

SIGN-BALANCED POSETS

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ABSTRACT. Let P be a finite partially ordered set with a fixed labeling. The *sign of a linear extension* of P is its sign when viewed as a permutation of the labels of the elements of P . Call P *sign-balanced* if the number of linear extensions of P of positive sign is the same as the number of linear extensions of P of negative sign. In this paper we determine when the posets in a particular class are sign-balanced. When posets in this class are not sign-balanced, we determine the difference between the number of positive linear extensions and the number of negative linear extensions. One special case of this class is the product of an m -chain with an n -chain, m and n both > 1 . In this case, we show P is sign-balanced if and only if $m = n \pmod{2}$.

1. INTRODUCTION

Let P be a partially ordered set (or *poset*) and fix a labeling of the elements of P . Then consider all the linear extensions of P as permutations of this labeling. An important property of this list of permutations is its statistic generating function for inversions.

More precisely, let f be a bijection from P to the set of integers $[n] = \{1, 2, \dots, n\}$. Call f a *labeling* of P . Let $L_f(P)$ denote the permutations of $[n]$ which correspond to linear extensions of P under the labeling f . Then

$$INV_{P,f}(q) = \sum_{\pi \in L_f(P)} q^{inv(\pi)}.$$

Björner and Wachs [1] have shown that if the Hasse diagram of P is a tree and f is a postorder labeling, then $INV_{P,f}(q)$ can be written as a product of q -binomial coefficients. However, little is known about the general case.

A substantial weakening of this problem is simply to evaluate $INV_{P,f}(-1)$. Since $|INV_{P,f}(-1)|$ is independent of the choice of f , we will usually write $INV_P(-1)$ with the understanding that the sign is determined by the choice of f . We might ask now which posets have $INV_P(-1) = 0$. We call such posets *sign-balanced*. A necessary condition for generating a list of linear extensions by transpositions is that the poset be sign-balanced (or that $|INV_P(-1)| = 1$). Pruesse and Ruskey [10] showed that posets with the property that every non-minimal element is greater than at least two minimal elements are sign-balanced. Indeed, this is the case for many of the natural combinatorially arising posets, such as the Boolean algebra and the partition lattice.

In this paper we completely resolve the problem of determining $|INV_P(-1)|$ when the Hasse diagram of P is a special kind of Ferrers diagram. A special case is when P is the product of two chains (so that the Hasse diagram is a rectangle).

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Suppose $P = [m] \times [n]$, $m, n > 1$. That is, P is the product of an m -chain with an n -chain. When both m and n are even, P is sign-balanced [11]. Ruskey conjectured [11] that when m and n are both odd and greater than 1, then P is also sign-balanced. Ruskey also conjectured [11] that when $m \not\equiv n \pmod{2}$, then P is not sign-balanced. In this paper we will give a unified proof of these three results, and will place these results into the context of posets whose Hasse diagrams are Ferrers diagrams. In fact, we will show that if $m \not\equiv n \pmod{2}$, then $|INV_P(-1)|$ is the number of standard Young tableaux of a certain shifted shape.

Here, briefly, is a sketch of the proof.

For a partition ρ , let P_ρ be the poset whose Hasse diagram is given by the Ferrers diagram of ρ . The value $|INV_{P_\rho}(-1)|$ is first reduced to a similar evaluation for the spin generating function for domino tableaux of shape ρ . *Spin* is a statistic on domino tableaux, and more generally on 2-ribbon tableaux, described by Carré and Leclerc [2], and extended to k -ribbon tableaux by Lascoux, Leclerc and Thibon [8]. Its generating function gives a generalization of the Hall-Littlewood symmetric functions [9].

A well-known result [15] states that there is a bijection between domino tableaux of shape ρ and pairs of standard tableaux of shapes α and β . Furthermore, the shapes α and β are completely determined by ρ . This decomposition is called the *2-quotient* of ρ . We will call the shape ρ *d-rectangular* if its 2-quotient is a pair of rectangles. Posets whose Hasse diagrams are d-rectangles are the ones that will be investigated in this paper.

We then use a result of Shimozono and White [14] that states that domino tableaux of d-rectangular shape are in one-to-one correspondence with standard Young tableaux of a special kind of shape called *semi-self-complementary*. This bijection sends the spin statistic to a natural statistic on semi-self-complementary shapes called *twist*. The statistic generating function for twist is the rectangle-tensor-rectangle case of the q -Littlewood-Richardson coefficient described in [2].

Finally, we evaluate the twist generating function at -1 . We give a complete evaluation for every d-rectangular shape by using a more general result due to Shimozono [12]. This result involves the Schur's Q symmetric functions and vertex operators [9]. We also give a combinatorial evaluation in the case of a product of two chains, using a sign-reversing involution.

We then have the following theorem.

Theorem 1. *Let P be the product of an m -chain with an n -chain. Then*

- i. P is sign-balanced if $m \equiv n \pmod{2}$, m and $n > 1$;
- ii. P is not sign-balanced if $m \not\equiv n \pmod{2}$, and, in fact, $|INV_P(-1)|$ is the number of standard shifted tableaux of shape

$$\left(\frac{m+n-1}{2}, \frac{m+n-3}{2}, \dots, \frac{|m-n|+3}{2}, \frac{|m-n|+1}{2} \right).$$

This paper is organized as follows. Section 2 gives the basic definitions associated with partitions and tableaux. Section 3 describes domino tableaux and their 2-quotients. Section 4 discusses rectangle partitions and the central ideas of d-rectangular, semi-self-complementary, quasi-self-complementary and twist. Section 5 discusses sign-balanced posets in general and, in particular, posets whose Hasse diagrams are Ferrers diagrams. Section 6 defines spin and reduces the sign-balance problem on P_ρ to a sign-balance problem for spin on domino tableaux. We also

state the result in [14] needed to convert the sign-balance problem for spin on d -rectangles to a sign-balance problem for twist on standard tableaux of semi-self-complementary shape. With the exception of the result in [14], the material up to this point is self-contained.

In Section 7 we describe a symmetric function identity due to Shimozono. This result and its proof require several results in [9], [13] and [16]. These results will be stated without proof. This identity produces a more general result than Theorem 1.

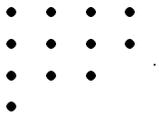
Finally, in Section 8 we give a combinatorial proof of Theorem 1 which uses a sign-reversing involution.

2. PARTITIONS AND TABLEAUX

In this section we will give the basic definitions for the combinatorial structures that arise in subsequent sections. Many of these standard definitions may be found in [9].

The sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t \geq 0)$ is called a *partition*. The *number of parts* of the partition λ , $l(\lambda)$, is the number of non-zero values. If $N = \sum_i \lambda_i$ then we say λ *partitions* N and we write $|\lambda| = N$. Another notation for partitions is to use an exponential form to denote the parts and their multiplicities. For example, the partition $(4, 4, 3, 1, 1, 1, 1, 1)$ is written $1^5 3 4^2$.

Yet another way of describing a partition is with a *Ferrers diagram*. A Ferrers diagram is an array of dots, left-justified, with the number of dots (or cells) in each row equal to the size of each part of the partition. For example, the Ferrers diagram for the partition $(4, 4, 3, 1)$ is



This pictorial description leads us to call partitions *shapes*.

If the Ferrers diagram of the partition β is contained in the Ferrers diagram of the partition α , but $\beta \neq \alpha$, we write $\beta \subset \alpha$. If equality is possible, we write $\beta \subseteq \alpha$.

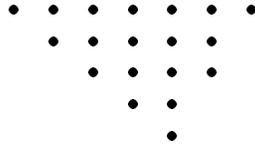
We will be combining partitions in several ways. For partitions α and β , write $\alpha \cup \beta$ to mean the partition whose parts are the parts of α and the parts of β . Write $\alpha + \beta$ to mean the partition $(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$. Write $\alpha \cap \beta$ to mean the partition whose Ferrers diagram is the intersection of the Ferrers diagrams of α and β . For example, if $\alpha = (4, 4, 3, 1)$ and $\beta = (3, 2, 2, 2, 2)$, then $\alpha \cup \beta = (4, 4, 3, 3, 2, 2, 2, 2, 1)$, $\alpha + \beta = (7, 6, 5, 3, 2)$, and $\alpha \cap \beta = (3, 2, 2, 1)$.

If α is a partition, then α' denotes the *conjugate* partition, obtained by transposing the Ferrers diagram of α . For instance, if $\alpha = (5, 4, 4, 1)$, then $\alpha' = (4, 3, 3, 3, 1)$.

Associated with each cell c in a Ferrers diagram is a set of cells called a *hook*. These are the cells below and in the same column as c , to the right of and in the same row as c , and c itself.

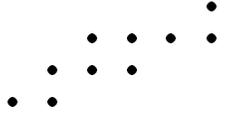
A partition in which all parts are distinct is called a *distinct partition*. For example, $(7, 5, 4, 2, 1)$ is a distinct partition. Such a partition can be described by a special kind of *shifted Ferrers diagram* or *shifted shape*. For example, the distinct

partition $(7, 5, 4, 2, 1)$ has this shifted shape:



A distinct partition may be viewed as a shape or a shifted shape. If λ is a distinct partition, we refer to the shape as λ and the shifted shape as $\nabla\lambda$. Thus, the example above shows $\nabla(7, 5, 4, 2, 1)$.

A *skew shape* is what we get when we remove the dots of one Ferrers diagram from another. If $\mu \subseteq \lambda$, the skew shape is denoted λ/μ . For example, here is the skew shape $(6, 6, 4, 2)/(5, 2, 1)$:



If the dots of a Ferrers diagram λ are replaced by numbers, what results is called a *tableau* of shape λ . A *standard tableau* (or standard Young tableau) is a tableau where the numbers increase across each row and down each column. Usually the numbers from 1 to N are used. If λ is a partition of N , then the number of standard tableaux of shape λ using the numbers $[N]$ is denoted f^λ . The special standard tableau with $1, 2, \dots, \lambda_1$ in the first row, $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ in the second row, etc., is called *superstandard*. If T is a tableau of shape λ , we say $sh(T) = \lambda$.

