AN INTRODUCTION TO THE SHOCK RESPONSE SPECTRUM
Revision Q

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INTRODUCTION

Spacecraft and launch vehicle components encounter mechanical shock from a variety of sources. Components must be designed and tested accordingly to ensure reliability.

For example, engineers must anticipate transportation and shipping shock. Consider an avionics component encased in foam packing material inside a shipping container. The container is placed on a truck which runs over a speed bump or a railroad track at an imprudent speed. The avionics component may receive a half-sine shock pulse. This type of pulse can be readily represented in the time domain by its duration and peak amplitude. Also, reproduction of this pulse in an environmental test lab is usually straightforward.

Eventually, the avionics component is integrated into a spacecraft. The component must now withstand a series of flight shock pulses. These pulses result from rocket motor ignition, staging, and deployment events. Linear shape charge and pyrotechnic devices are typically used to initiate staging events. These devices produce high-frequency, high-amplitude shock energy as a by-product. A typical pyrotechnic pulse is shown in Figure 1. The corresponding shock response spectrum is given in Figure 2.
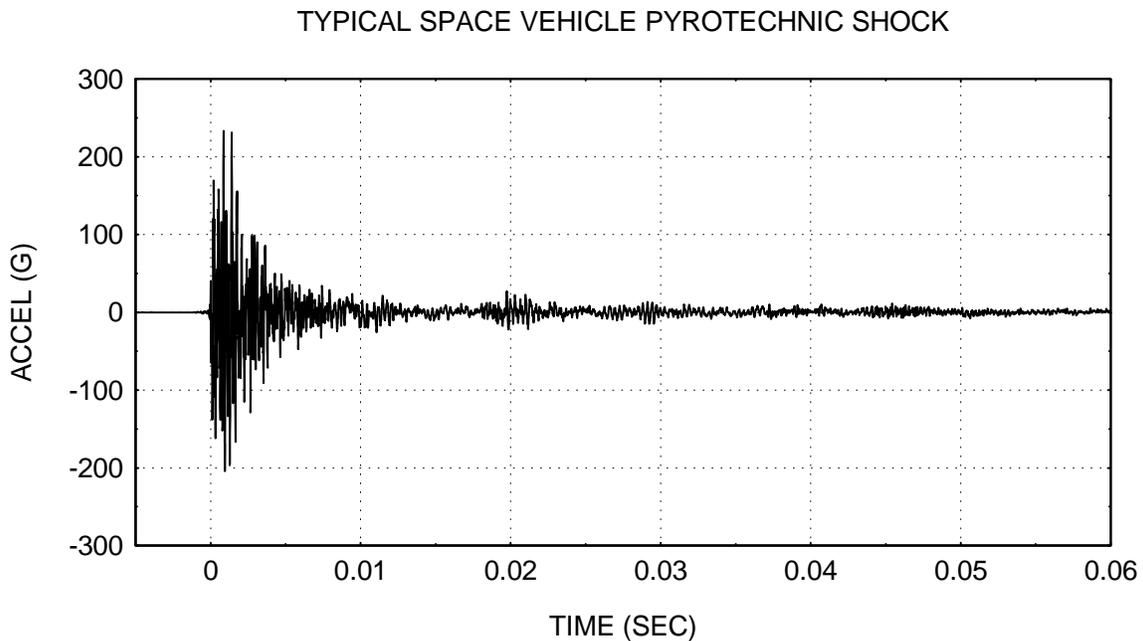

TYPICAL SPACE VEHICLE PYROTECHNIC SHOCK

Figure 1.

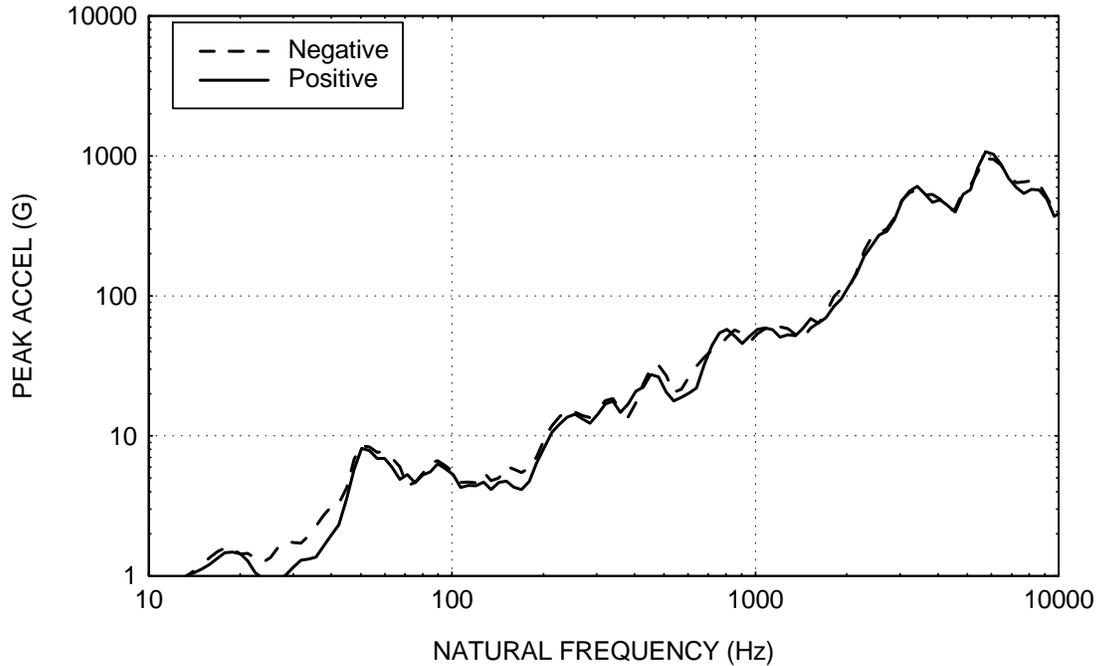SHOCK RESPONSE SPECTRA Q=10   TYPICAL SPACE VEHICLE PYROTECHNIC SHOCK

Figure 2. Shock Response Spectrum of a Pyrotechnic Input Pulse

Note that the shock response spectrum is displayed in terms of its positive and negative spectral curves. This format is sometimes used to evaluate the integrity of the pulse. Ideally, the two curves should be approximately equal, as is the case in this example. Note that the input pulse was measured during a pyrotechnic development test.

The pyrotechnic pulse in Figure 1 was measured during a nosecone fairing separation test performed on the ground. It is the result of ordnance detonation and the sudden release of strain energy from a clamp band.

