A General Class of Tensegrity Structures: Topology and Prestress Equilibrium Analysis

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Abstract

In this paper, we define a general class of tensegrity structures consisting of both compression members (ie bars) and tensile members (ie cables). For a given number N of bars, we define the topological structure which is necessary to establish a tensegrity. Necessary and sufficient conditions for prestress mechanical equilibria of the tensegrity are then provided in terms of a nonlinear function of the position and orientation of the bars, and the initial lengths of the cables.

Key Words: tensegrity, topology, equilibrium analysis, constrained particle dynamics, mechanical structure

1 Introduction

The word "tensegrity" is a contraction of the words "tension" and "integrity". A tensegrity structure was loosely defined by Fuller [2], [3] as a "structural relationship in which structural shape is guaranteed by the interaction between a continuous network of members in tension and a set of members in compression". In [8], a more scientific definition is offered: "A tensegrity system is a stable connection of axially-loaded members. A Class k tensegrity structure is one in which at most k compressive members are connected to any node." Compressive members are required to achieve stability, and Class 1 tensegrity structures have a continuous network of members in tension and a discontinuous network of members in compression. We shall refer to a compressive member as a *bar* and a tension member as a *cable*.

Such structures are mechanically stable because of the way in which the structures balance and distribute the mechanical stress, and not principally as a result of the strength of the individual components. The bars that make up the frameworks are connected into particular configurations (triangles, pentagons, hexagons) and are oriented so that each joint is constrained to a fixed position. An example of a Class 1 tensegrity is the "Needle Tower"

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sculpture of Kenneth Snelson which appears in the Krueller Mueller Museum, The Netherlands. This category of tensegrity structures was first invented by Snelson [1], who was a student of Fuller.

A *class C tensegrity* structure for $C > 1$ allows compressive members to be connected in a ball joint (so as not to apply torque from one member to another). All tensegrity structures have the property that even before the application of any external load, members of the structures are already in compression or tension; that is, they are prestressed. The "rigid" bars sustain compressive forces, while the "elastic" cables sustain tensile forces when they "stretch" from their rest lengths. In fact, it is this prestress that stabilizes the tensegrity structure. Examples of tensegrity structures have been studied in mathematics, biology, and engineering. The topology of the structures of Connelly and Bach [4] have subsystems consisting of continuous networks of bars, and therefore are of class greater than one. Group theory which allows a class of tensegrity structures to be defined with prescribed symmetries. Ingber [5], [6] has used a class 1 tensegrity model to demonstrate the existence of residual stress in a blood vessel.

In structural applications, tensegrity structures offer the possibility of considerable versatility if sensors and actuators are incorporated into the cables and bars. Furuya [13] and Motro [14] pointed out their versatility for space applications. More recently, Skelton and others [7], [8], [9] have analyzed the statics and dynamics of particular examples of the second category of tensegrity structures, which lays the foundation for further studies and applications.

This paper is concerned with the definition and analysis of a particular subclass of class 1 tensegrity structures which we define as the $(N, S; P_1, P_2, \ldots, P_M)$ class. This subclass has N bars which form a discontinuous network, and S cables which form a continuous network. The structure topology is based on the concept of a "stage" which (like the "Needle Tower" of Snelson) grows in "height" with the number of stages. The defined subclass has M stages with P_k bars in the kth stage. Both symmetric structures (where the number of bars per stage is constant) and non-symmetrical structures will be derived. Preliminary results [10] - [12] were earlier reported.

If an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure has N bars, its external geometry is characterized by the $2N$ coordinates of the "nodes", or "ends", of the bars. The resulting "shape" of the tensegrity then depends on: (i) the properties of the constituent components (that is, on the lengths of the bars and the rest lengths of the cables), (ii) the internal geometry (or topology) of how the bars and cables are connected, and (iii) on the existence of self stress, or pretension, which is necessary in order to provide rigidity for the structure. Since in will be shown that material characteristics (in particular, Hookes Law) are required to solve for internal forces, an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure belongs to a class of mechanical systems of indeterminant form.

In section 2, we begin by defining the topology of a single (i.e. $M = 1$) stage tensegrity, and then proceed to establish the topology of multistage tensegrity structures. Multistage structures which are both *symmetrical* (in which the number of bars per stage is constant) and *non-symmetrical* are developed, and in each case, the cable connections are specified. Since the end of each bar can possibly be connected by a cable to any one of the $2N - 2$ ends of the remaining $N - 1$ bars, it follows that a total of $0.5 \times (2N - 2)^N$ cable connections are possible, or equivalently, $(2N-2)^N$ cables per bar. In comparison, it is shown that for stable equilibrium, an $(N, S; P_1, P_2, \ldots, P_M)$ structure only requires between 3 and 4 cables per bar.

In section 3, we develop necessary and sufficient conditions for prestress mechanical equilibrium in the absence of external applied forces. These conditions are expressed in terms of the solution of a system of algebraic equations of the form $\mathbf{At} = \mathbf{0}$ for a column (i.e. non-square) matrix $\mathbf{A}$. However the problem is complicated by the fact that the components of $\mathbf{A}$ depend in a nonlinear way on the positions of the end points of the bars, and the fact that all components of the vector $\mathbf{t}$ must be strictly positive. Explicit analytical necessary and sufficient conditions on the geometry for a solution for the $N=2$ case are derived, but higher order cases require numerical solution.

2 Topology

We now proceed to define a general $(N, S; P_1, P_2, \ldots P_M)$ class of tensegrity structures consisting of N bars and S cables which are arranged into M stages with P_k bars in the kth stage such that

$$\sum_{k=1}^{M} P_k = N \tag{2.1}$$

An $(N=2, S=4; P=2)$ planar structure is illustrated in Fig. 1. In this structure, the bar 12 is of length L_{12}, and the bar 34 is of length L_{34}. The cables, or tensile elements, are illustrated as springs.

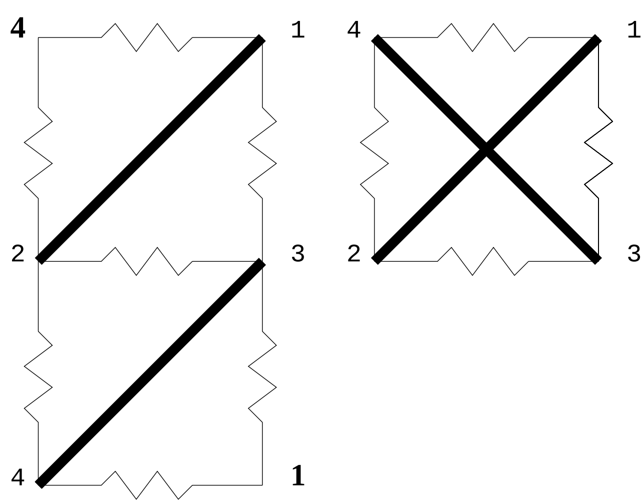

Figure 1: Topology of (2,4;2) tensegrity

As indicated in Fig. 1a, the structure is defined by first connecting cables [1,3], [2,3] and [2,4]. Then a cable is connected from node 1 to node 2 with midpoint **4**, and another cable is connected from node 3 to node 4 with midpoint **1**. Finally, after connecting midpoint **1** to the end of stick 12 at node 1, and midpoint **4** to node 4, the closed tensegrity in Fig. 1b results. Note that in the intermediate stage of Fig 1a, there appears to be a total of 7 cables.

However, once node 1 and midpoint **1**, and node 4 and midpoint **4** are made coincident, the number of cables reduces to $S = 4$. The resulting structure is single (i.e. $M = 1$) stage with $P_1 = N = 2$; that is, a $(2, 4; 2)$ tensegrity.

(2,4;2) Cable Connection Matrix: The resulting cable connections $[p_i, p_j]$ which connect node p_i to node p_j in the (2,4;2) structure are defined by:

$$[1, 3], \quad [1, 4]; \quad [2, 3], \quad [2, 4].$$

Note that the connections $[2m - 1, 2m]$ for $m = 1, 2$ define the bars.

Topological Equivalence: The actual coordinates of the nodes of the bars will depend on the lengths $\{L_{2m-1,2m};\ m = 1, 2\}$ of the bars, and the initial lengths $\{\ell^0_{pq}\}$ of the cables $[p, q]$. Consequently, there is an *equivalence class* of $(2, 4; 2)$ tensegrity structures. Specifically, the structure $(2, 4; 2)_1$ is said to be *topologically equivalent* to the structure $(2, 4; 2)_2$ if $(2, 4; 2)_1$ can be derived from $(2, 4; 2)_2$ by means of either a continuous change in the length of one or more bars, or a continuous change in the initial length of one or more cables subject to the condition that all cable tensions remain strictly positive, and no bars remain in contact.

2.1 Single Stage Structures

The planar $(2, 4; 2)$ structure is a "trivial" $(N, S; P_1, P_2\ \ldots\ , P_M)$ tensegrity in the sense that it has no "volume" and the bars "touch". The simplest 3-dimensional topology is provided by the $(N = 3, S = 9; P = 3)$ tensegrity whose realization is illustrated in Fig. 2.

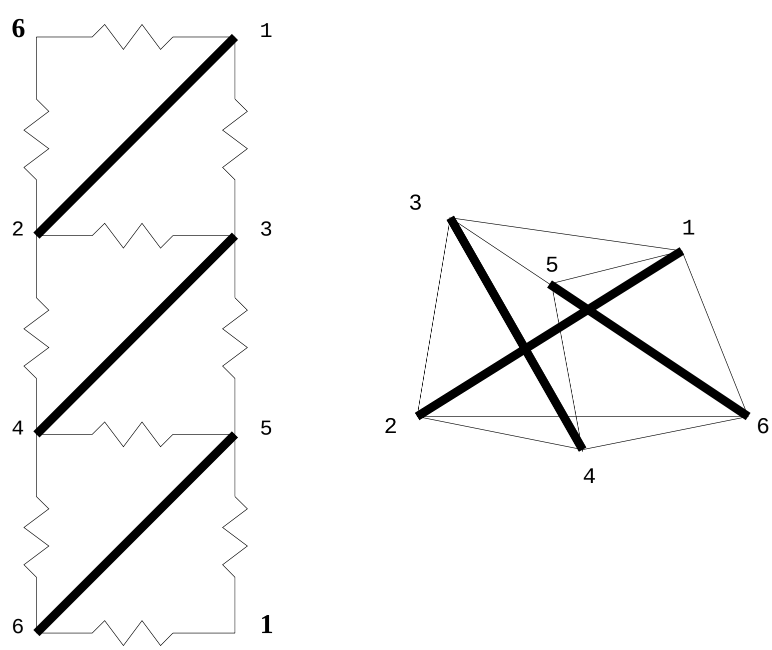

Figure 2: Topology of (3,9;3) tensegrity

To begin, first (as illustrated in Fig. 2a) take bar 12, bar 34 and bar 56 and connect the cables [1,3], [2,3], [2,4], [3,5] [4,5] and [4,6]. Then take another cable connecting node 5 to node 6 with midpoint **1**, and connect midpoint **1** to 1. Similarly, connect the midpoint **6** of the cable which connects node 1 to node 2 to node 6. This realization results in the 3-dimensional structure illustrated in Fig. 2b. Once again, as with the $(2, 4; 2)$ structure, the number S of cables is reduced during the realization process. That is, in Fig. 2a, the structure first appears to have 10 cables but the connection of node 1 to midpoint **1**, and node 6 to midpoint **6** reduces this number to $S = 9$ cables. (The cables from node 6 to midpoint **1**, and node 1 to midpoint **6** in Fig. 2a become identical in Fig. 2b.)

