Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems

Michael S. Branicky, *Member, IEEE*

***Abstract*—In this paper, we introduce some analysis tools for switched and hybrid systems. We first present work on stability analysis. We introduce *multiple Lyapunov functions* as a tool for analyzing Lyapunov stability and use *iterated function systems* (IFS) theory as a tool for Lagrange stability. We also discuss the case where the switched systems are indexed by an arbitrary compact set. Finally, we extend Bendixson's theorem to the case of Lipschitz continuous vector fields, allowing limit cycle analysis of a class of "continuous switched" systems.**

***Index Terms*—Hybrid systems, limit cycles, Lyapunov methods, nonlinear systems, stability, switched systems, variable-structure systems.**

I. Introduction

This paper develops some analysis tools applicable to the study of switched and hybrid systems. Below, we introduce the systems under study and summarize our contributions.

A. Switched Systems

We have in mind the following model as a prototypical example of a *switched system*

$$\dot{x}(t) = f_i(x(t)), \qquad i \in Q \simeq \{1, \cdots, N\} \tag{1}$$

where $x(t) \in R^n$. We add the following *switching rules*.

- Each f_i is globally Lipschitz continuous.
- The i's are picked in such a way that there are finite switches in finite time.

Switched systems are of "variable structure" or "multi-modal"; they are a simple model of (the continuous portion) of hybrid systems. The particular i at any given time may be chosen by some "higher process," such as a controller, computer, or human operator, in which case we say that the system is *controlled*. It may also be a function of time or state or both, in which case we say that the system is *autonomous*. In the latter case, we may really just arrive at a single (albeit complicated) nonlinear, time-varying equation. However, one might gain some leverage in the analysis of such systems by considering them to be amalgams of simpler systems.

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The author is with the Department of Electrical Engineering and Applied Physics, Case Western Reserve University, Cleveland, OH 44106-7221 USA (e-mail: branicky@alum.mit.edu).

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Models like (1) have been studied for stability [1]–[3]. However, those papers were predominantly concerned with the case where all the f_i are linear. Because of this, however, Peleties and his coworkers were able to derive switching laws based on generalized eigenvalue analysis of the constituent matrices. We discuss the general cases below.

We also discuss difference equations

$$x[k+1] = f_i(x[k+1]), \qquad i \in Q \simeq \{1, \cdots, N\} \tag{2}$$

where $x[k] \in R^n$. Here, we only add the assumption that each f_i is globally Lipschitz continuous. Again, these equations can be thought of as the "continuous" portion of the dynamics of hybrid systems combining difference equations and finite automata [4].

Finally, we also study *continuous* switched systems. A *continuous switched system* is a switching system with the additional constraint that the switched subsystems agree at the switching time. More specifically, consider (1) and suppose that at times t_j, $j = 1, 2, 3, \cdots$, there is a switch from $f_{k_{j-1}}$ to f_{k_j}. Then we require $f_{k_{j-1}}(x(t_j), t_j) = f_{k_j}(x(t_j), t_j)$. That is, we require that the vector field is continuous over time.

Throughout, R, R^+, Z, Z^+ denote the reals, nonnegative reals, integers, and nonnegative integers, respectively.

B. Hybrid Systems

Hybrid systems are those that inherently combine logical and continuous processes, usually coupled finite automata and differential equations [4]–[10]. Thus, the continuous dynamics is modeled by a differential equation

$$\dot{x}(t) = \xi(t), \qquad t \geq 0 \tag{3}$$

where $x(t)$ is the *continuous component* of the state taking values in some subset of a Euclidean space. $\xi(t)$ is a (controlled) vector field that generally depends on $x(t)$ and the aforementioned "logical" or "finite" dynamics.

As mentioned above, we consider two categories of switched systems [6].

- *Autonomous switching*: Here the vector field $\xi(\cdot)$ changes discontinuously when the state $x(\cdot)$ hits certain "boundaries."
- *Controlled switching*: Here $\xi(\cdot)$ changes abruptly in response to a control command, possibly with an associated cost.

A (continuous-time) *autonomous hybrid system* may be defined as follows:

$$\begin{aligned}\dot{x}(t) &= f(x(t), q(t)) \\ q(t) &= \nu(x(t), q(t^-))\end{aligned} \tag{4}$$

where $x(t) \in R^n$, $q(t) \in Q \simeq \{1, \cdots, N\}$. Here, $f(\cdot, q) : R^n \to R^n$, $q \in Q$, each globally Lipschitz continuous, is the *continuous dynamics* of (4), and $\nu : R^n \times Q \to Q$ is the *finite dynamics* of (4). Here, the notation t^- indicates that the finite state is piecewise continuous from the right. Thus, starting at $[x_0, i]$, the continuous state trajectory $x(\cdot)$ evolves according to $\dot{x} = f(x, i)$. If $x(\cdot)$ hits some $(\nu(\cdot, i))^{-1}(j)$ at time t_1, then the state becomes $[x(t_1), j]$, from which the process continues.

