
          if (dist[root][w] < 0 && p[w] == u && w != u){
          /* if w is an unvisited neighbor of u */
             dist[w][root] = dist[root][w] = dist[root][u] + 1;
             dist[u][w] = dist[w][u] = 1;
             ecc[w] = ecc[u] + 1;
             Enqueue(w, &head, &tail);

             for (z=0; z < n; z++)
                if ((dist[root][z] > 0 || z == root)&& z != w){
                  dist[z][w] = dist[w][z] = dist[u][z] + 1;
                  if (ecc[z] < dist[z][w])
                    ecc[z] = dist[z][w];
                }
          }
}

void ComputeSubtreeEcc(short *sub_ecc)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Recompute the eccentricity values with respect to each subtree.           */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j;

    for (i=0; i < n; i++)
        sub_ecc[i] = 0;

    for (j=0; j < n; j++)
        for (i=0; i < n; i++)
            if (ecc[j] * ecc[i] > 0 && dist[i][j] > sub_ecc[j])
              sub_ecc[j] = dist[i][j];

}

void RecomputeDistEcc(short a, short b)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Recompute the eccentricity values with respect to the entire tree.        */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j;
    short *subtree;


    if (NULL == (subtree = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory -- function RecomputeDistEcc()\n");
      exit(1);
    }

    for (j=0; j < n; j++) {
       subtree[j] = ecc[j];
```

```
      ecc[j] = 0;
    }

    dist[a][b] = dist[b][a] = 1;
    for (j=0; j < n; j++)
      for (i=0; i < n; i++) {
        if (subtree[j] < 0 && subtree[i] > 0)
          dist[j][i] = dist[j][a] + dist[b][i] + 1;
        else if (subtree[j] > 0 && subtree[i] < 0)
              dist[j][i] = dist[j][b] + dist[a][i] + 1;
        if (dist[j][i] > ecc[j])
          ecc[j] = dist[j][i];
      }
    free(subtree);
}

/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
double ComputeDCMST3(short *uu, short *vv)
{
    double wt, wt3;
    short u, v, w;
    short count, fail;
    short p2[MaxN];

    wt = wt3 = n * MaxWt;
    for (u=0; u < n-1; u++)
      for (v=u+1; v < n; v++)    /* for each edge (u,v) */
        if (G[u][v] > 0) {
        p2[u] = u;  /* make u the root of the tree */
        p2[v] = u;  /* initialize tree to edge (u,v) */
        wt = G[u][v];
        count = 2;
        fail = FALSE;
        for (w=0; w < n && !fail; w++)
          if (w != u && w != v)
            if (G[w][u] < G[w][v] && G[w][u] > 0) {
              p2[w] = u;
              wt += G[w][u];
              count++;
            }
            else if (G[w][v] > 0) {
                   p2[w] = v;
                   wt += G[w][v];
                   count++;
                 }
                 else fail = TRUE;

        if (wt < wt3 && count == n) {
          wt3 = wt;
          *uu = u;
          *vv = v;
          for (w=0; w < n; w++)
            p[w] = p2[w];
```

```
     }
   }

  root = *uu;
  return wt3;
}

double ComputeDCMST4(double mst3, short uu, short vv)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Assume DCMST3 has been computed, and that uu is the root, and that         */
/* the parent of every leaf is either uu or vv, and p[vv] = uu                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
  double wtu, wtv;
  short x, y, z;
  short p1[MaxN];
  short p2[MaxN];

  if (mst3 >= n * MaxWt)
   return mst3;
  for (x=0; x < n; x++)
    p2[x] = p1[x] = p[x];

  p1[uu] = uu;  /* this a potential root for DCMST(4) */
  p1[vv] = uu;
  wtu = mst3;
  for (x=0; x < n; x++)
    if (p[x] == vv && x != vv && x != uu) {
         z = vv;
         for (y=0; y < n; y++)
          if (p[y] == uu && y != uu && y != vv
          && G[x][y] < G[x][z] && G[x][y] > 0)
           z = y;
        p1[x] = z;
        wtu += G[x][z] - G[x][p[x]];
        }

  p2[vv] = vv;  /* this another potential root for DCMST(4) */
  p2[uu] = vv;
  wtv = mst3;
  for (x=0; x < n; x++)
    if (p[x] == uu && x != uu && x != vv ) {
       z = uu;
       for (y=0; y < n; y++)
        if (p[y] == vv && y != vv && y != uu
        && G[x][y] < G[x][z] && G[x][y] > 0)
         z = y;
       p2[x] = z;
       wtv += G[x][z] - G[x][p[x]];
       }

  if (wtv < wtu) {
   for (x=0; x < n; x++)
```

```
                  p[x] = p2[x];
      root = vv;
      return wtv;
    }
    else{
                for (x=0; x < n; x++)
                    p[x] = p1[x];
        root = uu;
                return wtu;
            }
}

void AddToHeapC(short u1, short u2)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Add edge (u1, u2) to heap C                                        */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    int wt;
    short i, done;

    wt = G[u1][u2];
    if (sizeC >= MaxN) {
      printf("Error: Heap C overflow\n");
      exit(1);
    }

    else {
       i = sizeC++;
       done = FALSE;
       while (!done)
            if (i == 0)
              done = TRUE;  /* at root */
            else if (wt <= G[C[(i-1)/2].u1][C[(i-1)/2].u2])
                   done = TRUE;
                 else {  /* move from parent to child */
                   C[i] = C[(i-1)/2];
                   i = (i-1)/2;
                 }

       C[i].u1 = u1;
       C[i].u2 = u2;
    }
}

void CreateHeapC(short bias)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Create heap C, and avoid duplicate entries.                       */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short u, close;
    QNode *headu, *tailu, *headv, *tailv;
    QNode *up, *vp;
```

```
    sizeC = 0;
    tailu = tailv = NULL;
    headu = headv = NULL;
    close = (diameter + 1)/2 + !(diameter & 1);

    if (bias != 0)
      for (u = 0; u < n; u++)
         if (abs(ecc[u]) == (diameter+1)/2 + bias)
           Enqueue(u, &headu, &tailu);
         else
           Enqueue(u, &headv, &tailv);
    else
      for (u = 0; u < n; u++)
         if (abs(ecc[u]) == (diameter+1)/2 + bias)
           Enqueue(u, &headu, &tailu);
         else
           if (abs(ecc[u]) == close)
             Enqueue(u, &headv, &tailv);

    if (headu != NULL && headv != NULL) {
      for (vp = headv; vp != NULL; vp = vp->next)
         for (up = headu; up != NULL; up = up->next)
            if (dist[vp->node][up->node] == 1)
              AddToHeapC(vp->node,up->node);

      if (bias != 0) {
        for (vp = headu; vp->next != NULL; vp = vp->next)
           for (up = vp->next; up != NULL; up = up->next)
              if (dist[vp->node][up->node] == 1)
              AddToHeapC(vp->node,up->node);
      }
      else  /* if bias == 0 */
        for (vp = headu; vp->next != NULL; vp = vp->next)
           for (up = vp->next; up != NULL; up = up->next)
              if (dist[vp->node][up->node] == 1 && abs(ecc[u]) == close)
                AddToHeapC(vp->node,up->node);
    }

    for (vp = headv; vp != NULL; ) {
       vp = vp->next;
       free(headv);
       headv = vp;
    }

    for (up = headu; up != NULL; ) {
       up = up->next;
       free(headu);
       headu = up;
    }
}

heap_type GetFromHeapC(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
```

```
/* Remove and return the edge with highest weight from heap C                        */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   short i, j;
   short done;
   heap_type ret, temp;

   if (sizeC == 0)  /* heap is empty */
     ret.u1 = ret.u2 = 0;

   else {
      done = FALSE;
      ret = C[0];
      temp = C[sizeC-1];
      sizeC--;
      i = 0;
      j = 1;  /* j is left child of i */
      while (j < sizeC && !done) {
           if (j < sizeC-1)
             if (G[C[j].u1][C[j].u2] < G[C[j+1].u1][C[j+1].u2])
               j++;  /* j points to larger child */
           if (G[temp.u1][temp.u2] >= G[C[j].u1][C[j].u2])
             done = TRUE;
           else {
             C[i] = C[j];  /* move child up */
             i = j; j = 2*j + 1;  /* move i and j down */
           }
      }
      C[i] = temp;
   }
   return ret;
}

void ERM2(short *move, short *aa, short *bb, short x, short y)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Edge Replacement method ERM2                                                        */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int ab_wt;
   short u;
   short *sub_ecc; /* eccentricity of node wrt to its subtree */
   QNode *a, *b;
   QNode *sub1_head, *sub1_tail;
   QNode *sub2_head, *sub2_tail;

   sub1_head = sub1_tail = NULL;
   sub2_head = sub2_tail = NULL;

   if (NULL == (sub_ecc = (short *)calloc(n,sizeof(short)))) {
     printf("Out of memory in ERM2.\n");
     exit(1);
   }
   ComputeSubtreeEcc(sub_ecc);
```

```
    for (u=0; u < n; u++)
       if (ecc[u] * ecc[x] > 0 && sub_ecc[u] <= sub_ecc[x])
         Enqueue(u, &sub1_head, &sub1_tail);
        else if (ecc[u] * ecc[y] > 0 && sub_ecc[u] <= sub_ecc[y])
              Enqueue(u, &sub2_head, &sub2_tail);

    *aa = *bb = -1;
    ab_wt = BIG_NUMBER;
    for (a = sub1_head; NULL != a; a = a->next)
       for (b = sub2_head; NULL != b; b = b->next)
          if ( G[a->node][b->node] < ab_wt &&
               G[a->node][b->node] > 0 &&
                 (abs(ecc[a->node]) < abs(ecc[x]) ||
                  abs(ecc[b->node]) < abs(ecc[y])) ) {
            *aa = a->node;
            *bb = b->node;
            ab_wt = G[a->node][b->node];
          }

    free(sub_ecc);
    if (*aa == -1 || *bb == -1) {  /* no replacement edge found */
      *aa = x;
      *bb = y;
      *move = TRUE;
    }

    a = sub1_head;
    while (NULL != sub1_head) {
         a = a->next;
         free(sub1_head);
         sub1_head = a;
    }

    a = sub2_head;
    while (NULL != sub2_head) {
         a = a->next;
         free(sub2_head);
         sub2_head = a;
    }
}

double IR2(double *IR2time, double dcmst_wt)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Iterative Refinement method IR2                                        */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    double time;
    short move, iterations;
    short origDiameter;
    short bias, eccu, eccv;
    heap_type *xy;
    short j, a, b;
```