(3,9;3) Cable Connection Matrix: Cable connections $[p_i, p_j]$ which connect node p_i to node p_j in the $(3, 9; 3)$ structure are defined by:

$$[1,3],\quad [1,5],\quad [1,6];\quad [2,3],\quad [2,4],\quad [2,6];\quad [3,5],\quad [4,5],\quad [4,6].$$

Note that the connections $[2m - 1, 2m]$ for $m = 1, 2, 3$ define the bars.

Single stage 3-dimensional structures $(N, S; N)$ for any value of $N \geq 3$ may be similarly realized. The distinguishing feature of this *equivalence class* of tensegrity structures is that both the base and the top form an N-sided polygon with the top rotated with respect to the base. For $N = 3$, the base (top) defined by the bar ends $\{2, 4, 6\}$, and the top (base) as defined by the bar ends $\{1, 3, 5\}$ form triangles. As illustrated Fig. 3, the base (top) of the general N bar structure is formed by the bar ends $\{2, 4, 6, \ldots, 2N\}$, while the top (base) is formed by the bar ends $\{1, 3, 5, \ldots, 2N-1\}$. It can also be seen that after initially beginning with $3N + 1$ cables, the 3-dimensional closure which occurs when midpoint **2N** is connected to node $2N$, and midpoint **1** is connected to node 1 results in only $3N$ cables. We summarize this result as follows.

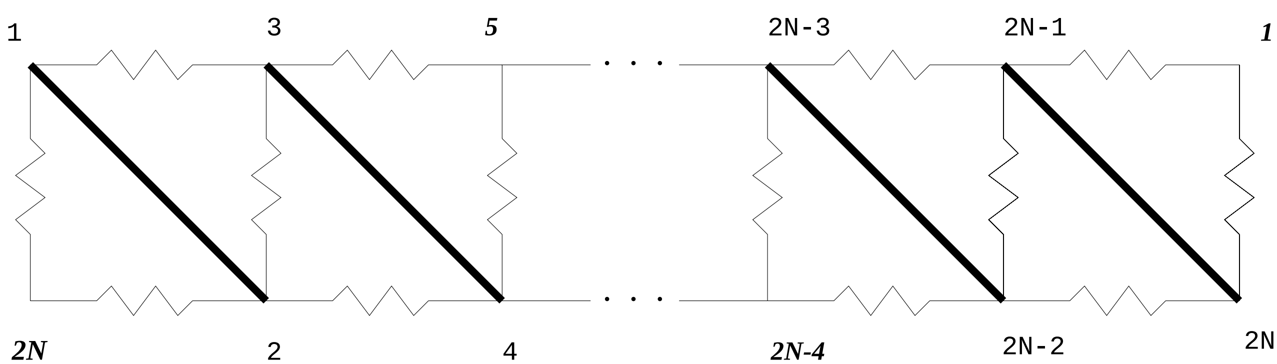

Figure 3: Planar topology of (N,3N;N) tensegrity

Theorem 2.1 *The single stage (ie $M = 1$) N bar tensegrity has a $(N, S = 3N; P_1 = N)$ structure. The bar ends which define both the base and top form a N-sided polygon. The $3N$ cable connections are defined by the connection matrix:*

$$[1, 2N - 1],\quad [1, 2N];\quad [2, 2N];$$

For $1 \leq m \leq N-1$

$$[2m+1, 2m-1], \quad [2m+1, 2m]; \quad [2m+2, 2m];$$

Note that the connections $[2m-1, 2m]$ for $1 \leq m \leq N$ define the bars.

As with a $(2, 4; 2)$ tensegrity, the actual coordinates of the nodes of the bars of an $(N, 3N; N)$ tensegrity structure will depend on the lengths $\{L_{2m-1,2m};\ 1 \leq m \leq N\}$ of the N bars, and the initial lengths of the cables $[p, q]$. Once again, an *equivalence class* of tensegrity structures is defined by means of a continuous change in the length of the bars or the initial cable lengths subject to the conditions that all cable tensions remain positive and no bars remain in contact.

2.2 Two Stage Structures

The simplest two stage structure is the $(N = 3, S = 9; P_1 = 2, P_2 = 1)$ tensegrity illustrated in Fig. 4. Stage 1 consists of the bar 12 and bar 34, while stage 2 consists of bar 56. The realization begins (as illustrated in Fig. 4a) by connecting cables [1,5], [2,5], [3,5] and [3,6]. Then, similar to the realization of single stage structures, midpoint **1** is connected to node 1, midpoint **2** is connected to node 2, midpoint **4** is connected to node 4, and midpoint **6** is connected to node 6 resulting in a 3-dimensional structure. Initially, in Fig. 3, there appears to be 12 cables which reduces to 9 after the connections of the various midpoints.

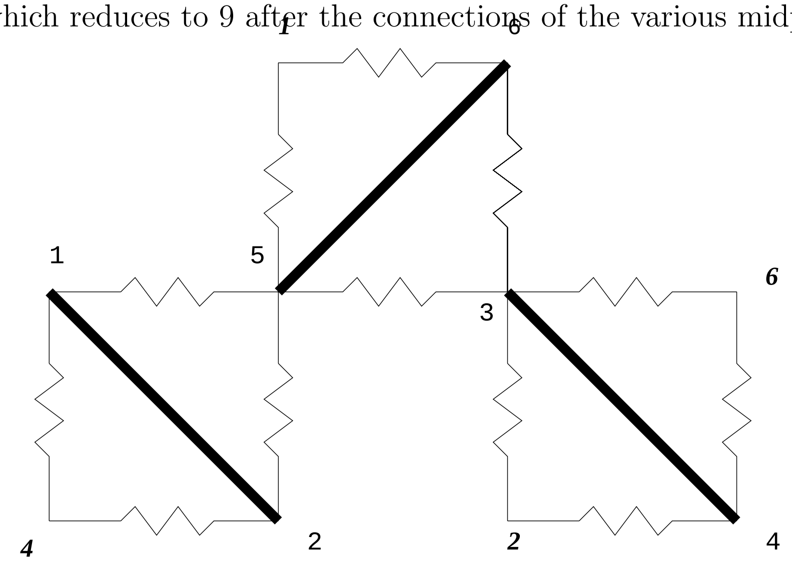

Figure 4: Planar topology of (3,9;2,1) tensegrity

(3,9;2,1) Cable Connection Matrix: Cable connections $[p_i, p_j]$ which connect node p_i to node p_j are defined by

$$[1, 4], \quad [1, 5], \quad [1, 6]; \quad [2, 3], \quad [2, 4], \quad [2, 5]; \quad [3, 5], \quad [3, 6], \quad [4, 6].$$

When compared with the single stage $(3, 9; 3)$ structure (which also has $N = 3$ bars and $S = 9$ cables), the $(3, 9; 2, 1)$ structure can be demonstrated after realization (by sticks and soft elastic bands) to have a greater *clockwise stiffness* in that if a clockwise torque is applied to the top while the base is held fixed, then there is more resistance from the $(3, 9; 2, 1)$

structure than for the $(3,9;3)$ structure. That is, the *different topologies* of the $(3,9;3)$ and the $(3,9;2,1)$ tensegrity structures result in *different mechanical properties.* (This fact is not evident from the connection matrices of the two structures, and the mechanical properties of $(N,S;P_1,P_2,\ \ldots\ ,P_M)$ tensegrities are not the subject of this paper.)

The two stage $(N=4,S=14;P_1=2,P_2=2)$, $(N=5,S=18;P_1=3,P_2=2)$ and $(N=6,S=24,P_1=3,P_2=3)$ structures are illustrated in Figs 5, 6 and 7 respectively. In the $(4,14;2,2)$ structure, the $4\times 4=16$ initial cables are reduced to $S=14$ cables after connecting midpoint **1** to node 1, midpoint **2** to node 2, midpoint **4** to node 4, midpoint **6** to node 6 and midpoint **8** to node 8. Similarly, the $(5,18;3,2)$ structure is reduced from $5\times 4=20$ to $S=18$ cables, while the $(6,24;3,3)$ requires the full $S=6\times 4=24$ cables.

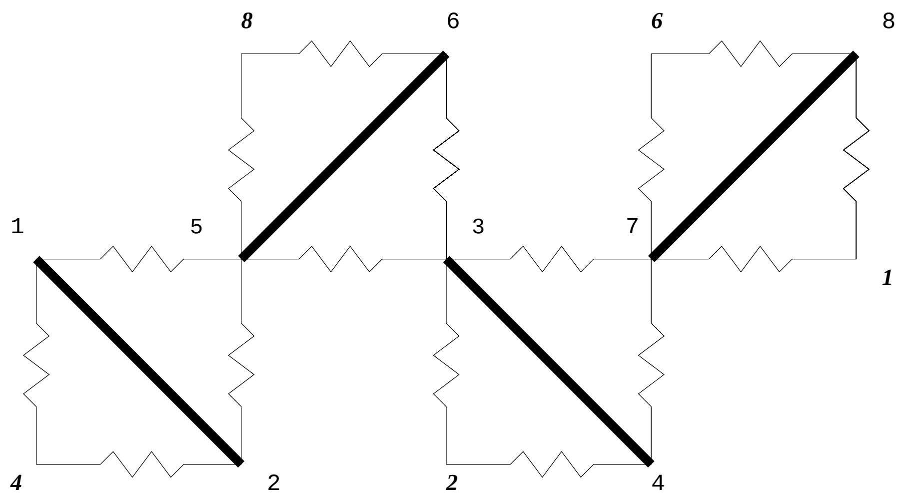

Figure 5: Planar topology of (4,14;2,2) tensegrity

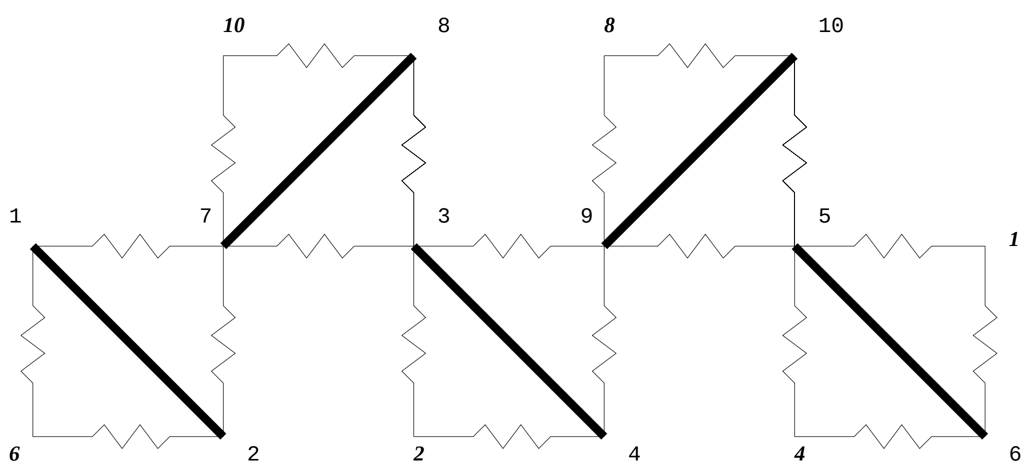

Figure 6: Planar topology of (5,18;3,2) tensegrity

2.3 M Stage Symmetric Structures

Both the $(4;14;2,2)$ and the $(6,24;3,3)$ structures are examples of *symmetrical structures* in that in both cases $P_1=P_2$. More generally, an $(N,S;P_1,P_2,\ \ldots\ P_M)$ tensegrity structure is said to be *symmetrical* if

