Reconstructing Ellipsoids from Its Three Projections

Temel Kayikcioglu*, Mahmut Ozer**, Ali Gangal*
*Department of Electrical and Electronics Engineering, KTU, Trabzon, Turkey
**Electrical Department, GOU, Tokat, Turkey
Phone (90) 462-3253232/2033 E-Mail: kayikci@eedec.bim.ktu.edu.tr

Abstract - In this study, a method for reconstructing ellipsoids from its three projection contours is described. The line integral projection of ellipsoid with uniform density is developed for any projection view. Our reconstruction algorithm starts from the detecting projection contours that has elliptic shape. Then three sets of estimated ellipse parameters are used in reconstructing the ellipsoid. The method takes into account background, noise and blurring. The performance of the method is evaluated for various types of background structures, different ellipsoid dimensions and different levels of noise. Result demonstrates the validity and usefulness of the proposed method.

I. INTRODUCTION

In this paper, we will consider the reconstruction of three-dimensional ellipsoid from its three projection contours. Such a reconstruction problem has received considerable attention in recent years and arises in many fields of science such as medical imaging and computer vision. In three-dimensional reconstruction of coronary arteries from projections the coronary arteries may be represented and visualized efficiently by a generalized cylinder model with elliptical cross-sections [1-2]. In other medical area's shape or volume of some anatomical organs such as heart and spine are modeled as ellipsoids [3]. In computer vision, simple familiar patterns such as ellipses provide strong features for camera orientation [4].

In [5] a three dimensional algebraic reconstruction technique utilizing uniform background and using more than three ideal projection images was applied to three dimensional ellipsoid shells. Karl in [6] represented ellipsoids as elements of the vector of symmetric matrices and provided simulation results using more than three ideal projections. In our work, we developed a method for reconstructing an ellipsoid from only three projection contours. Furthermore it also takes into account background intensity, noise and blurring.

The paper is organized as follows. Section II provides the development of line projection model based on general parametric equation of ellipsoid. In section III, we introduce a method to detect edges in projection images. This section also describes estimation of ellipse parameters from the detected edge points. A method developed for reconstruction of an ellipsoid from its three projection contours is described in Section IV. The performance of reconstruction method is demonstrated by computer simulations for different noise levels, different types of background, and different ellipsoid sizes. Results are given in Section V. Finally some conclusions of the proposed method will be given in Section VI.

II. PROJECTION MODEL

A line projection of an object is an integration of the three-dimensional density function of the object into a two-dimensional density function along the lines parallel to projection view. The model of imaging process is shown in Fig 1. The axis $u$ passes through origin and represents view direction. The view direction is given by angle pair $(\phi, \theta)$ where $\phi$ is the angle between the axis $z$ and $u$ and varies between $0^\circ$ and $90^\circ$. $\theta$ is the angle between the axis $x$ and the line formed by projection of the axis $u$ on the $x$-$y$ plane and varies between $0^\circ$ and $360^\circ$. The center of the image plane is on the $u$ axis. The image plane is aligned such that the horizontal axis $p$ is parallel to the axes $x$ and $u$. We wish to obtain line integral projection of a constant density ellipsoid onto the image plane.

Given a constant-density ellipsoid centered at, for the moment, $(0,0,0)$ in the 3-D Cartesian coordinates, $d(x,y,z)$, the density function of an ellipsoid, can be defined as

$$d(x,y,z) = \begin{cases} 
\mu, & \text{if } A x^2 + B y^2 + C z^2 + D x y + E x z + F y z \leq 1 \\
0, & \text{otherwise} 
\end{cases}$$

where $A \geq 0$, $B \geq 0$, $C \geq 0$, $D^2 - 4 A B \leq 0$, $E^2 - 4 A C \leq 0$, $F^2 - 4 B C \leq 0$ and $\mu$ is contrast coefficient. After introducing the new coordinate system as shown in Fig. 1, the line integral along $u$, $f(p,n,\phi,\theta)$, is then given by

$$f(p,n,\phi,\theta) = \int d(Sin(\phi)Cosp - Cos(\phi)Sinp) - Cos(\phi)Cosp, Sin(\phi)Sinp - Cos(\phi)Cosp, Cosp + Sinp)du$$