One of the purposes of this test was to measure shock levels at component mounting locations so that proper component test levels could be derived.

Note that pyrotechnic shock energy is a serious concern for electronic components. This energy could cause a crystal oscillator to shatter, for example. It could also cause a relatively large component, such as a DC-to-DC converter, to detach from a circuit board.

The pyrotechnic pulse in Figure 1 is a complex waveform. It tends to oscillate in a somewhat symmetric manner about the zero baseline. Its overall envelope has an exponential decay, although some lingering reverberation is also present.

Unlike the transportation shock pulse, this pyrotechnic pulse is too complicated to represent by a time domain mathematical function. Engineers need some other tool to understand the effect of the pyrotechnic shock on spacecraft components.

Furthermore, reproduction of the pyrotechnic pulse for a component qualification test would be virtually impossible. This is particularly true if a mechanical test method is used to simulate the pyrotechnic event. Another approach is needed for testing.

The shock response spectrum is a useful tool for estimating the damage potential of a shock pulse, as well as for test level specification. Note that MIL-STD-1540C and MIL-STD-810E require this format for certain shock environments.

The purpose of this report is to explain the shock response spectrum and to give a derivation of a calculation method. The calculation method is carried out in the time domain via a convolution integral.[1] The end result is represented in terms of the *natural frequency domain,* however, as shown in Figure 2.

SHOCK RESPONSE SPECTRUM MODEL

The shock response spectrum is a calculated function based on the acceleration time history. It applies an acceleration time history as a base excitation to an array of single-degree-of-freedom (SDOF) systems, as shown in Figure 3 Note that each system is assumed to have no mass-loading effect on the base input.
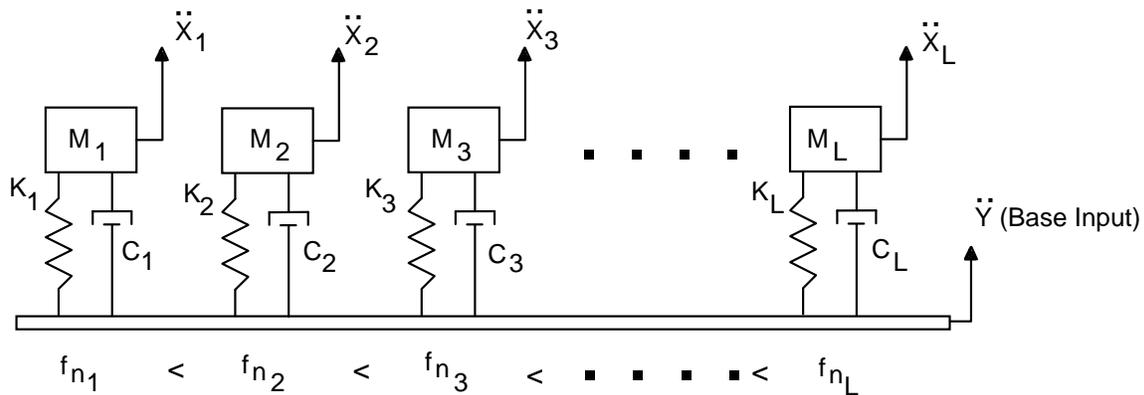

Figure 3. Shock Response Spectrum Model

$\ddot{Y}$ is the common base input for each system, and $\ddot{X}_i$ is the absolue response of each system to the input. The double‑dot denotes acceleration. M_i is the mass, C_i is the damping coefficient, and K_i is the stiffness for each system. f_{n_i} is the natural frequency for each system.

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[1] There is an equivalency between the convolution integral and the multiplication of Fourier transforms. Thus, the calculation process can be carried out in terms of Fourier transforms. This equivalent process requires additional calculation steps, however. The industry preference is thus the convolution method.

The damping of each system is typically assumed as 5%, which is equivalent to Q = 10. The natural frequency is an independent variable. Thus, the calculation is performed for a number of independent SDOF systems, each with a unique natural frequency.

Any arbitrary set of unique natural frequencies can be used for the shock response spectrum calculation. A typical scheme, however, is based on a proportional bandwidth, such as 1/6 octave. This means that each successive natural frequency is $2^{1/6}$ times the previous natural frequency. Thus, a sample set of N calculation frequencies in units of Hertz would be 10, 11.2, 12.6, 14.1, ... , $[10][2^{(1/6)(N-1)}]$. Note that the bandwidth increases with frequency.

Each of the natural frequencies in the above example represents the center frequency of an octave band. Continuing with the 1/6 octave format, the following relationships hold for the upper frequency fu and the lower frequency fl:

$$fl = [fc]/[2^{1/12}],$$
$$fu = [fc][2^{1/12}], \text{ where fc is the center frequency.}$$

The 14.1 Hz band thus has lower and upper limits of 13.3 Hz and 14.9 Hz, respectively.

The peak acceleration for a particular frequency is the exact value for that exact frequency, however.[2] For simplicity, the peak value may be assumed to be the same for any frequency in a given band, or the peak for an off-center frequency may be found by interpolation between two adjacent center frequencies.

HALF-SINE EXAMPLE

Each SDOF system has a unique time history response to a give base input. An example is shown in Figure 4. Note that the response calculation is carried out via a convolution integral, as explained later in the text.

The shock response spectrum is the peak absolute acceleration response of each SDOF system to the time history base input.[3] As an alternative, this function can be represented in terms of its peak positive and peak negative responses. The dimensions are peak response (G) versus natural frequency (Hz).

Figure 5 shows the shock response spectrum corresponding to the example in Figure 4.

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[2] In contrast, the level at a center frequency in a random vibration power spectral density represents the average mean square level across the band.

[3] Another shock response spectrum format is based on the relative displacement of the mass. An "acceleration" is then calculated by multiplying the relative displacement by ω_n^2, where ω_n, is the natural frequency in radians per second. This format is seldom used in the aerospace industry, however.

