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# THE MINIMAL INTEGRAL SEPARATOR OF A THRESHOLD GRAPH

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A graph is called threshold if there exists a real number $b$ and real numbers $a_j$ associated with its vertices $w_j$ such that $\sum_{j\in S} a_j \leqslant b$ holds iff $S$ is a stable (independent) set of vertices. The vector $\langle a_1, \ldots, a_n; b\rangle$ associated to a threshold graph is called an integral separator if $a_i + a_j \geqslant b+1$ for every edge $(w_i, w_j)$. A simple algorithm is presented to determine for a given threshold graph its (unique) integral separator which minimizes $b$.

Let $G$ be a loopless finite graph without multiple edges. If $w$ is a vertex of $G$, let $d(w)$ be the degree of $w$. The edge joining vertices $u$ and $w$ will be denoted as $(u, w)$.

Graph $G$ is said to have *property P* if for every two vertices $u, v$ such that $(u, v)$ is an edge, and for every pair of vertices $u^*, v^*$ with $d(u^*) \geqslant d(u)$ and $d(v^*) \geqslant d(v)$, $(u^*, v^*)$ is an edge. In this definition it is possible that $u^* = u$ or that $v^* = v$.

It has been shown in [1] that graph $G$ has property $P$ iff it is a threshold graph.

Suppose $G$ is a threshold graph with vertices $w_1, w_2, w_3, \ldots, w_n$. For $I \subseteq \{1, 2, \ldots, n\}$ let $S_I = \{w_i \mid i \in I\}$. Let $A = \langle a_1, a_2, \ldots, a_n\rangle$ be a real vector and let $b$ be a real number. The pair $[A; b]$ is said to *separate G integrally* if the following holds:

(1) $a_i \geqslant 0$ for $i = 1, \ldots, n$;

(2) $\sum_{i\in I} a_i \leqslant b$ iff $S_I$ is a stable (independent) set of vertices;

(3) $\sum_{i\in I} a_i \geqslant b+1$ iff $S_I$ is a non-stable set of vertices.

It was shown in [1] that a graph $G$ is threshold iff there exists a pair $[A; b]$ which separates $G$ integrally.

The following algorithm determines for a threshold graph $G$ a hyperplane $[A^*; b^*]$ which separates $G$ integrally and such that $b^*$ is minimum. It will also be shown that it is the unique hyperplane with minimum $b$.

**Algorithm A.**

*Step 0:* Relabel the vertices as $w_1, \ldots, w_n$ such that $d(w_1) \leqslant d(w_2) \leqslant \cdots \leqslant d(w_n)$.

*Step 1:* Let $t$ = minimum index such that $(w_t, w_{t+1})$ is an edge of $G$. [If no such $t$ exists let $a_i^* = 0$ for $i = 1$ to $n$ and let $b^* = 0$. Then exit from algorithm.]

*Step 2:* If $d(w_1) = 0$ let $a_1^* = 0$. If $d(w_1) \geqslant 1$ let $a_1^* = 1$.

*Step 3:* For $i = 2$ to $t$ if $d(w_i) = d(w_{i-1})$ then let $a_i^* = a_{i-1}^*$; if $d(w_i) > d(w_{i-1})$ then let $a_i^* = 1 + a_1^* + a_2^* + \cdots + a_{i-1}^*$.

*Step 4:* Let $b^* = a_1^* + a_2^* + \cdots + a_t^*$.

*Step 5:* For $i = t+1$ to $n$ let $s_i$ be the minimum index such that $(w_i, w_{s_i})$ is an edge. Then let $a_i^* = b^* - a_{s_i}^* + 1$.