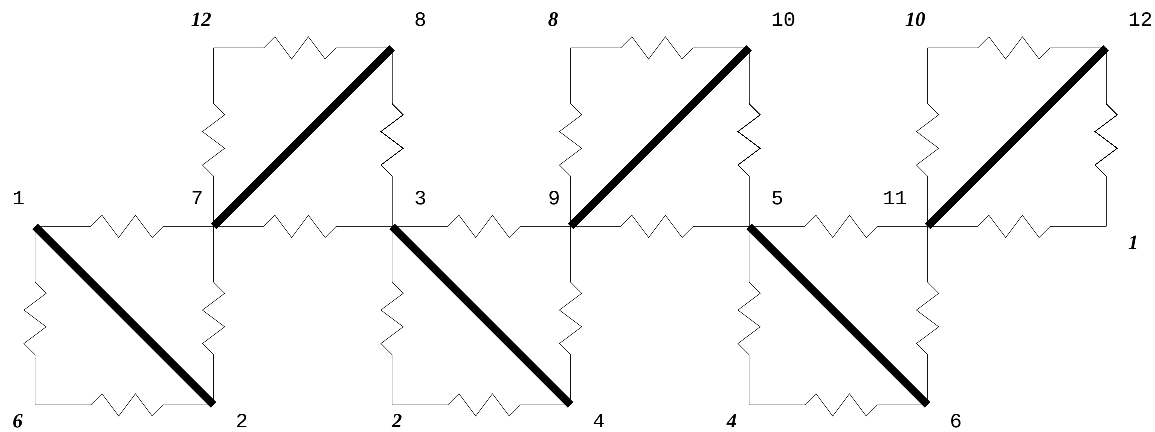

Figure 7: Planar topology of (6,24;3,3) tensegrity

$$P_k = P \quad \text{for all } k$$

For a symmetrical structure, it follows that $N = MP$. As described below, a symmetrical $(N, S; P_1, P_2, \ \dots \ , P_M)$ structure (except in the trivial case when $M = 1$) has the maximum number of 4 cables per bar.

Theorem 2.2 *A symmetrical* $(N = MP, S; P, P, \ \dots \ P)$ *tensegrity structure with* $M > 1$ *stages has* S *cables where*

$$S = \begin{cases} 4N-2 & ; \quad P = 2 \\ 4N & ; \quad P \geq 3 \end{cases}$$

The cable connections for the k*th stage where* $1 \leq k \leq M$ *are defined by:*
(i/a) For $k = 1$*;* $P = 2$

$$[1,4], \quad [1,5], \quad [1,7], \quad [1,8]; \quad [2,3], \quad [2,4], \quad [2,5]; \quad [3,5], \quad [3,6], \quad [3,7]; \quad [4,7].$$

and: (i/b) For $k = 1$*;* $P \geq 3$

$$[1,2P], \quad [1,2P+1], \quad [1,4P-1], \quad [1,4P]; \quad [2,3], \quad [2,4], \quad [2,2P], \quad [2,2P+1].$$

For $1 \leq m \leq P-2$*;*

$$[2m+1, 2P+2m-1], \quad [2m+1, 2P+2m], \quad [2m+1, 2P+2m+1];$$
$$[2m+2, 2m+3], \quad [2m+2, 2m+4], \quad [2m+2, 2P+2m+1].$$

and

$$[2P-1, 4P-3], \quad [2P-1, 4P-2], \quad [2P-1, 4P-1]; \quad [2P, 4P-1].$$

(ii) For $k = 2n < M$*;* $0 \le m \le P-2$*;*

$$[(4n-2)P+2m+1, 4nP+2m+2];$$
$$[(4n-2)P+2m+2, 4nP+2m+2], \quad [(4n-2)P+2m+2, 4nP+2m+3],$$
$$[(4n-2)P+2m+2, 4nP+2m+4].$$

and

$$[4nP-1, (4n+2)P]; \quad [4nP, 4nP+1], \quad [4nP, 4nP+2], \quad [4nP, (4n+2)P].$$

(iii) For $k = 2n+1 < M$*;*

$$[4nP+1, (4n+2)P+1], \quad [4nP+1, (6n+2)P-1], \quad [4nP+1, (6n+2)P];$$
$$[4nP+2, (4n+2)P+1].$$

For $1 \le m \le P-1$*;*

$$[4nP+2m+1, (4n+2)P+2m-1], \quad [4nP+2m+1, (4n+2)P+2m],$$
$$[4nP+2m+1, (4n+2)P+2m+1]; \quad [4nP+2m+2, (4n+2)P+2m+1].$$

(iv) a. For $k = 2n = M$*;* $P = 2$

$$[8n-3, 8n]; \quad [8n-2, 8n-1], \quad [8n-2, 8n].$$

and: (iv) b. For $k = 2n = M$*;* $P \ge 3$

$$[(4n-2)P+1, 4nP]; \quad [(4n-2)P+2, 4nP].$$

For $1 \le m \le P-1$*;*

$$[(4n-2)P+2m, (4n-2)P+2m+1], \quad [(4n-2)P+2m, (4n-2)P+2m+2].$$

(v) a. For $k = 2n+1 = M$*;* $P = 2$

$$[8n+1, 8n+3], \quad [8n+1, 8n+4]; \quad [8n+2, 8n+3].$$

and: (v) b. For $k = 2n+1 = M$*;* $P \ge 3$

$$[4nP+1, 4nP+3], \quad [4nP+1, (4n+2)P-1], \quad [4nP+1, (4n+2)P];$$
$$[4nP+2, 4nP+3].$$

For $1 \le m \le P-2$*;*

$$[4nP+2m+1, 4nP+2m+3]; \quad [4nP+2m+2, 4nP+2m+3].$$

2.4 M Stage Non-symmetrical Structures

A *non-symmetrical* $(N, S; P_1, P_2, \ldots P_M)$ tensegrity structure is one for which

$$P_k \neq P_\ell \ \text{ for some } k, \ell$$

Non-symmetrical structures of N bars have *less* than $4N$ cables. For example, the $(5, 18; 3, 2)$ structure in Fig. 6 has $S = 18 < 20 (= 4N)$ cables. The following result gives both the precise number of cables for each structure, and provides conditions for the allowable number of bars P_k in stage k.

Theorem 2.3 *An $(N, S; P_1, P_2, \ldots P_M)$ tensegrity structure exists when the following two conditions are satisfied:*

1.
$$P_1 > 1 \text{ and } P_k \geq 1 \text{ for } k \geq 2$$
and when $P_r = 1$, then $M = r$.

2.
$$|P_{k+1} - P_k| = 1 \ \text{ for } 1 \leq k \leq M - 1$$

The number S of cables for all $(N, S; P_1, P_2 \ \ldots \ , P_M)$ structures is bounded according to:

$$3N \leq S \leq 4N$$

In particular, for: (a) symmetrical structures

$$S = \begin{cases} 3N & ; \ M = 1, \ P \geq 3 \\ 4N - 2 & ; \ M \geq 2, P = 2 \\ 4N & ; \ M \geq 2, P \geq 3 \end{cases}$$

and for: (b) non-symmetrical structures (and so $M \geq 2$)

$$S = 4N - L; L = \sum_{k=1}^{M} \ell_k$$

where

$$\begin{aligned} \ell_1 &= 1 \ \text{ if } \ P_1 = 2 \text{ or } P_2 = P_1 - 1 \\ \ell_k &= 1 \ \text{ for } \ 2 \leq k \leq M-1 \text{ if } \ P_k = P_{k-1} + 1 \text{ and } P_{k+1} = P_{k-1} \\ \ell_M &= \begin{cases} 1 & ; \ \text{if } \ P_M = 2 \text{ or } P_M = P_{M-1} + 1 \\ 2 & ; \ \text{if } \ P_M = 1 \end{cases} \end{aligned}$$

An $(N, S; P_1, P_2, \ \ldots \ , P_M)$ structure is equivalent to an $(N, S; Q_1, Q_2, \ \ldots \ , Q_M)$ structure when

$$P_k = Q_{M+1-k} \ ; \ 1 \leq k \leq M$$

The case when $S = 3N$ is provided in Theorem 2.1, and the other symmetrical cases are provided in Theorem 2.2. (A symmetrical $(6, 24; 3, 3)$ structure is illustrated in Fig.7.) The proof of the result for non-symmetrical structures is by inspection and realization. The typical situation when $\ell_1 = 1$ is demonstrated by the $(3, 9; 2, 1)$ structure in Fig.4, the $(4, 14; 2, 2)$ structure in Fig.5, and the $(5, 18; 3, 2)$ structure in Fig.6. The typical situation when $\ell_k = 1$ for some $2 \leq k \leq M - 1$ is demonstrated by the $(7, 25; 2, 3, 2)$ structure illustrated in Fig.8. The typical situation when $\ell_M = 1$ is demonstrated by the $(4, 14; 2, 2)$ structure and the $(5, 18; 2, 3)$ structure where the $(5, 18; 2, 3)$ structure is identical with the $(5, 18; 3, 2)$ structure in Fig.5. The typical situation when $\ell_M = 2$ is demonstrated by the $(3, 9; 2, 1)$ structure.