Clearly, this is an instantiation of autonomous switching. Switchings that are a fixed function of time may be taken care of by adding another state dimension, as usual. This definition is closely related to the so-called differential automata in [10]; it is a simplified view of the hybrid systems models in [4]–[6], and [9]. We do not discuss here restrictions on ν which lead to finite switches in finite time. For a discussion of this, see [11], [6], [4], and [10].

By a (continuous-time) *controlled hybrid system* we have in mind a system of the form

$$\begin{aligned}\dot{x}(t) &= f(x(t), q(t), u(t)) \\ q(t) &= \nu(x(t), q(t^-), u(t))\end{aligned} \tag{5}$$

where everything is as above except that $u(t) \in R^m$, with f and ν modified appropriately.

Next we give an example in which we have suppressed the finite dynamics.

Example 1.1: A simplified model of a manual transmission is given by [4]

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= [-a(x_2) + u]/(1 + v)\end{aligned}$$

where x_1 is the ground speed, x_2 is the engine RPM, $u \in [0, 1]$ is the throttle position, and $v \in \{1, 2, 3, 4\}$ is the gear shift position. The function a is positive for positive argument.

Likewise, we can define discrete-time autonomous and controlled hybrid systems by replacing the ODE's above with difference equations. In this case, (4) represents a simplified view of some of the models in [4].

C. Continuation Program for Hybrid Systems

The additional constraint of the continuous switched system introduced above leads to a simpler class of systems to consider. At the same time, it is not overly restrictive since many switching systems naturally satisfy this constraint. Indeed they may even arise from the discontinuous logic present in hybrid systems. These can result in discontinuous control inputs, which, after passing through a dynamical system (e.g., actuator dynamics), yield switching controls at the plant level that preserves continuity in the derivative.

More generally, we may approximate the finite dynamics of a hybrid system by considering singular perturbations that are "continuations" of them

$$\begin{aligned}\dot{x} &= f(x, z) \\ \epsilon\dot{z} &= [\nu(x, z) - z].\end{aligned}$$

Here, we have shown a linear interpolation of the dynamics, although others are possible, including appending states so that the continuation of the discrete transitions are well defined [12], [13]. If one can analyze such a system, then one can conclude properties of the resulting hybrid system via comparison—and vice versa!

This "continuation" idea for analyzing hybrid systems is originally due to Branicky and Mitter [14]. More discussion of this program, comparison tools in support of it, and a detailed example yielding novel analysis results may be found in [15] and [14].

D. Paper Organization

Section II introduces multiple Lyapunov functions (MLF's) as a tool for analyzing Lyapunov stability of switched systems. In Section III iterative function systems are presented as a tool for proving Lagrange stability and positive invariance. We also address the case where $\{1, \cdots, N\}$ in (1) and (2) is replaced by an arbitrary compact set. In Section IV, we extend Bendixson's theorem to the case of Lipschitz continuous vector fields. This gives us a tool for analyzing the existence of limit cycles of continuous switched systems. We conclude with some discussion.

II. Stability via Multiple Lyapunov Functions

In this section, we discuss Lyapunov stability of switched and hybrid systems via MLF's. The idea here is that even if we have Lyapunov functions for each system f_i individually, we need to impose restrictions on switching to guarantee stability. Indeed, it is easy to construct examples of two globally exponentially stable systems and a switching scheme that sends all trajectories to infinity.[1]

Example 2.1: Consider $f_i(x) = A_i x$ where

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}.$$

Then $\dot{x} = f_i(x)$ is globally exponentially stable for $i = 1, 2$. But the switched system using f_1 in the second and fourth quadrants and f_2 in the first and third quadrants is unstable. See Figs. 1–3, which plot 1 s of trajectories for f_1, f_2, and the switched system starting from $(1, 0)$, $(0, 1)$, and $(10^{-6}, 0)$, respectively.

We assume the reader is familiar with basic Lyapunov theory in continuous and discrete time [18]. We let $S(r)$, $B(r)$, and $\bar{B}(r)$ represent the sphere, ball, and closed ball of Euclidean radius r about the origin in R^n, respectively.

Below, we deal with systems that switch among vector fields (respectively, difference equations) over time or regions of state space. One can associate with such a system the following

[1] Such examples appear to be "classical." Similar examples showing stable systems constructed from unstable ones appear in [16]; Åström had qualitatively similar ones in [17].

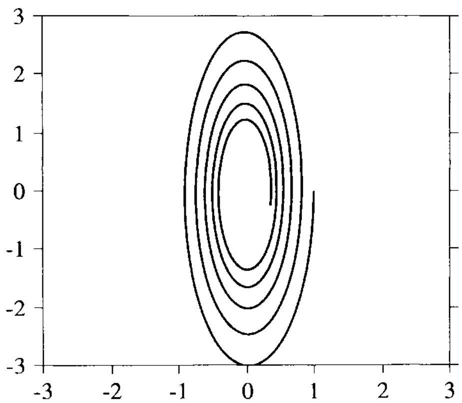

Fig. 1. Trajectory of f_1.