BASE INPUT HALF-SINE
PULSE (11 msec, 50 G)

RESPONSE (fn = 30 Hz, Q=10)

RESPONSE (fn = 80 Hz, Q=10)

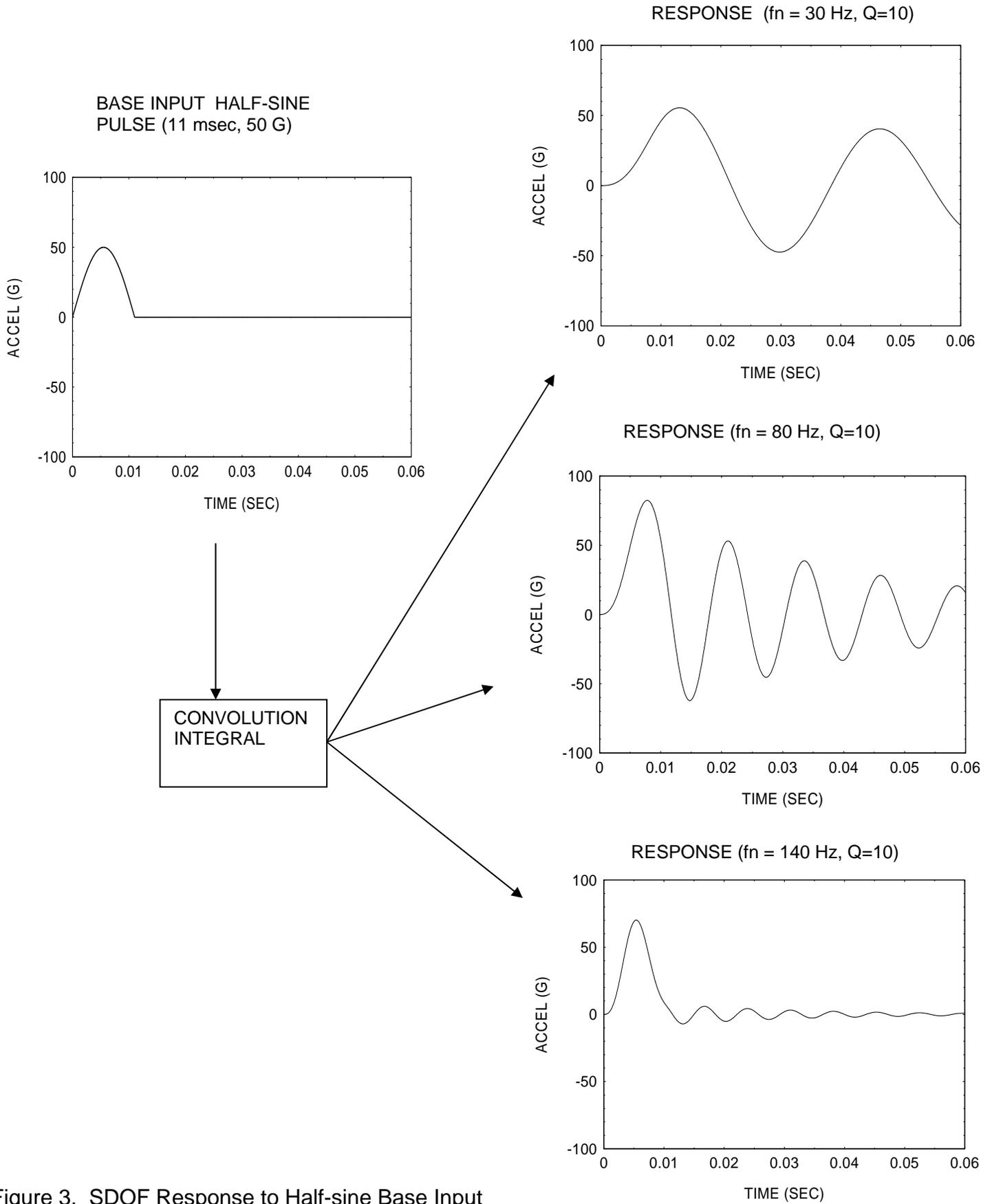CONVOLUTION
INTEGRAL

RESPONSE (fn = 140 Hz, Q=10)