Similarly we have shifted tableaux, skew tableaux, standard shifted tableaux, and standard skew tableaux. If λ is a partition of N with distinct parts, the number of standard shifted tableaux of shape $\nabla\lambda$ using the numbers $[N]$ is denoted g^λ .

3. DOMINO TABLEAUX

We now describe a special kind of tableau called a domino tableau. A *domino* is a special kind of skew shape. This skew shape consists of two dots in the same row or same column. If they are in the same row, it is called a *horizontal domino*. If they are in the same column, it is called a *vertical domino*.

A *domino tableau* is a tableau such that the entries in each row and in each column are weakly increasing, and such that the cells containing any given number form a domino. For example, here is a domino tableau of shape $(6, 6, 3, 3, 2)$:

1	2	4	4	7	8
1	2	6	6	7	8
3	3	9			
5	5	9			
10	10				

We let Dom_ρ represent the set of domino tableaux of shape ρ . For certain ρ (e.g., $\rho = (3, 2, 1)$), this set is empty. Shapes for which Dom_ρ is not empty are said to have *empty 2-core*.

Domino tableaux are in one-to-one correspondence with pairs of standard tableaux, as described by the following theorem [15].

Theorem 2. *There is a one-to-one correspondence between domino tableaux D , using the set $[n]$, and pairs of standard tableaux, (U, V) , which together use the set $[n]$. Furthermore, the shape of the domino tableau determines the shapes of the standard tableaux.*

Theorem 2 appears in [5] and [15], but dates back to Nakayama and Robinson.

We illustrate here this bijection. This description of the bijection appears in [2] and in [3]. Label each domino in D either 0 or 1 according to whether the lattice distance between the upper or right cell of the domino and the main diagonal is even or odd. Similarly label each diagonal of D either 0 or 1 according to whether its lattice distance to the main diagonal is even or odd.

Now delete all dominoes labeled 1. The remaining entries on diagonals labeled 0 are the same as the entries of the diagonals of U . Deleting dominoes labeled 0 and retaining diagonals labeled 1 produces V .

In our example above, first deleting the dominoes labeled 1 gives

1			7	
1		6 6	7	
		9		
5	5	9		

The diagonals labeled 0 then produce this tableau

1	6	7
5	9	

First deleting the dominoes labeled 0 gives

	2	4	4		8
	2				8
3	3				
10	10				

The diagonals labeled 1 then yield this tableau

2	4	8
3		
10		

It is not too difficult to see that this is a bijection and that different domino tableaux of the same shape give the same shapes for the corresponding U and V . We write $D = U * V$ to denote this decomposition, and $\rho = \mu * \nu$ to denote the corresponding decomposition of the shape of D into the shapes of U and V . The pair (U, V) (resp. (μ, ν)) is called the *2-quotient* of D (resp. ρ).

Our main interest in domino tableaux will be when the 2-quotient is a pair of rectangles.

4. RECTANGLES AND SELF-COMPLEMENTARY SHAPES

The Hasse diagram of a product of two chains is a rectangular Ferrers diagram and the linear extensions of the product of two chains are in one-to-one correspondence to standard tableaux of rectangular shape. Therefore, this section is devoted to some definitions and properties related to rectangular Ferrers diagrams.

If a rectangular Ferrers diagram is to be the shape of a domino tableau, then one of the dimensions must be even. We can construct a domino tableau with odd-by-odd rectangular shape if we omit the lower-right corner cell. This motivates the definition of a *quasi-rectangle*. The 2-quotient of a quasi-rectangle is a pair of rectangles which are either equal or almost equal.

More generally, if the 2-quotient of a shape is a pair of rectangles, we call the shape *d-rectangular*.

As will be seen in later sections, a certain special kind of shape associated with a pair of rectangles plays a central role. These shapes are called *semi-self-complementary*. These shapes are so named because they consist of an inner rectangle and then two complementary shapes to the right and below.

If the two rectangles are equal or almost equal, as in the case of the 2-quotient of a quasi-rectangle, then we call the corresponding semi-self-complementary shapes *quasi-self-complementary*. Such shapes are either self-complementary or self-complementary with respect to a rectangle with a “hole” in the center.

Here are the details of these definitions.

An $M \times N$ *rectangle* is the partition N^M . An $M \times N$ *quasi-rectangle* is the $M \times N$ rectangle if M and N are not both odd and is the partition $N^{M-1}(N-1)$ if M and N are both odd.

Proposition 3. *If ρ is an $M \times N$ quasi-rectangle, with $\rho = \alpha * \beta$, then α is an $\lfloor \frac{M}{2} \rfloor \times \lceil \frac{N}{2} \rceil$ rectangle and β is an $\lceil \frac{M}{2} \rceil \times \lfloor \frac{N}{2} \rfloor$ rectangle.*

Proof. This follows from the definition of the 2-quotient. □

Proposition 3 motivates the following definition. Let (α, β) be a pair of rectangles. The partition ρ is (α, β) *d-rectangular* if $\rho = \alpha * \beta$. Thus, d-rectangular partitions are partitions whose 2-quotient is a pair of rectangles. Proposition 3 says that quasi-rectangles are d-rectangular.

Suppose $\beta = n^m$ is a rectangle and suppose $\lambda \subseteq \beta$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$. We call the partition λ^c the β -*complement* of λ or the $m \times n$ -*complement* of λ if

$$\lambda^c = (n - \lambda_m, n - \lambda_{m-1}, \dots, n - \lambda_1).$$

We say λ is β *self-complementary* or $m \times n$ *self-complementary* if $\lambda^c = \lambda$.

For example, if $\beta = (5, 5, 5, 5)$ and $\lambda = (4, 2, 1)$, then $\lambda^c = (5, 4, 3, 1)$. Also, $(4, 4, 1, 1)$ is β self-complementary.

A generalization of the idea of self-complementary partitions, called *semi-self-complementary*, plays a key role in the rest of this paper. Semi-self-complementary partitions depend upon two rectangles.

Let α and β be two rectangles. Let D be the cellwise union of the Ferrers diagrams of α and β . We call the shape λ (α, β) *semi-self-complementary* if

- i. The partition λ contains the cells in D .
- ii. The skew shape λ/D consists of at most two shapes, one (μ) to the right of D and one (ν) below D .
- iii. The shapes μ and ν are $\alpha \cap \beta$ -complementary.

See Figure 1 and Figure 2.

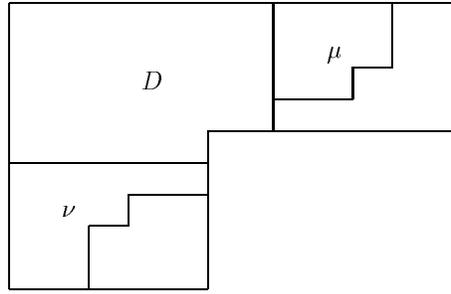


FIGURE 1. A semi-self-complementary shape

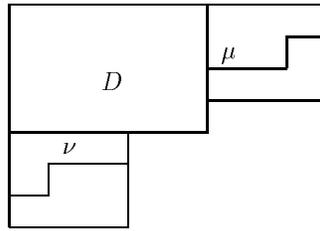


FIGURE 2. Another semi-self-complementary shape

To place semi-self-complementary shapes into some context, if α and β are rectangles and if $c_{\alpha,\beta}^\lambda$ is a Littlewood-Richardson coefficient, then

$$c_{\alpha,\beta}^\lambda = \begin{cases} 1 & \text{if } \lambda \in C_{\alpha,\beta} \\ 0 & \text{otherwise.} \end{cases}$$

Here are some examples of semi-self-complementary shapes. If $\alpha = 6^4$ and $\beta = 4^5$, then let $\lambda = (10, 8, 7, 6, 4, 4, 3, 2, 0)$ with $\mu = (4, 2, 1, 0)$ and $\nu = (4, 3, 2, 0)$. Figure 3 shows the shape λ with μ indicated with $+$ and ν with $*$.

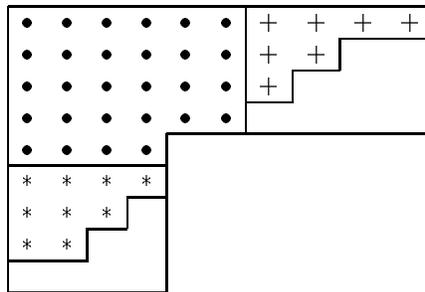
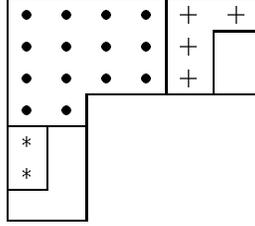


FIGURE 3. A $(6^4, 4^5)$ semi-self-complementary shape

If $\alpha = 2^4$ and $\beta = 4^3$, then let $\lambda = (6, 5, 5, 2, 1, 1, 0)$ with $\mu = (2, 1, 1)$ and $\nu = (1, 1, 0)$. See Figure 4.

The next proposition describes some important special cases.

FIGURE 4. A $(2^4, 4^3)$ semi-self-complementary shape

Proposition 4. *Suppose λ is (α, β) semi-self-complementary. Then we have*

- i. *If $\alpha = \beta = n^m$, then λ is $2m \times 2n$ self-complementary.*
- ii. *If $\alpha = n^m$ and $\beta = (n-1)^m$, then λ is $2m \times (2n-1)$ self-complementary.*
- iii. *If $\alpha = n^m$ and $\beta = n^{m+1}$, then λ is $(2m+1) \times 2n$ self-complementary.*
- iv. *If $\alpha = n^m$ and $\beta = (n-1)^{m+1}$, then λ is “self-complementary” inside the $(2m+1) \times (2n-1)$ rectangle with a “hole” in the center cell. That is, $\lambda_{m+1} = n-1$*
- v. *If $\beta = \emptyset$, then $\lambda = \alpha$.*
- vi. *The shape λ is also (β, α) semi-self-complementary.*

The four special cases in Proposition 4, Case (i), Case (ii), Case (iii) and Case (iv), motivate the following definition. The partition λ is $M \times N$ *quasi-self-complementary* if λ is an $M \times N$ self-complementary shape if M and N are not both odd, and with the added condition that $\lambda_{(M+1)/2} = (N-1)/2$ if M and N are both odd.