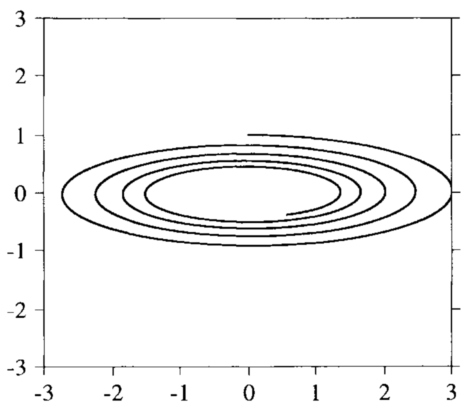

Fig. 2. Trajectory of f_2.

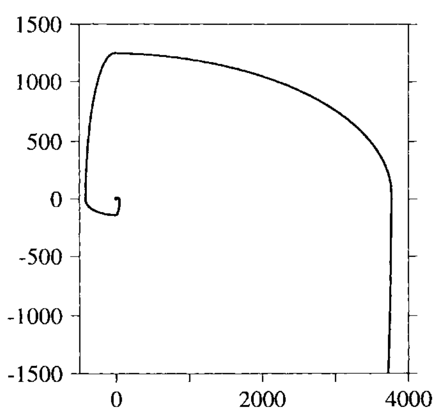

Fig. 3. Trajectory of switched system.

(*anchored*) *switching sequence*, indexed by an initial state, x_0:

$$S = x_0; \quad (i_0, t_0), (i_1, t_1), \cdots, (i_N, t_N), \cdots. \tag{6}$$

The sequence may or may not be infinite. In the finite case, we may take $t_{N+1} = \infty$, with all further definitions and results holding. However, we present in the sequel only in the infinite case to ease notation. The switching sequence, along with (1), completely describes the trajectory of the system according to the following rule: (i_k, t_k) means that the system evolves according to $\dot{x}(t) = f_{i_k}(x(t), t)$ for $t_k \leq t < t_{k+1}$. We denote this trajectory by $x_S(\cdot)$. Throughout, we assume that the switching sequence is *minimal* in the sense that $i_j \neq i_{j+1}$, $j \in Z_+$.

We can take projections of this sequence onto its first and second coordinates, yielding the sequence of indexes

$$\pi_1(S) = x_0; \quad i_0, i_1, \cdots, i_N, \cdots$$

and the sequence of switching times

$$\pi_2(S) = x_0; \quad t_0, t_1, \cdots, t_N, \cdots$$

respectively. Suppose S is a switching sequence as in (6). We denote by $S \mid i$ the endpoints of the times that system i is active in both the continuous- and discrete-time cases. The *interval completion* $\mathcal{I}(T)$ of a strictly increasing sequence of times $T = t_0, t_1, \cdots, t_N, \cdots$ is the set

$$\bigcup_{j \in Z_+} [t_{2j}, t_{2j+1}].$$

Hence, $\mathcal{I}(S \mid i)$ is the set of times that the ith system is active (up to a set of measure zero in the continuous-time case). Finally, let $\mathcal{E}(T)$ denote the *even sequence* of $T : t_0, t_2, t_4, \cdots$.

Below, we say that V is a *candidate Lyapunov function* if V is a continuous positive definite function (about the origin, 0) with continuous partial derivatives. Note this assumes $V(0) = 0$. We also use the following.

Definition 2.2: Given a strictly increasing sequence of times T in R (respectively, Z), we say that V is *Lyapunov-like* for function f and trajectory $x(\cdot)$ (respectively, $x[\cdot]$) over T if:

- $\dot{V}(x(t)) \leq 0$ (respectively, $V(x[t+1]) \leq V(x[t])$) for all $t \in \mathcal{I}(T)$;
- V is monotonically nonincreasing on $\mathcal{E}(T)$.

Theorem 2.3: Suppose we have candidate Lyapunov functions V_i, $i = 1, \cdots, N$ and vector fields $\dot{x} = f_i(x)$ (respectively, difference equations $x[k+1] = f_i(x[k])$) with $f_i(0) = 0$ for all i. Let $\mathcal{S}$ be the set of all switching sequences associated with the system.

If for each $S \in \mathcal{S}$ we have that for all i, V_i is Lyapunov-like for f_i and $x_S(\cdot)$ over $S \mid i$, then the system is stable in the sense of Lyapunov.

Proof: In each case, we do the proofs only for $N = 2$.

- *Continuous-time*: Let $R > 0$ be arbitrary. Let $m_i(\alpha)$ denote the minimum value of V_i on $S(\alpha)$. Pick $r_i < R$ such that in $B(r_i)$ we have $V_i < m_i(R)$. This choice is possible via the continuity of V_i. Let $r = \min(r_i)$. With this choice, if we start in $B(r)$, either vector field alone will stay within $B(R)$.