**Example.** Let $G$ be the graph in Fig. 1. Table 1 shows how the algorithm worked.

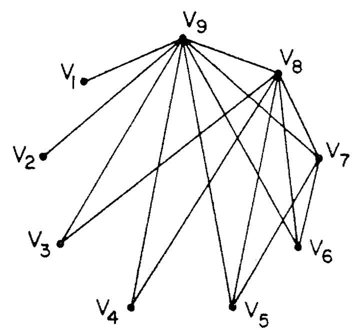


Fig. 1

Table 1

| | $V_1$ | $V_2$ | $V_3$ | $V_4$ | $V_5$ | $V_6$ | $V_7$ | $V_8$ | $V_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $d(V_i)$ | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 6 | 8 |
| $a_i^*$ | 1 | 1 | 3 | 3 | 9 | 9 | 18 | 24 | 26 |
| defining step of algorithm | 2 | 3 | 3 | 3 | 3 | 3 | 5 | 5 | 5 |

$t = 6$ as defined in step 1.
$b^* = 26$ as defined in step 4.

**Proposition 1.** *$[A; b]$ as constructed in algorithm A does separate the threshold graph $G$ integrally.*

**Proof.** Assume that the vertices have already been relabeled such that $d(w_1) \leqslant d(w_2) \leqslant \cdots \leqslant d(w_n)$.

*Case 0:* Algorithm A exited at step 1 after labeling $a_i^* = 0$ for $i = 1$ to $n$.

*Claim*: $G$ has no edges. Else consider edge $(w_i, w_j)$ of $G$ such that $i < j$. Then $i \leqslant n-1$ and $j \leqslant n$. From this it follows that $d(w_{n-1}) \geqslant d(w_i)$ and $d(w_n) \geqslant d(w_j)$. But $G$ has property $P$. Thus $(w_{n-1}, w_n)$ is an edge. Thus in algorithm A $t \leqslant n-1$. This contradicts that the algorithm exited at step 1. Hence $G$ has no edges. It follows from the definitions that $[A^*, b^*]$ does separate $G$ integrally in this case.

Now assume that algorithm A exited at step 5 with $[A^*; b^*]$ which does not separate $G$ integrally.

*Case 1:* There exists a stable set $S_I$ such that $\sum_{i\in I} a_i^* > b^*$. Let $I = \{j_1, j_2, \ldots, j_k\}$ with $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k$. If $j_k \leqslant t$ then $\sum_{i\in I} a_i^* \leqslant \sum_{i=1}^{t} a_i^* = b^*$ and we have a contradiction. Thus we may assume that $j_k \geqslant t+1$. Let $q = s_{j_k}$ as chosen in step 5 of algorithm A. Thus $(w_q, w_{j_k})$ is an edge of $G$. If $q \leqslant j_{k-1}$ then $d(w_q) \leqslant d(w_{j_{k-1}})$. Since $G$ has property $P$, this would mean that $(w_{j_{k-1}}, w_{j_k})$ is an edge of $G$, contradicting that $S_I$ is stable. Hence we may assume that $q > j_{k-1}$. But now by construction of $[A^*, b^*]$ we have:

$$\sum_{i\in I} a_i^* \leqslant a_{j_k}^* + \sum_{i=1}^{q-1} a_i^* = a_{j_k}^* + (a_q^* - 1) = b^*.$$

Thus for all stable sets $S_I$ the proposition is true.

*Case 2:* There exists a non-stable set $S_I$ such that $\sum_{i\in I} a_i^* < b^* + 1$.

Then $S_I$ contains vertices $w_i$, $w_j$ such that $(w_i, w_j)$ is an edge. Assume that $i < j$. If $j \leqslant t$ then $i \leqslant t-1$ and $d(w_i) \leqslant d(w_{t-1})$ and $d(w_j) \leqslant d(w_t)$. Since $G$ has property $P$, this would imply that $(w_{t-1}, w_t)$ is an edge, which is a contradiction. Hence we may assume that $j > t$. Then by the choice of $S_j$ in step 5 it follows that $i \geqslant s_j$. But then

$$\sum_{h\in I} a_h^* \geqslant a_i^* + a_j^* \geqslant a_{s_j}^* + a_j^* = b^* + 1.$$

Thus the proposition is ture.