For example, if $M = 4$ and $N = 7$, then $(6, 5, 2, 1)$ is quasi-self-complementary (and self-complementary), while if $M = 5$ and $N = 7$, then $(6, 5, 3, 2, 1)$ is quasi-self-complementary.

Proposition 5. *If λ is $M \times N$ quasi-self-complementary, then λ is $(\lfloor \frac{M}{2} \rfloor \times \lceil \frac{N}{2} \rceil, \lceil \frac{M}{2} \rceil \times \lfloor \frac{N}{2} \rfloor)$ semi-self-complementary.*

If $M \not\equiv N \pmod{2}$, then there is a special quasi-self-complementary shape which we call *stairstep*. It is

$$stst(M, N) = \left(\frac{M+N-1}{2}, \frac{M+N-3}{2}, \dots, \frac{|M-N|+1}{2} \right).$$

Relating the definitions semi-self-complementary and quasi-self-complementary back to the definitions of d-rectangular and quasi-rectangular, if ρ is d-rectangular, then $\rho = \alpha * \beta$ for two rectangles α and β , from which we can construct a collection of (α, β) semi-self-complementary shapes. If ρ is quasi-rectangular, then the (α, β) semi-self-complementary shapes are also quasi-self-complementary.

Next we define the statistic *twist* on (α, β) semi-self-complementary shapes. Suppose $\lambda \in C_{\alpha, \beta}$ with associated μ and ν . Then

$$tw(\lambda) = |\nu|.$$

In the previous examples, $tw(10, 8, 7, 6, 4, 4, 3, 2, 0) = 9$ and $tw(6, 5, 5, 2, 1, 1, 0) = 2$.

If α is an $m \times n$ rectangle, let

$$Str(\alpha) = (m+n-1, m+n-3, \dots, |m-n|+1),$$

which is a distinct partition. For example, if $\alpha = 5^3$, then $Str(\alpha) = (7, 5, 3)$.

A theorem of Worley [18] states that the number of standard tableaux of shape α is equal to the number of standard shifted tableaux of shape $\nabla Str(\alpha)$. That is,

$$(1) \quad f^\alpha = g^{Str(\alpha)}.$$

This theorem is proved using a bijection in [4].

Proposition 6.

- i. If α is a rectangle, then $Str(\alpha') = Str(\alpha)$.
- ii. If α and β are rectangles, then λ is (α, β) semi-self-complementary if and only if λ' is (α', β') semi-self-complementary.
- iii. If α and β are rectangles and λ is (α, β) semi-self-complementary, then $tw(\lambda) + tw(\lambda') = |\alpha \cap \beta|$.

If α and β are both rectangles, we say they are *strict-distinct* if $Str(\alpha)$ and $Str(\beta)$ do not have a common part. For example, 6^3 and 5^7 are strict-distinct, but 6^3 and 5^6 are not.

We conclude this section by identifying the quasi-rectangles whose 2-quotient is strict-distinct.

Proposition 7. *If ρ is an $M \times N$ quasi-rectangle, M and N both > 1 , with $\rho = \alpha * \beta$, then α and β are strict-distinct if and only if $M \not\equiv N \pmod{2}$. Furthermore, if α and β are strict-distinct, then $Str(\alpha) \cup Str(\beta) = stst(M, N)$.*

Proof. If M and N are both even, then $\alpha = \beta$ and so $Str(\alpha) = Str(\beta)$.

If M and N are both odd, then

$$Str(\alpha) = \left(\frac{M+N}{2} - 1, \frac{M+N}{2} - 3, \dots, \left| \frac{M-N}{2} - 1 \right| + 1 \right)$$

and

$$Str(\beta) = \left(\frac{M+N}{2} - 1, \frac{M+N}{2} - 3, \dots, \left| \frac{M-N}{2} + 1 \right| + 1 \right),$$

which have common parts as long as M and N both > 1 .

If M is odd and N is even, then

$$Str(\alpha) = \left(\frac{M+N-1}{2} - 1, \frac{M+N-1}{2} - 3, \dots, \left| \frac{M-N-1}{2} \right| + 1 \right)$$

and

$$Str(\beta) = \left(\frac{M+N-1}{2}, \frac{M+N-1}{2} - 2, \dots, \left| \frac{M-N+1}{2} \right| + 1 \right).$$

Therefore α and β are strict-distinct. If $M > N$, then the smallest part is in $Str(\alpha)$ and is $\frac{M-N+1}{2}$. If $M < N$, then the smallest part is in $Str(\beta)$ and is $\frac{N-M+1}{2}$. It follows that $Str(\alpha) \cup Str(\beta) = stst(M, N)$.

The case where M is even and N is odd is similar. □

5. SIGN-BALANCED POSETS

Let P be a finite partially ordered set (or poset) with n elements. Label the elements of P with the set $[n]$. Call this labeling f . The labeling f may be thought of as a bijection from P to the set $[n]$. Each linear extension of P then is a permutation of this labeling. Denote by $L_f(P)$ the set of permutations which arise as linear extensions of P . For such a permutation π denote by $inv(\pi)$ the number

of inversions in π , that is, the number of pairs of numbers in π which are out of order. Then construct the generating function for these inversions:

$$(2) \quad \text{INV}_{P,f}(q) = \sum_{\pi \in L_f(P)} q^{\text{inv}(\pi)}.$$

Observe that $|\text{INV}_{P,f}(-1)|$ will not depend upon f . For this reason, we write $\text{INV}_P(-1)$. If

$$\text{INV}_P(-1) = 0,$$

then we say the poset is *sign-balanced*. The *sign of a linear extension* of P with labeling f is then the sign of the corresponding permutation $\pi \in L_f(P)$.

Very little seems to be known about $\text{INV}_{P,f}(q)$ for an arbitrary labeling f . Björner and Wachs [1] show that when the Hasse diagram of P is a forest and the labeling is a postorder labeling, then $\text{INV}_{P,f}(q)$ has a simple product form.

However, many combinatorially occurring posets are sign-balanced. This follows from the following proposition [10].

Proposition 8. *If every non-minimal element of the poset P is greater than at least two minimal elements, then P is sign-balanced.*

Proof. Let $x_1, x_2, x_3, \dots, x_n$ be a linear extension of P . Then $x_2, x_1, x_3, \dots, x_n$ is also a linear extension. Therefore, swapping the first two elements in the linear extension is a sign-reversing involution. \square

For example, this proposition can be applied to the Boolean algebra or the partition lattice.

Now suppose P is a product of two chains. That is, let $P = [m] \times [n]$ where $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. The Hasse diagram of P is then a Ferrers diagram of shape n^m , and the linear extensions of P may be regarded as standard tableaux of shape n^m .

More generally, let P_ρ be the poset whose Hasse diagram is the shape ρ , so that its linear extensions may be regarded as standard tableaux of shape ρ . This motivates us to define the sign of a standard tableau T , $\text{sign}(T)$, as the sign of the permutation obtained by reading the entries of T row-by-row, from left to right. This sign agrees with the sign of the corresponding linear extension of P_ρ when the fixed labeling of P_ρ is the superstandard tableau. The question of when such P_ρ are sign-balanced then becomes a question of when the standard tableaux of shape ρ are sign-balanced.

There are two natural sign-reversing involutions on the set of standard tableaux of shape ρ , a partition of N . Let T be such a standard tableau. The first involution, which we call α , pairs 1 with 2, 3 with 4, 5 with 6, etc. Then simply find the smallest such pair such that the two numbers are in different rows and columns of T . Then swap these two numbers in T to form $\alpha(T)$. If N is even, then the fixed points of α are the domino tableaux of shape ρ . If N is odd, then the fixed points are the domino tableaux of shape μ , where μ is a partition of $N - 1$, $\mu \subseteq \rho$.

The second involution, β , pairs 2 and 3, 4 and 5, etc. Again, find the smallest such pair such that the two numbers are in different rows and columns of T and swap these two numbers to form $\beta(T)$. If N is odd, then the fixed points are domino skew tableaux of shape $\rho/(1)$. If N is even, then the fixed points are domino skew tableaux of shape $\mu/(1)$, where μ is a partition of $N - 1$, $\mu \subseteq \rho$.

Both of these involutions are sign-reversing because they produce linear extensions which differ by a transposition.

For example, if

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 8 & \\ 5 & 7 & & & & \\ 6 & & & & & \end{array},$$

then

$$\alpha(T) = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & \\ 5 & 8 & & & & \\ 6 & & & & & \end{array}$$

and

$$\beta(T) = \begin{array}{cccccc} 1 & 2 & 3 & 5 & 8 & \\ 4 & 7 & & & & \\ 6 & & & & & \end{array}.$$

We next determine the sign of the fixed points of these two involutions. For a domino (skew) tableau, let $ov(T)$ be the number of vertical dominoes in odd columns and let $ev(T)$ be the number of vertical dominoes in even columns. Let $v(T)$ be the number of vertical dominoes in T .

Also, suppose N is odd (resp. even), μ is a partition of $N - 1$, and T is a domino tableau of shape μ (resp. $\mu/(1)$), $\mu \subseteq \rho$. Then let $rv(T)$ be the number of vertical dominoes in T whose uppermost cell is in the same row as the single cell ρ/μ .

Proposition 9. *If T is a fixed point of α , then*

$$\text{sign}(T) = \begin{cases} (-1)^{ev(T)} & \text{if } N \text{ is even;} \\ (-1)^{ev(T)+rv(T)} & \text{if } N \text{ is odd.} \end{cases}$$

If T is a fixed point of β , then

$$\text{sign}(T) = \begin{cases} (-1)^{ev(T)} & \text{if } N \text{ is odd;} \\ (-1)^{ev(T)+rv(T)} & \text{if } N \text{ is even.} \end{cases}$$

Proof. Horizontal dominoes do not contribute to the sign because inversions involving horizontal dominoes come in pairs.