Figure 3. SDOF Response to Half-sine Base Input

SHOCK RESPONSE SPECTRUM Q=10 BASE INPUT: HALF-SINE PULSE (11 msec, 50 G)
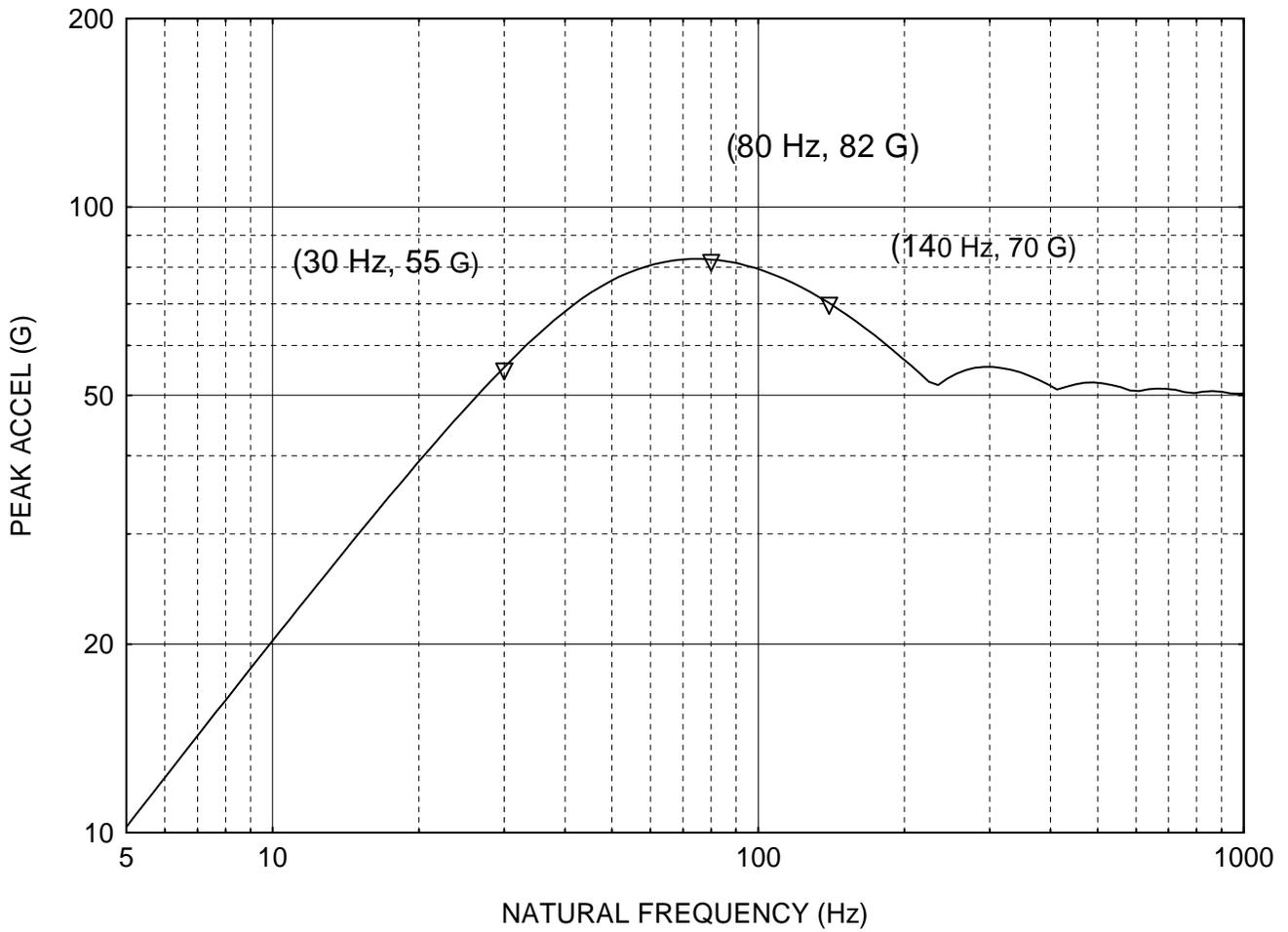

Figure 5. Corresponding Shock Response Spectrum

Note that coordinate pairs are given explicitly for three natural frequency cases. Each of these coordinates represents the peak absolute response for one of the examples in Figure 4.

DERIVATION

Newton's law can be applied to a free-body diagram of an individual system, as shown in Figure 6.

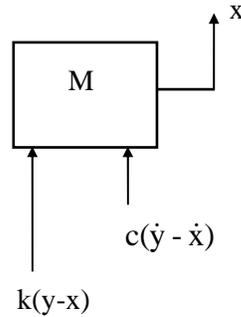

Figure 6. Free-body Diagram

A summation of forces yields the following governing differential equation of motion:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{1}$$

A relative displacement can be defined as $z = x-y$. The following equation is obtained by substituting this expression into equation (1):

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{2}$$

Additional substitutions can be made as follows,

$$\omega_n^2 = \frac{k}{m} \tag{3}$$

$$2\xi\omega_n = \frac{c}{m} \tag{4}$$

Note that ξ is the damping ratio, and that ω_n is the natural frequency in radians per second. Furthermore, ξ is often represented by the amplification factor Q, where $Q=1/(2\xi)$.

Substitution of these terms into equation (2) yields an equation of motion for the relative response

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -\ddot{y}(t) \tag{5}$$

Equation (5) does not have a closed-form solution for the general case in which $\ddot{y}(t)$ is an arbitrary function. A convolution integral approach must be used to solve the equation. The convolution integral is then transformed into a series for the case where $\ddot{y}(t)$ is in the form of digitized data. Furthermore, the series is converted to a digital recursive filtering relationship to expedite the calculation. A complete derivation is given in Appendix A. The resulting formula for the absolute acceleration is

$$\ddot{x}_i =$$

$$+ 2\exp[-\xi\omega_n\Delta t]\cos[\omega_d\Delta t]\,\ddot{x}_{i-1}$$

$$- \exp[-2\xi\omega_n\Delta t]\ddot{x}_{i-2}$$

$$+ 2\xi\omega_n\Delta t\,\ddot{y}_i$$

$$+ \omega_n\Delta t\exp[-\xi\omega_n\Delta t]\left\{\left[\frac{\omega_n}{\omega_d}\left(1-2\xi^2\right)\right]\sin[\omega_d\Delta t]-2\xi\cos[\omega_d\Delta t]\right\}\ddot{y}_{i-1}$$

$$(6)$$

Equation (6) was used to calculate the shock response spectrum in Figure 2. Note that this equation must be used for each natural frequency.

ALTERNATE CALCULATION METHOD

An alternate method for calculating the shock response spectrum is given in Appendix B. The alternate method is based on Fourier transforms.

SAMPLING RATE CRITERIA

A rule-of-thumb states that the sampling rate for the input time history should be at least ten times greater than the highest shock response spectrum calculation frequency.

The Institute of Environmental Sciences and Technology has published the "IES Handbook for Dynamic Data Acquisition and Analysis." This reference gives the following guideline:

Unlike other spectral quantities evolving from the discrete Fourier transform computations, the SRS is essentially a time domain quantity. Hence, the digital sampling rate given by Rs=1/(delta t), introduces errors beyond those associated with aliasing about the Nyquist frequency. Thus, Rs must be high enough to accurately describe the response of the SRS oscillators. To minimize potential error, it is recommended that the SRS computations be performed with a sampling rate of Rs $\geq$ 10 fh, where fh is the highest natural frequency of the SRS computation.

Likewise, the sampling rate should be at least ten times greater than the highest spectral component in the input data. This is a data acquisition issue. An analog lowpass filter can be used in the data stream to ensure that this criterion is met. Note that the filter should be analog so that the filtering is performed prior to digitization.

ERROR SOURCES

Baseline Shift

Numerous problems can affect the quality of accelerometer data during pyrotechnic shock events. A baseline shift, or zero shift, in the acceleration time history is perhaps the most common error source.

Chu notes in Reference 2 that this shift can be of either polarity and of unpredictable amplitude and duration. He has identified six causes of zero shift:

> a. Overstressing of sensing elements
> b. Physical movement of sensor parts
> c. Cable noise
> d. Base strain induced errors
> e. Inadequate low-frequency response
> f. Overloading of signal conditioner.

Accelerometer resonant ringing is a special example of causes "a" and "e." This is a particular problem if the accelerometer has a piezoelectric crystal as its sensing element.

A piezoelectric accelerometer may have its amplification factor Q well above 30 dB at resonance. This resonance may be excited by high-frequency pyrotechnic shock energy. Resonant ringing causes higher element stresses than expected.

Chu notes that this may cause the signal conditioner to overload, as follows

> When a signal conditioner attempts to process this signal, one of its stages is driven into saturation. Not only does this clipping distort the in-band signals momentarily, but the overload can partially discharge capacitors in the amplifier, causing a long time-constant transient.

This overload causes zero shift in the acceleration time history. This shift distorts the low-frequency portion of the shock response spectrum.

Piersol gives criteria for determining whether a signal has a zero shift in References 3 and 4. Briefly, it involves integrating the acceleration time history to obtain a velocity time history. The velocity time history is then inspected for any unexpected net change.