```
if (NULL == (xy = (heap_type *)malloc(sizeof(heap_type)))) {
  printf ("Out of memory -- function IR2(): xy\n");
  exit(1);
}
if (NULL == (C = (heap_type *)calloc(n, sizeof(heap_type)))) {
  printf ("Out of memory -- function IR2(): C[]\n");
  exit(1);
}
if (NULL == (ecc = (short *)calloc(n, sizeof(short)))) {
  printf ("Out of memory -- function IR2(): ecc[]\n");
  exit(1);
}

StartTiming();
iterations = 0;
ComputeDistEcc();
diameter = FindMax(ecc, n);
origDiameter = diameter;
StopTiming(time);
printf("\nIn IR2:\n");
printf("Iteration: %d,  Diameter: %d,  DCMST Weight: %0.0lf,  Time: %0.4f \n",
        iterations,diameter,dcmst_wt,time);

move = FALSE;
bias = 0;
while (diameter > k && bias < (diameter+1)/2
       && iterations < maxIterations) {
  iterations++;
  if (move && 0 == sizeC) {
    move = FALSE;
    bias++;
    }
  else if (!move)
         bias = 0;
  if (!move && sizeC == 0)
    CreateHeapC(bias);
  eccu = eccv = 0;
  *xy = GetFromHeapC();

  if (xy->u1 != 0)
    do{
      for (j=0; j < n; j++)
         if (dist[j][xy->u2] > dist[j][xy->u1])
           ecc[j] = -(abs(ecc[j]));  /* j is in subtree 1 */
         else
             ecc[j] = abs(ecc[j]);   /* j is in subtree 2 */

      FindMaxEccInSub(&eccu,&eccv);
      if (eccu != eccv)
        *xy = GetFromHeapC();
    } while(xy->u1 != 0 && eccu != eccv);
```

```
      if (eccu != eccv)
        move = TRUE;
      else { /*
            if (useERM1)
              ERM1(&move,&a,&b,bias,xy->u1,xy->u2);
            else  */
              ERM2(&move,&a,&b,xy->u1,xy->u2);

          if (!((a == xy->u1 && b == xy->u2)||(b == xy->u1 && a == xy->u2))) {
          /* recompute dist and ecc only if tree changes */
            RecomputeDistEcc(a,b);
            diameter = FindMax(ecc, n);
          }
      }
    }

    dcmst_wt = FindMST(dist);  /* reorganize the tree as a rooted tree */
    StopTiming(time);
    *IR2time = time;
    printf("Final Iteration: %d,  Diameter: %d,  DCMST Weight: %0.0lf,  Time: %0.4lf\n",
            iterations,diameter,dcmst_wt,time);
    free(xy);
    free(C);
    free(ecc);

    return(dcmst_wt);
}

/*=====================================================================*/
main(int argc, char *argv[])
{
  extern char *optarg;
  int m;  /* number of edges in the graph */
  short graph_type, uu, vv;
  short seed;
  short IR1d_hi, IR1d_low;
  double IR1d_sum, IR1d_sumsq, ave_dens;
  double dcmst3, dcmst4, factor;
  double dcmstIR1_wt, dcmstIR2_wt;
  double dcmst3_low, dcmst3_high, dcmst4_low, dcmst4_high;
  double dcmst4_sum, dcmst4_sumsq, dcmst3_sum, dcmst3_sumsq;
  double IR1time,IR2time,totalIR1time, totalIR2time, total_time;
  double dcmstIR1_low, dcmstIR1_high, dcmstIR1_sum, dcmstIR1_sumsq;
  double dcmstIR2_low, dcmstIR2_high, dcmstIR2_sum, dcmstIR2_sumsq;
  double dcmst_ratio_low, dcmst_ratio_high, dcmst_ratio_sum, dcmst_ratio_sumsq;
  short trial, num_trials, fails, big_k, c;
  short upper_bound_fails;

  n = 10;
  m = 45;
  k = 5;
  dens = 100;
  seed = 7;
```

```
num_trials = 1;
graph_type = 0;

while ((c = getopt(argc, argv, "n:k:d:es:t:p" )) != EOF )
  switch ( c ) {
    case 'n': n = atoi (optarg); break;
    case 'k': k = atoi (optarg); break;
    case 'd': dens = atoi (optarg); break;
    case 'p': graph_type = 1; break;  /* has Hamiltonian MST */
    case 'e': graph_type = 2; break;  /* Euclidean */
    case 's': seed = atoi(optarg); break;
    case 't': num_trials = atoi(optarg); break;
    case '?':
    default:
      printf("usage: %s [-n NumberOfNodes] [-t NumberOfTrials]", argv[0]);
      printf("[-k DesiredDiameter] [-p] [-e] [-s RandomSeed] ");
      printf("[-d GraphDensity]\n");
            exit(1);
  }

if (n > MaxN) {
  printf ( "%s: Maximum number of vertices is %d\n", argv[0], MaxN);
  exit(1);
}
if (k < 4) {
  printf ( "Minimum diameter bound is 4\n");
  exit(1);
}

printf ("\nn: %d \t k: %d \t Density: %d%% \t Seed: %d \t Trials: %d \n",
        n, k, dens, seed, num_trials);
switch (graph_type) {
   case 0: printf("Graph Type: Random\n"); break;
   case 1: printf("Graph Type: Randomly generated and has Hamiltonian MST\n");
           break;
   case 2: printf("Graph Type: \n"); break;
   default: printf("Invalid Graph Type\n"); exit(1);
}

fails = 0;
big_k = 0;
upper_bound_fails = 0;
totalIR1time = 0;
totalIR2time = 0;
ave_dens = 0;

IR1d_low = 0x7fff;
IR1d_hi = 0;
IR1d_sum = IR1d_sumsq = 0;

dcmstIR1_low = BIG_NUMBER;
dcmstIR1_high = 0;
dcmstIR1_sum = dcmstIR1_sumsq = 0;
```

```
dcmstIR2_low = BIG_NUMBER;
dcmstIR2_high = 0;
dcmstIR2_sum = dcmstIR2_sumsq = 0;

dcmst_ratio_low = BIG_NUMBER;
dcmst_ratio_high = 0;
dcmst_ratio_sum = dcmst_ratio_sumsq = 0;

dcmst3_low = BIG_NUMBER;
dcmst3_high = 0;
dcmst3_sum = dcmst3_sumsq = 0;

dcmst4_low = BIG_NUMBER;
dcmst4_high = 0;
dcmst4_sum = dcmst4_sumsq = 0;

if (NULL == (p = (short *)calloc(n, sizeof(short)))) {
   printf("Out of memory in main().\n");
   exit(1);
}

trial = num_trials;
AllocateMemory();
srandom(seed);
while (trial > 0) {
          trial--;
  switch (graph_type) {
     case 0: GenerateRandomGraph(G,&m); break;
     case 1: GenerateRandomHamGraph(&m); break;
     case 2: GenerateEuclideanGraph(); break;
     default: printf("Invalid Graph Type\n"); exit(1);
  }

  original_mstwt = FindMST(dist);
  dcmst3 = ComputeDCMST3(&uu, &vv);
  if (dcmst3 < n * MaxWt) {
    factor = dcmst3/original_mstwt;
    if (factor < dcmst3_low)
      dcmst3_low = factor;
    if (factor > dcmst3_high)
      dcmst3_high = factor;
    dcmst3_sum += factor;
    dcmst3_sumsq += factor * factor;

    dcmst4 = ComputeDCMST4(dcmst3, uu, vv);
    factor = dcmst4/original_mstwt;
    if (factor < dcmst4_low)
      dcmst4_low = factor;
    if (factor > dcmst4_high)
      dcmst4_high = factor;
    dcmst4_sum += factor;
    dcmst4_sumsq += factor * factor;
```

```
}
else upper_bound_fails++;

printf("\n======================== TRIAL #%d ============================\n",
       num_trials-trial);
printf("\nNumber of edges = %d,  Density = %0.2lf%%\n",
       m, (double)m/(double)(n*n-n)*200);
if (dcmst3 < n * MaxWt)
  printf("\nUpper Bound Weights: DCMST(3) = %0.0lf,  DCMST(4) = %0.0lf\n",
         dcmst3, dcmst4);
else printf("Upper bound is not available.\n");

ave_dens += (double)m/(double)(n*n - n) * 2;
maxIterations = 500;
dcmstIR1_wt = IR1(&IR1time);
factor = dcmstIR1_wt/original_mstwt;
if (factor < dcmstIR1_low)
  dcmstIR1_low = factor;
if (factor > dcmstIR1_high)
  dcmstIR1_high = factor;
dcmstIR1_sum += factor;
dcmstIR1_sumsq += factor * factor;
if (IR1d_hi < diameter)
  IR1d_hi = diameter;
if (IR1d_low > diameter)
  IR1d_low = diameter;
IR1d_sum += diameter;
IR1d_sumsq += diameter * diameter;

maxIterations = 10000;
dcmstIR2_wt = IR2(&IR2time, dcmstIR1_wt);
if (diameter <= k) {
  totalIR1time += IR1time;
  totalIR2time += IR2time;
  factor = dcmstIR2_wt/original_mstwt;
  if (factor < dcmstIR2_low)
    dcmstIR2_low = factor;
  if (factor > dcmstIR2_high)
    dcmstIR2_high = factor;
  dcmstIR2_sum += factor;
  dcmstIR2_sumsq += factor * factor;

  factor = dcmstIR2_wt/dcmstIR1_wt;
  if (factor < dcmst_ratio_low)
    dcmst_ratio_low = factor;
  if (factor > dcmst_ratio_high)
    dcmst_ratio_high = factor;
  dcmst_ratio_sum += factor;
  dcmst_ratio_sumsq += factor * factor;
}
else {
   fails++;
   if (diameter > big_k)
```