  Now, pick $\rho_i < r$ such that in $B(\rho_i)$ we have $V_i < m_i(r)$. Set $\rho = \min(\rho_i)$. Thus, if we start in $B(\rho)$, either vector field alone will stay in $B(r)$. Therefore, whenever the other is first switched on we have $V_i(x(t_1)) < m_i(R)$, so that we will stay within $B(R)$.
- *Discrete-time*: Let $R > 0$ be arbitrary. Let $m_i(\alpha, \beta)$ denote the minimum value of V_i on the closed annulus $\bar{B}(\beta) - B(\alpha)$. Pick $R_0 < R$ so that none of the f_i can jump out of $B(R)$ in one step. Pick $r_i < R_0$ such that in $B(r_i)$ we have $V_i < m_i(R_0, R)$. This choice is possible via the continuity of V_i. Let $r = \min(r_i)$. With this choice, if we start in $B(r)$, either equation alone will stay within $B(R)$.

Pick $r_0 < r$ so that none of the f_i can jump out of $B(r)$ in one step. Now, pick $\rho_i < r_0$ such that in $B(\rho_i)$ we have $V_i < m_i(r_0, r)$. Set $\rho = \min(\rho_i)$. Thus, if we start in $B(\rho)$, either equation alone will stay in $B(r_0)$, and hence $B(r)$. Therefore, whenever the other is first switched on we have $V_i(x(t_1)) < m_i(R_0, R)$ so that we will stay within $B(R_0)$ and hence $B(R)$.

The proofs for general N require N sets of concentric circles constructed as the two were in each case above. □

Some remarks are in order.

- The case $N = 1$ is the usual theorem for Lyapunov stability [18]. Also, compare Figs. 4 and 5, both of which depict the continuous-time case. Fig. 6 depicts MLF's over time.
- The theorem also holds if the f_i are time-varying.
- It is easy to see that the theorem does not hold if $N = \infty$, and we leave it to the reader to construct examples.
- It is not hard to generalize our MLF theory to the case of different equilibria, which is generally the case in hybrid systems. For example, under a Lyapunov-like switching rule, after all controllers have been switched in at level α_i, the set $\bigcup_i V_i^{-1}(\alpha_i)$ is invariant.
- It is not hard to extend the presented theorems to consider variations such as: 1) relaxing *the first part* of our Lyapnov-like definition by allowing the Lyapunov functions to increase in each active region if the gain from "switch in" to "switch out" is given by a positive definite function [19], or, more provocatively and 2) allowing increases over energy V_i at its switching times with a similar constraint from initial value to final limit.
- After we proved the theorem above, we became aware of the related work in [20]. There, Pavlidis concludes stability of differential equations containing impulses by introducing a positive definite function which decreases during the occurrence of an impulse and remains constant or decreases during the "free motion" of the system. Hence, it is a special case of our results.
- The stabilization strategies proposed in [21], e.g., choosing at each time the minimum of several Lyapunov functions, clearly satisfies our switching condition.
- Sliding modes may be taken care of by defining each sliding mode and its associated sliding dynamics as an additional system to which we can switch. We then merely check the conditions as before.
- In proving stability, we can use more Lyapunov functions than constituent systems (see [22] for an example where this is necessary) by simply introducing new discrete substates with the same continuous dynamics but different Lyapunov functions.
- When the dynamics are piecewise affine, computational tests may be used to compute appropriate switching conditions [2] (eigenvalue analysis) or to find Lyapunov functions that prove stability [23] [linear matrix inequalities (LMI's)].

Example 2.4: Pick any line through the origin. Going back to Example 2.1 and choosing to use f_1 above the line and f_2 below it, the resulting system is globally asymptotically stable. The reason is that each system is strictly stable linear and hence diminishes $V_i = x^T P_i x$ for some $P_i > 0$. However, since switchings occur on a line through the origin, we are assured that on switches to system i, V_i is lower energy than when it was last switched out; see Fig. 7.

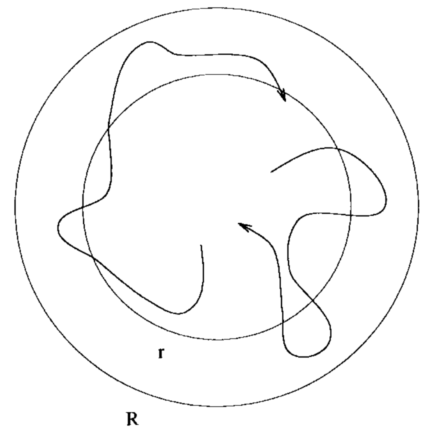

Fig. 4. Lyapunov stability.

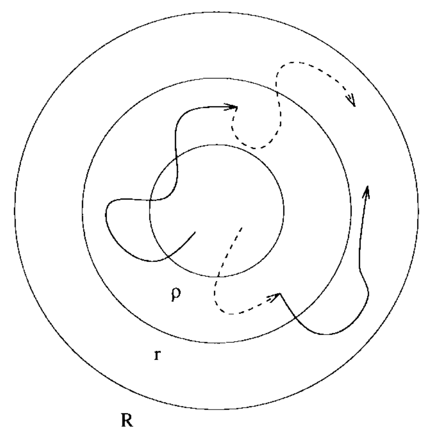

Fig. 5. Multiple Lyapunov stability, $N = 2$.