Let δ be a vertical domino in T and let i be the row of the lowermost dot of δ . Then δ will contribute to the sign of T according to the number of vertical dominoes to the left of δ and containing a dot in row i . These dominoes will appear in alternating odd and even columns, starting with an odd column. Thus, δ will contribute -1 to the sign of T if and only if δ is in an even column. See Figure 5.

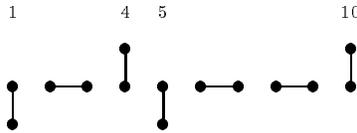


FIGURE 5. Dominoes in a row

For the special cases where rv is involved, we must consider the contribution of the largest letter N in the tableau, which appears in location ρ/μ . All dominoes to

the left and below this location do not contribute to the sign, since their inversions with N come in pairs. The only contributors will be the dominoes counted by rv . \square

Note that if ρ is quasi-rectangular, then $rv(T) = 0$.

6. SPIN AND TWIST

Spin is a simple statistic on ribbon tableaux whose generating function generalizes the Hall-Littlewood symmetric functions. For a domino tableau, D , *spin* is defined by $sp(D) = v(D)/2$, i.e., half the number of vertical dominoes. For fixed shape ρ , let s^* be the maximum spin of all domino tableaux of shape ρ . Then the *cospin* of D of shape ρ is $cosp(D) = s^* - sp(D)$. Cospin is integral, as is guaranteed by the following easily established proposition.

Proposition 10. *If S and T are domino tableaux of the same shape, then $ov(S) - ev(S) = ov(T) - ev(T)$. That is, $ov(S) - ev(S)$ is a constant depending only on the shape of the tableau.*

Lemma 11. *If S and T are domino tableaux of the same shape, then $cosp(S) + ev(S) = cosp(T) + ev(T)$. That is, $cosp(S) + ev(S)$ is a constant depending only on the shape of the tableau.*

Proof. We have

$$\begin{aligned} cosp(S) - cosp(T) &= s^* - sp(S) - s^* + sp(T) \\ &= v(T)/2 - v(S)/2 \\ &= ev(T)/2 + ov(T)/2 - ev(S)/2 - ov(S)/2 \\ &= (ov(T) - ev(T))/2 + ev(T) - (ov(S) - ev(S))/2 - ev(S) \\ &= ev(T) - ev(S). \end{aligned}$$

\square

Now define $d(\rho) = cosp(D) + ev(D)$ for any domino tableau D of shape ρ .

Theorem 12. *For every partition ρ of an even number, and for every partition ρ of an odd number for which $rv(T) = 0$ for all T of shape ρ , we have*

$$INV_{P_\rho}(-1) = (-1)^{d(\rho)} \sum_{D \in Dom_\rho} (-1)^{cosp(D)}.$$

Proof. This theorem follows immediately from Lemma 11 and Proposition 9 (using the involution α). \square

Note that P_ρ is sign-balanced if $|\rho|$ is even and ρ does not have empty 2-core. Also note that if $|\rho|$ is odd, a similar theorem can be proved using skew domino tableaux of shape $\rho/(1)$ and the involution β . We leave it to the reader to provide details.

Shimozono and White [14] describe a bijection between (α, β) d -rectangular domino tableaux and (α, β) semi-self-complementary standard tableaux. This bijection sends cospin to twist. The following is a corollary in [14].

Theorem 13. *There is a bijection ψ from (α, β) semi-self-complementary standard tableaux to (α, β) d -rectangular domino tableaux such that*

$$cosp(\psi(Q)) = tw(sh(Q)).$$

The bijection in Theorem 13 gives an explicit description of the statistic for the Carré-Leclerc q -Littlewood-Richardson coefficients [2].

Theorem 12 and Theorem 13 provide the bridge between the poset sign-balance problem described in the previous section and the proofs given in subsequent sections.

Corollary 14. *If ρ is an (α, β) d -rectangle, then*

$$|INV_{P_\rho}(-1)| = \left| \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{tw(\lambda)} f^\lambda \right|.$$

Our goal then will be to evaluate

$$\sum_{\lambda \in C_{\alpha, \beta}} (-1)^{tw(\lambda)} f^\lambda.$$

In the next section, we evaluate this sum for all rectangle pairs using symmetric functions and vertex operators. In Section 8 we evaluate the sum for pairs of rectangles which are the 2-quotient of quasi-rectangles, using a sign-reversing involution.

7. A SYMMETRIC FUNCTION PROOF

In this section we give a complete evaluation of the sum in Corollary 14. This evaluation is based on a symmetric function identity due to Shimozono [12]. The proof of this identity is outlined below. This section is not completely self-contained. Where proofs and definitions have been omitted, appropriate references have been provided.

To describe the identity and its proof, we need to introduce two sets of symmetric functions and a class of operators on these functions. The reader is referred to Macdonald [9] for background on symmetric functions.

Let $X = \{x_1, x_2, \dots\}$ be a set of indeterminates. Let $Q_\rho(X)$ denote Schur's Q symmetric functions described in [9, III.8]. These symmetric functions are the Hall-Littlewood Q symmetric functions evaluated at $t = -1$. They are defined for partitions ρ with distinct parts. More generally, let $Q_{\nu/\rho}(X)$ denote the skew Schur Q function, with ν and ρ both partitions with distinct parts.

Let $S_\rho(X)$ be the modified Schur function described in [9, III (4.5)], evaluated at $t = -1$.

Lemma 15.

- i. *If ρ is a partition with k parts, then $S_\rho(X) = Q_{\rho+\delta/\delta}(X)$, where*

$$\delta = (k-1, k-2, \dots, 1).$$

- ii. *If α is a rectangle with k parts, then $Q_{\alpha+\delta/\delta}(X) = Q_{Str(\alpha)}(X)$.*
 iii. *The $Q_\lambda(X)$, where λ ranges over partitions with distinct parts, are linearly independent.*
 iv. *If ρ is a partition, then $S_\rho(X) = S_{\rho'}(X)$.*

Proof. Part (i) follows from the combinatorial descriptions of $Q_{\mu/\rho}$ and S_ρ given in [9, III (8.16'')] and Ex. III.8.7a]. Part (ii) is a consequence of the shifted Littlewood-Richardson rule in [17], described in [9, III (8.18)]. Part (iii) is [9, III (8.9)]. Part (iv) is immediate from [9, Ex. III.8.7a]. \square

The class of operators we need are the *vertex operators* of Jing [6], B_i , i an integer, described in [9, Ex. III.8.8]. More generally [16], we need B_ρ for ρ a partition. While these operators are described for general t , we will only need them when $t = -1$. Let b_i and b_ρ denote these specializations. We will avoid the precise definitions of B_ρ and b_ρ and simply describe the properties, many of which are in [9], that we will need.

Lemma 16.

i. For partition ρ , $b_\rho(1) = S_\rho(X)$.

ii. For partition ρ with k distinct parts,

$$b_{\rho_1} \circ b_{\rho_2} \circ \cdots \circ b_{\rho_k}(1) = Q_\rho(X).$$

iii. For $i + j \neq 0$, $b_i \circ b_j = -b_j \circ b_i$.

iv. The b_ρ can be expanded as sums of compositions of the b_i . That is, if ρ has k parts, then

$$b_\rho = \sum_{\gamma=(\gamma_1, \gamma_2, \dots, \gamma_k)} A_{\gamma, \rho} b_{\gamma_1} \circ b_{\gamma_2} \circ \cdots \circ b_{\gamma_k},$$

where the γ_i and the $A_{\gamma, \rho}$ are integers. Furthermore, if $\rho_1 \geq l(\rho)$ then the γ_i are non-negative.

Proof. Part (i) follows from [16]. More generally, $B_\rho(1) = S_\rho(X; t)$. Part (ii) is from [9, Ex. III.8.8(5)]. More generally, for any distinct partition ρ ,

$$B_{\rho_1} \circ B_{\rho_2} \circ \cdots \circ B_{\rho_k}(1) = Q_\rho(X; t).$$

Part (iii) is from [9, Ex. III.8.8(C1)]. Part (iv) follows from [9, Ex. III.8.8(4)]. The γ 's which appear can be described exactly and can be used to give a definition of the B_ρ . \square

Partitions for which $\rho_1 \geq l(\rho)$ (as in Lemma 16, Part (iv)) will be called *flat*. Partitions for which $\rho_1 \leq l(\rho)$ will be called *tall*.

Suppose $\alpha = l^k$ and $\beta = n^m$ are rectangles. If $l \geq n$, we say (α, β) is a *dominant pair of rectangles*. For $\lambda \in C_{\alpha, \beta}$, let $\text{cotw}(\lambda) = |\alpha \cap \beta| - \text{tw}(\lambda)$.

Lemma 17. *Suppose (α, β) is a dominant pair of rectangles. The operator*

$$B_\alpha \circ B_\beta - \sum_{\lambda \in C_{\alpha, \beta}} t^{\text{cotw}(\lambda)} B_\lambda$$

annihilates the symmetric function 1.

Proof. From [16], the operator

$$B_\alpha \circ B_\beta - \sum_{\lambda} K_{\lambda; (\alpha, \beta)}(t) B_\lambda$$

annihilates the symmetric function 1, where $K_{\lambda; (\alpha, \beta)}(t)$ is the Shimozono-Weyman generalized Kostka polynomial [13]. However, from [13],

$$K_{\lambda; (\alpha, \beta)}(t) = \begin{cases} t^{\text{cotw}(\lambda)} & \text{if } \lambda \text{ is semi-self-complementary} \\ 0 & \text{if } \lambda \text{ is not semi-self-complementary} \end{cases}$$

for a pair of rectangles (α, β) . In fact, this identity, in conjunction with [14], shows that the generalized Kostka polynomial is the Carré-Leclerc q -Littlewood-Richardson coefficient [2]. That is,

$$K_{\lambda; (\alpha, \beta)}(t) = t^{|\alpha \cap \beta|} c_{\alpha, \beta}^\lambda(1/t).$$

□

We now proceed to the symmetric function identity [12].