```
      big_k = diameter;
  }
}

printf("\n\n~~~~~~~~~~~~~~~~~~~ OVERALL RESULTS:~~~~~~~~~~~~~~~~~~~~~\n");
printf("\nAverage Density = %0.2lf%%\n", ave_dens/num_trials * 100);
if (upper_bound_fails < num_trials) {
  printf("\n1) The Upper Bounds: (weight as a factor of MST weight)\n");
  printf("Upper Bound Sucess Rate = %0.2f%%\n",
        100*((double)(num_trials-upper_bound_fails))/((double)num_trials));
  printf("\nDCMST(3):\n Weight Range [%0.4lf,%0.4lf]\n",
        dcmst3_low,dcmst3_high);
  if (num_trials > 1) {
    num_trials -= upper_bound_fails;
    printf(" Weight Mean = %0.4lf, Standard Deviation = %0.4lf\n",
          dcmst3_sum/((double)num_trials),
          sqrt((dcmst3_sumsq - (dcmst3_sum*dcmst3_sum)/
          ((double)num_trials))/((float)num_trials-1.0)));
    num_trials += upper_bound_fails;
  }
  printf("\nDCMST(4):\n Weight Range [%0.4lf, %0.4lf]\n",
        dcmst4_low,dcmst4_high);
  if (num_trials > 1) {
    printf(" Weight Mean = %0.4lf, Standard Deviation = %0.4lf\n",
          dcmst4_sum/((double)num_trials),
          sqrt((dcmst4_sumsq - (dcmst4_sum*dcmst4_sum)/
          ((double)num_trials))/((float)num_trials-1.0)));
  }
}
else printf("No upperbounds available.\n");

printf("\n\n2)The Composite Iterative Refinement Algorithm:\n");
printf("There was %d successful trials out of %d. (failure rate = %0.2f%%)\n",
       num_trials-fails, num_trials, 100*(float)fails/(float)num_trials);
if (fails > 0)
  printf("The largest diameter of a fail attempt was %d \n", big_k);

printf("\nIR1:  Final tree weight as a factor of MST weight:\n");
printf(" Weight Range [%0.4lf, %0.4lf]\n",dcmstIR1_low,dcmstIR1_high);
if (num_trials > 1) {
  printf(" Weight Mean = %0.4lf, Standard Deviation = %0.4lf\n",
        dcmstIR1_sum/((double)num_trials),
        sqrt((dcmstIR1_sumsq - (dcmstIR1_sum*dcmstIR1_sum)/
        ((double)num_trials))/((float)num_trials-1.0)));
}
printf(" Diameter Range [%d, %d]\n", IR1d_low, IR1d_hi);
if (num_trials > 1) {
  printf(" Diameter Mean = %0.4lf, Standard Deviation = %0.4lf\n",
        IR1d_sum/((double)num_trials),
        sqrt((IR1d_sumsq - (IR1d_sum*IR1d_sum)/
        ((double)num_trials))/((float)num_trials-1.0)));
}
```

```
  num_trials -= fails;
  printf("\nStatistics for the %d successful trials are reported below:\n",
        num_trials);
  printf("\nIR2:  DCMST(%d) weight as a factor of MST weight:\n", k);
  printf(" Weight Range [%0.4lf, %0.4lf]\n",dcmstIR2_low,dcmstIR2_high);
  if (num_trials > 1) {
    printf(" Weight Mean = %0.4lf, Standard Deviation = %0.4lf\n",
          dcmstIR2_sum/((double)num_trials),
          sqrt((dcmstIR2_sumsq - (dcmstIR2_sum*dcmstIR2_sum)/
          ((double)num_trials))/((float)num_trials-1.0)));
  }
  printf("\nDCMST(%d) weight as a factor of weight of DCMST from IR1:\n", k);
  printf(" Weight Range [%0.4lf, %0.4lf]\n",dcmst_ratio_low,dcmst_ratio_high);
  if (num_trials > 1) {
    printf(" Weight Mean = %0.4lf, Standard Deviation = %0.4lf\n",
          dcmst_ratio_sum/((double)num_trials),
          sqrt((dcmst_ratio_sumsq - (dcmst_ratio_sum*dcmst_ratio_sum)/
          ((double)num_trials))/((float)num_trials-1.0)));
  }

  if (num_trials > 0 && totalIR1time+totalIR2time > 0) {
    printf("\nAverage IR1 Time = %0.4lf = %0.2lf%%\n",
        totalIR1time/num_trials, totalIR1time*100/(totalIR1time+totalIR2time));
    printf("Average IR2 Time = %0.4lf = %0.2lf%%\n",
        totalIR2time/num_trials, totalIR2time*100/(totalIR1time+totalIR2time));
    printf("Average Total CIR Time = %0.4lf\n",
          (totalIR1time+totalIR2time)/num_trials);
  }

  printf("\n");
}

/*****************************************   End of File   *****************************************/
```

APPENDIX C

PROGRAM CODE FOR THE ONE-TIME-TREE-CONSTRUCTION ALGORITHM

```
/*********************************************************************************************
*  One-time-tree-construction code (written in C) to find approximate solutions for
*  the Diameter-Constrained MST problem.
*
*********************************************************************************************/

#include <stdio.h>
#include <math.h>
#include <sys/time.h>
#include <sys/resource.h>

#define MaxWt 10000
#define MaxN  4000
#define MaxMSTWt MaxWt/100
#define boolean short
#define MaxStart  50       /* maximum number of start nodes to select */
#define NUM_TO_CHECK 20   /* suggested number of start nodes to select */

#define TRUE 1
#define FALSE 0
#define BIG_NUMBER 0x7fffffff

#define MY_SEC (((after.ru_utime.tv_sec*1000000+after.ru_utime.tv_usec) \
-(before.ru_utime.tv_sec*1000000+before.ru_utime.tv_usec))/1000000.0)

#define StartTiming() getrusage(who,&before)
#define StopTiming(time) getrusage(who,&after); time = MY_SEC

struct rusage before,after;
long hsec, husec;
int who=RUSAGE_SELF;

short n;
short k;            /* the diameter constraint */
short dens;         /* density of incomplete graph */
boolean *inTree;
short root;
int **G1;    /* the original graph */
int **G;     /* the graph after the labels are changed */
short dist[MaxN][MaxN];  /* the distance between each pair of nodes */
short *parent;
struct QNode_type {
          short node;
          struct QNode_type *next;
          };
typedef struct QNode_type QNode;


void AllocateMemory(void)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Allocate memory for the arrays G, G1, parent and dist.  This Memory                    */
/* will not be freed until the program exits.                                               */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
```

```
{
   int i;

   if (NULL == (G = (int **)calloc(n, sizeof(int)))) {
     printf("Out of memory -- function AllocateMemory() -- G\n");
     exit(1);
   }
   for (i=0; i < n; i++)
      if (NULL == (G[i] = (int *)calloc(n, sizeof(int)))) {
        printf("Out of memory -- function AllocateMemory() -- G[%d]\n", i);
        exit(1);
      }

   if (NULL == (G1 = (int **)calloc(n, sizeof(int)))) {
     printf("Out of memory -- function AllocateMemory() -- G1\n");
     exit(1);
   }
   for (i=0; i < n; i++)
      if (NULL == (G1[i] = (int *)calloc(n, sizeof(int)))) {
        printf("Out of memory -- function AllocateMemory() -- G1[%d]\n", i);
        exit(1);
      }

   if (NULL == (parent = (short *)calloc(n, sizeof(short)))) {
     printf("Out of memory -- function AllocateMemory() -- parent\n");
     exit(1);
   }
}

void Span(int **graph)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Generate a random spanning tree to make the graph connected.                        */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   int i, u, v;
   short *tree;
   short *rest;

   if (NULL == (tree = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in Span(): tree[].\n");
      exit(1);
   }
   if (NULL == (rest = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in Span(): rest[].\n");
      exit(1);
   }

   for (i=0; i < n; i++)
      tree[i] = i;

   /* Put the nodes of the graph in a random order. */
   for (i=0; i < n-1; i++)   {
      u = rand() % (n - i);
```

```
       rest[i] = tree[u];
       for (v=u; v < n-i-1; v++)
          tree[v] = tree[v+1];
    }
    /* tree[0] contains the node leftover */

    tree[1] = rest[n-2];
    graph[tree[0]][tree[1]] = random() % MaxWt + 1;
    graph[tree[1]][tree[0]] = random() % MaxWt + 1;

    for (i=2; i < n; i++) {
       u = tree[rand()%i];  /* select an arbitrary node from the tree */
       v = rest[n-i-1];     /* select an "arbitrary" node not in the tree */
       tree[i] = v;         /* add new node to the tree */
       graph[u][v] = random() % MaxWt + 1;
       graph[v][u] = random() % MaxWt + 1;
    }

    free(tree);
    free(rest);
}

void GenerateRandomGraph(int **graph, int *m)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Generate a random graph of n nodes, using the given density.              */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j;
    short d;

    /* Initialize the diagonal of the graph to zero (no self loops). */
    for (i=0; i < n; i++)
       graph[i][i] = 0;
    for (i=0; i < n; i++)
       for (j = i+1; j < n; j++)
          graph[i][j] = graph[j][i] = 0;

    *m = 0;
    Span(graph);
    for (i=0; i < n; i++)
       for (j=0; j < i; j++) {
          d = rand() % 100;
          if (d < dens && graph[i][j] == 0)
            graph[i][j] = random() % MaxWt + 1;
          if (graph[i][j] > 0) {  /* if there is an edge between i and j */
            graph[j][i] = graph[i][j];
            *m = *m + 1;
          }
          else
              graph[i][j] = graph[j][i] = 0;
       }

    for (i=0;i<n;i++)
```

```
        for (j = i+1; j < n; j++)
            dist[i][j] = dist[j][i] = graph[i][j];
}

void GenerateRandomHamGraph(int *m)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Generate a complete graph with random edge weights, where the MST is         */
/* forced to be a Hamiltonian path                                                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j, u;
    short *node, *p, d;

    if (NULL == (node = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in GenerateHamRandomGraph() -- node[].\n");
      exit(1);
    }
    if (NULL == (p = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in GenerateHamRandomGraph() -- p[].\n");
      exit(1);
    }

    /* Put the nodes of the graph in a random order. */
    for (i=0; i < n; i++)
        node[i] = i;
    for (i=0; i < n-1; i++) {
        u = rand() % (n - i);
        p[i] = node[u];
        for (j=u; j < n-i-1; j++)
            node[j] = node[j+1];
    }
    p[n-1] = node[0];

    for (i=0; i < n; i++)
        for (j = i; j < n; j++)
            G1[i][j] = 0;

    /* Generate the Hamiltonian path MST */
    for (i=0; i < n-1; i++)
                G1[p[i]][p[i+1]] = G1[p[i+1]][p[i]] = random() % MaxMSTWt + 1;