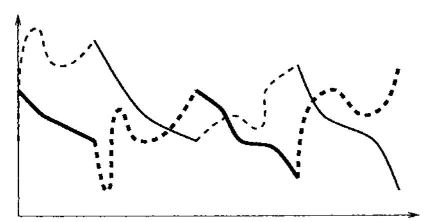

Fig. 6. Multiple Lyapunov stability, $N = 2$. Lyapunov function values versus time. Solid/dotted denotes corresponding system active/inactive.

Example 2.5: Consider the following system, inspired from one in [22]: $\dot{x}(t) = A_i(t)x(t)$ with

$$A_1 = \begin{bmatrix} -1 & -100 \\ 10 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 10 \\ -100 & 1 \end{bmatrix}$$

with the switching rule that we go from system i to j on hitting the sets $c_{i,j}^T x(t) = 0$ *in the second and fourth quadrants* where

$$c_{1,2} = [4, 3], \qquad c_{2,1} = [3, 4].$$

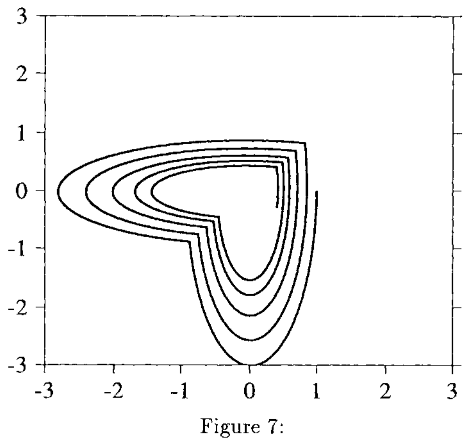

Fig. 7. Switching on a line through the origin.

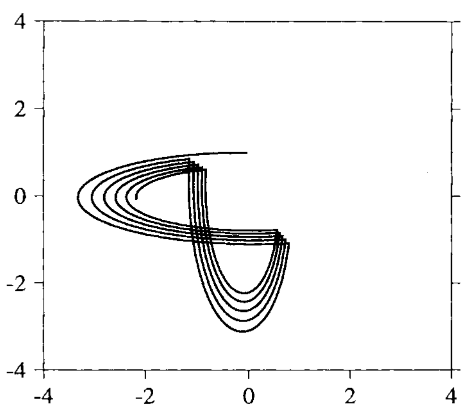

Fig. 8. A switching system requiring hybrid states and MLF's.

An example trajectory is shown in Fig. 8. There, the dynamics alternate between going counterclockwise along a short, fat ellipse and then clockwise along a tall, skinny one.

It is clear that the conic switching region: 1) is attractive; 2) leads to a hybrid system; and 3) admits no single quadratic Lyapunov function that can be used to show stability (e.g., the system intersects a whole range of lines through the origin at greater distances on the clockwise portion than the counterclockwise one).

However, it is also easy to see that energy is decreasing at switching times (just consider the switching lines through the origin and note we get closer). Lyapunov functions showing this may be computed using LMI's which encode the conditions of our theorem [23].

It is possible to use different conditions on the V_i to ensure stability. For instance, consider the following.

Definition 2.6: If there are candidate Lyapunov functions V_i corresponding to f_i for all i, we say they satisfy the *sequence nonincreasing condition* for a trajectory $x(\cdot)$ if

$$V_{i_{j+1}}(x(t_{j+1})) < V_{i_j}(x(t_j)).$$

This is a stronger notion than the Lyapunov-like condition used above.

The sequence nonincreasing condition is used in the stability (version of the asymptotic stability) theorem of [2]. Thus that theorem is a special case of the continuous-time version of Theorem 2.3 above. Moreover, the proof of asymptotic stability in [2] is flawed since it only proves state convergence and not state convergence plus stability, as required. It can be fixed using our theorem.

Now, consider the case where the index set is an arbitrary compact set

$$\dot{x} = f(x, \lambda), \quad \lambda \in K, \quad \text{compact}. \tag{7}$$

Here, $x \in R^n$ and f is globally Lipschitz in x, continuous in λ. For brevity, we only consider the continuous-time case. Again, we assume finite switches in finite time.

As above, we may define a switching sequence

$$S = x_0;\ (\lambda_0, t_0), (\lambda_1, t_1), \cdots, (\lambda_N, t_N), \cdots$$

with its associated projection sequences.

Theorem 2.7: Suppose we have candidate Lyapunov functions $V_\lambda \equiv V(\cdot, \lambda)$ and vector fields as in (7) with $f(0, \lambda) = 0$, for each $\lambda \in K$. Also, $V : R^n \times K \to R_+$ is continuous. Let $\mathcal{S}$ be the set of all switching sequences associated with the system.