Theorem 18 (Shimozono). *Let (α, β) be a dominant pair of strict-distinct rectangles. Let $inv(\alpha, \beta)$ be the number of transpositions required to sort $Str(\alpha) \cup Str(\beta)$. Then*

$$Q_{Str(\alpha) \cup Str(\beta)}(X) = (-1)^{inv^+(\alpha, \beta)} \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{cotw(\lambda)} S_{\lambda}(X),$$

where

$$inv^+(\alpha, \beta) = \begin{cases} inv(\alpha, \beta) & \text{if } \alpha \text{ is flat} \\ inv(\alpha, \beta) + |\alpha \cap \beta| & \text{if } \alpha \text{ is tall and } (\alpha', \beta') \text{ is dominant.} \\ inv(\beta, \alpha) + |\alpha \cap \beta| & \text{if } \alpha \text{ is tall and } (\beta', \alpha') \text{ is dominant} \end{cases}$$

If (α, β) is a dominant pair of rectangles which are not strict-distinct, then

$$\sum_{\lambda \in C_{\alpha, \beta}} (-1)^{cotw(\lambda)} S_{\lambda}(X) = 0.$$

Proof. Suppose α is a flat rectangle with k parts. We first show that

$$(3) \quad b_{\alpha} = b_{Str(\alpha)_1} \circ b_{Str(\alpha)_2} \circ \cdots \circ b_{Str(\alpha)_k}.$$

By Lemma 16, part (iv),

$$b_{\alpha} = \sum_{\gamma=(\gamma_1, \gamma_2, \dots, \gamma_k)} A_{\gamma, \alpha} b_{\gamma_1} \circ b_{\gamma_2} \circ \cdots \circ b_{\gamma_k},$$

where the γ_i are non-negative integers. Using Lemma 16, Part (iii), these compositions may be sorted so that

$$(4) \quad b_{\alpha} = \sum_{\mu} A'_{\mu, \alpha} b_{\mu_1} \circ b_{\mu_2} \circ \cdots \circ b_{\mu_k},$$

where the sum is over partitions with distinct parts. We now show that $A'_{\mu, \alpha} = 0$ for all $\mu \neq Str(\alpha)$ and that $A'_{Str(\alpha), \alpha} = 1$. Apply Equation (4) to the symmetric function 1. By Lemma 16, Part (i), the left-hand side is $S_{\alpha}(X)$. By Lemma 16, Part (ii), the right-hand side is

$$\sum_{\mu} A'_{\mu, \alpha} Q_{\mu}(X),$$

where the sum is over partitions with distinct parts. By Lemma 15, Parts (i) and (ii), the left-hand side is $Q_{Str(\alpha)}(X)$. But by Lemma 15, Part (iii), the Q are linearly independent, so that

$$A'_{\mu, \alpha} = \begin{cases} 0 & \text{if } \mu \neq Str(\alpha) \\ 1 & \text{if } \mu = Str(\alpha), \end{cases}$$

as desired.

Again, assume α is flat with k parts. Specialize $t = -1$ in Lemma 17, and apply this operator to the symmetric function 1 to obtain

$$b_{\alpha} \circ b_{\beta}(1) = \sum_{\lambda} (-1)^{cotw(\lambda)} b_{\lambda}(1).$$

By Lemma 16, Part (i), this becomes

$$b_\alpha(S_\beta(X)) = \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} S_\lambda(X).$$

Now apply Lemma 15, Parts (i) and (ii), to $S_\beta(X)$ as before and substitute Equation (3) for b_α to obtain

$$b_{Str(\alpha)_1} \circ b_{Str(\alpha)_2} \circ \cdots \circ b_{Str(\alpha)_k}(Q_{Str(\beta)}(X)) = \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} S_\lambda(X).$$

Next, use Lemma 16, Part (ii), to give

$$\begin{aligned} b_{Str(\alpha)_1} \circ b_{Str(\alpha)_2} \circ \cdots \circ b_{Str(\alpha)_k} \circ b_{Str(\beta)_1} \circ b_{Str(\beta)_2} \circ \cdots \circ b_{Str(\beta)_l}(1) \\ = \sum_{\lambda} (-1)^{\text{cotw}(\lambda)} S_\lambda(X). \end{aligned}$$

If $Str(\alpha)$ and $Str(\beta)$ have a common part, then by Lemma 16, Part (iii), the left-hand side is 0. Otherwise, use Lemma 16, Part (iii), to sort the b 's, and reapply Lemma 16, Part (ii), to get

$$Q_{Str(\alpha) \cup Str(\beta)}(X) = (-1)^{\text{inv}(\alpha, \beta)} \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} S_\lambda(X).$$

When α is tall, we use Lemma 15, Part (iv) and Proposition 6. Observe that if (α, β) is a dominant pair and α is tall, then either (α', β') is a dominant pair and α' is flat or (β', α') is a dominant pair and β' is flat. In the former case,

$$\begin{aligned} Q_{Str(\alpha) \cup Str(\beta)}(X) &= Q_{Str(\alpha') \cup Str(\beta')}(X) \\ &= (-1)^{\text{inv}(\alpha', \beta')} \sum_{\lambda' \in C_{\alpha', \beta'}} (-1)^{\text{cotw}(\lambda')} S_{\lambda'}(X) \\ &= (-1)^{\text{inv}(\alpha, \beta) + |\alpha \cap \beta|} \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} S_\lambda(X) \\ &= (-1)^{\text{inv}^+(\alpha, \beta)} \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} S_\lambda(X), \end{aligned}$$

while in the latter case,

$$\begin{aligned} Q_{Str(\alpha) \cup Str(\beta)}(X) &= Q_{Str(\beta') \cup Str(\alpha')}(X) \\ &= (-1)^{\text{inv}(\beta', \alpha')} \sum_{\lambda' \in C_{\beta', \alpha'}} (-1)^{\text{cotw}(\lambda')} S_{\lambda'}(X) \\ &= (-1)^{\text{inv}(\beta, \alpha) + |\alpha \cap \beta|} \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} S_\lambda(X) \\ &= (-1)^{\text{inv}^+(\alpha, \beta)} \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} S_\lambda(X). \end{aligned}$$

□

We then have the following corollary.

Corollary 19. *If (α, β) is a pair of strict-distinct rectangles, then*

$$g^{Str(\alpha) \cup Str(\beta)} = (-1)^{\text{inv}^+(\alpha, \beta)} \sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} f^\lambda.$$

If (α, β) is a pair of rectangles which are not strict-distinct, then

$$\sum_{\lambda \in C_{\alpha, \beta}} (-1)^{\text{cotw}(\lambda)} f^\lambda = 0.$$

Proof. Equate the square-free terms in Theorem 18. \square

Now combine Corollary 19 with Corollary 14.

Corollary 20. *If ρ is (α, β) d -rectangular, then*

$$|INV_{P_\rho}(-1)| = \begin{cases} g^{\text{Str}(\alpha) \cup \text{Str}(\beta)} & \text{if } \alpha \text{ and } \beta \text{ are strict-distinct} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we restate Theorem 1.

Theorem 21. *If P is the product of an M -chain and an N -chain, then*

$$|INV_P(-1)| = \begin{cases} 0 & \text{if } M = N \pmod{2}, M \text{ and } N > 1 \\ g^{\text{stst}(M, N)} & \text{if } M \neq N \pmod{2}. \end{cases}$$

Proof. Since $P = P_\rho$ for quasi-rectangular shape ρ , apply Corollary 20 and Proposition 7. \square

Note that Equation (1) is another special case of Corollary 19.

8. A SIGN-REVERSING INVOLUTION

Our goal in this section will be to find an involution on standard tableaux of $M \times N$ quasi-self-complementary shape which reverses the parity of twist.

One such sign-reversing involution can be easily described: move the largest entry in the tableau to the corresponding ‘‘complementary’’ position.

What results is another quasi-self-complementary standard tableau, but one whose spin is one more or one less than that of the original.

This idea is not sufficient to cancel completely tableaux with opposite signs. What happens if the largest value is in a corner whose corresponding complementary position is in the same row or column? However, this simple idea forms the basis for the general sign-reversing involution. In fact, the shape obtained under the sign-reversing involution will be exactly the shape obtained using this idea.

8.1. Corners and Notches. Let λ be an $M \times N$ quasi-self-complementary shape. A cell in λ is called a *corner* if the cell to its right and the cell directly below are both not in λ . Also, a cell not in λ is called a *notch* if (1) it is in the first row or the cell directly above is in λ and (2) if it is in the first column or the cell directly to the left is in λ .

The corners pair off with the notches in the complementary positions. If a is a corner and b its corresponding notch, we call the pair (a, b) a *corner-notch pair*. Suppose (a, b) is a corner-notch pair and a and b are in the same row or column. Then we call the pair *inadmissible*. Otherwise, the pair is *admissible*.

These definitions are illustrated in Figure 6. In this example, $M = 5$, $N = 6$ and $\lambda = (5, 5, 3, 1, 1)$. The corner-notch pairs are marked \circ , \circ_1 and \circ_2 . The corner-notch pair \circ_2 is inadmissible. The other two are admissible.

If (a, b) is an admissible corner-notch pair and a is in a row above b , we say a is *above* b and b is *below* a . Otherwise, b is above a and a is below b .

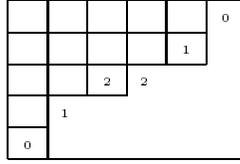


FIGURE 6. Corner-notch pairs

Suppose (a, b) is an admissible corner-notch pair. Without loss of generality, assume a is above b . Also, let i be the row of a and j the column of b . The *hook of (a, b)* is the hook defined by the cell (i, j) .

Proposition 22. *Let λ be $M \times N$ quasi-self-complementary.*

- i. *If M and N are both even, then every corner-notch pair in λ is admissible.*
- ii. *If M and N are both odd, then there is exactly one inadmissible corner-notch pair in λ .*
- iii. *If $M \not\equiv N \pmod{2}$, then there is at most one inadmissible corner-notch pair in λ .*

Quasi-self-complementary shapes for which there is an inadmissible corner-notch pair will be called *reducible*. Thus, Proposition 22 states that if M and N are both even, then there are no $M \times N$ quasi-self-complementary reducible shapes, while if M and N are both odd, then every $M \times N$ quasi-self-complementary shape is reducible.