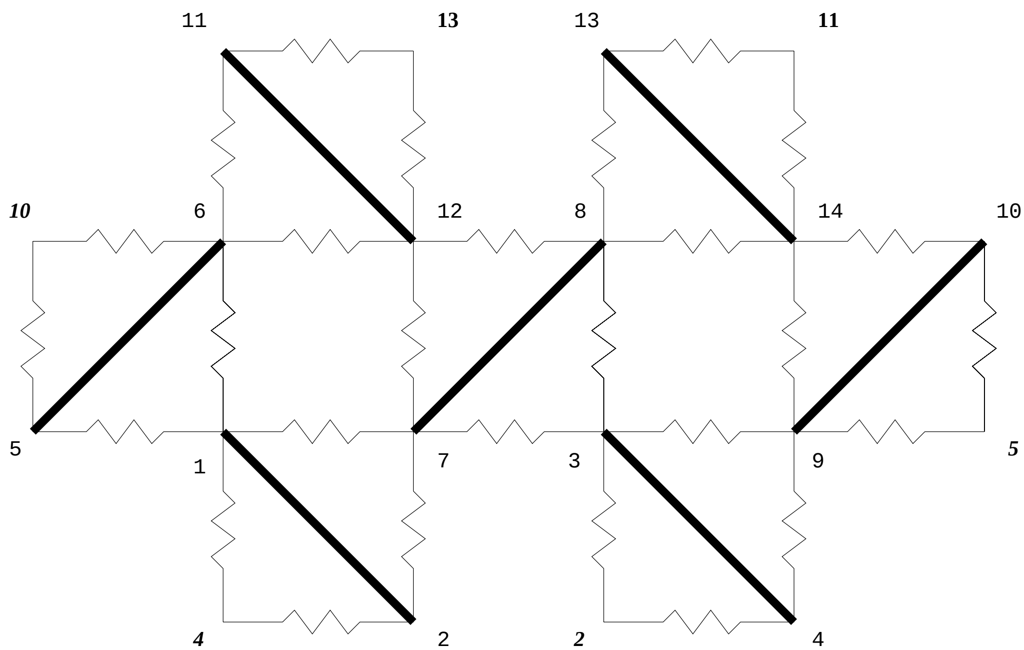

Figure 8: Planar topology of (7,25;2,3,2) tensegrity

For a given number N of bars, Theorem 2.3 may be used to determine all possible $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structures. For $N = 3$, the only structures are: $(3, 9; 3)$ and $(3, 9; 2, 1)$. For the non-symmetrical structure $(3, 9; 2, 1)$, we have from Theorem 2.3 that $\ell_1 = 1, \ell_2 = 2$ and so $S = 4 \times 3 - 3 = 9$. All possible structures for $3 \leq N \leq 10$ are provided in Table 2.1. Note that even though any number of additional cables may be added from any node to any other node, such additional cables are unnecessary in order to maintain stable equilibrium of the structure.

No of Struts	*Structures*
3	$(3, 9; 3)$, $(3, 9; 2, 1)$.
4	$(4, 12; 4)$, $(4, 14; 2, 2)$.
5	$(5, 15; 5)$, $(5, 18; 3, 2)$.
6	$(6, 18; 6)$, $(6, 24; 3, 3)$, $(6, 21; 3, 2, 1)$, $(6, 22; 2, 2, 2)$.
7	$(7, 21; 7)$, $(7, 27; 4, 3)$, $(7, 26; 3, 2, 2)$, $(7; 24; 2, 2, 2, 1)$, $(7, 26; 2, 3, 2)$.
8	$(8, 24; 8)$, $(8, 32; 4, 4)$, $(8, 31; 3, 3, 2)$, $(8, 31; 3, 2, 3)$
9	$(9, 27; 9)$, $(9, 35; 5, 4)$, $(9, 35; 4, 3, 2)$, $(9, 36; 3, 3, 3)$, $(9, 33; 3, 2, 1)$.
10	$(10, 30; 10)$, $(10, 40; 5, 5)$, $(10, 39; 4, 3, 3)$, $(10, 38; 4, 3, 2, 1)$, $(10, 39; 3, 4, 3)$, $(10, 38; 3, 2, 2, 3)$.

Table 2.1: Tensegrity structures with $3 \leq N \leq 10$ bars

Two further ways in which Theorem 2.3 can be applied are detailed as follows: (i) Consider the $M = 10$ stage $(33, S; 3, 2, 3, 4, 5, 4, 3, 2, 3, 4)$ structure. Then

$$\ell_1 = \ell_5 = \ell_{10} = 1; \quad \text{otherwise } \ell_k = 0$$

Hence $L = 3$ and so $S = 4N - L = 129$.

(ii) Consider the $M = 9$ stage $(22, S; 3, 2, 2, 2, 3, 4, 3, 2, 1)$ structure. Then

$$\ell_1 = \ell_6 = 1; \;\; \ell_9 = 2 \;\; \text{otherwise } \ell_k = 0$$

Hence $L = 4$ and so $S = 4N - L = 84$.

As for the case of a symmetrical tensegrity structure, a recursive algorithm can also be developed to define the cable connections of an M stage non-symmetrical tensegrity structure.

3 Prestress Equilibrium: Algebraic Conditions

In this section, we derive necessary and sufficient conditions on the position (x_k, y_k, z_k) of each bar end point (or node) for $1 \leq k \leq 2N$ in order that an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure is in mechanical equilibrium in the absence of external forces. In order that the topology defined in Section 2 results in a class D tensegrity, it is necessary to impose the *additional constraint* that the tensions in *all* cables are *strictly positive*.

Two Force Problem: In the absence of external forces, the forces on each bar in an $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity structure are only applied at *two* nodes (node $2m - 1$ and node $2m$ of the mth bar). Then the force system on each bar defines a *two force problem*. Consequently, both the resultant force $\mathbf{t}_{2m-1}$ at node $2m - 1$, and the resultant force $\mathbf{t}_{2m}$ at node $2m$, are directed along the bar $[2m-1, 2m]$. Furthermore, since each bar is in mechanical equilibrium $\mathbf{t}_{2m-1} + \mathbf{t}_{2m} = \mathbf{0}$. That is, for $1 \leq m \leq N$, we have

$$\mathbf{t}_{2m-1} = \gamma_{2m-1}\mathbf{h}_{2m-1} \quad ; \quad \mathbf{t}_{2m} = -\mathbf{t}_{2m-1} \tag{3.2}$$

for some scalar γ_{2m-1} where

$$\mathbf{h}_k^T = [x_k - x_{k+1} \quad y_k - y_{k+1} \quad z_k - z_{k+1}] \tag{3.3}$$

However after taking the inner product of both sides of (3.2) with $\mathbf{h}_{2m-1}$, it follows that

$$\gamma_{2m-1} = \frac{\mathbf{h}_{2m-1}^T \mathbf{t}_{2m-1}}{L_{2m-1,2m}^2} \tag{3.4}$$

where the length $L_{2m-1,2m}$ of the bar $[2m-1, 2m]$ is given by

$$L_{2m-1,2m} = ||\mathbf{h}_{2m-1}||_2$$

where $||\cdot||_2$ denotes the Euclidean norm.

3.1 Planar $N = 2$ Strut Tensegrity

Consider two possible $(2,4;2)$ structures defined in the xy-plane where node k has coordinates (x_k, y_k) with the corresponding force diagram as illustrated in Fig.9. We assume the bars are arbitrarily thin (and so don't touch) so that the only forces that are exerted on the system occur at nodes 1,2,3 and 4. We adopt the convention that the (vector) tension $\mathbf{t}_{k\ell}$ in the cable connecting node k to node ℓ is in the direction from node k to node ℓ. Consequently,

$$\mathbf{t}_{k\ell} = -\mathbf{t}_{\ell k} \tag{3.5}$$

and so in writing down the force balance equations, we only need to use tensions $\mathbf{t}_{pq}$ for $p < q$.

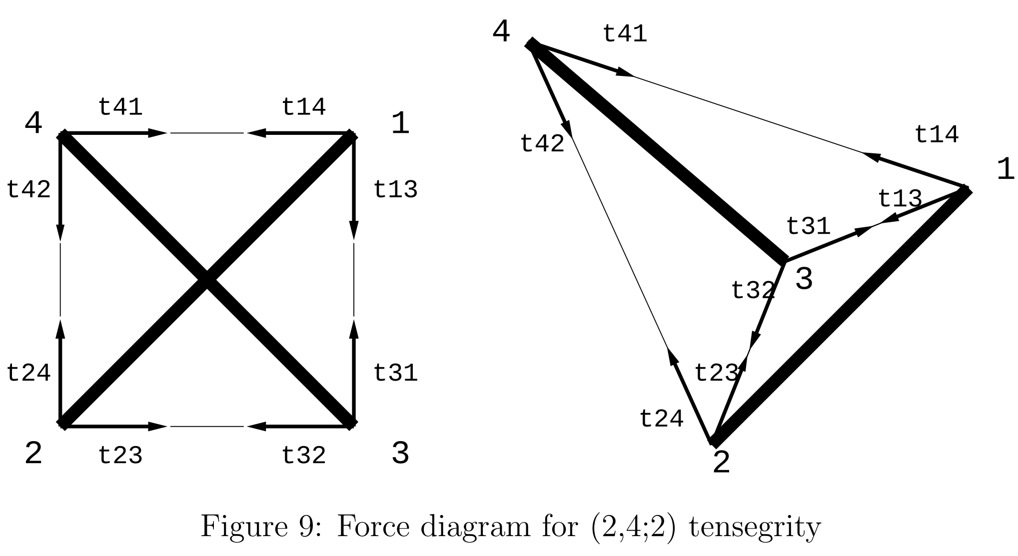

Figure 9: Force diagram for (2,4;2) tensegrity

Force Balance Equations: From Fig. 9, the force balance equation on bar 12 is:

$$\mathbf{t}_1 + \mathbf{t}_2 = \mathbf{0} \tag{3.6}$$

where $\mathbf{t}_1$ (or $\mathbf{t}_2$) is the resultant force applied at node 1 (or node 2), and

$$\mathbf{t}_1 = \mathbf{t}_{13} + \mathbf{t}_{14} \quad ; \quad \mathbf{t}_2 = \mathbf{t}_{23} + \mathbf{t}_{24} \tag{3.7}$$

Likewise, the force balance equation on bar 34 is:

$$\mathbf{t}_3 + \mathbf{t}_4 = \mathbf{0} \tag{3.8}$$

where $\mathbf{t}_3$ (or $\mathbf{t}_4$) is the resultant force applied at node 3 (or node 4), and using (3.5)

$$\begin{aligned} \mathbf{t}_3 &= \mathbf{t}_{31} + \mathbf{t}_{32} = -\mathbf{t}_{13} - \mathbf{t}_{23} \\ \mathbf{t}_4 &= \mathbf{t}_{41} + \mathbf{t}_{42} = -\mathbf{t}_{14} - \mathbf{t}_{24} \end{aligned} \tag{3.9}$$

Since the forces on each bar are only applied in the xy-plane at two nodes (nodes 1 and 2 for bar 12, or nodes 3 and 4 for bar 34), we have from (3.2), (3.4) that

$$\begin{aligned} \mathbf{t}_1 &= \left(\frac{\mathbf{h}_1^T \mathbf{t}_1}{L_{12}^2} \right) \mathbf{h}_1 \quad ; \quad \mathbf{t}_2 = -\mathbf{t}_1 \\ \mathbf{t}_3 &= \left(\frac{\mathbf{h}_3^T \mathbf{t}_3}{L_{34}^2} \right) \mathbf{h}_3 \quad ; \quad \mathbf{t}_4 = -\mathbf{t}_3 \end{aligned} \tag{3.10}$$

where

$$\mathbf{h}_1^T = [x_1 - x_2 \quad y_1 - y_2] \quad ; \quad \mathbf{h}_3^T = [x_3 - x_4 \quad y_3 - y_4] \tag{3.11}$$

Note that since $\mathbf{t}_1 = \mathbf{t}_{13} + \mathbf{t}_{14}$ is not directed along $\mathbf{h}_1$, the orientation in Fig.9b is actually *not* possible.