    *m = 0;
    for (i=0; i < n; i++)
        for (j = i+1; j < n; j++) {
            d = rand() % 100;
            if (d < dens && G1[i][j] == 0)
              G1[i][j] = random() % (MaxWt-MaxMSTWt) + MaxMSTWt + 1;
            if (G1[i][j] > 0) {  /* if there is an edge between i and j */
              G1[j][i] = G1[i][j];
              *m = *m + 1;
            }
            else
                G1[i][j] = G1[j][i] = 0;
```

```
      }

   for (i=0; i < n; i++)
      dist[i][i] = 0;
   for (i=0; i < n; i++)
      for (j=i+1; j < n; j++)
         dist[i][j] = dist[j][i] = G1[i][j];

   free(node);
   free(p);
}

double FindMST(int **graph)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute the minimum spanning tree using Prim's algorithm.                    */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
   double wt=0;
   short i, j;
   short *cheap, *p1;
   short nextEdge, nextNode;

   if (NULL == (p1 = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in FindMST(): p1[].\n");
      exit(1);
   }
   if (NULL == (cheap = (short *)calloc(n, sizeof(short)))) {
      printf("Out of memory in FindMST(): cheap[].\n");
      exit(1);
   }

   for (i=0; i < n; i++) {
      inTree[i] = FALSE;
      if (graph[0][i] > 0)
        cheap[i] = graph[0][i];
      else
          cheap[i] = MaxWt + 1;
      p1[i] = 0;
   }

   inTree[0] = TRUE;
   parent[0] = 0;
   root = 0;

   for (i=1; i < n; i++) {
      nextEdge = MaxWt;
      nextNode = -1;

      for (j=1; j < n; j++)
         if (!(inTree[j]) && (nextEdge > cheap[j])) {
               nextEdge = cheap[j];
               nextNode = j;
         }
```

```
        inTree[nextNode] = TRUE;
        wt += graph[nextNode][p1[nextNode]];
        parent[nextNode] = p1[nextNode];

        for (j=0; j < n; j++)
           if (cheap[j] > graph[j][nextNode] && graph[j][nextNode] > 0) {
                 cheap[j] = graph[j][nextNode];
                 p1[j] = nextNode;
           }
    }

    free(p1);
    free(cheap);
    return wt;
}

void Enqueue(short u, QNode **head, QNode **tail)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Insert node u at the tail of the queue.                                     */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    QNode *oldtail;

    oldtail = *tail;
    if(NULL==(*tail=(struct QNode_type *)malloc(sizeof(struct QNode_type)))){
      printf ("Out of memory -- function Enqueue()\n");
      exit(1);
    }
    (*tail)->node = u;

    if (oldtail != NULL)
      oldtail->next = *tail;
    else *head = *tail;

    (*tail)->next = NULL;
}

int Dequeue(QNode **head, QNode **tail)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Delete and return the node at the head of the queue.                       */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    QNode *oldhead;
    int ret;

    if (*head == NULL)
      return(-1);

    ret = (*head)->node;
    oldhead = *head;
    *head = (*head)->next;
    if (*head == NULL)
```

```
      *tail = NULL;
    oldhead->next = NULL;
    free(oldhead);

    return(ret);
}

short RemoveFromQ(QNode **prev, QNode **child)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Remove a node from the middle of the queue.                                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    QNode *temp;

    if (*prev != *child) {
      temp = *child;
      (*prev)->next = (*child)->next;
      *child = (*child)->next;
      temp->next = NULL;
      free(temp);
      return 1;
    }
    else {
       temp = *child;
       *prev = *child = (*child)->next;
       free(temp);
       return 0;
    }
}

short Diameter(short nn)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute diameter using Leaf Deletion Method                                */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
  short i, dia;
  short leaf_count = 0, que_size = 0;
  short num_cross_point = 0; /* tree level, starting from the leaves */
  short leaf_node;
  short *label, *par;
  short front, rear, crossed;
  short *deg, *cross_pointer;
  QNode *head, *tail, *child, *prev;
  QNode **child_h, **child_t;

  if (nn==1) return 0;
  else if (nn==2) return 1;

  if (NULL == (label = (short *)calloc(n, sizeof(short)))) {
     printf("Out of memory -- function Diameter(): label[].\n");
     exit(1);
  }
  if (NULL == (par = (short *)calloc(n, sizeof(short)))) {
```

```
   printf("Out of memory -- function Diameter(): par[].\n");
   exit(1);
}
if (NULL == (deg = (short *)calloc(n, sizeof(short)))) {
   printf("Out of memory -- function Diameter(): deg[].\n");
   exit(1);
}
if (NULL == (cross_pointer = (short *)calloc(n-1, sizeof(short)))) {
   printf("Out of memory in Diameter(): cross_pointer[].\n");
   exit(1);
}
if (NULL == (child_h = (QNode **)calloc(n, sizeof(QNode*)))) {
   printf("Out of memory in Diameter(): child_h[].\n");
   exit(1);
}
if (NULL == (child_t = (QNode **)calloc(n, sizeof(QNode*)))) {
   printf("Out of memory in Diameter(): child_t[].\n");
   exit(1);
}

/* Since partial trees (with non-consecutive labels) could be
   passed as input, we re-label the nodes to make them consecutive */
front = 1;
label[0] = 0;
for (i=1; i < n; i++)
          if (inTree[i]) {
              label[i] = front;
              front++;
          }

/* Computing node degrees */
for (i=0; i < n; i++)
           deg[i] = 0;
for (i=0; i < n; i++)
   child_h[i] = child_t[i] = NULL;
for (i=0; i < n; i++)
          if (inTree[i]) {
              deg[label[i]] += 1;
              deg[label[parent[i]]] += 1;
              par[label[i]] = label[parent[i]];
      if (label[i] != root)
        Enqueue(label[i], &child_h[par[label[i]]], &child_t[par[label[i]]]);
          }

deg[root] -= 2;

front =  rear = 0;
head = tail = NULL;

for (i=0; i < nn; i++)
   if (deg[i] == 1) {
             Enqueue(i, &head, &tail);
     rear++;
```

```
    leaf_count++;
    que_size++;
  }

cross_pointer[num_cross_point] = que_size - 1;
crossed = FALSE;
par[root] = -1;

if (que_size != nn-1)      /* not STAR, i.e., n-1 nodes are leaf nodes. */
  /* que_size takes care of the star  */
  /* leaf count takes care of the path */

    while (que_size <= nn)  {
        leaf_node  = Dequeue(&head, &tail);
        deg[leaf_node] = 0;
        front++;
        leaf_count--;

        if (leaf_node != root) {
          deg[par[leaf_node]]--;
          if (deg[par[leaf_node]] == 1)  {
            Enqueue(par[leaf_node], &head, &tail);
            rear++;
            leaf_count++;
            que_size++;
          }
        }
        /* decrement children's deg */
        prev = child = child_h[leaf_node];
        while (child != NULL) {
              deg[child->node]--;
              if (deg[child->node] == 1) {
                Enqueue(child->node, &head, &tail);
                rear++;
                leaf_count++;
                que_size++;
              }

              if (deg[child->node] <= 1) {
                if (RemoveFromQ(&prev, &child) == 0)
                /* return 0 for child is at beginning of queue */
                  child_h[leaf_node] = child;
              }
              else {
                prev = child;
                child = child->next;
              }
        }

        if (front > cross_pointer[num_cross_point]) {
          num_cross_point++;
          cross_pointer[num_cross_point] = rear - 1;
          crossed = TRUE;
```

```
            }

            if (crossed == TRUE && rear-front <= 2)  break;
            /* Detection of Path after deleting all leaf nodes */

            crossed = FALSE;
       }

    if (num_cross_point == 0 && que_size == nn-1)  /* STAR case */
      dia = 2;

    if (num_cross_point > 0 && leaf_count == 2)     /* a path remains */
      dia = 2*(num_cross_point) + (n - (que_size - 2)) - 1;

    if ( (num_cross_point > 0) && (que_size - front) == 1)  /* node remains */
      dia = 2 * num_cross_point;

    tail = head;
    while (NULL != head) {
         tail = tail->next;
         free(head);
         head = tail;
    }

    for (i = 0; i < nn; i++) {
       child_t[i] = child_h[i];
       while (NULL != child_h[i]) {
            child_t[i]= (child_t[i])->next;
            free(child_h[i]);
            child_h[i] = child_t[i];
       }
    }

    free(label);
    free(par);
    free(deg);
    free(cross_pointer);
    free(child_h);
    free(child_t);

    return dia;
}

int FindDCMST()
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Compute a DCMST from a given source node: Node 0                                                   */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j, h;
    int *cheap;
    double wt=0;
    short *par;
    short nextEdge, nextNode;
```

```
short *max;   /* eccentricity values of the nodes in the current tree */
short nodes;  /* number of nodes currently in the tree */
boolean success;

if (NULL == (cheap = (int *)calloc(n, sizeof(int)))) {
   printf("Out of memory -- function FindDCMST(): cheap[].\n");
   exit(1);
}
if (NULL == (par = (short *)calloc(n, sizeof(short)))) {
   printf("Out of memory -- function FindDCMST(): par[].\n");
   exit(1);
}
if (NULL == (max = (short *)calloc(n, sizeof(short)))) {
   printf("Out of memory -- function FindDCMST(): max[].\n");
   exit(1);
}

root = 0;
parent[0] = 0;
for (i=0; i < n; i++)
   for (j=0; j < n; j++)
            dist[i][j] = -1;
dist[0][0] = 0;

for (i=0; i < n; i++) {
   inTree[i] = FALSE;
   if (G[0][i] > 0)
     cheap[i] = G[0][i];
   else cheap[i] = MaxWt + 1;
   par[i] = 0;
   max[i] = 0;
}
inTree[0] = TRUE;
nodes = 1;

for (i=1; i < n; i++) {
   nextEdge = MaxWt + 1;
   nextNode = -1;

   for (j=0; j < n; j++)
      if (!(inTree[j]) && (nextEdge > cheap[j])) {
        nextEdge = cheap[j];
        nextNode = j;
      }
   if (nextNode < 0)  /* failed to connect node j to the tree */
     return -1;
   inTree[nextNode] = TRUE;
   nodes++;
   wt += nextEdge;
   parent[nextNode] = par[nextNode];
   if (G[nextNode][par[nextNode]] < 0)
     return -1;
```