We call the process of moving the corner cell to the notch in an admissible corner-notch pair a *swap*. We write $Sw_{(a,b)}(\lambda)$ to denote the resulting shape. The shape $Sw_{(a,b)}(\lambda)$ is also $M \times N$ quasi-self-complementary and the pair (b, a) will be an admissible corner-notch pair in $Sw_{(a,b)}(\lambda)$.

We use similar notation to describe moving the tableau entry in the corner cell to the corresponding notch. That is, if T is a standard tableau of $M \times N$ quasi-self-complementary shape λ and (a, b) is an admissible corner-notch pair, then $Sw_{(a,b)}(T)$ is the tableau produced by moving the entry in cell a to cell b . Also, $Sw_{(a,b)}(\lambda)$ is the shape of $Sw_{(a,b)}(T)$. If $Sw_{(a,b)}(T)$ is a standard tableau, then we say that T can be *1-swapped* at (a, b) . If T cannot be 1-swapped at any of its admissible corner-notch pairs, then we say T is *1-fixed*.

For example, if $M = 5$, $N = 6$, and $\lambda = (6, 5, 3, 1, 0)$, then λ is reducible, since the corner-notch pair in row 3 is inadmissible. Furthermore, the tableau

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 9 & 12 & 14 \\ \hline 3 & 4 & 8 & 11 & 13 & \\ \hline 5 & 10 & 15 & & & \\ \hline 7 & & & & & \\ \hline & & & & & \\ \hline \end{array}$$

can be 1-swapped, whereas the tableau

$$T' = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 7 & 9 & 10 \\ \hline 3 & 4 & 8 & 11 & 13 & \\ \hline 5 & 14 & 15 & & & \\ \hline 12 & & & & & \\ \hline & & & & & \\ \hline \end{array}$$

is 1-fixed.

Proposition 23. *If the $M \times N$ quasi-self-complementary shape is not reducible, then every standard tableau of shape λ has at least one admissible corner-notch pair (a, b) at which T can be 1-moved.*

Proof. Find the largest entry in T , which must be in the corner of an admissible corner-notch pair (a, b) . \square

8.2. The Even-by-Even Case. If M and N are both even, we define the map $\Phi_{M \times N}$ from $M \times N$ quasi-self-complementary standard tableaux to $M \times N$ quasi-self-complementary standard tableaux as follows. Find the largest entry in T and move it to its complementary position.

Theorem 24. *If M and N are both even, then $\Phi_{M \times N}$ is an involution on standard tableaux of $M \times N$ quasi-self-complementary shape. Furthermore, $\Phi_{M \times N}$ has no fixed points and reverses the parity of twist.*

Proof. The map $\Phi_{M \times N}$ is well-defined by Proposition 22 and Proposition 23. It is clearly an involution which reverses twist parity. \square

Corollary 25. *If P is the product of an M -chain and an N -chain, with M and N both even, then P is sign-balanced.*

For example, let $M = 4$, $N = 6$ and $\lambda = (5, 3, 3, 1)$. Let

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 7 & 8 & 11 \\ \hline 3 & 4 & 9 & & \\ \hline 5 & 6 & 10 & & \\ \hline 12 & & & & \\ \hline \end{array}.$$

Then

$$\Phi_{M \times N}(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 7 & 8 & 11 & 12 \\ \hline 3 & 4 & 9 & & & \\ \hline 5 & 6 & 10 & & & \\ \hline & & & & & \\ \hline \end{array}.$$

8.3. The Odd-by-Odd Case. If M and N are not both even, and if T can be 1-moved, then the same involution as in the even-by-even case can be used. However, if T is 1-fixed, then a more complicated move is required. Note that by Proposition 23 this means that the shape of T is reducible.

Let λ be a reducible $M \times N$ quasi-self-complementary shape. Let $m = \lfloor \frac{M}{2} \rfloor$. Then let

$$R(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_m - 1, \lambda_{m+2}, \lambda_{m+3}, \dots, \lambda_M).$$

That is, $R(\lambda)$ is obtained from λ by removing one cell from each row in the upper half, and one cell from each column in the left half. We call $R(\lambda)$ the *reduction* of λ .

For example, if $M = 4$, $N = 5$ and $\lambda = (5, 3, 2, 0)$, then $R(\lambda) = (4, 2, 0)$. If $M = 5$, $N = 7$ and $\lambda = (7, 5, 3, 2, 0)$, then $R(\lambda) = (6, 4, 2, 0)$.

By Proposition 22, R is not defined when M and N are both even and is not defined for some quasi-self-complementary shapes when $M \not\equiv N \pmod{2}$. However, it is defined for every quasi-self-complementary shape when M and N are both odd.

Proposition 26.

- i. *The reduction operator R is a bijection from reducible $M \times N$ quasi-self-complementary shapes to all $(M - 1) \times (N - 1)$ quasi-self-complementary shapes.*
- ii. *Suppose λ and μ are two reducible $M \times N$ quasi-self-complementary shapes and (a, b) is an admissible corner-notch pair in λ such that $\mu = Sw_{(a,b)}(\lambda)$. Then there exists an admissible corner-notch pair (c, d) of $R(\lambda)$ such that*

$$Sw_{(c,d)} \circ R(\lambda) = R(\mu).$$

- iii. *Conversely, if (c, d) is an admissible corner-notch pair of $R(\lambda)$, with*

$$R(\mu) = Sw_{(c,d)} \circ R(\lambda),$$

then there exists an admissible corner-notch pair (a, b) of λ such that $\mu = Sw_{(a,b)}(\lambda)$.

Proof. For Part (i), we show how to construct the inverse map of R . Suppose μ is an $(M - 1) \times (N - 1)$ quasi-self-complementary shape where one of M or N is odd. Let

$$\lambda = (\mu_1 + 1, \mu_2 + 1, \dots, \mu_m + 1, \left\lfloor \frac{N}{2} \right\rfloor, \mu_{m+1}, \mu_{m+2}, \dots, \mu_{M-1}).$$

We must verify that λ is an $M \times N$ quasi-self-complementary reducible partition.

First, since λ is quasi-self-complementary,

$$(5) \quad \lambda_i + \lambda_{M+1-i} = \mu_i + 1 + \mu_{M-i} = N$$

for $1 \leq i \leq m$.

For M and N both odd, we have

$$\lambda_m = \mu_m + 1 \geq \frac{N-1}{2} + 1 > \lambda_{m+1}$$

and

$$\lambda_{m+2} = \mu_{m+1} \leq \frac{N-1}{2} = \lambda_{m+1},$$

which show λ is a partition. It is quasi-self-complementary because of Equation (5) and the fact that $\lambda_{m+1} = \frac{N-1}{2}$. And it must be reducible since M and N are both odd.

If M is odd and N is even, then

$$\lambda_m = \mu_m + 1 \geq \frac{N}{2} + 1 > \lambda_{m+1}$$

and

$$\lambda_{m+2} = \mu_{m+1} \leq \frac{N}{2} = \lambda_{m+1},$$

from which it follows that λ is a partition. It is quasi-self-complementary because of Equation (5) and $\lambda_{m+1} = \frac{N}{2}$. It is reducible because from above $\lambda_m > \lambda_{m+1}$.

Finally, for M even and N odd,

$$\lambda_m = \mu_m + 1 = \frac{N-1}{2} + 1 > \lambda_{m+1}$$

and

$$\lambda_{m+2} = \mu_{m+1} \leq \frac{N-1}{2} = \lambda_{m+1},$$

which imply that λ is a partition. It is quasi-self-complementary by Equation (5)

and

$$\lambda_m + \lambda_{m+1} = \left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{2} \right\rfloor = N.$$

It is reducible since $\lambda_m = \frac{N+1}{2}$.

Parts (ii) and (iii) follow because if (a, b) is a corner-notch pair and the number of rows between the row of a and the row of b is k , then R will produce a corner-notch pair whose “row distance” is $k - 1$. Thus, R will destroy corner-notch pairs at the center (i. e., inadmissible), but preserve all the others. Conversely, corner-notch pairs in the image of R will be further apart in “row distance” in the preimage. \square

Parts (ii) and (iii) of Proposition 26 say that there is a bijection between all admissible corner-notch pairs of $R(\lambda)$ and those of λ whose swaps are reducible shapes.

For example, let $M = 5$, $N = 6$ and $\lambda = (6, 4, 3, 2, 0)$. Then λ is reducible and $R(\lambda) = (5, 3, 2, 0)$. Also there are three admissible corner-notch pairs of λ , which, when swapped, produce the shapes $(6, 5, 3, 1, 0)$, $(5, 4, 3, 2, 1)$ and $(6, 3, 3, 3, 0)$. The first two of these are reducible while the third is not. In fact, $R(6, 5, 3, 1, 0) = (5, 4, 1, 0)$ and $R(5, 4, 3, 2, 1) = (4, 3, 2, 1)$. Finally, $R(\lambda)$ has two admissible corner-notch pairs, which, when swapped, produce $(5, 4, 1, 0)$ and $(4, 3, 2, 1)$.

In the odd-by-odd case, every $M \times N$ quasi-self-complementary shape is reducible, so the $M \times N$ quasi-self-complementary shapes are in one-to-one correspondence with the $(M - 1) \times (N - 1)$ quasi-self-complementary shapes. Also, the admissible corner-notch pairs of λ are in one-to-one correspondence with corner-notch pairs of $R(\lambda)$.

We now characterize the 1-fixed standard tableaux.

Lemma 27. *Let T be a 1-fixed standard tableau of $M \times N$ quasi-self-complementary shape λ . Then the entries in the skew shape $\lambda/R(\lambda)$, as shown in Figure 7, must satisfy these inequalities:*

$$(6) \quad a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < b_4 < a_4 < \dots$$

Conversely, if T is a standard tableau of $M \times N$ reducible quasi-self-complementary shape λ which satisfies the inequalities (6), then T is 1-fixed.