To continue, express the tension $\mathbf{t}_{pq}$ in the cable $[p, q]$ connecting node p to node q in the form

$$\mathbf{t}_{pq} = \alpha_{pq} \mathbf{e}_{pq} \tag{3.12}$$

where α_{pq} is the *magnitude*, and $\mathbf{e}_{pq}$ is a unit vector *directed* from node p to node q; that is,

$$\mathbf{e}_{pq} \triangleq \begin{bmatrix} \cos\theta_{pq} \\ \sin\theta_{pq} \end{bmatrix} = \frac{1}{\ell_{pq}} \begin{bmatrix} x_p - x_q \\ y_p - y_q \end{bmatrix} \tag{3.13}$$

where the length ℓ_{pq} of the cable from node p to node q is given by

$$\ell_{pq} \triangleq \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2} \tag{3.14}$$

Then from *Hookes Law*, we have that α_{pq} in (3.12) is given by

$$\alpha_{pq} = k_{pq}(\ell_{pq} - \ell_{pq}^0) \tag{3.15}$$

where $k_{pq} > 0$ is the *spring constant* of the cable $[p,q]$, and ℓ^0_{pq} is the *initial length* of cable $[p,q]$.

Recall that $[p,q]$ is a cable connection unless $\{p = 2m-1, q = 2m; m = 1,2\}$, and $[2m-1, 2m]$ defines a bar. In order to distinguish bars from cables, we henceforth define

$$L_{2m-1,2m} = \ell_{2m-1,2m} \quad ; \quad 1 \le m \le N \tag{3.16}$$

Theorem 3.1 *Consider a planar truss with nodes 1,2,3 and 4 as illustrated in Fig.9 in which node k has coordinates* (x_k, y_k)*. Define the unit vectors* $\mathbf{e}_{pq}$ *by (3.13) and the vectors* $\{\mathbf{l}, \mathbf{l}^0\}$ *by*

$$\mathbf{l}^T = [\ell_{13} \quad \ell_{14} \quad \ell_{23} \quad \ell_{24}] \quad ; \quad (\mathbf{l}^0)^T = [\ell^0_{13} \quad \ell^0_{14} \quad \ell^0_{23} \quad \ell^0_{24}] \tag{3.17}$$

Then:

1. *Given the length and orientation* $\{L_{12}, \mathbf{e}_{12}\}$ *of bar 12, the geometry in Fig.9 is uniquely determined by the 4 unit vectors* $\{\mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}\}$ *according to*

$$\mathbf{G}_1 \mathbf{l} = \mathbf{b}_1 \quad ; \quad \ell_{km} > 0 \tag{3.18}$$

where the 4×4 *matrix* $\mathbf{G}_1$*, and the 4-vector* $\mathbf{b}_1$ *are given by*

$$\mathbf{G}_1 = \begin{bmatrix} \mathbf{e}_{13} & \mathbf{0} & -\mathbf{e}_{23} & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_{14} & \mathbf{0} & -\mathbf{e}_{24} \end{bmatrix} \quad ; \quad \mathbf{b}_1 = \begin{bmatrix} L_{12}\mathbf{e}_{12} \\ L_{12}\mathbf{e}_{12} \end{bmatrix} \tag{3.19}$$

2. *The geometry defined by (3.18), (3.19) defines a* $(2,4;2)$ *tensegrity if there exists some vector* $\mathbf{l}^0$ *in (3.17) with components* ℓ^0_{km} *satisfying* $0 < \ell^0_{km} < \ell_{km}$ *for all* k, m *such that:*

$$\mathbf{A}\mathbf{K}\mathbf{l}^0 = \mathbf{A}\mathbf{K}\mathbf{l} \tag{3.20}$$

where $\mathbf{K} = \mathrm{diag}\{k_{13}, k_{14}, k_{23}, k_{24}\}$ *is a diagonal positive definite matrix, and*

$$\mathbf{A} = \begin{bmatrix} \mathbf{H}_1\mathbf{e}_{13} & \mathbf{H}_1\mathbf{e}_{14} & \mathbf{0} & \mathbf{0} \\ \mathbf{e}_{13} & \mathbf{e}_{14} & \mathbf{e}_{23} & \mathbf{e}_{24} \end{bmatrix} \quad ; \quad \mathbf{H}_1 = \mathbf{I} - \mathbf{e}_{12}\mathbf{e}_{12}^T \tag{3.21}$$

The proof for part 1 follows directly from geometrical considerations. To prove part 2, first observe from (3.10), (3.11) that $\mathbf{t}_1 + \mathbf{t}_2 = \mathbf{0}$ where

$$\mathbf{t}_1 = (\mathbf{e}_{12}^T \mathbf{t}_1)\mathbf{e}_{12} = (\mathbf{e}_{12}\mathbf{e}_{12}^T)\mathbf{t}_1 \quad ; \quad \mathbf{e}_{12} = \frac{\mathbf{h}_1}{L_{12}}$$

Hence

$$\mathbf{H}_1 \mathbf{t}_1 = \mathbf{0} \quad ; \quad \mathbf{t}_1 + \mathbf{t}_2 = \mathbf{0}$$

The result then follows from (3.7) and (3.15).

Note that the geometric condition in part 1 of Theorem 3.1 can be stated in terms of other sets of initial data. For example, given the length and orientation $\{L_{12}, \mathbf{e}_{12}, L_{34}, \mathbf{e}_{34}\}$ of both bars, the tensegrity geometry is then uniquely determined by the initial vectors $\{\mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}\}$ according to

$$\mathbf{G}_2\mathbf{l} = \mathbf{b}_2 \quad ; \quad \ell_{km} > 0$$

where

$$\mathbf{G}_2 = \begin{bmatrix} \mathbf{e}_{13} & \mathbf{0} & -\mathbf{e}_{23} & \mathbf{0} \\ -\mathbf{e}_{13} & \mathbf{e}_{14} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad ; \quad \mathbf{b}_2 = \begin{bmatrix} L_{12}\mathbf{e}_{12} \\ L_{34}\mathbf{e}_{34} \end{bmatrix}$$

For any two p-dimensional vectors $\{\mathbf{v}, \mathbf{w}\}$ with their kth components given by v_k, w_k respectively, we shall adopt the notation:

$$\mathbf{0} < \mathbf{v} < \mathbf{w} \tag{3.22}$$

to mean that: $0 < v_k < w_k$ for all $1 \leq k \leq p$. We then have the following consequence of Theorem 3.1.

Corollary 3.1 *Suppose* $\mathbf{l} > \mathbf{0}$ *satisfies (3.18), and* $\mathbf{A}$ *is defined by (3.21). Then there exists a solution* $\mathbf{l}^0$ *of (3.20) such that* $\mathbf{0} < \mathbf{l}^0 < \mathbf{l}$ *if and only if there exists a solution* $\mathbf{p} > \mathbf{0}$ *of*

$$\mathbf{A}\mathbf{p} = \mathbf{0} \tag{3.23}$$

In order to prove this result, observe that if $\mathbf{l} - \mathbf{l}^0 = \alpha\mathbf{K}^{-1}\mathbf{p}$, then (3.23) implies (3.20) is satisfied for all α. Furthermore, if all components of $\mathbf{p}$ are positive, then since all components of $\mathbf{l}$ in (3.18) are positive, the parameter $\alpha > 0$ may be chosen sufficiently small such that all components of $\mathbf{l}^0$ are positive. If $p_k \leq 0$ for some k, then $\ell^0_{km} \geq \ell_{km}$ for cable $[k, m]$ which implies the tension in this cable is either zero or negative, and therefore does not define a valid prestressed tensegrity structure. This completes the proof.

We now consider the solution of (3.23) for some $\mathbf{p} > \mathbf{0}$. To begin, assume without any loss of generality that bar 12 is oriented such that $\mathbf{e}_{12}^T = [-1 \;\; 0]$ with $\{x_1 = y_1 = 0; x_2 > 0, y_2 = 0\}$. Then in (3.21), we have

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -y_3/\ell_{13} & -y_4/\ell_{14} & 0 & 0 \\ -x_3/\ell_{13} & -x_4/\ell_{14} & (x_2 - x_3)/\ell_{23} & (x_2 - x_4)/\ell_{24} \\ -y_3/\ell_{13} & -y_4/\ell_{14} & -y_3/\ell_{13} & -y_4/\ell_{24} \end{bmatrix} \tag{3.24}$$

Solutions of (3.23) for $\mathbf{p} \neq \mathbf{0}$ when $\mathbf{A}$ is given by (3.24) are therefore of the form

$$\mathbf{p}^T = [1 \;\; p_2 \;\; p_3 \;\; p_4]$$

where

$$p_2 = -\left(\frac{\ell_{14}}{\ell_{13}}\right)\left(\frac{y_3}{y_4}\right) \quad ; \quad p_3 = -\left(\frac{\ell_{13}}{\ell_{24}}\right)\left(\frac{y_4}{y_3}\right) p_4$$

$$\left[\left(\frac{\ell_{13}}{\ell_{24}}\right)\left(\frac{y_4}{y_3}\right)\left(\frac{x_2 - x_3}{\ell_{23}}\right) - \left(\frac{x_2 - x_4}{\ell_{24}}\right)\right] p_4 = \left(\frac{x_4}{\ell_{13}}\right)\left(\frac{y_3}{y_4}\right) - \left(\frac{x_3}{\ell_{13}}\right)$$

Consequently, we have the following result.