Therefore, the line projection of the ellipsoid with contrast satisfies the following condition

![Image of projection geometry](https://example.com/projection_image.png)

Fig. 1. Projection geometry
The equation of the projection contour, which is essential in 3D reconstruction, is as follows

$$a(p-p_c)^2 + b(p-p_c)(\eta-\eta_c) + c(\eta-\eta_c)^2 = 1$$  \hspace{1cm} (14)$$

The components involved in the imaging chain, as well as the structure of imaging geometry, two distortion sources, i.e., blurring and noise, degrade the image. We model the blurring effect as a two-dimensional convolution of the ideal projection image profile with a two-dimensional Gaussian point spread function with standard deviations \(\sigma_p\) and \(\sigma_n\) in horizontal and vertical directions, respectively. Noise is modeled as zero mean white Gaussian noise \(w(p,n)\) with standard deviation \(\sigma_w\) and blurring caused by the background structure is modeled by a two-dimensional polynomial of the third order and represented by \(b(p,n)\). The observed intensity distribution of an ellipsoid for a particular viewing angle \((\phi, \theta)\) is then given by following expression:

$$i(p, \eta) = f(p, \eta, \phi, \theta) * * g(p, \eta) + b(p, \eta) + w(p, n)$$  \hspace{1cm} (15)$$

where ** (double asterisk) denotes two-dimensional convolution. The projection contours, in other words, ellipse parameters are obtained from this model.

**II. DETECTION OF PROJECTION CONTOURS**

Now we present a method to detect contours in projection images. Since projection contours have elliptic shape, the problem converted to estimation of ellipse parameters. Estimation of ellipse parameters requires determination of five parameters. Three of these, namely \(a, b, \) and \(c\), are associated with its shape while other two, namely \(p_c\) and \(n_c\), are associated with its center. The proposed scheme for estimating these parameters consists of three stages. In the first stage, the edge points in the projection image are estimated. In the second stage, the location of the center of ellipsoid is computed by averaging estimated edge pairs. Finally, we use a nonlinear least squares estimation algorithm for computing shape parameters \(a, b, \) and \(c\), from translated edge points.

Many methods exist for detecting edges in a given image. Some of them are general in the sense that they are developed independent of application context while others are developed for particular applications, and utilize a priori knowledge. In this present study, we have used the latter due to the fact that it yields much more accurate edge detection for model based applications. In detecting edges two-dimensional model in (15) is considered a one-dimensional model with variable \(p\) for each \(n\) and vice-versa and a nonlinear estimation algorithm to detect the edges is applied to one-dimensional projection profile. Thus, computations in detecting edges simplify considerably when one-dimensional model is used. Details of the proposed method are as follows. For sake of generality, let \(x\) represent variable \(p\) or \(n\). The projection profile along the \(x\) axis in the image, due to the fact that any cross-section of an ellipsoid is always an ellipse, is given by [1]

$$f(x) = \begin{cases} \mu \sqrt{\alpha_x^2 - (x-c_x)^2} & \text{for } c_x - \alpha_x \leq x \leq c_x + \alpha_x \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (16)$$

where \(\alpha_x\) is the semi-axis of an ellipse, \(c_x\) is the projection of the center of ellipsoid on \(x\) axis, and \(\mu\) is the maximum observed intensity.
where $c_x$ is the location of the midpoint of the line lying in the boundaries of the ellipse. The quantities $c_x, c_y, c_z$ correspond to the left and right edge points in the case of the horizontal lines and the upper and lower edge points in the case of the vertical lines, respectively. Consequently, the observed intensity distribution along the axis $x$, based on two-dimensional observed intensity distribution (15), may be expressed as follows:

$$i(x) = f(x)^* g(x) + b(x) + w(x)$$  

where the asterisk denotes the convolution, $f(x)$ denotes the projection profile along the axis $x$, $g(x)$ denotes one-dimensional Gaussian point spread function with standard deviation $\sigma_x$, $b(x)$ denotes the background along the axis $x$ and represented by a polynomial of the third order, associated with parameters $b_0$ through $b_3$, and $w(x)$ denotes the zero mean white Gaussian noise with standard deviation $\sigma_w$. To estimate the parameters $\mu_x, \sigma_x, a_x, b_y, b_x, b_z$, the Marquardt-Levenberg nonlinear least-squares estimation technique is applied to fit the model $i(x)$ to observed projection profile. Details of implementation of this algorithm are beyond the scope of this paper[1]. To detect edge points in the horizontal lines the algorithm is applied to all horizontal lines within the projection contour. The same procedure is repeated for all vertical profiles. After having detected edge points, next step is to estimate the ellipse parameters from these. The center coordinates of the ellipse ($p_x, p_y, p_z$) is computed by averaging edge points, say $(p_i, n_i)$. In order to estimate the shape parameters $a, b$, and $c$, the set of translated edge points ($p_i - p_x, n_i - p_y$) in the Cartesian coordinates is first represented by a set of $(r_i, \alpha_i)$ in the spherical coordinates by using following equations

$$p_i - p_x = r_i \cos \alpha_i \quad \text{and} \quad n_i - p_y = r_i \sin \alpha_i$$  

In the spherical coordinates equation (14) can be written as

$$r(\alpha) = \left(a \cos^2 \alpha + b \sin \alpha \cos \alpha + c \sin^2 \alpha \right)^{0.5}$$  

To estimate the parameters $a, b$ and $c$, again the Marquardt-Levenberg nonlinear estimation algorithm is used to fit the above equation into the set of the translated edge points.

**IV. RECONSTRUCTION METHOD**

In the previous section we described a method for detecting the projection contour. Now we present the method of reconstructing an ellipsoid from contours in three projection images. An ellipsoid completely determined by nine parameters. Three of them gives the location of the center in the three-dimensional space. The remaining, namely $A, B, C, D, E$ and $F$, are associated with its shape in the three-dimensional space. Since geometrical relationship among three image planes is known, the location of the center point can be obtained from back projections of the estimated ellipse centers. We propose the following method to estimate shape parameters of the ellipsoid, $A, B, C, D, E$ and $F$ from three set of elliptical parameters $a, b,$ and $c$.

Equations (4) through (6) give the relationships between shape parameters of an ellipsoid and shape parameters of its projection contour for each view. Let subscript $i$ indicate the $i$-th projection view. After a slight manipulation they can be rewritten in following forms

$$a_i \cdot s_i - (s_i^3 - s_i^2) = 0$$  

$$b_i \cdot s_i - 2(s_i^3 - s_i^2) = 0$$  

$$c_i \cdot s_i - (s_i^3 - s_i^2) = 0$$

where $i=1,2,3$. We define an objective function as the unweighted sum of squares of the left sides of the equations given above

$$h(A,B,C,D,E,F) = \sum_{i=1}^{3} \left[ b_i \cdot s_i - (s_i^3 - s_i^2) \right] + [b_i \cdot s_i - (s_i^3 - s_i^2)]$$

Computing shape parameters can therefore be reduced to minimization of the above objective function with respect to the shape parameters. Without any error, $h$ would be zero. Close examination of equations (4) through (6) reveals that function is also zero when all shape parameters simultaneously equal to zero. In the estimation algorithm, this problem can be avoided by selecting good initial estimates. Many techniques are available to compute the minimum of this function respect to the shape parameters. A quasi-Newton method is the most powerful one for minimization of such a multivariate function

**V. SIMULATION STUDIES**

Precise reconstruction of an ellipsoid from its three projection contours requires accurate detection of projection contours. However, errors in the detection process occur due to the several factors, such as non-uniformity of the background, noise introduced by the imaging components and the size of the ellipsoid. Consequently, these factors yield erroneous ellipsoid parameters. In this section effects of these factors on the ellipsoidal parameters are examined.