**Proposition 2.** *Let $[A; b]$ be any hyperplane that separates $G$ integrally, where $A = \langle a_1, a_2, \ldots, a_n \rangle$. Then for all $i$ from 1 to $t$ it is true that $a_i \geqslant a_i^*$.*

**Proof.** Once again assume $d(w_1) \leqslant d(w_2) \leqslant \cdots \leqslant d(w_n)$. If $d(w_1) = 0$ than $a_1^* = 0$ which is minimum by definition. Else there exists $w_j$ such that $(w_1, w_j)$ is an edge. Thus in any hyperplane $[A; b]$ which separates $G$ integrally we must have that $a_j \leqslant b$ and $a_1 + a_j \geqslant b+1$. This implies that $a_1 \geqslant 1$. Thus $a_1 \geqslant a_1^* = 1$.

Assume inductively that $a_i^*$ is minimum for $i = 1$ to $k-1$ for $k \leqslant t$. It will be shown that $a_k^*$ is also minimum.

Suppose $d(w_{k-1}) = d(w_k)$. Then since $G$ has property $P$, $w_{k-1}$ and $w_k$ are adjacent to the same other vertices. By symmetry and by the induction hypothesis $a_k \geqslant a_{k-1}^*$. Since $a_{k-1}^* = a_k^*$ we have that $a_k \geqslant a_k^*$ and that $a_k^*$ is thus minimum.

Suppose instead that $d(w_{k-1}) < d(w_k)$. Choose $q$ to be the minimum index such that $w_q$ is adjacent to $w_k$ but not to $w_{k-1}$. Since $G$ has property $P$, $w_q$ is not adjacent to any $w_i$ for $i = 1, \ldots, k-1$; it is also true that no two vertices in $S = \{w_1, w_2, \ldots, w_{k-1}\}$ are adjacent. Thus in any hyperplane $[A; b]$ we have

$$a_q + a_k \geqslant b + 1 \quad \text{and}$$

$$\sum_{i=1}^{k-1} a_i + a_q \leqslant b.$$

It follows that

$$a_k \geq 1 + \sum_{i=1}^{k-1} a_i \geq 1 + \sum_{i=1}^{k-1} a_i^* = a_k^*.$$

**Corollary.** *The value for $b^*$ is also minimum.*

**Proof.** $\{w_1, w_2, \ldots, w_t\}$ is a stable set. Thus

$$b \geq \sum_{i=1}^{t} a_i \geq \sum_{i=1}^{t} a_i^* = b^*.$$

**Proposition 3.** *The algorithm constructs the unique $[A; b]$ which separates $G$ integrally with minimum $b$.*

**Proof.** Suppose $[A; b^*]$ separates $G$ integrally. Since

$$b^* \geq \sum_{i=1}^{t} a_i \geq \sum_{i=1}^{t} a_i^* = b^*$$

it follows that $a_i = a_i^*$ for $i = 1$ to $t$.

For $i = t+1, t+2, \ldots, n$ we have that

$$a_i + a_{s_i}^* \geq b^* + 1$$

$$a_i + \sum_{j=1}^{s_i-1} a_j^* \leq b^*.$$

By construction

$$\sum_{j=1}^{s_i-1} a_j^* + 1 = a_{s_i}^*.$$

Thus $a_i = b^* - a_{s_i}^* + 1 = a_i^*$.

**Proposition 4.** *The hyperplane $[A^*, b^*]$ is also the solution to the following linear program:*

$$\min b$$

$$\text{s.t.} \sum_{i=1}^{t} a_i \leq b \tag{1}$$

*and, for $j = t+1$ to $n$,*

$$a_j + a_{s_j} \geq b + 1 \tag{2}$$

$$a_j + \sum_{i=1}^{s_j-1} a_i \leq b \tag{3}$$

*where $s_j$ and $t$ are chosen as in the algorithm.*

**Proof.** By (2) and property $P$ for any non-stable set $S_I$, $\sum_{i\in I} a_i \geqslant b+1$. If $S_I$ is stable then either $I \subseteq \{1,2,3,\ldots,t\}$ or else $I \subseteq \{1,2,\ldots,s_j-1,j\}$ for some $j$. In either case by (1) and (3) we must have that $\sum_{i\in I} a_i \leqslant b$.

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## Reference

[1] V. Chvátal and P.L. Hammer, Aggregation of Inequalities in Integer Programming, Ann. Discrete Math. 1 (1977) 145–162.