Corollary 3.2 *The structure illustrated in Fig.9 where node k has coordinates (x_k, y_k) with $\{x_1 = y_1 = 0; x_2 > 0, y_2 = 0\}$, and the length of the cable $[k, m]$ is ℓ_{km} is a $(2, 4; 2)$ tensegrity if and only if*

$$y_3 y_4 < 0 \tag{3.25}$$

$$\left[\left(\frac{\ell_{13}}{\ell_{24}}\right)\left|\frac{y_4}{y_3}\right|\left(\frac{x_2 - x_3}{\ell_{23}}\right) + \left(\frac{x_2 - x_4}{\ell_{24}}\right)\right]\left[\left(\frac{x_4}{\ell_{13}}\right)\left|\frac{y_3}{y_4}\right| + \left(\frac{x_3}{\ell_{13}}\right)\right] > 0$$

If we examine Fig.9 (with bar 12 re-orientated to comply with the assumptions in Corollary 3.2), we see that the condition $y_3 y_4 < 0$ implies that the nodes 3 and 4 *cannot* be on the same side of bar 12; that is, the configuration in Fig.9b can never be a $(2, 4; 2)$ tensegrity structure. (This statement is a confirmation of the earlier observation that the resultant forces $\mathbf{t}_1$ and $\mathbf{t}_2$ in Fig.9b do not point along the vector $\mathbf{h}_1$.)

As evident from Corollary 3.2, the condition $y_3 y_4 < 0$ is only necessary, but not sufficient. However, if in addition, we have $\{x_2 > x_4 > 0;\ x_2 > x_3 > 0\}$ (which is the case in Fig.9a), it follows that the resulting configuration is *always* a valid $(2, 4; 2)$ tensegrity for *some* choice of bar lengths, *some* choice of spring constants $\{k_j\}$, and *some* choice of initial cable lengths $\{\ell^0_{ij}\}$.

More specifically, given the coordinates $\mathbf{q}^T = [x_k\ y_k]$ of the 4 nodes that results in a solution $\mathbf{p} > \mathbf{0}$ where $\mathbf{p}^T = [1\ p_2\ p_3\ p_4]$, we have

$$\mathbf{l}^0 = \mathbf{l} - \alpha \mathbf{K}^{-1}\mathbf{p}$$

for some sufficiently small $\alpha > 0$ such that $\mathbf{0} < \mathbf{l}^0 < \mathbf{l}$; that is, the initial cable lengths of the $(2, 4; 2)$ tensegrity are given by

$$\begin{aligned} \ell^0_{13} &= \ell_{13} - \alpha k_{13}^{-1} \ ; \ \ell^0_{14} = \ell_{14} - \alpha k_{14}^{-1} p_2 \\ \ell^0_{23} &= \ell_{23} - \alpha k_{23}^{-1} p_3 \ ; \ \ell^0_{24} = \ell_{24} - \alpha k_{24}^{-1} p_4 \end{aligned}$$

and the magnitudes $\{\alpha_{ij}\}$ of the tensions in the cables by

$$\alpha_{13} = \alpha \ ; \ \alpha_{24} = \alpha p_2 \ ; \ \alpha_{23} = \alpha p_3 \ ; \ \alpha_{24} = \alpha p_4$$

Hence scaling *all* the cable tensions (by adjusting α) does not change the shape of the tensegrity; only the pretension.

It also follows from Corollary 3.2 that the conditions $\{x_4 > x_2 > 0;\ x_3 > x_2 > 0\}$ can never result in a $(2, 4; 2)$ tensegrity. The case $\{x_2 > x_4 > 0;\ x_3 \geq x_2 > 0\}$ is more interesting. Given $y_3 y_4 < 0$, a necessary and sufficient condition is then

$$|x_2 - x_3| < \left(\frac{\ell_{23}}{\ell_{13}}\right)\left|\frac{y_3}{y_4}\right|(x_2 - x_4)$$

It can be shown that the 4×4 matrix $\mathbf{A}$ in (3.24) has rank 3 for all valid $(2, 4; 2)$ tensegrity structures which implies that the null space of $\mathbf{A}$ has dimension 1. For 3-dimensional tensegrity structures, we will see that necessary and sufficient conditions for a valid tensegrity can also be expressed in the form of the existence of a solution $\mathbf{p} > \mathbf{0}$ of a system of equations $\mathbf{A}\mathbf{p} = \mathbf{0}$ where $\mathbf{A}$ (which is a nonlinear function of the coordinates of the ends of the bars) is in general non-square with a null space of dimension greater than 1.

3.2 Single Stage Structures

Consider now the force diagram for the $(N, 3N; N)$ tensegrity structure. The force balance equation on bar $[2k-1, 2k]$ is:

$$\mathbf{t}_{2k-1} + \mathbf{t}_{2k} = \mathbf{0} \quad ; \quad k = 1, 2, \ \ldots \ , N \tag{3.26}$$

where $\mathbf{t}_m$ is the resultant force applied at node m with coordinates $\mathbf{q}_m^T = [x_m \ \ y_m \ \ z_m]$. Based on the earlier results for the (2,4;2) tensegrity, and the fact that the resultant forces on each bar of the $(N, 3N; N)$ structure define a two force problem for each bar, we have the following results.

Lemma 3.1 *Suppose the $(N, 3N; N)$ structure is in mechanical equilibrium in the absence of external forces. Then the resultant force $\mathbf{t}_k$ on the bar at node k with coordinates $\mathbf{q}_k^T = [x_k \ \ y_k \ \ z_k]$ is given by: for $1 \leq m \leq N$*

$$\mathbf{H}_{2m-1}\mathbf{t}_{2m-1} = \mathbf{0} \quad ; \quad \mathbf{t}_{2m} = -\mathbf{t}_{2m-1} \tag{3.27}$$

where

$$\mathbf{H}_{2m-1} = \mathbf{I} - \mathbf{e}_{2m-1,2m}\mathbf{e}_{2m-1,2m}^T \tag{3.28}$$

where the unit vector $\mathbf{e}_{kn}$ is defined by

$$\mathbf{e}_{kn}^T = \frac{1}{\ell_k}[x_k - x_n \ \ y_k - y_n \ \ z_k - z_n] \tag{3.29}$$

with

$$\ell_{kn} = \sqrt{(x_k - x_n)^2 + (y_k - y_n)^2 + (z_k - z_n)^2}$$

Lemma 3.2 *Given the lengths $\{L_{2m-1,2m};\ 1 \leq m \leq N\}$ of the bars in (3.16) and orientations $\{\mathbf{e}_{2m-1,2m};\ 1 \leq m \leq N\}$ of the bars, the truss geometry of an $(N, 3N; N)$ structure is uniquely determined by the unit vectors $\{\mathbf{e}_{kn}\}$ for all k, n except $\{k = 2m-1, n = 2m\}$ when $\ell_{pq} > 0$ for all p, q as follows:*

$$
\begin{aligned}
\ell_{1,2N}\mathbf{e}_{1,2N} - \ell_{2,2N}\mathbf{e}_{2,2N} &= L_{1,2}\mathbf{e}_{1,2} \\
\ell_{1,3}\mathbf{e}_{1,3} - \ell_{2,3}\mathbf{e}_{2,3} &= L_{1,2}\mathbf{e}_{1,2} \\
-\ell_{2N-2,2N-1}\mathbf{e}_{2N-2,2N-1} + \ell_{2N-2,2N}\mathbf{e}_{2N-2,2N} &= L_{2N-1,2N}\mathbf{e}_{2N-1,2N} \\
-\ell_{1,2N-1}\mathbf{e}_{1,2N-1} + \ell_{1,2N}\mathbf{e}_{1,2N} &= L_{2N-1,2N}\mathbf{e}_{2N-1,2N}
\end{aligned} \tag{3.30}
$$

and for $1 \le m \le N-2$

$$
\begin{aligned}
-\ell_{2m,2m+1}\mathbf{e}_{2m,2m+1} + \ell_{2m,2m+2}\mathbf{e}_{2m,2m+2} &= L_{2m+1,2m+2}\mathbf{e}_{2m+1,2m+2} \\
\ell_{2m+1,2m+3}\mathbf{e}_{2m+1,2m+3} - \ell_{2m+2,2m+3}\mathbf{e}_{2m+2,2m+3} &= L_{2m+1,2m+2}\mathbf{e}_{2m+1,2m+2}
\end{aligned} \tag{3.31}
$$

Now from the cable connections defined in Theorem 2.1 (see also Fig.3), we have that the resultant forces $\{\mathbf{t}_m\}$ are given by:

$$
\begin{aligned}
\mathbf{t}_1 &= \mathbf{t}_{13} + \mathbf{t}_{1,2N-1} + \mathbf{t}_{1,2N} \\
\mathbf{t}_2 &= \mathbf{t}_{23} + \mathbf{t}_{2,4} + \mathbf{t}_{2,2N} \\
\mathbf{t}_{2N-1} &= -\mathbf{t}_{2N-3,2N-1} - \mathbf{t}_{2N-2,2N-1} - \mathbf{t}_{1,2N-1} \\
\mathbf{t}_{2N} &= -\mathbf{t}_{1,2N} - \mathbf{t}_{2,2N} - \mathbf{t}_{2N-2,2N}
\end{aligned} \tag{3.32}
$$

and for $2 \le m \le N-1$ by:

$$
\begin{aligned}
\mathbf{t}_{2m-1} &= -\mathbf{t}_{2m-3,2m-1} - \mathbf{t}_{2m-2,2m-1} + \mathbf{t}_{2m-1,2m+1} \\
\mathbf{t}_{2m} &= -\mathbf{t}_{2m-2,2m} + \mathbf{t}_{2m,2m+1} + \mathbf{t}_{2m,2m+2}
\end{aligned} \tag{3.33}
$$

We then have the next result.