Example

Saturated Signal

An accelerometer signal from a pyrotechnic shock test is shown in Figure 7. The velocity

time history and the shock response spectrum are also shown. Note that the velocity signal diverges from the zero baseline. Also, there is a wide margin between the positive and negative spectral curves, particularly near 100 Hz. These problems indicate accelerometer saturation.
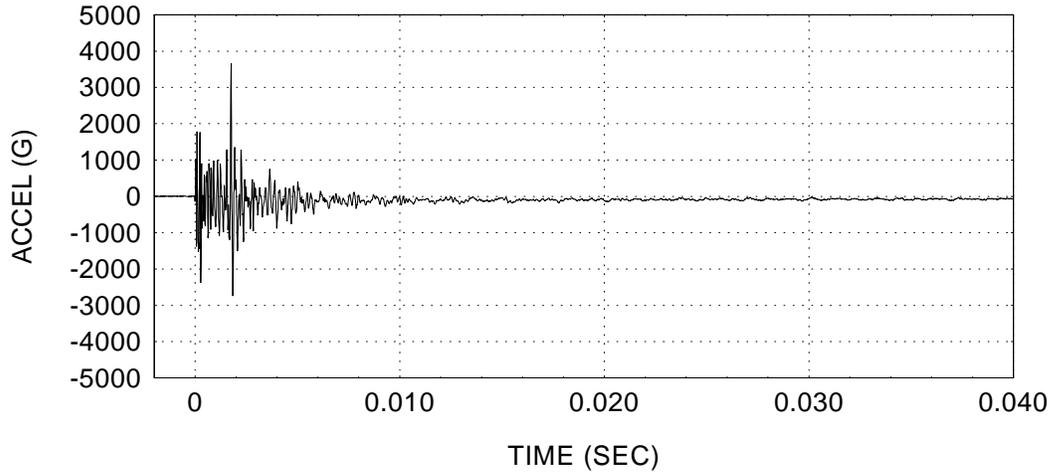
Good Signal

A similar set of plots for a good signal is given in Figure 8. Note that the velocity signal oscillates about the zero baseline. Also, the positive and negative spectral curves are nearly equal.

Solution

An analog lowpass filter can be placed before the first input stage of the signal conditioner to prevent overloading the electronics. Note that this filter can also be used to meet the sample rate criterion, as previously mentioned.

A more effective solution, however, is to use an accelerometer which has a mechanical low pass filter. The purpose of this filter is to prevent excitation of the accelerometer natural frequency.

PYROTECHNIC SHOCK ACCELERATION TIME HISTORY SATURATED SIGNAL

PYROTECHNIC SHOCK VELOCITY TIME HISTORY SATURATED SIGNAL
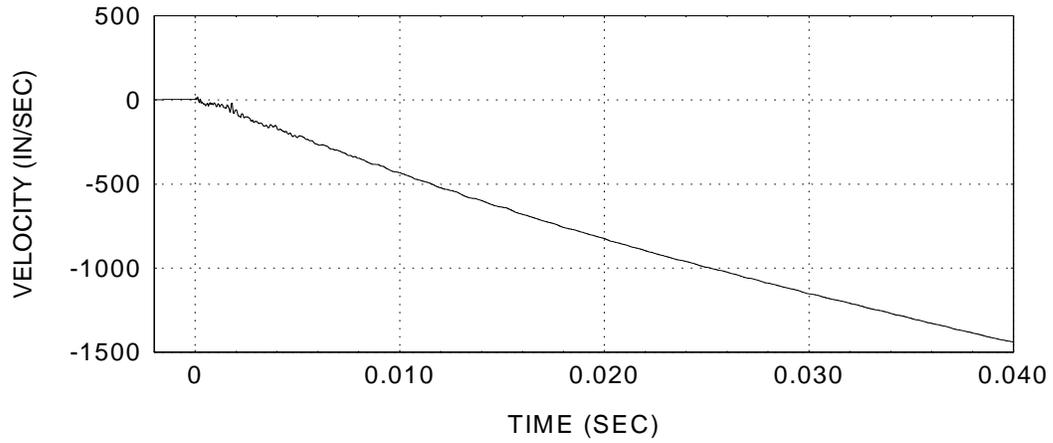
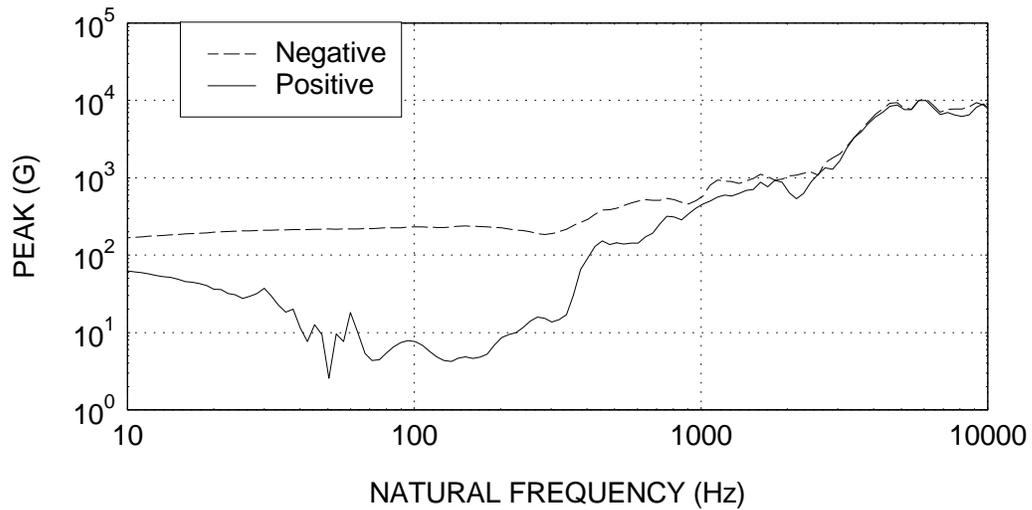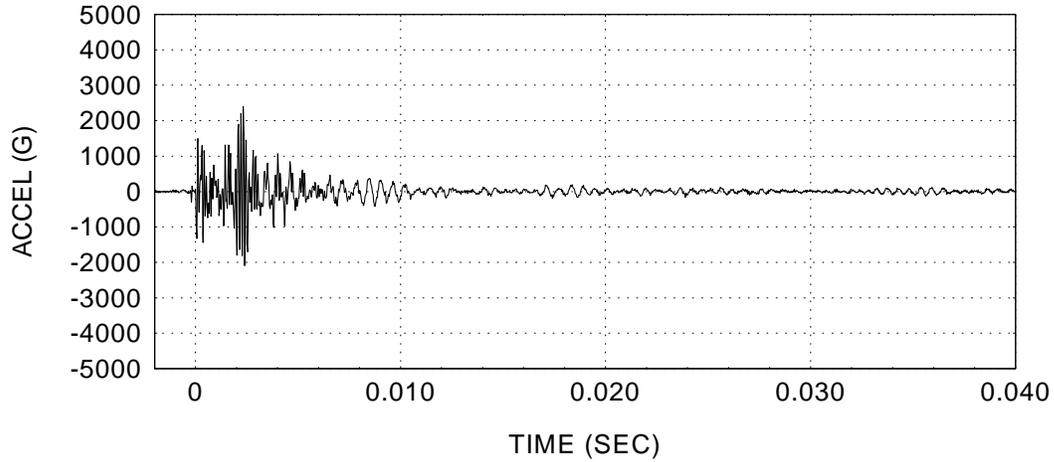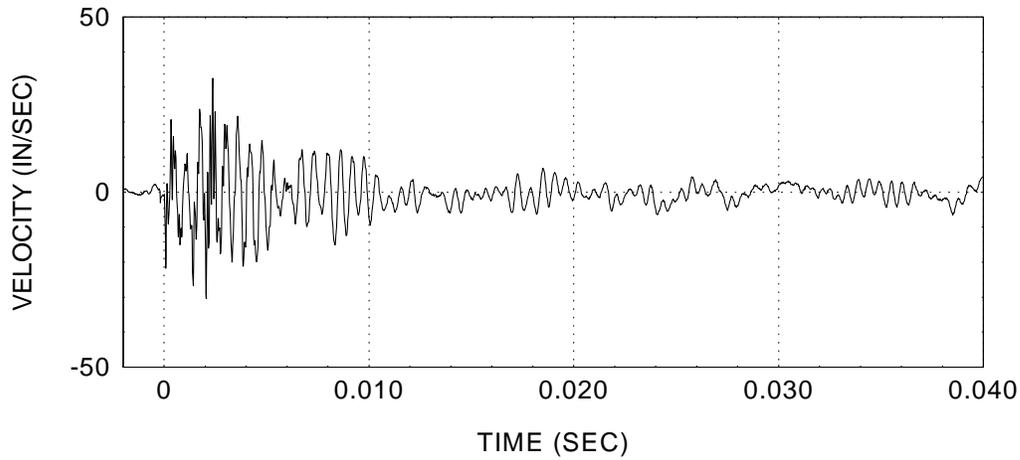SHOCK RESPONSE SPECTRUM Q=10   PYROTECHNIC SHOCK SATURATED DATA