```
   /* Update dist and max */
   /* Part 1:  Copy values into the new node */
   for (j=0; j < n; j++) {
       dist[nextNode][j] = dist[par[nextNode]][j];
       if (dist[par[nextNode]][j] > -1)
             dist[nextNode][j]++;
   }
   dist[nextNode][nextNode] = 0;
   max[nextNode] = 1+max[par[nextNode]];
   dist[par[nextNode]][nextNode] = 1;
   if (max[par[nextNode]] < 1)
       max[par[nextNode]] = 1;

   /* Part 2:  Update all tree nodes' values for nextNode */
   for (j=0; j < n; j++)
      if (inTree[j] && j!=nextNode && j!=par[nextNode]) {
        dist[j][nextNode] = 1+dist[j][par[nextNode]];
        if (max[j]<dist[j][nextNode])
           max[j] = dist[j][nextNode];
      }

   /* Now update cheap & par */
   for (j=0; j < n; j++)
       if (!inTree[j])
             if ((1+max[par[j]]) > k) {
           /* We've increased the tree diameter.  This means */
           /* we need to recompute a new parent for this node */
           cheap[j] = MaxWt + 1;
           success = FALSE;
           for (h=0; h < n; h++)
              if (inTree[h] && (1+max[h] <= k) &&
                 (cheap[j] > G[h][j]) && (G[h][j] > 0)) {
                cheap[j] = G[h][j];
                par[j] = h;
                success = TRUE;
              }
           if (!success)
              return -1;
         }

         else
        /* We only need to look at the new node */
           if (1+max[nextNode] <= k && cheap[j] > G[nextNode][j]
                && G[nextNode][j] > 0) {
              cheap[j] = G[nextNode][j];
              par[j] = nextNode;
          }
}  /* end for(i...) */

free(cheap);
free(par);
free(max);
if (nodes == n)
```

```
      return wt;
    else
      return -1;
}

void SelectNodes(short *start_node)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Find the smallest stars (their centers are to be used as source nodes)        */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    boolean found;
    short i, j, min_node[MaxStart];
    int min_weight[MaxStart], curr_weight;
    int max_index;
    int sort_list[MaxN];
    int h,l,temp;

    for (i=0; i < MaxStart; i++) {
       min_weight[i] = 0;
       min_node[i] = i;
    }
    max_index = 0;

    for (i=0; i < MaxStart; i++) {
       for (j=1;j<n;j++)
          min_weight[i] += G1[i][j];
       if (min_weight[i] > min_weight[max_index])
         max_index = i;
    }

    for (i=MaxStart; i < n; i++) {
       curr_weight = 0;
       for (j=0;j<n;j++)
          sort_list[j] = G1[i][j];
       for (h=0;h<n-1;h++)
          for (l=h+1;l<n;l++)
             if (sort_list[h] > sort_list[l]) {
               temp = sort_list[h];
               sort_list[h] = sort_list[l];
               sort_list[l] = temp;
             }

               for (j=1; j < MaxStart; j++)
                  curr_weight += G1[i][j];
       if (curr_weight < min_weight[max_index]) {
                 min_weight[max_index] = curr_weight;
                 min_node[max_index] = i;
                 for (j=0;j<MaxStart;j++)
                            if (min_weight[j] > min_weight[max_index])
                              max_index = j;
       }
    }
```

```
    for (i=0;i<MaxStart;i++)
    {
       start_node[i] = min_node[i];
    }
}

void AddToHeapD(short *heapD, short *gdeg, double *wt, short *sizeD, short u)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Add node u to heapD                                                 */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, done;

    if (*sizeD >= MaxN) {
      printf("Error: Heap D overflow\n");
      exit(1);
    }

    else {
       i = (*sizeD)++;
       done = FALSE;
       while (!done)
            if (i == 0)
              done = TRUE;  /* at root */
            else if ((gdeg[u] < gdeg[heapD[(i-1)/2]]) ||
                     ( (gdeg[u] == gdeg[heapD[(i-1)/2]]) &&
                       (wt[u] >= wt[heapD[(i-1)/2]]) ))
                   done = TRUE;
                 else {  /* move from parent to child */
                   heapD[i] = heapD[(i-1)/2];
                   i = (i-1)/2;
                 }

       heapD[i] = u;
    }
}

short GetFromHeapD(short *heapD, short *gdeg, double *wt, short *sizeD)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Remove and return the node with highest degree from heapD          */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i, j;
    short done;
    short ret, temp;

    if (*sizeD == 0)  /* heap is empty */
      ret = -1;

    else {
       done = FALSE;
       ret = heapD[0];
       temp = heapD[*sizeD - 1];
```

```
       (*sizeD)--;
       i = 0;
       j = 1;  /* j is left child of i */

       while (j < *sizeD && !done) {
            if (j < *sizeD - 1)
              if ((gdeg[heapD[j]] < gdeg[heapD[j+1]]) ||
                  ( (gdeg[heapD[j]] == gdeg[heapD[j+1]]) &&
                    (wt[heapD[j]] > wt[heapD[j+1]]) ))
                j++;  /* j points to larger child */
            if ((gdeg[temp] > gdeg[heapD[j]]) ||
                ( (gdeg[temp] == gdeg[heapD[j]]) &&
                  (wt[heapD[temp]] <= wt[heapD[j]]) ))
              done = TRUE;
            else {
              heapD[i] = heapD[j];  /* move child up */
              i = j; j = 2*j + 1;  /* move i and j down */
            }
       }
       heapD[i] = temp;
    }
    return ret;
}

void SelectHiDegNodes(short *start_node)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Find the smallest-degree nodes in G. Break a tie using weight.                   */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    boolean found;
    short *gdeg, *heapD;
    short i, j, sizeD;
    double *wt;

    if (NULL == (gdeg = (short *)calloc(n, sizeof(short)))) {
       printf("Out of memory -- function SelectHiDegNodes(): gdeg[].\n");
       exit(1);
    }
    if (NULL == (heapD = (short *)calloc(n, sizeof(short)))) {
       printf("Out of memory -- function SelectHiDegNodes(): heapD[].\n");
       exit(1);
    }
    if (NULL == (wt = (double *)calloc(n, sizeof(double)))) {
       printf("Out of memory -- function SelectHiDegNodes(): wt[].\n");
       exit(1);
    }


    for (i=0; i < n; i++) {
       gdeg[i] = 0;
       wt[i] = 0;
    }
```

```
    for (i=0; i < n-1; i++)
       for (j = i+1; j < n; j++)
          if (G1[i][j] > 0) {
             gdeg[i]++;
             gdeg[j]++;
             wt[i] += G1[i][j];
             wt[j] += G1[i][j];
          }

    sizeD = 0;
    for (i=0; i < n; i++)
       AddToHeapD(heapD, gdeg, wt, &sizeD, i);
    for (i=0; i < MaxStart && i < n; i++)
       start_node[i] = GetFromHeapD(heapD, gdeg, wt, &sizeD);

    free(gdeg);
    free(heapD);
    free(wt);
}

double ComputeDCMST3(short *uu, short *vv)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    double wt, wt3;
    short u, v, w;
    short count, fail;
    short p2[MaxN];

    wt = wt3 = (n-1) * MaxWt + 1;
    for (u=0; u < n-1; u++)
       for (v=u+1; v < n; v++)     /* for each edge (u,v) */
          if (G1[u][v] > 0) {
            p2[u] = u;  /* make u the root of the tree */
            p2[v] = u;  /* initialize tree to edge (u,v) */
            wt = G1[u][v];
            count = 2;
            fail = FALSE;
            for (w=0; w < n && !fail; w++)
               if (w != u && w != v)
                 if (G1[w][u] < G1[w][v] && G1[w][u] > 0) {
                   p2[w] = u;
                   wt += G1[w][u];
                   count++;
                 }
                 else if (G1[w][v] > 0) {
                        p2[w] = v;
                        wt += G1[w][v];
                        count++;
                      }
                      else fail = TRUE;

            if (wt < wt3 && count == n) {
              wt3 = wt;
```

```
              *uu = u;
              *vv = v;
              for (w=0; w < n; w++)
                 parent[w] = p2[w];
            }
          }

    root = *uu;
    return wt3;
}

double ComputeDCMST4(double mst3, short uu, short vv)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Assume DCMST3 has been computed, and that uu is the root, and that             */
/* the parent of every leaf is either uu or vv, and parent[vv] = uu              */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    double wtu, wtv;
    short x, y, z;
    short p1[MaxN];
    short p2[MaxN];

    for (x=0; x < n; x++)
       p2[x] = p1[x] = parent[x];

    p1[uu] = uu;  /* this a potential root for DCMST(4) */
    p1[vv] = uu;
    wtu = mst3;
    for (x=0; x < n; x++)
       if (parent[x] == vv && x != vv && x != uu) {
         z = vv;
         for (y=0; y < n; y++)
            if (parent[y] == uu && y != uu && y != vv
                && G1[x][y] < G1[x][z] && G1[x][y] > 0)
              z = y;
       p1[x] = z;
       wtu += G1[x][z] - G1[x][parent[x]];
       }

    p2[vv] = vv;  /* this another potential root for DCMST(4) */
    p2[uu] = vv;
    wtv = mst3;
    for (x=0; x < n; x++)
       if (parent[x] == uu && x != uu && x != vv ) {
     z = uu;
     for (y=0; y < n; y++)
        if (parent[y] == vv && y != vv && y != uu
            && G1[x][y] < G1[x][z] && G1[x][y] > 0)
          z = y;
     p2[x] = z;
     wtv += G1[x][z] - G1[x][parent[x]];
     }
```

```
    if (wtv < wtu) {
      for (x=0; x < n; x++)
          parent[x] = p2[x];
      root = vv;
      return wtv;
    }
    else{
        for (x=0; x < n; x++)
            parent[x] = p1[x];
        root = uu;
        return wtu;
    }
}

boolean Find(short key, short *list, short nn)
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
/* Use sequential search to find a given short integer in the given list.        */
/* Return TRUE if found and FALSE if not found.                                    */
/*~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
{
    short i;

    for (i=0; i < nn; i++)
       if (key == list[i])
         return TRUE;
    return FALSE;
}