Lemma 3.3 *Define the matrices* $\mathbf{A}_1$ *and* $\{\mathbf{A}_k;\ k = 2N-1\}$ *by*

$$
\begin{aligned}
\mathbf{A}_1 &= \begin{bmatrix} \mathbf{H}_1\mathbf{e}_{13} & \mathbf{H}_1\mathbf{e}_{1,2N-1} & \mathbf{H}_1\mathbf{e}_{1,2N} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{e}_{13} & \mathbf{e}_{1,2N-1} & \mathbf{e}_{1,2N} & \mathbf{e}_{23} & \mathbf{e}_{24} & \mathbf{e}_{2,2N} \end{bmatrix} \\
\mathbf{A}_k &= \begin{bmatrix} -\mathbf{H}_k\mathbf{e}_{1,k} & -\mathbf{H}_k\mathbf{e}_{k-2,k} & -\mathbf{H}_k\mathbf{e}_{k-1,k} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{e}_{1,k} & -\mathbf{e}_{k-2,k} & -\mathbf{e}_{k-1,k} & -\mathbf{e}_{1,k+1} & -\mathbf{e}_{2,k+1} & -\mathbf{e}_{k-1,k+1} \end{bmatrix}
\end{aligned} \tag{3.34}
$$

and $\{\mathbf{A}_k;\ k = 2m-1;\ 2 \le m \le N-1\}$ *by:*

$$
\mathbf{A}_k = \begin{bmatrix} -\mathbf{H}_k\mathbf{e}_{k-2,k} & -\mathbf{H}_k\mathbf{e}_{k-1,k} & \mathbf{H}_k\mathbf{e}_{k,k+2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{e}_{k-2,k} & -\mathbf{e}_{k-1,k} & \mathbf{e}_{k,k+2} & -\mathbf{e}_{k-1,k+1} & \mathbf{e}_{k+1,k+2} & \mathbf{e}_{k+1,k+3} \end{bmatrix}
$$

where $\mathbf{H}_{2k-1}$ *is defined by (3.28),*

Also, define the positive definite diagonal matrices $\mathbf{K}_1$ *and* $\mathbf{K}_{2N-1}$ *by*

$$\begin{aligned}\mathbf{K}_1 &= diag\{k_{13}, k_{1,2N-1}, k_{1,2N}, k_{23}, k_{24}, k_{2,2N}\} \\ \mathbf{K}_r &= diag\{k_{1,r}, k_{r-2,r}, k_{r-1,r}, k_{1,r+1}, k_{2,r+1}, k_{r-1,r+1}\} \ ; \ r = 2N-1\end{aligned} \tag{3.35}$$

and $\{\mathbf{K}_n;\ n = 2m-1;\ 2 \leq m \leq N\}$ *by:*

$$\mathbf{K}_n = diag\{k_{n-2,n}, k_{n-1,n}, k_{n,n+2}, k_{n-1,n+1}, k_{n+1,n+2}, k_{n+1,n+3}\}$$

Also, define the vectors $\mathbf{l}_1$ *and* $\mathbf{l}_{2N-1}$ *by*

$$\begin{aligned}\mathbf{l}_1^T &= [\ell_{13}\ \ell_{1,2N-1}\ \ \ell_{1,2N}\ \ \ell_{23}\ \ \ell_{24}\ \ \ell_{2,2N}] \\ \mathbf{l}_r^T &= [\ell_{1,r}\ \ell_{r-2,r}\ \ \ell_{r-1,r}\ \ \ell_{1,r+1}\ \ \ell_{2,r+1}\ \ell_{r-1,r+1}] \ ; \ r = 2N-1\end{aligned} \tag{3.36}$$

and the vectors $\{\mathbf{l}_n;\ n = 2m-1;\ 2 \leq m \leq N\}$ *by:*

$$\mathbf{l}_n^T = [\ell_{n-2,n}\ \ \ell_{n-1,n}\ \ \ell_{n,n+2}\ \ \ell_{n-1,n+1}\ \ell_{n+1,n+2}\ \ \ell_{n+1,n+3}]$$

Similarly, define the vectors $\mathbf{l}_1^0$ *and* $\mathbf{l}_{2N-1}^0$ *by*

$$\begin{aligned}(\mathbf{l}_1^0)^T &= [\ell_{13}^0\ \ \ell_{1,2N-1}^0\ \ \ell_{1,2N}^0\ \ \ell_{23}^0\ \ \ell_{24}^0\ \ \ell_{2,2N}^0] \\ (\mathbf{l}_r^0)^T &= [\ell_{1,r}^0\ \ \ell_{r-2,r}^0\ \ \ell_{r-1,r}^0\ \ \ell_{1,r+1}^0\ \ \ell_{2,r+1}^0\ \ell_{r-1,r+1}^0] \ ; \ r = 2N-1\end{aligned} \tag{3.37}$$

and the vectors $\{\mathbf{l}_n^0;\ n = 2m-1;\ 2 \leq m \leq N\}$ *by*

$$(\mathbf{l}^0)_n^T = [\ell_{n-2,n}^0\ \ \ell_{n-1,n}^0\ \ \ell_{n,n+2}^0\ \ \ell_{n-1,n+1}^0\ \ \ell_{n+1,n+2}^0\ \ \ell_{n+1,n+3}^0]$$

Then the geometry defined by Lemma 3.2 establishes a valid $(N, 3N; N)$ *tensegrity if the system of equations: for* $1 \leq m \leq N$

$$\mathbf{A}_{2m-1}\mathbf{K}_{2m-1}\mathbf{l}_{2m-1}^0 = \mathbf{A}_{2m-1}\mathbf{K}_{2m-1}\mathbf{l}_{2m-1} \tag{3.38}$$

has a solution in which all components of $\mathbf{l}_{2m-1}^0$ *are positive for all* $1 \leq m \leq N$*.*

Extending Corollary 3.1, we then have the next result.

Theorem 3.2 *Suppose* $\mathbf{l}_{2m-1} > \mathbf{0}$ *satisfies the geometric constraints in (3.30), (3.31), and* $\mathbf{A}_{2m-1}$ *is defined by lemma 3.3. Then there exists a solution* $\mathbf{l}_{2m-1}^0$ *of (3.38) such that* $\mathbf{0} < \mathbf{l}_{2m-1}^0 < \mathbf{l}_{2m-1}$ *if and only if there exists a solution* $\mathbf{p}_{2m-1} > \mathbf{0}$ *of the equation*

$$\mathbf{A}_{2m-1}\mathbf{p}_{2m-1} = \mathbf{0} \tag{3.39}$$

Unfortunately, conditions for mechanical equilibrium are complicated by the fact that the components of $\{\mathbf{l}_m, \mathbf{l}_n\}$ (and $\{\mathbf{l}_m^0, \mathbf{l}_n^0\}$) are not necessarily independent. The degree of independence is expressed in the following result.

Lemma 3.4 *1. The vectors $\{\mathbf{l}_p, \mathbf{l}_q\}$ have no components in common except when $\{p = 1, q = 2N-1\}$ and $\{p = 2m-1, q = 2m+1;\ 2 \leq m \leq N-1\}$*

2. *The vectors $\{\mathbf{l}_1, \mathbf{l}_{2N-1}\}$ have components $\{\ell_{1,2N-1}, \ell_{1,2N}, \ell_{2,2N}\}$ in common*

3. *The vectors $\{\mathbf{l}_{2m-1}, \mathbf{l}_{2m+1}\}$ for $2 \leq m \leq N-1$ have components $\{\ell_{2m-1,2m+1}, \ell_{2m,2m+1}, \ell_{2m,2m+2}\}$ in common*

Definition 3.1 *For an N bar tensegrity, define the $6N$-dimensional vector $\tilde{\mathbf{l}}$ to be the concatenation of the N 6-dimensional vectors $\{\mathbf{l}_{2m-1};\ 1 \leq m \leq N\}$; that is*

$$\tilde{\mathbf{l}}^T = [\mathbf{l}_1^T \ \ \mathbf{l}_3^T \ \ \ldots \ \ \mathbf{l}_{2N-3}^T \ \ \mathbf{l}_{2N-1}^T] \tag{3.40}$$

Then (via lemma 3.4), define the $3N$-dimensional vector $\mathbf{l}$ whose components are the $3N$ cables of the $(N, 3N; N)$ tensegrity structure by progressing down the components of $\tilde{\mathbf{l}}$, and eliminating any component of $\tilde{\mathbf{l}}$ that already appears in $\mathbf{l}$.

Note that by the time we progress to $\mathbf{l}_{2N-1}$ in $\tilde{\mathbf{l}}$, all components of $\mathbf{l}_{2N-1}$ will have already been included in $\mathbf{l}$. For example, when $N = 4$, we have from (3.40) that

$$\tilde{\mathbf{l}}^T = [\mathbf{l}_1^T \ \ \mathbf{l}_3^T \ \ \mathbf{l}_5^T \ \ \mathbf{l}_7^T]$$

where

$$\mathbf{l}_1^T = [\ell_{13} \ \ \ell_{17} \ \ \ell_{18} \ \ \ell_{23} \ \ \ell_{24} \ \ \ell_{28}] \ , \ \mathbf{l}_3^T = [\ell_{13} \ \ \ell_{23} \ \ \ell_{35} \ \ \ell_{24} \ \ \ell_{45} \ \ \ell_{46}]$$

$$\mathbf{l}_5^T = [\ell_{35} \ \ \ell_{45} \ \ \ell_{57} \ \ \ell_{46} \ \ \ell_{67} \ \ \ell_{68}] \ , \ \mathbf{l}_7^T = [\ell_{17} \ \ \ell_{57} \ \ \ell_{67} \ \ \ell_{18} \ \ \ell_{28} \ \ \ell_{68}]$$

Then the corresponding $3N = 12$ dimensional vector $\mathbf{l}$ is given by

$$\mathbf{l} = [\mathbf{l}_1^T, \ \ell_{35}, \ \ell_{45}, \ \ell_{46}, \ \ell_{57}, \ \ell_{67}, \ \ell_{68}] \tag{3.41}$$

Theorem 3.3 *Consider a 3-dimensional truss consisting of N bars in which the $2N$ end points have coordinates $\mathbf{q}^T = [x_m \ \ y_m \ \ z_m]$. Define the unit vectors $\{\mathbf{e}_{kn}\}$ by (3.29), and the $3N$-dimensional vector $\mathbf{l}^0$ as the initial length of the $3N$-dimensional vector $\mathbf{l}$ defined in Definition 3.1. Then:*

1. *Given the lengths and orientations $\{L_{2m-1,2m}, \mathbf{e}_{2m-1,2m};\ 1 \leq m \leq N\}$ of the N bars, a truss geometry is uniquely determined by the $S = 3N$ unit vectors $\{\mathbf{e}_{kn}\}$ which define the cable orientations according to*

$$\mathbf{G}\mathbf{l} = \mathbf{b} \ \ ; \ \ \mathbf{l} > \mathbf{0} \tag{3.42}$$

where the $6N \times 3N$ matrix $\mathbf{G}$ is defined by (3.30), (3.31).