Figure 7. Saturated Data Set

PYROTECHNIC SHOCK ACCELERATION TIME HISTORY GOOD SIGNAL

PYROTECHNIC SHOCK VELOCITY TIME HISTORY GOOD SIGNAL

SHOCK RESPONSE SPECTRUM Q=10   PYROTECHNIC SHOCK   GOOD DATA
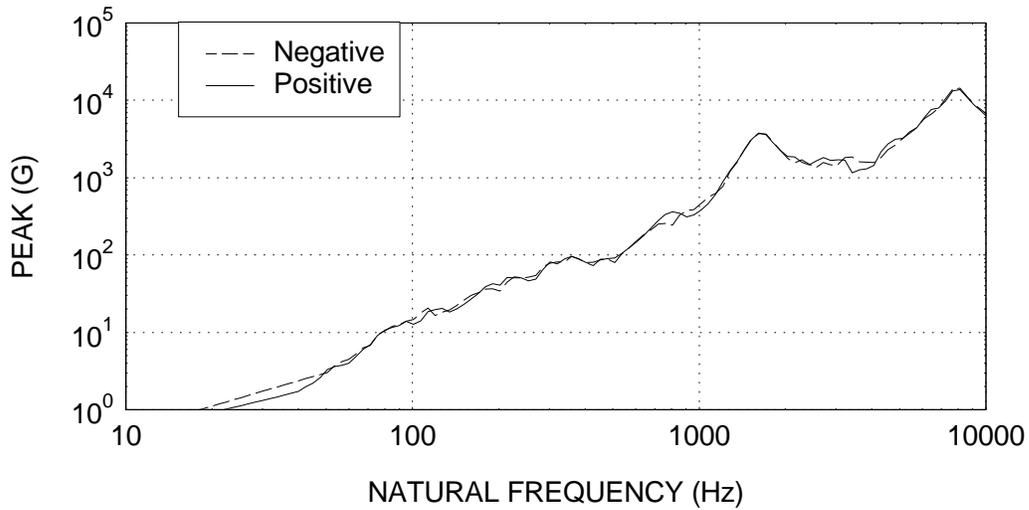
Figure 8. Good Data Set

SHOCK RESPONSE SPECTRUM SLOPE

Constant Velocity Line

The slope on a shock response spectrum curve is considered as a constant velocity line if it is equal to 6 dB/octave.

A one-octave difference in frequency means that the higher frequency is twice the lower frequency.

An amplitude difference of 6 dB means that the higher amplitude is twice the lower amplitude. Note that the difference in dB between two amplitudes A and B is calculated by

$$\Delta dB = 20 \log \left[\frac{A}{B} \right] \qquad (7)$$

Note that the log function is base ten.

The slope N of a line on a log-log plot is given by

$$N = \frac{\log \left[\dfrac{A_2}{A_1} \right]}{\log \left[\dfrac{f_2}{f_1} \right]} \qquad (8)$$

where (f_1, A_1) and (f_2, A_2) are the two endpoint coordinates of the line.

The slope is a constant velocity line if N=1, which corresponds to 6 dB/octave.
Pyrotechnic shock response spectra tend to have an overall slope which represents a constant velocity line. Local peaks, however, may have much higher slopes. The slope may transition to a flat plateau in the higher frequency domain.

Velocity Formula

This section explains the calculation of velocity from acceleration. Harmonic oscillation is typically represented as follows

$$x(t) = X \exp(j\omega t) \tag{9}$$

where

      $x(t)$ is the displacement as a function of time
      X is the displacement amplitude
      j is the square root of -1
      ω is the oscillation frequency

The velocity and acceleration terms are derived as follows

$$\dot{x}(t) = j\omega X \exp(j\omega t) \tag{10}$$

$$\ddot{x}(t) = -\omega^2 X \exp(j\omega t) \tag{11}$$

Substitute equation (9) into (11).

$$\ddot{x}(t) = j\omega \dot{x}(t) \tag{12}$$

The j term indicates phase. The magnitude relationship between acceleration and velocity is

$$\left| \ddot{x}(t) \right| = \omega \left| \dot{x}(t) \right| \tag{13}$$

The frequency term can be converted as follows

$$\left| \ddot{x}(t) \right| = 2\pi f \left| \dot{x}(t) \right| \tag{14}$$

The velocity magnitude is thus

$$\left| \dot{x}(t) \right| = \frac{1}{2\pi f} \left| \ddot{x}(t) \right| \tag{15}$$

Note that equation (15) is approximate for a transient response.