/*=============================================================================*/
int main(int argc, char *argv[])
{
  extern char *optarg;
  int m;  /* number of edges in the graph */
  short graph_type;
  short i, j, c;
  short *label, *start_node;
  short seed, diameter;
  short trial, num_trials;
  short iter, first;
  short vv, uu;
  short upper_bound_fails;
  short hfails, rfails, afails;
  short trial_hfails[MaxStart], trial_rfails[MaxStart], trial_afails;
  boolean hsuccess, asuccess;
  boolean use_all, use_rand, use_heur;

  double ave_dens, total_afails, total_hfails19, total_hfails49, total_first;
  double original_mstwt, factor, mstwt;
  double dcmst3, dcmst3_low, dcmst3_high, dcmst3_sum, dcmst3_sumsq;
  double dcmst4, dcmst4_low, dcmst4_high, dcmst4_sum, dcmst4_sumsq;
  double heur_time_sum, rand_time_sum, ottc_time_sum;  /* all trials */
  double heur_time, rand_time, ottc_time;  /* trial time */
  double time, heur_overhead;
```

```
double heur_best_weight[MaxStart], rand_best_weight[MaxStart];
double ottc_best_weight;
double heur_worst_weight, rand_worst_weight, ottc_worst_weight;
double heur_best_low19, heur_best_high19;
double heur_best_low49, heur_best_high49;
double heur_best_sum[MaxStart], heur_best_sumsq19, heur_best_sumsq49;
double heur_worst_low, heur_worst_high, heur_worst_sum;
double heur_worst_sumsq;
double rand_best_low19, rand_best_high19;
double rand_best_low49, rand_best_high49;
double rand_best_sum[MaxStart], rand_best_sumsq19, rand_best_sumsq49;
double rand_worst_low, rand_worst_high, rand_worst_sum;
double rand_worst_sumsq;
double ottc_best_low, ottc_best_high, ottc_best_sum, ottc_best_sumsq;
double ottc_worst_low, ottc_worst_high, ottc_worst_sum;
double ottc_worst_sumsq;

n = 50;
m = 1225;
k = 5;
dens = 100;
seed = 7;
graph_type = 0;  /* random graph */
num_trials = 1;
use_all = FALSE;  /* don't use all n nodes as start nodes */
use_rand = FALSE;  /* don't use random start nodes */
use_heur = FALSE;   /* don't select start nodes */

while ((c = getopt (argc, argv, "n:k:d:pes:t:arh" )) != EOF )
  switch ( c ) {
    case 'n': n = atoi ( optarg ); break;
    case 'k': k = atoi ( optarg ); break;
    case 'd': dens = atoi (optarg); break;
    case 'p': graph_type = 1; break;  /* has Hamiltonian MST */
    case 'e': graph_type = 2; break;  /* Euclidean */
    case 's': seed = atoi ( optarg ); break;
    case 't': num_trials = atoi(optarg); break;
    case 'a': use_all = TRUE; break;
    case 'r': use_rand = TRUE; break;
    case 'h': use_heur = TRUE; break;
    case '?':
    default:
       printf("usage: %s [-n NumberOfNodes] [-t NumberOfTrials] ", argv[0]);
       printf("[-k DesiredDiameter]\n\t[-p] [-e] [-u] [-s RandomSeed]");
       printf("\t[-a] [-r] [-h] [-d GraphDensity]\n");
              exit(1);
  }

if (use_all)
  use_rand = TRUE;
if (n > MaxN  || n < MaxStart) {
  printf ("%s: Number of vertices must be between %d and %d\n",
```

```
            argv[0], MaxStart, MaxN);
    exit(1);
  }
  if (k < 4) {
    printf ("Minimum diameter bound is 4\n");
    exit(1);
  }

  printf ("\nn: %d \t k: %d \t Density: %d%% \t Seed: %d  \t ",
          n, k, dens, seed);
  printf("Trials = %d\n",num_trials);
  switch (graph_type) {
     case 0: printf("Graph Type: Random\n"); break;
     case 1: printf("Graph Type: Randomly generated and has Hamiltonian MST\n");
             break;
/*   case 2: printf("Graph Type: Euclidean\n"); break;   */
     default: printf("Invalid Graph Type\n"); exit(1);
  }

  if (NULL == (inTree = (short *)calloc(n, sizeof(short)))) {
    printf("Out of memory in FindMST(): inTree[].\n");
    exit(1);
  }
  if (NULL == (label = (short *)calloc(n, sizeof(short)))) {
    printf("Out of memory in FindMST(): label[].\n");
    exit(1);
  }
  if (NULL == (start_node = (short *)calloc(n, sizeof(short)))) {
    printf("Out of memory in FindMST(): start_node[].\n");
    exit(1);
  }

  total_first = 0;
  total_hfails19 = 0;
  total_hfails49 = 0;
  total_afails = 0;
  upper_bound_fails = 0;
  trial_afails = 0;

  for (i = 0; i < MaxStart; i++) {
     trial_hfails[i] = 0;
     trial_rfails[i] = 0;
  }

  trial = num_trials;
  ave_dens = 0;

  dcmst3_low = BIG_NUMBER;
  dcmst3_high = 0;
  dcmst3_sum = 0;
  dcmst3_sumsq = 0;
  dcmst4_low = BIG_NUMBER;
  dcmst4_high = 0;
```

```
  dcmst4_sum = 0;
  dcmst4_sumsq = 0;
  heur_time_sum = 0;
  rand_time_sum = 0;
  ottc_time_sum = 0;

  heur_best_low19 = BIG_NUMBER;
  heur_best_high19 = 0;
  heur_best_sumsq19 = 0;
  heur_best_low49 = BIG_NUMBER;
  heur_best_high49 = 0;
  heur_best_sumsq49 = 0;
  heur_worst_low = BIG_NUMBER;
  heur_worst_high = 0;
  heur_worst_sum = 0;
  heur_worst_sumsq = 0;

  rand_best_low19 = BIG_NUMBER;
  rand_best_high19 = 0;
  rand_best_sumsq19 = 0;
  rand_best_low49 = BIG_NUMBER;
  rand_best_high49 = 0;
  rand_best_sumsq49 = 0;
  rand_worst_low = BIG_NUMBER;
  rand_worst_high = 0;
  rand_worst_sum = 0;
  rand_worst_sumsq = 0;

  for (i = 0; i < MaxStart; i++) {
     rand_best_sum[i] = 0;
     heur_best_sum[i] = 0;
  }

  ottc_best_low = BIG_NUMBER;
  ottc_best_high = 0;
  ottc_best_sum = 0;
  ottc_best_sumsq = 0;
  ottc_worst_low = BIG_NUMBER;
  ottc_worst_high = 0;
  ottc_worst_sum = 0;
  ottc_worst_sumsq = 0;

  AllocateMemory();
  srandom(seed);
  while (trial > 0) {
          trial--;
    switch (graph_type) {
       case 0: GenerateRandomGraph(G1,&m); break;
       case 1: GenerateRandomHamGraph(&m); break;
/*     case 2: GenerateEuclideanGraph(); break;  */
       default: printf("Invalid Graph Type\n"); exit(1);
    }
```

```
    heur_time = 0;
    rand_time = 0;
    ottc_time = 0;
    hfails = 0;
    rfails = 0;
    afails = 0;

    original_mstwt = FindMST(G1);
    diameter = Diameter(n);
    dcmst3 = ComputeDCMST3(&uu, &vv);
    if (dcmst3 < (n-1) * MaxWt + 1) {
      factor = dcmst3/original_mstwt;
      if (factor < dcmst3_low)
        dcmst3_low = factor;
      if (factor > dcmst3_high)
        dcmst3_high = factor;
      dcmst3_sum += factor;
      dcmst3_sumsq += factor * factor;

      dcmst4 = ComputeDCMST4(dcmst3, uu, vv);
      factor = dcmst4/original_mstwt;
      if (factor < dcmst4_low)
        dcmst4_low = factor;
      if (factor > dcmst4_high)
        dcmst4_high = factor;
      dcmst4_sum += factor;
      dcmst4_sumsq += factor * factor;
    }
    else upper_bound_fails++;

    printf("\n======================== TRIAL #%d ============================\n",
           num_trials-trial);
    printf("\nNumber of edges = %d,  Density = %0.2lf%%\n",
           m, (double)m/(double)(n*n-n)*200);
    printf("\nMST Weight = %0.0lf,  Diameter = %d\n",
           original_mstwt, diameter);
    if (dcmst3 < (n-1) * MaxWt + 1)
      printf("\nUpper Bound Weights: DCMST(3) = %0.0lf,  DCMST(4) = %0.0lf\n",
             dcmst3, dcmst4);
    else printf("Upper bound is not available.\n");

    ave_dens += (double)m/(double)(n*n - n) * 2;
    heur_worst_weight = 0;
    heur_best_weight[0] = BIG_NUMBER;
    rand_worst_weight = 0;
    rand_best_weight[0] = BIG_NUMBER;
    ottc_worst_weight = 0;
    ottc_best_weight = BIG_NUMBER;

/*--------- heur --------------------*/
    if (use_heur) {
      StartTiming();
      if (dens == 100)
```

```
    SelectNodes(start_node);
  else SelectHiDegNodes(start_node);

  StopTiming(time);
  heur_time += time;
  heur_overhead = time;

  for (iter = hsuccess = 0;
       iter < MaxStart; iter++) {
    StartTiming();
    /* Letting each node be the starting node for Prim's algorithm */
    for (i=0; i < n; i++)
       label[i] = i;
    label[0] = start_node[iter];
    label[start_node[iter]] = 0;

    for (i=0; i < n; i++)
       for (j=0; j < n; j++)
          G[label[i]][label[j]] = G1[i][j];

    mstwt = FindDCMST();
    StopTiming(time);
    heur_time += time;

    if (mstwt > 0) {  /* sucessful attempt ?? */
      diameter = Diameter(n);
      if (diameter > k)
        mstwt = -1;
    }
    if (mstwt > 0 && iter == 0)
      heur_best_weight[0] = mstwt;
    else if (mstwt > 0 && heur_best_weight[iter-1] > mstwt)
           heur_best_weight[iter] = mstwt;
         else heur_best_weight[iter] = heur_best_weight[iter-1];

    if (mstwt > 0) {  /* sucessful attempt */
      hsuccess = TRUE;

      if (heur_worst_weight < mstwt)
        heur_worst_weight = mstwt;
    }
    else hfails++;
    if (!hsuccess)
      trial_hfails[iter]++;
    if (iter == NUM_TO_CHECK-1)
      total_hfails19 += hfails;
  }  /* end for iter */

  if (!hsuccess)
    printf("Heur failed in all %d attemps\n", MaxStart);
  printf("Time taken by last NSM1 iteration = %0.4lf\n", time);
}
```