2. *The geometry described by (3.42) defines a valid $(N, 3N; N)$ tensegrity if there exists some vector* $\mathbf{l}^0$ *such that*

$$\mathbf{AKl}^0 = \mathbf{AKl} \quad ; \quad \mathbf{0} < \mathbf{l}^0 < \mathbf{l} \tag{3.43}$$

where $\mathbf{K}$ *is a* $3N \times 3N$ *positive definite diagonal matrix whose nth diagonal component is the spring constraint of the cable whose length is the nth component of* $\mathbf{l}$*, and* $\mathbf{A}$ *is a* $2N \times 3N$ *matrix defined from lemma 3.3 so that (3.43) is equivalent to the system of equations (3.38) when expressed in terms of* $\mathbf{l}$*.*

3. *A vector* $\mathbf{l}^0$ *exists in (3.43) if and only if there exists a vector* $\mathbf{p}$ *such that*

$$\mathbf{Ap} = \mathbf{0} \quad ; \quad \mathbf{p} > \mathbf{0} \tag{3.44}$$

For example, when $N = 4$, $3N = 12$ and the 12-dimensional vector $\mathbf{l}$ is given by (3.41). The corresponding 24×12 matrix $\mathbf{G}$ in (3.42) is then given by

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{e}_{18} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{28} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{e}_{13} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{23} & \mathbf{e}_{24} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{35} & -\mathbf{e}_{45} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{45} & \mathbf{e}_{46} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{57} & -\mathbf{e}_{67} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{67} & \mathbf{e}_{68} \\ \mathbf{0} & -\mathbf{e}_{17} & \mathbf{e}_{18} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and the corresponding 24×12 matrix $\mathbf{A}$ in (3.43) is given by

$$\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2]$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{H}_1\mathbf{e}_{13} & \mathbf{H}_1\mathbf{e}_{17} & \mathbf{H}_1\mathbf{e}_{18} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{e}_{13} & \mathbf{e}_{17} & \mathbf{e}_{18} & \mathbf{e}_{23} & \mathbf{e}_{24} & \mathbf{e}_{28} \\ -\mathbf{H}_3\mathbf{e}_{13} & \mathbf{0} & \mathbf{0} & -\mathbf{H}_3\mathbf{e}_{23} & \mathbf{0} & \mathbf{0} \\ -\mathbf{e}_{13} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{23} & -\mathbf{e}_{24} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{H}_7\mathbf{e}_{17} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{e}_{17} & -\mathbf{e}_{18} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{28} \end{bmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_3\mathbf{e}_{35} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{e}_{35} & \mathbf{e}_{45} & \mathbf{e}_{46} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{H}_5\mathbf{e}_{35} & -\mathbf{H}_5\mathbf{e}_{45} & \mathbf{0} & \mathbf{H}_5\mathbf{e}_{57} & \mathbf{0} & \mathbf{0} \\ -\mathbf{e}_{35} & -\mathbf{e}_{45} & -\mathbf{e}_{46} & \mathbf{e}_{57} & \mathbf{e}_{67} & \mathbf{e}_{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{H}_7\mathbf{e}_{57} & -\mathbf{H}_7\mathbf{e}_{67} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_{57} & -\mathbf{e}_{67} & -\mathbf{e}_{68} \end{bmatrix}$$

3.3 M Stage Structures

The algebraic results just presented in Theorem 3.3 for the single stage $(N, 3N; N)$ tensegrity structure can be extended to the case of both M stage symmetrical and non-symmetrical structures based on the respective cable connection matrices. However, a formal proof for the existence of an analytical solution to (3.44) (or the equivalent equation for M stage structures) is not available.

The dimension of the null space of $\mathbf{A}$ for a valid tensegrity is also of interest since this property has implications on the conditions for mechanical prestress. Specifically, as was developed for the planar tensegrity in section 3.1, once a solution $\mathbf{p} > \mathbf{0}$ in the null space of $\mathbf{A}$ is found, the initial length of the jth cable that results from the given position and orientation of the tensegrity (which in fact determines components of the matrix $\mathbf{A}$) is given by

$$\ell_j^0 = \ell_j - \alpha k_j^{-1} p_j$$

where k_j is the spring constant of the jth cable, $p_j > 0$ is the jth component of $\mathbf{p}$, and $\alpha > 0$ is sufficiently small so as to guarantee that $0 < \ell_j^0 < \ell_j$ for all $1 \leq j \leq S$ where S is the number of cables. Once $\alpha > 0$ is selected, the corresponding magnitude α_j of the tension in the jth cable is given by

$$\alpha_j = \alpha p_j$$

If the null space of $\mathbf{A}$ has dimension 1, then all tensions can only be scaled in proportion by adjusting α. However, if the null space of $\mathbf{A}$ has dimension greater than 1, and two solutions $\{\mathbf{p}_1, \mathbf{p}_2\}$ are *independent*, then other tension patterns are possible, and can be selected to satisfy other properties. For example, one may be interested in choosing the solution $\mathbf{p} > \mathbf{0}$ such that the stiffness characteristics of the structure in a particular direction are maximized subject to a given geometry and given cable spring constants. However, this problem is not for consideration in this paper.

3.4 Numerical Solution of Prestress Parameters

Necessary and sufficient conditions for an N bar truss to be a tensegrity structure are equivalent to the existence of a vector $\mathbf{p} > \mathbf{0}$ such that

$$\mathbf{A}(\mathbf{q}_1, \mathbf{q}_2, \ \ldots, \ \mathbf{q}_{2N})\mathbf{p} = \mathbf{0} \tag{3.45}$$

for (some) column matrix $\mathbf{A}$. As explicitly indicated in (3.45), the matrix $\mathbf{A}$ is a nonlinear function of the coordinates $\{\mathbf{q}_k^T = [x_k\ y_k\ z_k]\ ;\ 1 \leq k \leq 2N\}$ which define the location of the $2N$ nodes which are the ends of the bars. As was adopted in Adhikari [8], a solution of (3.45) can be found using an iterative Newton-Raphson based approach.

Besides the coordinates of the ends of the bars, the key parameters which define a valid $(N, S; P_1, P_2, \ \ldots \ , P_M)$ prestressed tensegrity are the (positive) magnitudes of the cable tensions, and the (positive) initial cable lengths which must be all less than the prestressed lengths calculated from the coordinates of the ends of the bars. In particular, a valid tensegrity structure will not result if the initial cable lengths are too large. For example, if the bar lengths L_{12}, L_{34} of the $(2, 4; 2)$ tensegrity illustrated in Fig.9a are given by $L_{12} = L_{34} = 1$, then there will be no prestressed mechanical equilibrium condition if all the initial cable lengths are (say) of length 10.

Therefore in determining a valid structure, the initial cable lengths must be selected "sufficiently small". However whether or not the selection is "small enough" can only be determined once the iterative equations have converged, and all the prestressed cable lengths determined. Each (valid) selection of initial cable lengths results in a different set of coordinates for the end points of the bars, and so defines one member of the particular equivalence class of tensegrity structures.

One approach for finding one element in an equivalence class is to carry out the numerical computation by first setting all the initial cable lengths which connect point p to point q to zero. Once mechanical equilibrium is established, let the vector $\mathbf{l}_{0,\infty}$ define the lengths of the corresponding prestressed cable lengths. Then a *valid* (i.e nonzero) vector $\mathbf{l}^0$ of initial cable lengths is given by

$$\mathbf{l}^0 = (1 - \alpha)\mathbf{l}_{0,\infty}$$

for any $0 < \alpha < 1$.

In order to see this result, observe that since the initial cable lengths are all zero, $\mathbf{p}_0 = \mathbf{K}\mathbf{l}_{0,\infty}$ belongs to the null space of the corresponding matrix $\mathbf{A}_0$ whose components are determined by the coordinates of the bar ends. Then $\mathbf{A}_0\mathbf{K}\mathbf{l}_{0,\infty} = \mathbf{0}$ implies

$$\mathbf{A}_0\mathbf{K}\mathbf{l}^0 = \mathbf{A}_0\mathbf{K}\mathbf{l}_{0,\infty}$$

Other solutions $\mathbf{p}$ in the null space of $\mathbf{A}_0$ which are *independent* of $\mathbf{p}_0$ can also be found by direct numerical analysis of the null space of $\mathbf{A}_0$.

Results of computation which give the state of prestress mechanical equilibrium for a number of tensegrity structures may be viewed using a VRML viewer from the web address: `http://anusf.anu.edu.au/Vizlab/viz_showcase/williamson_darrell/`.
The assistance of Drew Whitehouse of the Australian National University Supercomputer Facility in writing the code and creating the visualization is gratefully acknowledged.

4 Conclusions

This paper has provided some preliminary analysis for a category of tensegrity systems which we have classified as $(N, S; P_1, P_2, \ldots, P_M)$. This category is characterized by N compressive elements (called bars), S tensile elements (called cables) and M stages with P_k bars per stage. The bars form a discontinuous network in compression, while the cables form a continuous network in tension. The "Needle Tower" sculpture of Snelson (which has inspired the last 50 years of interest in tensegrity systems) is of the type $(MP, 4MP; P, P, \ldots, P)$ structure with M stages.

This category $(N, S; P_1, P_2, \ldots, P_M)$ (for $N \geq 3$) was defined topologically as a 3-dimensional closure of a 2-dimensional lattice configuration, and the existence as a valid tensegrity system was inferred by the construction procedure. In effect, the mathematical existence of a valid $(N, S; P_1, P_2, \ldots, P_M)$ tensegrity system was shown to be equivalent to solving a non-square system of algebraic equations of the form $\mathbf{Ap} = \mathbf{0}$ for some $\mathbf{p} > \mathbf{0}$ in which the matrix $\mathbf{A}$ is a nonlinear function of the coordinates of the ends of the bars.

The category $(N, S; P_1, P_2, \ldots, P_M)$ included both symmetrical and non-symmetrical structures, but the permitted number of bars P_k in stage k can only either increase or decrease by one with respect to the number of bars P_{k-1} in the previous stage. Specific results quantified this constraint, and proved that the number of cables S that were required to establish the structure averaged somewhere between 3 and 4 per bar. The minimum number of 3 was required for a single stage structure, whereas the maximum number of 4 was required for the symmetrical case when the number of bars per stage was constant (and greater than 2).

Using the results of the paper, it is possible to determine all tensegrity structures having a given number N of bars. The cable connection matrix which provides all the required cable connections for such structures is also available. Each particular (valid) choice of $\{P_1, P_2 \ldots, P_M\}$ where $N = \sum P_k$ in fact defines an equivalence class of structures by a continuous change in the length of one or more bars and the continuous change in the initial length of one or more cables.

Even though numerical studies have not found a tensegrity system $(N, S; P_1, P_2, \ldots, P_M)$ that does *not exist*, the existence of an *analytical* solution was only established for the (planar) $(2, 4; 2)$ system, and the establishment of analytical solutions for higher order systems remains an open problem. The relevance of the dimension of the null space of $\mathbf{A}$ for a valid tensegrity is also of interest. Thus to date, a numerical solution of the system of equations $\mathbf{Ap} = \mathbf{0}$ for some $\mathbf{p} > \mathbf{0}$ for mechanical equilibrium is required.

Besides the problem of deriving the analytical form of the matrix $\mathbf{A}$, a direct solution to this problem requires an iterative adjustment of the coordinates of the ends of the bars which define the components of the matrix $\mathbf{A}$. Hence not only must $\mathbf{A}$ have a non-trivial null space, but in addition, there must be a vector in this null space which has all positive components. Since $\mathbf{A}$ itself does not necessarily have all positive components, this latter condition is difficult from both an analytical and a computational point of view,

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