Half-Sine Pulse Slope Example

Base Input

An acceleration half-sine pulse is shown in Figure 9. The amplitude of the pulse is 1 G.
The duration is 11 milliseconds. The net velocity change of this input pulse is 2.7 in/sec,
as calculated by integration.

BASE INPUT: 1 G, 11 msec HALF-SINE PULSE
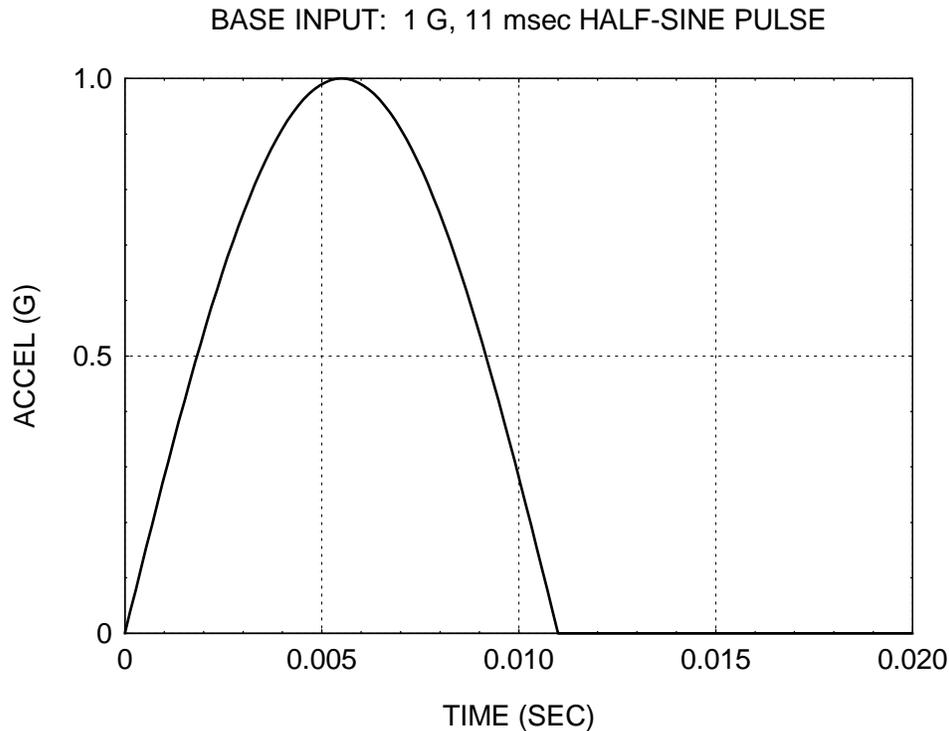
Figure 9. Half-Sine Base Input Pulse

Acceleration and Velocity Time History Responses for a Half-Sine Base Input

Consider a single-degree-of-freedom system with a natural frequency of 10 Hz, and an
amplification factor of Q=10. The acceleration time history response of this system to the
base input pulse is shown in Figure 10. The absolute peak acceleration response is about
0.4 G.

Note that a full set of acceleration response time histories must be generated to obtain the
peak response for each natural frequency. Again, these peak acceleration values determine
the shock response spectrum.

The absolute and relative velocity responses are shown in Figure 11.   Equation (15)
assumes oscillation about a zero baseline for both acceleration and velocity. Thus, the
relative velocity curve response should be used for this analysis. The peak velocity value
is thus approximately 2.5 in/sec.

INPUT AND RESPONSE FOR HALF-SINE PULSE
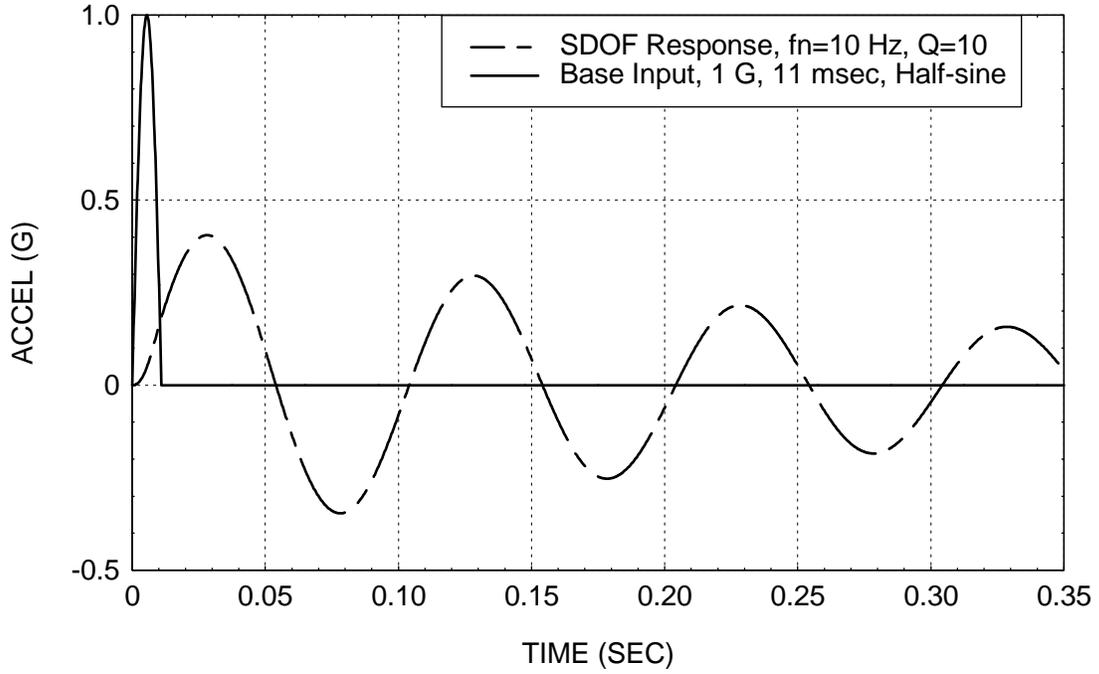
Figure 10. Half-Sine Base Input and Acceleration Response