Proof. The converse is immediate, since the inequalities (6) produce tableau violations when a 1-swap is attempted.

We have $a_0 < a_1$, $b_1 < b_2$, $a_2 < a_3$, etc., by the standard tableau conditions. We have $a_1 < b_1$, for otherwise we could move the a_1 to the notch outside the b_1 . We have $b_2 < a_2$, for otherwise we could move the b_2 into the notch outside the a_2 . Continuing in this fashion gives the result. \square

If T is 1-fixed, then remove the entries from the end of each of the first $m = \lfloor \frac{M}{2} \rfloor$ rows and from the bottom of the first $n = \lfloor \frac{N}{2} \rfloor$ columns and call the resulting tableau $R(T)$, or the *reduction* of T . The reduction of T has shape $R(\lambda)$.

In both the odd-by-odd case and the odd-by-even case, we will need a “reduction condition.” We describe this next. Let T and T' be two tableaux satisfying the following conditions.

- i. The tableau T is a 1-fixed standard tableau of $M \times N$ quasi-self-complementary shape λ .
- ii. The tableau T' has shape $Sw_{(a,b)}(\lambda)$ for some admissible corner-notch pair (a, b) . Without loss of generality, assume a is above b .
- iii. The tableaux T and T' are equal for all entries not in the hook of (a, b) .
- iv. The tableau $R(T')$ is standard.

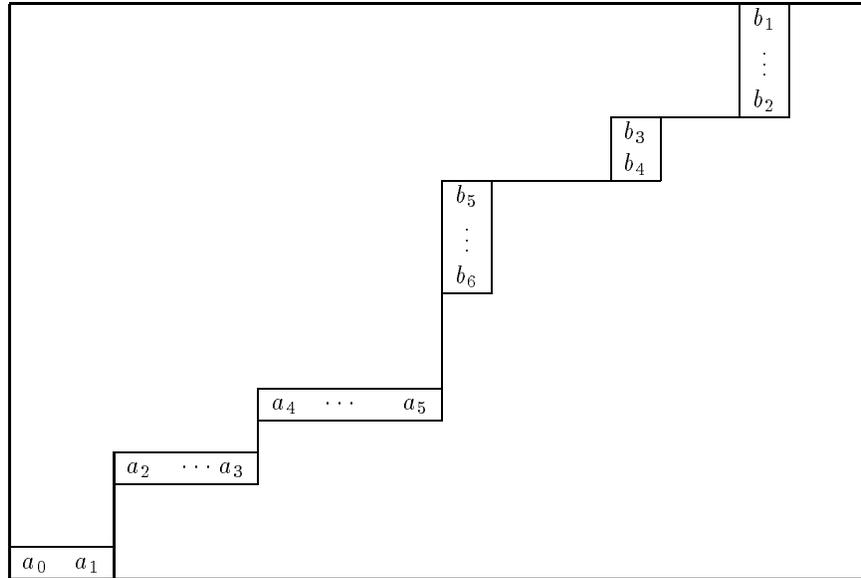


FIGURE 7. 1-Fixed inequalities

- v. If x and y are the last two entries in the row of a in T and z is the last entry in the column of b in T , then x and y are the last two entries in the column of b in T' and z is the last entry in the row of a in T' .

The last condition is shown in Figure 8. If these conditions are satisfied, we say T and T' are *joined*.

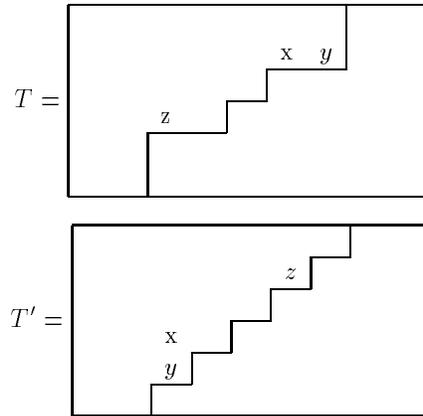


FIGURE 8. 2-pair conditions

Lemma 28. *If T and T' are joined, then T' is standard.*

Proof. Since $R(T')$ is standard and T and T' agree outside the hook of (a, b) , we only need to verify tableau inequalities around y and z . We indicate what can happen

$$\begin{array}{cc}
L = \begin{array}{c} u_2 \quad p \quad u_3 \\ u_1 \quad z \quad \text{---} \\ u_0 \quad \text{---} \quad \circ \end{array} & U = \begin{array}{c} v_3 \quad v_2 \quad v_1 \\ q \quad x \quad y \quad \text{---} \\ v_4 \quad \text{---} \quad v_5 \quad \text{---} \quad \circ \end{array} \\
L' = \begin{array}{c} u_2 \quad p' \quad u_3 \\ u_1 \quad x \quad \text{---} \\ u_0 \quad \text{---} \quad y \quad \text{---} \end{array} & U' = \begin{array}{c} v_3 \quad v_2 \quad v_1 \\ q' \quad z \quad \text{---} \\ v_4 \quad \text{---} \quad v_5 \quad \text{---} \end{array}
\end{array}$$

FIGURE 9. Joined tableaux

$$\begin{array}{cc}
L = \begin{array}{c} u_2 \quad p \quad u_3 \\ u_1 \quad z \quad u_4 \\ u_0 \quad \text{---} \quad \circ \end{array} & U = \begin{array}{c} v_3 \quad v_2 \quad v_1 \\ q \quad x \quad y \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \circ \end{array} \\
L' = \begin{array}{c} u_2 \quad p' \quad u_3 \\ u_1 \quad x \quad u_4 \\ u_0 \quad \text{---} \quad y \quad \text{---} \end{array} & U' = \begin{array}{c} v_3 \quad v_2 \quad v_1 \\ q' \quad z \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \end{array}
\end{array}$$

FIGURE 10. Joined tableaux

in two figures. In these figures, U refers to the upper portion of T containing the corner a , L refers to the lower portion of T containing the notch b , and U' and L' refer to the same regions in T' . We use \circ to indicate the corner-notch pair (a, b) .

We refer first to Figure 9. By Lemma 27, $u_0 < y < z < v_5 < u_3$. Then using the fact that z is larger than all the entries in the hook of (a, b) , we have

$$\begin{array}{ccc}
y > u_0 & u_3 > p' & z > q' \\
z > v_2 \quad \text{and} & z < v_5, &
\end{array}$$

which are the required tableau conditions.

Now referring to Figure 10, by Lemma 27, $u_0 < y < z < u_4$. Then using the fact that z is larger than all the entries in the hook of (a, b) , we have

$$\begin{array}{ccc}
y > u_0 & x < u_4 \\
z > q' \quad \text{and} & z > v_2,
\end{array}$$

which are the required tableau conditions. □

Lemma 29. *If T and T' are joined, then T' is 1-fixed.*

Proof. Using the same notation as in the previous proof, we refer to Figure 11. We use the labels \circ , \scriptsize 1 and \scriptsize 2 to indicate corner-notch pairs. The corner-notch pair (a, b) is \circ .

Since T is 1-fixed, by Lemma 27 we have from Figure 11

$$\cdots < y < z < u_3 < v < \cdots.$$

But then by the converse stated in Lemma 27 (and Figure 11), we have that T' is 1-fixed. □

$$\begin{aligned}
 L &= \begin{array}{cc} p_1 & u_1 \\ p_2 & u_2 \\ z & u_3 \end{array} \Big|_1 \\
 &\quad \Big|_0 \\
 U &= \begin{array}{ccc} q & x & y \\ v & \Big|_1 & \Big|_0 \end{array} \\
 \\
 L' &= \begin{array}{cc} p'_1 & u_1 \\ p'_2 & u_2 \\ x & u_3 \end{array} \Big|_1 \\
 &\quad \Big|_0 \\
 U' &= \begin{array}{ccc} q' & z & \Big|_0 \\ v & \Big|_1 & \Big|_2 \end{array}
 \end{aligned}$$

FIGURE 11. Joined tableaux

We may now describe the complete involution in the odd-by-odd case, which we call $\Phi_{M \times N}$. We require that M and N both be > 1 .

If T has a 1-swap, then swap the largest such corner entry.

If T is 1-fixed, then by Proposition 26, $R(\lambda)$ is an even-by-even self-complementary shape. Let (a', b') be the corner-notch pair to be swapped in the construction of

$$\Phi_{(M-1) \times (N-1)} \circ R(T).$$

Let (a, b) be the corner-notch pair which corresponds to (a', b') under the bijection in Proposition 26. Without loss of generality, assume a is above b . Let c be the cell to the left of a and let d be the cell above b . Now form $\Phi_{M \times N}(T)$ by placing the entry in cell a into cell b and by swapping the entries in cell c and cell d .

For example, let $M = 5$, $N = 7$ and $\lambda = (7, 5, 3, 2, 0)$. If

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 6 & 7 & 8 & 9 \\ 4 & 5 & 12 & 15 & 16 & & \\ 10 & 13 & 17 & & & & \\ 11 & 14 & & & & & \\ & & & & & & \end{array}$$

Then cell a' contains 15, cell a contains 16, cell c contains 15 and cell d contains 17. Thus

$$\Phi_{M \times N}(T) = \begin{array}{cccccc} 1 & 2 & 3 & 6 & 7 & 8 & 9 \\ 4 & 5 & 12 & 17 & & & \\ 10 & 13 & 15 & & & & \\ 11 & 14 & 16 & & & & \\ & & & & & & \end{array}$$

Theorem 30. *If M and N are both odd with M and $N > 1$, then $\Phi_{M \times N}$ is an involution on standard tableaux of $M \times N$ quasi-self-complementary shape. Furthermore, $\Phi_{M \times N}$ has no fixed points and reverses the parity of twist.*

Proof. If T can be 1-swapped, then the result is clear. If T is 1-fixed, let $T' = \Phi_{M \times N}(T)$. Then T and T' are joined. The tableau T' is then standard by Lemma 28. The map $\Phi_{M \times N}$ is an involution because T' is 1-fixed by Lemma 29 and because $\Phi_{(M-1) \times (N-1)}$ is an involution. \square

Corollary 31. *If P is the product of an M -chain and an N -chain, with M and N both odd and > 1 , then P is sign-balanced.*

8.4. The Odd-by-Even Case. The building blocks of the sign-reversing involution $\Phi_{M \times N}$ if $M \not\equiv N \pmod{2}$ are Lemmas 28 and 29.