If for each $S \in \mathcal{S}$ we have that for all λ, V_λ is Lyapunov-like for f_λ and $x_S(\cdot)$ over $S \mid \lambda$ and the V_λ satisfy the sequence nonincreasing condition for $x_S(\cdot)$, then the system is stable in the sense of Lyapunov.

Proof: We present the proof in the case that K is sequentially compact, which is automatic if K is a metric space. The general case follows with little change from the argument below by using countable compactness and nets instead of sequences (see [24] and [25] for definitions).

The Lyapunov-like and sequence nonincreasing constraints are such that if $\pi_1(S) = x_0;\ \lambda_0, \lambda_1, \lambda_2, \cdots$, then the state $x(t)$ will remain within the set

$$R_{V(x_0,\lambda_0)} \equiv \bigcup_{\lambda \in K} \{x \mid V(x, \lambda) < V(x_0, \lambda_0)\}.$$

Next, note that if x_0 lies in

$$I_\epsilon \equiv \left\{ x \,\middle|\, \sup_{\lambda \in K} V(x, \lambda) < \epsilon \right\}$$

then the state will remain in R_ϵ.

Thus, it remains to show that given any $\epsilon > 0$, there exist ϵ', $\delta > 0$ such that

$$B(\delta) \subset I_{\epsilon'} \cap B(\epsilon) \subset R_{\epsilon'} \cap B(\epsilon) \subset B(\epsilon).$$

Letting m denote the minimum of V on $S(\epsilon) \times K$, $\epsilon' = m/2$ satisfies the last equation. Now $I_{\epsilon'}$ contains the origin $0 \in R^n$. Suppose there is no open ball about zero in $I_{\epsilon'}$. Then for each $n \in Z_+$, there exists y_n such that

$$\|y_n\| \le 1/n, \quad \sup_{\lambda \in K} V(y_n, \lambda) \ge \epsilon'.$$

Further, we may take each of the y_n distinct. Let $\lambda_n \in K$ be the point at which the sup above is attained. Since $\bar{B}(\epsilon') \times K$ is sequentially compact, there is a subsequence $\{(y_{i_k}, \lambda_{i_k})\}$ converging to $(0, \lambda^*)$ with $V(y_{i_k}, \lambda_{i_k}) \ge \epsilon'$, a contradiction to the continuity of V and the assumption that $V(0, \lambda) = 0$ for all $\lambda \in K$. □

This theorem is a different generalization of the aforementioned theorems of [20] and [2].

III. Stability via Iterated Function Systems

In this section, we study iterated function systems (IFS) theory as a tool for Lagrange stability. We begin with some background from [26]–[28].

Definition 3.1: Recall that a *contractive function* f is one such that there exists $s < 1$ where $d(f(x), f(y)) \leq sd(x, y)$, for all x, y.

An IFS is a complete metric space and a set $\{f_i\}_{i \in I}$ of contractive functions such that I is a compact space and the map $(x, i) \mapsto f_i(x)$ is continuous.

The image of a compact set X under an IFS is the set $Y = \bigcup_{i \in I} f_i(X)$. It is compact. Now suppose W is an IFS. Let $S(W)$ be the semigroup generated by W under composition. For example, if $W = \{f, g\}$, then

$$S(W) = f, g, f \circ f, f \circ g, g \circ f, g \circ g, \cdots.$$

Now, define A_W to be the closure of the fixed points of $S(W)$. We have the following.

Theorem 3.2: Suppose $W = \{w_i\}_{i \in I}$ is an IFS on X. Then:

- A_W is compact;
- $A_W = \bigcup_{i \in I} w_i(A_W)$;
- for all $x \in X$

$$A_W = \bigcup_{\sigma} \left\{ \lim_{n \to \infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x) \right\}$$

where $\sigma = (\sigma_1, \sigma_2, \cdots)$, $\sigma_i \in I$.

The relevance of this theorem is twofold.

- A_W is an invariant set under the maps $\{w_i\}_{i \in I}$.
- All points approach A_W under iterated composition of the maps $\{w_i\}_{i \in I}$.

Clearly, this theory can be applied in the case of a set of contractive discrete maps indexed by a compact set (usually finite). Thus, it is directly applicable to systems of the form (2).

Example 3.3: The following IFS is well known:

$$F_i(x) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} x + \begin{bmatrix} 0.5 \cdot 1_{\{i>1\}} \\ 0.5 \cdot 1_{\{i<3\}} \end{bmatrix}, \qquad i = 1, 2, 3.$$

Its limit set is the well-known Sierpinski triangle, as shown in Fig. 9.