```
/*--------- rand --------------------*/
    if (use_rand) {
      for (iter = asuccess = 0;
           iter < MaxStart || (iter < n && asuccess == FALSE); iter++) {
          StartTiming();
          /* Letting each node be the starting node for Prim's algorithm */
          for (i=0; i < n; i++)
             label[i] = i;
          label[0] = iter;
          label[iter] = 0;

          for (i=0; i < n; i++)
             for (j=0; j < n; j++)
                G[label[i]][label[j]] = G1[i][j];

          mstwt = FindDCMST();
          StopTiming(time);
          rand_time += time;
          if (mstwt > 0) {  /* sucessful attempt ?? */
            diameter = Diameter(n);
            if (diameter > k)
               mstwt = -1;
          }
          if (use_all)
            ottc_time += time;

          if (mstwt > 0 && iter == 0)
            rand_best_weight[0] = mstwt;
          else if (mstwt > 0 && rand_best_weight[iter-1] > mstwt)
                rand_best_weight[iter] = mstwt;
               else rand_best_weight[iter] = rand_best_weight[iter-1];

          if (mstwt > 0) {  /* successful attempt */
            asuccess = TRUE;
            if (rand_worst_weight < mstwt)
              rand_worst_weight = mstwt;
            if (use_all) {
              if (ottc_best_weight > mstwt)
                ottc_best_weight = mstwt;
              if (ottc_worst_weight < mstwt)
                ottc_worst_weight = mstwt;
            }
          }
          else {
            rfails++;
            afails++;
          if (!asuccess && iter < MaxStart)
            trial_rfails[iter]++;
          }
        }  /* end if(...) and for(iter...) */
       if (!asuccess) {
         printf("All iterations of this trial have failed.\n");
         trial_afails++;
```

```
      }
      printf("Time taken by last NSM2 iteration = %0.4lf\n", time);
    }

/*--------- ottc with all start nodes --------------------*/
    if (use_all) {
      if (use_rand) {
        if (rfails > MaxStart)
          first = rfails + 1;
        else
            first = MaxStart;
      }
      for (iter = first; iter < n; iter++) {
          StartTiming();
          /* Letting each node be the starting node for Prim's algorithm */
          for (i=0; i < n; i++)
             label[i] = i;
          label[0] = iter;
          label[iter] = 0;

          for (i=0; i < n; i++)
             for (j=0; j < n; j++)
                G[label[i]][label[j]] = G1[i][j];

                  DisplayGraph(G);

          mstwt = FindDCMST();
          StopTiming(time);
          ottc_time += time;
          if (mstwt > 0) {  /* sucessful attempt ?? */
            diameter = Diameter(n);
            if (diameter > k)
              mstwt = -1;
          }
          if (mstwt > 0) {  /* sucessful attempt */
            diameter = Diameter(n);
            if (ottc_best_weight > mstwt)
              ottc_best_weight = mstwt;
            if (ottc_worst_weight < mstwt)
              ottc_worst_weight = mstwt;
          }
          else afails++;
        }  /* end if(...) and for(iter...) */
        printf("Time taken by last OTTC iteration = %0.4lf\n", time);
    }

/*--------- totals for this trial --------------------*/
    if (use_heur) {
      total_hfails49 += hfails;
      heur_time_sum += heur_time;
      if (hsuccess) {
        for (i = 0; i < MaxStart; i++)
           if (heur_best_weight[i] < BIG_NUMBER)
```

```
        heur_best_sum[i] += heur_best_weight[i] / original_mstwt;
    heur_worst_sum += heur_worst_weight / original_mstwt;
    heur_worst_sumsq += heur_worst_weight * heur_worst_weight
                    / (original_mstwt * original_mstwt);
    heur_best_sumsq19 += heur_best_weight[NUM_TO_CHECK-1]
                     * heur_best_weight[NUM_TO_CHECK-1]
                     / (original_mstwt * original_mstwt);
    heur_best_sumsq49 += heur_best_weight[MaxStart-1]
                     * heur_best_weight[MaxStart-1]
                     / (original_mstwt * original_mstwt);
    if (heur_best_low19 > heur_best_weight[NUM_TO_CHECK-1] / original_mstwt)
      heur_best_low19 = heur_best_weight[NUM_TO_CHECK-1] / original_mstwt;
    if (heur_best_low49 > heur_best_weight[MaxStart-1] / original_mstwt)
      heur_best_low49 = heur_best_weight[MaxStart-1] / original_mstwt;
    if (heur_best_high19 < heur_best_weight[NUM_TO_CHECK-1] / original_mstwt)
      heur_best_high19 = heur_best_weight[NUM_TO_CHECK-1] / original_mstwt;
    if (heur_best_high49 < heur_best_weight[MaxStart-1] / original_mstwt)
      heur_best_high49 = heur_best_weight[MaxStart-1] / original_mstwt;
    if (heur_worst_low > heur_worst_weight / original_mstwt)
      heur_worst_low = heur_worst_weight / original_mstwt;
    if (heur_worst_high < heur_worst_weight / original_mstwt)
      heur_worst_high = heur_worst_weight / original_mstwt;
  }

  printf("\nHeuristic (NSM1):\n");
  printf("   Total Time = %0.5lf = Iterations' Time + Overhead\n",heur_time);
  printf("   Average Time per Iteration = %0.5lf,",
         (heur_time-heur_overhead)/MaxStart);
  printf("  Total Overhead = %0.4lf\n", heur_overhead);
  printf("%d Start:  Best DCMST (Solution) Weight = %0.0f\n",
        NUM_TO_CHECK, heur_best_weight[NUM_TO_CHECK-1]);
  printf("%d Start:  Best DCMST (Solution) Weight = %0.0f,",
        MaxStart, heur_best_weight[MaxStart-1]);
  printf("  Worst DCMST Weight = %0.0f\n", heur_worst_weight);
}

if (use_rand) {
  total_first += (rfails + 1);
  rand_time_sum += rand_time;
  if (rfails < MaxStart) {
    for (i = 0; i < MaxStart; i++)
      if (rand_best_weight[i] < BIG_NUMBER)
        rand_best_sum[i] += rand_best_weight[i]/ original_mstwt;
    rand_worst_sum += rand_worst_weight / original_mstwt;
    rand_worst_sumsq += rand_worst_weight * rand_worst_weight
                    / (original_mstwt * original_mstwt);
    rand_best_sumsq19 += rand_best_weight[NUM_TO_CHECK-1]
                    * rand_best_weight[NUM_TO_CHECK-1]
                    / (original_mstwt * original_mstwt);
    rand_best_sumsq49 += rand_best_weight[MaxStart-1]
                    * rand_best_weight[MaxStart-1]
                    / (original_mstwt * original_mstwt);
    if (rand_best_low19 > rand_best_weight[NUM_TO_CHECK-1]/original_mstwt)
```

```
      rand_best_low19 = rand_best_weight[NUM_TO_CHECK-1] / original_mstwt;
    if (rand_best_low49 > rand_best_weight[MaxStart-1] / original_mstwt)
      rand_best_low49 = rand_best_weight[MaxStart-1] / original_mstwt;
    if (rand_best_high19 < rand_best_weight[NUM_TO_CHECK-1] / original_mstwt)
      rand_best_high19 = rand_best_weight[NUM_TO_CHECK-1] / original_mstwt;
    if (rand_best_high49 < rand_best_weight[MaxStart-1] / original_mstwt)
      rand_best_high49 = rand_best_weight[MaxStart-1] / original_mstwt;
    if (rand_worst_low > rand_worst_weight / original_mstwt)
      rand_worst_low = rand_worst_weight / original_mstwt;
    if (rand_worst_high < rand_worst_weight / original_mstwt)
      rand_worst_high = rand_worst_weight / original_mstwt;
  }

  printf("\nRandom (NSM2):\n");
  printf("The first successful iteration was #%d.\n", rfails+1);
  if (rfails < MaxStart)
    factor = rand_time/((float)MaxStart);
  else factor = rand_time/((float)rfails+1.0);
  printf("   Total Time = %0.5lf,  Average Time per Iteration = %0.5lf\n",
         rand_time, factor);
  printf("%d Start:  Best DCMST (Solution) Weight = %0.0f\n",
        NUM_TO_CHECK, rand_best_weight[NUM_TO_CHECK-1]);
  printf("%d Start:  Best DCMST (Solution) Weight = %0.0f,",
        MaxStart, rand_best_weight[MaxStart-1]);
  printf("  Worst DCMST Weight = %0.0f\n", rand_worst_weight);
}

if (use_all) {
  total_afails += afails;
  ottc_time_sum += ottc_time;
  if (asuccess) {
    if (ottc_best_weight < BIG_NUMBER)
      ottc_best_sum += ottc_best_weight / original_mstwt;
    ottc_worst_sum += ottc_worst_weight / original_mstwt;
    ottc_worst_sumsq += ottc_worst_weight * ottc_worst_weight
                      / (original_mstwt * original_mstwt);
    ottc_best_sumsq += ottc_best_weight * ottc_best_weight
                      / (original_mstwt * original_mstwt);
    if (ottc_best_low > ottc_best_weight / original_mstwt)
      ottc_best_low = ottc_best_weight / original_mstwt;
    if (ottc_best_high < ottc_best_weight / original_mstwt)
      ottc_best_high = ottc_best_weight / original_mstwt;
    if (ottc_worst_low > ottc_worst_weight / original_mstwt)
      ottc_worst_low = ottc_worst_weight / original_mstwt;
    if (ottc_worst_high < ottc_worst_weight / original_mstwt)
      ottc_worst_high = ottc_worst_weight / original_mstwt;
  }

  printf("\nUsing All Nodes:\n");
  printf("There was %d successful iterations.  Success Rate = %0.2lf%%\n",
         n-afails, 100*((double)(n-afails))/((float)n));
  printf("   Total Time = %0.5lf,  Average Time per Iteration = %0.5lf\n",
         ottc_time, ottc_time/n);
```