In the first experiment, we examine the effects of different levels of noise in estimating the ellipsoid parameters. The projection views are selected as $(0°, 90°)$, $(90°, 0°)$ and $(20°, 40°)$. All quantities are fixed except the level of noise. The shape parameters of the ellipsoid are fixed at 1,2,1,2,1 and 1, respectively. It is centered at the origin. The contrast coefficient $\mu$ is selected as 2 for all three projection views. Each ideal projection image is convoluted with a two-dimensional Gaussian point spread function with standard deviations $\sigma_p=\sigma_n=1$ pixel. The resulting images of each view are superimposed on three different real background images with standard deviation of approximately 6 gray levels. After adding white
Gaussian noise at different levels, the resulting images are digitized into 60x60x8 bit image arrays. The SNR’s calculated for $\sigma_w=1$ are 24.75 dB, 24.71 dB and 23.72 dB for the projection images of views at (0°,90°), (90°,0°) and (20°,40°), respectively. For $\sigma_w=2$ the SNR’s are 18.73 dB, 18.68 dB and 17.69 dB for the corresponding projection images, respectively. The SNR was computed as 10 log (mean square of projection without background / variance of noise). For each view 50 images are generated. Typical projection images and detected edge points are shown in Fig. 2. The averages and standard deviations of shape parameters of the ellipsoid and of the distance between the center point of the estimated and actual ellipses are shown in Table I. The table reveals that the reconstruction algorithm results in, due to non uniformity of background structures, slightly biased parameter estimates.

![Fig. 2. First row: typical noisy projection images for a) (0°,90°), b) (90°,0°) and c) (20°,40°) views. Second row: corresponding estimated edge points.](image)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Noise level, $\sigma_w$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>0.97</td>
<td>0.98±0.016</td>
<td>0.98±0.023</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1.96</td>
<td>1.96±0.016</td>
<td>1.96±0.022</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0.99</td>
<td>0.99±0.012</td>
<td>0.99±0.018</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1.89</td>
<td>1.90±0.033</td>
<td>1.90±0.046</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>0.95</td>
<td>0.96±0.033</td>
<td>0.96±0.046</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>0.92</td>
<td>0.92±0.032</td>
<td>0.92±0.046</td>
<td></td>
</tr>
<tr>
<td>Distance(pixel)</td>
<td>view (0°,90°)</td>
<td>0.25</td>
<td>0.19±0.12</td>
<td>0.22±0.13</td>
</tr>
<tr>
<td>view(90°,0°)</td>
<td>0.44</td>
<td>0.43±0.008</td>
<td>0.42±0.04</td>
<td></td>
</tr>
<tr>
<td>view(20°,40°)</td>
<td>0.50</td>
<td>0.44±0.11</td>
<td>0.38±0.13</td>
<td></td>
</tr>
</tbody>
</table>

In the second experiment, the effects of the different background on the ellipsoidal parameters are examined. For this simulation the same data employed in the previous experiment are used. Different background images for each projection are used. The standard deviations of the selected background images are roughly 6 gray levels. For each view and set of background structures 50 projection images are generated. The results show that the average parameter estimates are not sensitive to the shape of the backgrounds. However, the standard deviations for the parameter estimates are slightly sensitive to the background structures.

Finally, our objective is to examine the effects of the size of the ellipsoid on estimation of ellipsoidal parameters. For this experiment, the size of the ellipsoid used in the first experiment are changed by using a scaling constant. Fifty projection images are generated for each size. It has been observed that the expanded ellipsoid gives better results compared to those of the compressed ellipsoid.

VI. CONCLUSION

In this paper, we have proposed a method to reconstruct ellipsoids from its three projection contours which have elliptic shape. The method mainly consists of estimation of the parameters of the ellipsoid, and then computation of the ellipsoid parameters from the set of parameters associated with its three projection contours.

The performance of the method has been evaluated for different levels of noise, different ellipsoid dimensions and various background structures. Simulation results indicate that satisfactory detection of projection contours is very crucial for the success of the reconstruction method.

We have applied the proposed reconstruction method to line projected images of ellipsoid. However, the method is not restricted to that kind of projection models. The scope of the reconstruction method extends to most imaging disciplines such as orthogonal silhouette projections.

REFERENCES