To obtain contractive maps while switching among differential equations requires a little thought. Assume there is some lower limit T on the interswitching time. Now, notice that for any interswitching time $r \geq T$, there is a decomposition into smaller intervals as follows:

$$r = \sum_{i=1}^{M} t_i, \quad t_i \in [T, 2T].$$

Proof: Let $k = \lfloor r/(2T) \rfloor$ and $q = r - 2Tk$. Now, $2T > q \geq 0$. If $q = 0$, the decomposition is $t_i = 2T$, $i = 1, \cdots, k$. If $2T > q \geq T$, the decomposition is $t_i = 2T$, $i = 1, \cdots, k$; $t_{k+1} = q$, the first equation not applying if $k = 0$. Finally, if $T > q > 0$, then (we must have $k \geq 1$ since $r \geq T$) $2T > q + T > T$, so the decomposition is $t_i = 2T$, $i = 1, \cdots, k-1$; $t_k = T$; $t_{k+1} = T + q$, the first equation not applying if $k = 1$. □

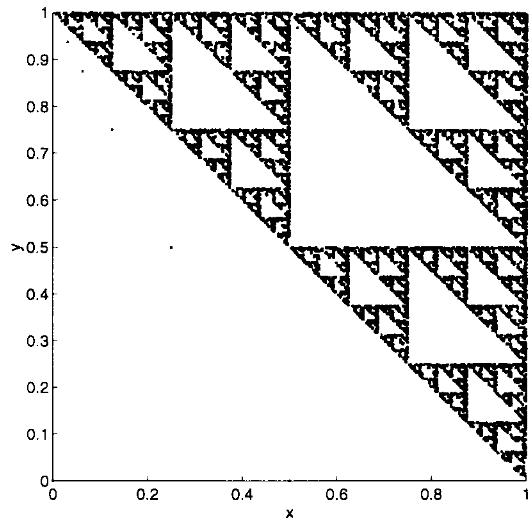

Fig. 9. Example IFS: Sierpinski triangle (10 000 points).

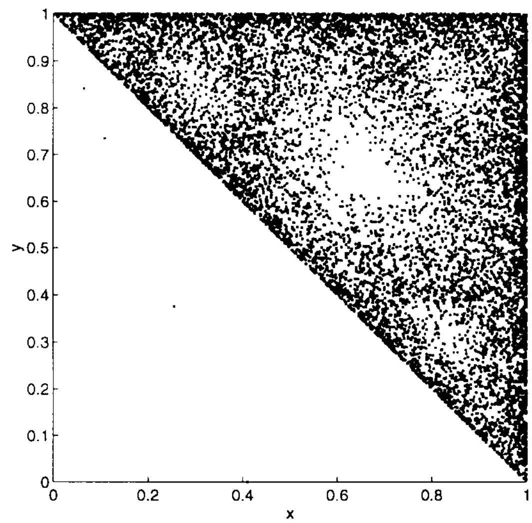

Fig. 10. Example differential IFS: Sierpinski-like triangle, $T = 2/3$ (10 000 points).

Therefore, we can convert switching among vector fields into an IFS by letting $I = \bigcup_{j=1,\cdots,N} j \times [T, 2T]$. In particular, we see that for each i, if it is active for a time $r \geq T$, we can write the solution in that interval as $\phi_r^i(x) = (\bigcirc_{j=1}^{M} \phi_{t_j}^i)(x)$, where ϕ_t^i is the fundamental solution for f_i acting for time t. Thus the switching sequence can be converted to an iterated composition of maps indexed by the compact set I.

The other interesting point about IFS theory is that the different vector fields (or difference equations) need not have the same equilibrium point. This is important as it appears to be the usual case in switched and hybrid systems.

Example 3.4: Starting with Example 3.3, we consider a corresponding *differential IFS* (DIFS), with

$$F_i(x) = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix} x + \begin{bmatrix} \alpha \cdot 1_{\{i>1\}} \\ \alpha \cdot 1_{\{i<3\}} \end{bmatrix}, \qquad i = 1, 2, 3.$$

Its limit set can be computed for different values of α and T. Fig. 10 depicts it for $\alpha = \ln 2$ and $T = 2/3$.

In conclusion, in IFS we have a tool for analyzing the Lagrange stability and computing the invariant sets of switched systems of the form (1) and (2). The resulting sets A_W are reminiscent of those for usual IFS (see [26]), as seen in comparing Figs. 9 and 10. The reader may consult [26] for algorithms to compute such invariant sets.

IV. Limit Cycle Analysis

Suppose we are interested in the existence of limit cycles of continuous switched systems in the plane. The traditional tool for such analysis is Bendixson's theorem. But under our model, systems typically admit vector fields that are Lipschitz, with no other smoothness assumptions. Bendixson's theorem, as it is traditionally stated (e.g., [29] and [30]), requires continuously differentiable vector fields and is thus not of use in general. Therefore, we offer an extension of Bendixson's theorem to the more general case of Lipschitz continuous vector fields. Its proof is based on results in geometric measure theory (which are discussed in the Appendix).