```
      printf("   Best DCMST (Solution) Weight = %0.0f,", ottc_best_weight);
      printf("  Worst DCMST Weight = %0.0f\n", ottc_worst_weight);
    }
  }  /* end while */

printf("\n\n~~~~~~~~~~~~~~~~~~~~~ OVERALL RESULTS: ~~~~~~~~~~~~~~~~~~~~~~~~~~");
printf("\nAverage Density = %0.2lf%%\n", ave_dens/num_trials * 100);
printf("\n ** Spanning tree weight is reported as a factor of MST weight. **\n");
if (upper_bound_fails < num_trials) {
  printf("\nUpper Bounds:\n");
  printf("Upper Bound Sucess Rate = %0.2f%%\n",
          100*((double)(num_trials-upper_bound_fails))/((double)num_trials));
  printf("   DCMST(3): Range [%0.4lf, %0.4lf]\n",dcmst3_low,dcmst3_high);
  if (num_trials > 1)
    printf("             Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
            dcmst3_sum/((double)num_trials),
            sqrt((dcmst3_sumsq - (dcmst3_sum*dcmst3_sum)/
            ((double)num_trials))/((float)num_trials-1.0)));

  printf("   DCMST(4): Range [%0.4lf, %0.4lf]\n",dcmst4_low,dcmst4_high);
  if (num_trials > 1)
    printf("             Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
            dcmst4_sum/((double)num_trials),
            sqrt((dcmst4_sumsq - (dcmst4_sum*dcmst4_sum)/
            ((double)num_trials))/((float)num_trials-1.0)));
}
else printf("No upperbounds available\n");

printf("\nActual first successful iteration (on average) is #%0.2lf\n",
        total_first/((double)(num_trials-trial_afails)));
if (use_heur) {
  printf("\nHeuristic (NSM1):\n");
  printf("\n%d Start Nodes:\n", NUM_TO_CHECK);
  printf("Trial Success rate = %0.2lf%% \n",
        100 - 100*trial_hfails[NUM_TO_CHECK-1] / ((double)num_trials));
  printf("There was %0.2lf successful iterations (out of %d) on average. \n",
          NUM_TO_CHECK - total_hfails19/((double)num_trials), NUM_TO_CHECK);
  printf("Iteration Success rate = %0.2lf%%\n",
        100 - 100*total_hfails19 / ((double)(num_trials * NUM_TO_CHECK)));
  printf("   Best Weight (Solution) Range [%0.4lf, %0.4lf]\n",
        heur_best_low19, heur_best_high19);
  if (num_trials > 1 && trial_hfails[NUM_TO_CHECK-1] < num_trials)
    printf(" * Best Weight (Solution) Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          heur_best_sum[NUM_TO_CHECK-1]/
          ((double)(num_trials-trial_hfails[NUM_TO_CHECK-1])),
          sqrt((heur_best_sumsq19 -
          (heur_best_sum[NUM_TO_CHECK-1] * heur_best_sum[NUM_TO_CHECK-1])/
          ((double)(num_trials-trial_hfails[NUM_TO_CHECK-1])))/
          ((float)(num_trials-trial_hfails[NUM_TO_CHECK-1]-1))));

  printf("\n%d Start Nodes:\n", MaxStart);
  printf("Trial Success rate = %0.2lf%% \n",
        100 - 100*trial_hfails[MaxStart-1] / ((double)num_trials));
```

```
  printf("There was %0.2lf successful iterations (out of %d) on average. \n",
          MaxStart - total_hfails49/((double)num_trials), MaxStart);
  printf("Iteration Success rate = %0.2lf%%\n",
        100 - 100*total_hfails49 / ((double)(num_trials * MaxStart)));
  printf("   Best Weight (Solution) Range [%0.4lf, %0.4lf]\n",
        heur_best_low49, heur_best_high49);
  if (num_trials > 1 && trial_hfails[MaxStart-1] < num_trials)
    printf(" * Best Weight (Solution) Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          heur_best_sum[MaxStart-1]/((double)(num_trials-trial_hfails[MaxStart-1])),
          sqrt((heur_best_sumsq49 -
          (heur_best_sum[MaxStart-1]*heur_best_sum[MaxStart-1])/
          ((double)(num_trials-trial_hfails[MaxStart-1])))/
          ((float)(num_trials-trial_hfails[MaxStart-1]-1))));
  printf("   Worst Weight Range [%0.4lf, %0.4lf]\n",heur_worst_low,heur_worst_high);
  if (num_trials > 1 && trial_hfails[MaxStart-1] < num_trials) {
    printf("   Worst Weight Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          heur_worst_sum/((double)(num_trials-trial_hfails[MaxStart-1])),
          sqrt((heur_worst_sumsq - (heur_worst_sum*heur_worst_sum)/
          ((double)(num_trials-trial_hfails[MaxStart-1])))/
          ((float)(num_trials-trial_hfails[MaxStart-1]-1))));
    printf("\n   Average Time = %0.4lf\n", heur_time_sum/((float)num_trials));
  }
}
if (use_rand) {
  printf("\nRandom (NSM2):\n");
  printf("\n%d Start Nodes:\n", NUM_TO_CHECK);
  printf("Trial Success rate = %0.2lf%% \n",
        100 - 100*trial_rfails[NUM_TO_CHECK-1] / ((double)num_trials));
  printf("   Best Weight (Solution) Range [%0.4lf, %0.4lf]\n",
        rand_best_low19, rand_best_high19);
  if (num_trials > 1 && trial_rfails[NUM_TO_CHECK-1] < num_trials)
    printf(" * Best Weight (Solution) Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          rand_best_sum[NUM_TO_CHECK-1]/
          ((double)(num_trials-trial_rfails[NUM_TO_CHECK-1])),
          sqrt((rand_best_sumsq19 -
          (rand_best_sum[NUM_TO_CHECK-1] * rand_best_sum[NUM_TO_CHECK-1])/
          ((double)(num_trials-trial_rfails[NUM_TO_CHECK-1])))/
          ((float)(num_trials-trial_rfails[NUM_TO_CHECK-1]-1))));

  printf("\n%d Start Nodes:\n", MaxStart);
  printf("Trial Success rate = %0.2lf%% \n",
        100 - 100*trial_rfails[MaxStart-1] / ((double)num_trials));
  printf("   Best Weight (Solution) Range [%0.4lf, %0.4lf]\n",
        rand_best_low49, rand_best_high49);
  if (num_trials > 1 && trial_rfails[MaxStart-1] < num_trials)
    printf(" * Best Weight (Solution) Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          rand_best_sum[MaxStart-1]/
          ((double)(num_trials-trial_rfails[MaxStart-1])),
          sqrt((rand_best_sumsq49 -
          (rand_best_sum[MaxStart-1] * rand_best_sum[MaxStart-1])/
          ((double)(num_trials-trial_rfails[MaxStart-1])))/
          ((float)(num_trials-trial_rfails[MaxStart-1]-1))));
  printf("   Worst Weight Range [%0.4lf, %0.4lf]\n",rand_worst_low,rand_worst_high);
```

```
  if (num_trials > 1 && trial_rfails[MaxStart-1] < num_trials) {
    printf("   Worst Weight Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          rand_worst_sum/((double)(num_trials-trial_rfails[MaxStart-1])),
          sqrt((rand_worst_sumsq - (rand_worst_sum*rand_worst_sum)/
          ((double)(num_trials-trial_rfails[MaxStart-1])))/
          ((float)(num_trials-trial_rfails[MaxStart-1]-1))));
    printf("\n   Average Time = %0.4lf\n", rand_time_sum/((float)num_trials));
  }
}
if (use_all) {
  printf("\nOTTC using all nodes:\n");
  printf("Trial Success rate = %0.2lf%%\n",
        100 - 100*trial_afails / ((double)num_trials));
  printf("There was %0.2lf successful iterations (out of %d) on average. \n",
          n - total_afails/((double)num_trials), n);
  printf("Iteration Success rate = %0.2lf%%\n",
        100 - 100*total_afails / ((double)(num_trials * n)));
  printf("   Best Weight (Solution) Range [%0.4lf, %0.4lf]\n",
        ottc_best_low, ottc_best_high);
  if (num_trials > 1 && trial_afails < num_trials)
    printf(" * Best Weight (Solution) Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          ottc_best_sum/((double)num_trials),
          sqrt((ottc_best_sumsq - (ottc_best_sum*ottc_best_sum)/
          ((double)(num_trials-trial_afails)))/((float)(num_trials-trial_afails-1))));

  printf("   Worst Weight Range [%0.4lf, %0.4lf]\n",ottc_worst_low,ottc_worst_high);
  if (num_trials > 1 && trial_afails < num_trials) {
    printf("   Worst Weight Mean = %0.4lf,  Standard Deviation = %0.5lf\n",
          ottc_worst_sum/((float)(num_trials-trial_afails)),
          sqrt((ottc_worst_sumsq - (ottc_worst_sum*ottc_worst_sum)/
          ((double)(num_trials-trial_afails)))/((float)(num_trials-trial_afails-1))));
    printf("\n   Average Time = %0.4lf\n", ottc_time_sum/((float)num_trials));
  }
}
printf("\n");

/*~~~~~~~~~~~~~~~~~ The 50 iteration results: ~~~~~~~~~~~~~~~~~~~~~~~~~~~*/
printf("\nResults for Start Nodes: 1 to %d \n", MaxStart);
if (use_heur) {
  printf("\nHeuristic (NSM1):\n");
  printf("Trial Success rate (precentage):\n");
  for (i=0; i < MaxStart; i += 10) {
     for (j=0; j < 10 && j+i < MaxStart; j++)
        printf("%5.2lf ", 100 - 100*trial_hfails[i] / ((double)num_trials));
     printf("\n");
  }
  printf("\n");
  if (num_trials > 1 && trial_hfails[MaxStart-1] < num_trials)
    printf("   Best Weight (Solution) Mean: \n");
    for (i=0; i < MaxStart; i++)
       printf("%0.2lf \n", heur_best_sum[i]/((double)(num_trials-trial_hfails[i])));
  printf("\n");
}
```

```
if (use_rand) {
  printf("\nRandom (NSM2):\n");
  printf("Trial Success rate (precentage):\n");
  for (i=0; i < MaxStart; i += 10) {
     for (j=0; j < 10 && j+i < MaxStart; j++)
        printf("%5.2lf ", 100 - 100*trial_rfails[i] / ((double)num_trials));
     printf("\n");
  }
  printf("\n");
  if (num_trials > 1 && trial_rfails[MaxStart-1] < num_trials)
    printf("   Best Weight (Solution) Mean: \n");
    for (i=0; i < MaxStart; i++)
      printf("%0.2lf \n",rand_best_sum[i]/((double)(num_trials-trial_rfails[i])));
  printf("\n");
}
}

/**********************   End of File  ******************************/
```

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