In the odd-by-even (or even-by-odd) case, the shapes may or may not be reducible, by Proposition 22. If the shape is not reducible, or if the tableau can be 1-swapped, then the even-by-even involution may be applied. However, if the shape is reducible and the tableau is 1-fixed, then something similar to the odd-by-odd case must be done.

Let T be a tableau of $M \times N$ quasi-self-complementary shape λ , with $M \not\equiv N \pmod{2}$. Find the smallest j such that $R^j(T)$ is not 1-fixed.

Let (a', b') be the corner-notch pair to be swapped in the construction of

$$\Phi_{(M-j) \times (N-j)} \circ R^j(T).$$

Let (a, b) be the corner-notch pair of λ which corresponds to (a', b') under the iteration of the bijection in Proposition 26. Without loss of generality, assume a is above b . Let c be the j cells to the left of cell a in T and let d be the j cells above cell b . Now form $\Phi_{M \times N}(T)$ by moving the entry in cell a into cell b and by swapping the entries in cells c with the entries in cells d .

For example, let $M = 6$, $N = 7$ and $\lambda = (7, 5, 4, 3, 2, 0)$. If

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 9 & 14 \\ \hline 6 & 7 & 11 & 15 & 18 & & \\ \hline 8 & 10 & 19 & 21 & & & \\ \hline 12 & 13 & 20 & & & & \\ \hline 16 & 17 & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

then

$$\Phi_{M \times N}(T) = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 8 & 12 & 16 & \\ \hline 3 & 7 & 11 & 15 & 18 & & \\ \hline 4 & 10 & 19 & 21 & & & \\ \hline 5 & 13 & 20 & & & & \\ \hline 9 & 17 & & & & & \\ \hline 14 & & & & & & \\ \hline \end{array}.$$

In this example, $j = 4$

Theorem 32. *If $M \not\equiv N \pmod{2}$, then $\Phi_{M \times N}$ is an involution on standard tableaux of $M \times N$ quasi-self-complementary shape. Furthermore, $\Phi_{M \times N}$ reverses the parity of twist.*

Proof. The proof is by induction on M and N . When one of M or N is 1, the result is clear.

Suppose T is a 1-fixed standard tableau of $M \times N$ quasi-self-complementary shape. We must show $T' = \Phi_{M \times N}(T)$ is also standard and that $\Phi_{M \times N}(T') = T$. For this last part, it suffices to show $R^i(T')$ is 1-fixed for $0 \leq i < j$, for then $\Phi_{M \times N}$ will pick the same j and the same entries to move.

Since $R(T)$ is $(M-1) \times (N-1)$ quasi-self-complementary, by induction we have that each

$$R^i \circ \Phi_{(M-i) \times (N-i)} \circ R(T),$$

for $0 \leq i < j-1$, is standard and 1-fixed. Therefore each $R^i \circ \Phi_{M \times N}(T)$, for $0 < i < j$, is standard and 1-fixed. It therefore suffices to prove that T' is standard

and 1-fixed. But now we can easily check that T and T' are joined, so the result follows from Lemmas 28 and 29. \square

8.5. The Fixed Points. Finally, we identify the fixed points in the case that $M \not\equiv N \pmod{2}$. An immediate consequence of Proposition 23 and the construction of $\Phi_{M \times N}$ is that if T is a fixed point and has shape λ , then $R^j(\lambda)$ is reducible for every j .

Proposition 33. *If $M \not\equiv N \pmod{2}$, then $stst(M, N)$ is the unique $M \times N$ quasi-self-complementary shape λ such that $R^j(\lambda)$ is reducible for every j .*

Proof. The proof is by induction on M and N . If M or N is 1, the result is obvious.

It is clear that $stst(M, N)$ is reducible and that

$$(7) \quad R(stst(M, N)) = stst(M - 1, N - 1).$$

Therefore $stst(M, N)$ has the desired property. We must show that it is the only shape having that property. We proceed by induction, using Proposition 26 as our tool. Suppose λ is another shape such that $R^j(\lambda)$ is reducible for every j . Then $R(\lambda)$ must also have this property. But then by induction $R(\lambda) = stst(M - 1, N - 1)$. Recall from Proposition 26 that R is a bijection from reducible $M \times N$ quasi-self-complementary shapes to all $(M - 1) \times (N - 1)$ quasi-self-complementary shapes. By Equation (7), we have $\lambda = stst(M, N)$. \square

The fixed points of $\Phi_{M \times N}$ must then be tableaux of shape $stst(M, N)$ which satisfy the inequalities in Lemma 27 at each iteration of R .

Theorem 34. *The fixed points of $\Phi_{M \times N}$ are in one-to-one correspondence with standard shifted tableaux of shape $\nabla stst(M, N)$.*

Proof. Let T be a fixed point of $\Phi_{M \times N}$. Then $sh(T) = stst(M, N)$ and the inequalities in Lemma 27 are satisfied for each $R^i(T)$.

We now describe the required bijection Ψ recursively. If M or N is 1, then let $\Psi(T) = T$.

For M and N both > 1 , let $t = (M + N - 1)/2$ and $Z = \frac{MN}{2} + 1$. Label the entries of T which are not in $R(T)$ according to Figure 12. Note that the entries labeled a_1, \dots, a_t are the entries in $stst(M, N)/stst(M - 1, N - 1)$. Also, the entries labeled b_1, \dots, b_{t-1} are the entries in $stst(M - 1, N - 1)/stst(M - 2, N - 2)$.

Construct the first row of $\Psi(T)$ as shown in Figure 13. Fill the rest of $\Psi(T)$ by applying Ψ to $R(T)$ recursively. We now verify the tableau conditions on $\Psi(T)$. By Lemma 27, the first row will increase. The second row of $\Psi(T)$ will be as shown in Figure 13. The column inequalities thus follow from the tableau inequalities on T satisfied by a_1, \dots, a_t and b_1, \dots, b_{t-1} .

To verify that this is a bijection, we must check that a shifted tableau T' will yield a fixed point by reversing this process. But the inequalities of Lemma 27 are satisfied because the first row of T' is increasing, and the tableau conditions on T follow from the tableau conditions on T' (see again Figures 12 and 13). \square

Corollary 35. *The number of fixed points of $\Phi_{M \times N}$ is 0 if $M \equiv N \pmod{2}$ with M and N both > 1 and is $g^{stst(M, N)}$ if $M \not\equiv N \pmod{2}$.*

Corollary 35 is a restatement of Theorem 1.

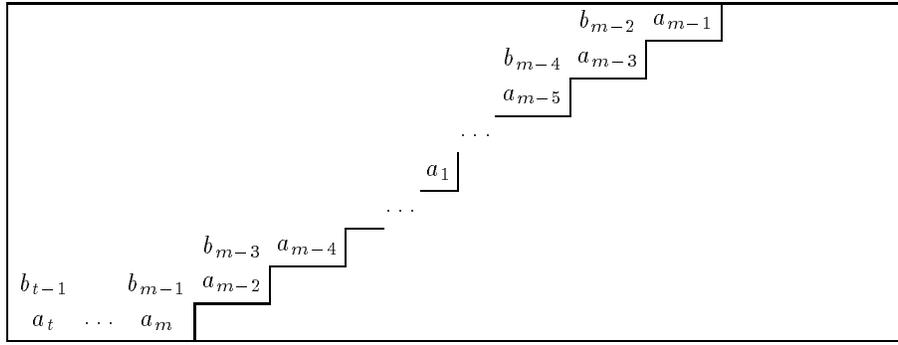


FIGURE 12. A staircase tableau

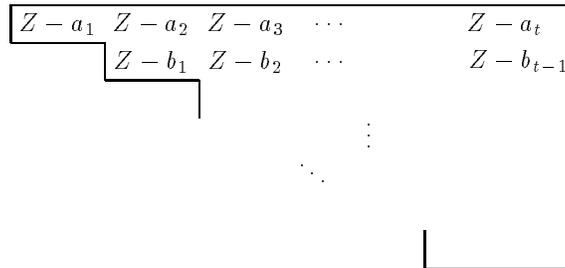


FIGURE 13. The corresponding shifted tableau

9. REMARKS

The involution Φ (and many of the concepts in Section 8) may be extended to semi-self-complementary shapes in an obvious way. However, complete cancellation does not result. It would be interesting to find generalizations of Φ and Ψ which prove Corollary 35. Even more enticing from a combinatorial point of view is the identity in Theorem 18.

The proof in Section 7 leaves many unanswered questions. For instance, why are two different arguments necessary to evaluate

$$b_{\mu_1} \circ b_{\mu_2} \circ \cdots \circ b_{\mu_k}?$$

Can the requirement that α be flat be dropped? Can the generalized Kostka number be shown to be the same as the Carré-Leclerc Littlewood-Richardson coefficient without going through twist? These questions further indicate the need for a more transparent proof of Corollary 35.

There is strong empirical evidence that a proof of Corollary 20 using the spin representations of the symmetric group may exist.

If the shape λ is a partition with all even parts, each part with even multiplicity, then standard tiling arguments and the involution β can be used to show P_λ is sign-balanced. The odd case appears to be more complicated. Many examples of partitions with all odd parts, each part with odd multiplicity, are sign-balanced. There are also sign-balanced examples where λ is a partition of an odd number so

that the sign of the tableau must be modified by the statistic rv as in Theorem 9. However, there are also examples which are not sign-balanced, including partitions with odd distinct parts.

We have been able to prove that one other class of P_λ is sign-balanced. If the 2-quotient of λ is (μ, ν) and one of these partitions is (1) while the other is self-conjugate, then $INV_{P_\lambda}(-1)$ can be completely determined. The techniques used are entirely different from those described in this paper.

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