Theorem 4.1 (Extension of Bendixson's Theorem): Suppose D is a simply connected domain in R^2 and $f(x)$ is a Lipschitz continuous vector field on D such that the quantity $\nabla f(x)$ (the *divergence* of f, which exists almost everywhere) defined by

$$\nabla f(x) = \frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_2}(x_1, x_2)$$

is not zero almost everywhere over any subregion of D and is of the same sign almost everywhere in D. Then D contains no closed trajectories of

$$\dot{x}_1(t) = f_1[x_1(t), x_2(t)] \tag{8}$$

$$\dot{x}_2(t) = f_2[x_1(t), x_2(t)]. \tag{9}$$

Proof: See the Appendix. □

Finally, we give an example which shows the necessity of Lipschitz continuity of the vector fields.

Example 4.2: Consider Example 2.1. Note that if the roles of A_1 and A_2 are interchanged, then the resulting system is asymptotically stable. Thus, continuity of solutions and the intermediate value theorem imply that there exists $\lambda \in (0, 1)$ such that $f_1 = \lambda A_2 + (1 - \lambda)A_1$ and $f_2 = \lambda A_1 + (1 - \lambda)A_2$ results in a closed trajectory. Yet, $\nabla f_1 < 0$ and $\nabla f_2 < 0$.

V. Conclusion

In both the MLF and IFS cases, the stability results are sufficiency conditions on the continuous dynamics and switching. We do not consider this a drawback since Lyapunov theory is almost always used in its sufficiency form; also, the use of such conditions in *design* of provably stable control laws is an important area of future research (cf. the way Lyapunov theory is used to design stable adaptive control laws). This work, then, represents the rudiments of a stability theory of the systems in (1) and (2) and, in turn, of hybrid systems. We also discussed the case where $\{1, \cdots, N\}$ in (1) and (2) is replaced by an arbitrary compact set.

For future directions, we offer the following brief treatment. In searching for necessary and sufficient stability criteria, the theory in [32] appears helpful. As far as IFS, we have yet to explore their full potential. For instance, we can state IFS theorems analogous to Theorem 2.3, namely, in which the maps need only be contractions on the points (time periods) on which they are applied. Finally, if there is no lower limit T on the interswitching time, then we are not assured to have a contraction mapping. However, as long as we have only finite switches in finite time, one expects that the trajectories should be well-behaved (e.g., invoke continuity of ODE solutions and take convex hulls).

Appendix

The proof of Bendixson's theorem depends, in a critical way, on Green's theorem. The usual statement of Green's theorem says [31] that a C^1 vector field $f(x)$ on a compact region A in R^n with C^1 boundary B satisfies

$$\int_B f(x) \cdot \mathbf{n}(A, x)\, d\sigma = \int_A \nabla f(x)\, d\mathcal{L}^n x$$

where $\mathbf{n}(A, x)$ is the exterior unit normal to A at x, $d\sigma$ is the element of area on B, and $\mathcal{L}^n$ is the Lebesgue measure on R^n. It is possible, however, to treat more general regions and vector fields that are merely Lipschitz continuous. A general extension is the so-called Gauss–Green–Federer theorem given in [31]. Even the statement of this theorem requires the development of a bit of the language of geometric measure theory. We state a relaxed version of this theorem that is still suitable for our purposes. In the final formula, ∇f exists almost everywhere because a Lipschitz continuous function is differentiable almost everywhere.

Theorem A.1 (Relaxation of Gauss–Green–Federer): Let A be a compact region of R^n with C^1 boundary B. Then for any Lipschitz vector field $f(x)$

$$\int_B f(x) \cdot \mathbf{n}(A, x)\, d\sigma = \int_A \nabla f(x)\, d\mathcal{L}^n x.$$

Now we can prove our version of Bendixson's theorem.

Proof of Theorem 4.1: The proof is similar to that of Bendixson's theorem in [30, pp. 31–32]. Suppose, for contradiction, that J is a closed trajectory of (8) and (9). Then at each point $x \in J$, the vector field $f(x)$ is tangent to J. Then $f(x) \cdot \mathbf{n}(S, x) = 0$ for all $x \in J$, where S is the area enclosed by J. But by Theorem A.1

$$0 = \int_J f(x) \cdot \mathbf{n}(A, x)\, dl = \int_S \nabla f(x)\, d\mathcal{L}^2 x.$$

Therefore, we must have either: 1) $\nabla f(x)$ is zero almost everywhere, or 2) the sets $\{x \in S \mid \nabla f(x) < 0\}$ and $\{x \in S \mid \nabla f(x) > 0\}$ both have positive measure. But if S is a subset of D, neither can happen. Hence, D contains no closed trajectories of (8) and (9). □

Acknowledgment

The work in Sections II and III was begun in [33] and continued in [34]. In personal discussions, Prof. W. S. Newman essentially conjectured Theorem 2.3 in the continuous-time setting. Our extension of Bendixson's theorem was used to prove no limit cycles exist in a realistic aircraft control

problem [15]. Figs. 1–3, 7, and 8 were produced by the simulation tool Omsim, from Lund Institute of Technology, using hybrid systems macros written by the author; see [35] for details.

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Michael S. Branicky (S'92–M'95), for photograph and biography, see p. 45 of the January 1998 issue of this TRANSACTIONS.