SDOF VELOCITY RESPONSE (fn=10 Hz, Q=10)
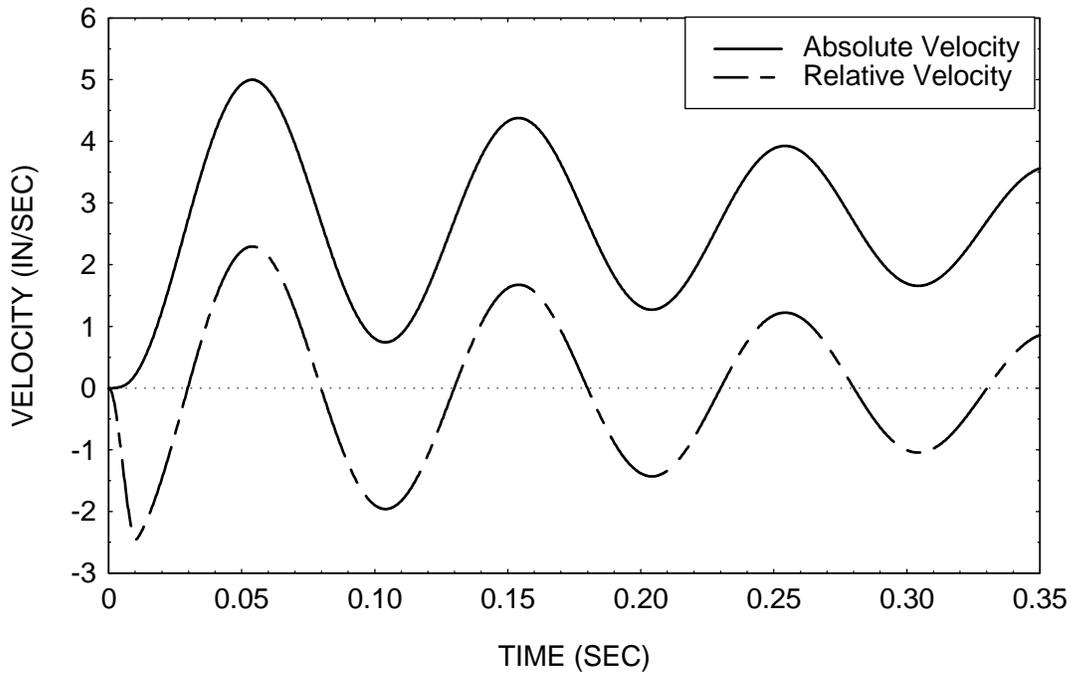
Figure 11. Velocity Response to Half-Sine Pulse

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Shock Response Spectrum of a Half-Sine Base Input

A full shock response spectrum for the base input is given in Figure 12. Note that the shock response spectrum amplitude is 0.4 G at 10 Hz, as expected from Figure 10. Also, note that the initial slope of the spectrum is 6 dB/octave. This slope represents a constant velocity of 2.5 in/sec, in agreement with Figure 11.

ABSOLUTE ACCELERATION SHOCK RESPONSE SPECTRUM
1 G, 11 msec HALF-SINE SHOCK BASE INPUT
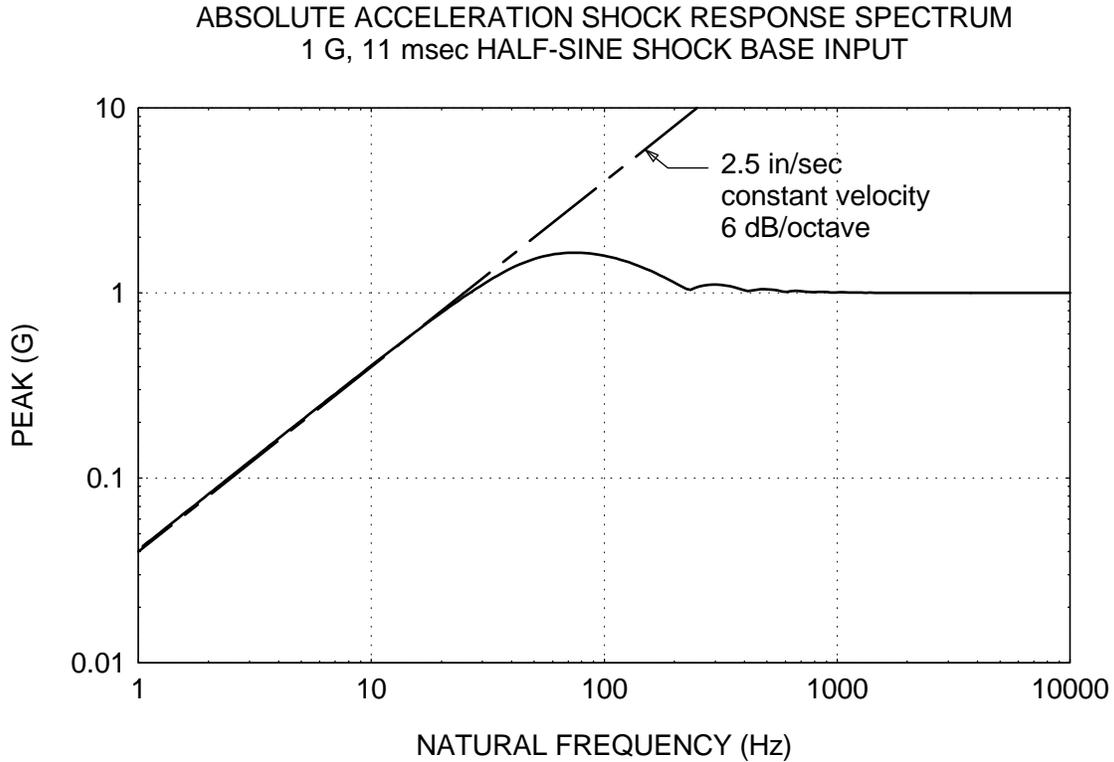
Figure 12. Shock Response Spectrum of Half-Sine Pulse

The constant velocity of 2.5 in/sec is verified via equation (15) for three cases in Table 1.

Table 1. Peak Velocity for Half-Sine SRS		
Natural Frequency (Hz)	SRS Peak (G)	Peak Response Velocity (in/sec)
1	0.04	2.5
5	0.2	2.5
10	0.4	2.5

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Pyrotechnic Pulse Slope Example

Acceleration and Velocity Responses to Pyrotechnic Base Input

Recall the pyrotechnic shock pulse from Figure 1. The response of 50 Hz system with an amplification factor of Q=10 is shown in Figure 13. The peak response magnitude is about 8.5 for both the positive and negative polarities.

The velocity response of the same system is shown in Figure 14. The peak velocity is about 10.5 in/sec

SDOF ACCELERATION RESPONSE (fn=50 Hz, Q=10) TO PYROTECHNIC SHOCK BASE INPUT
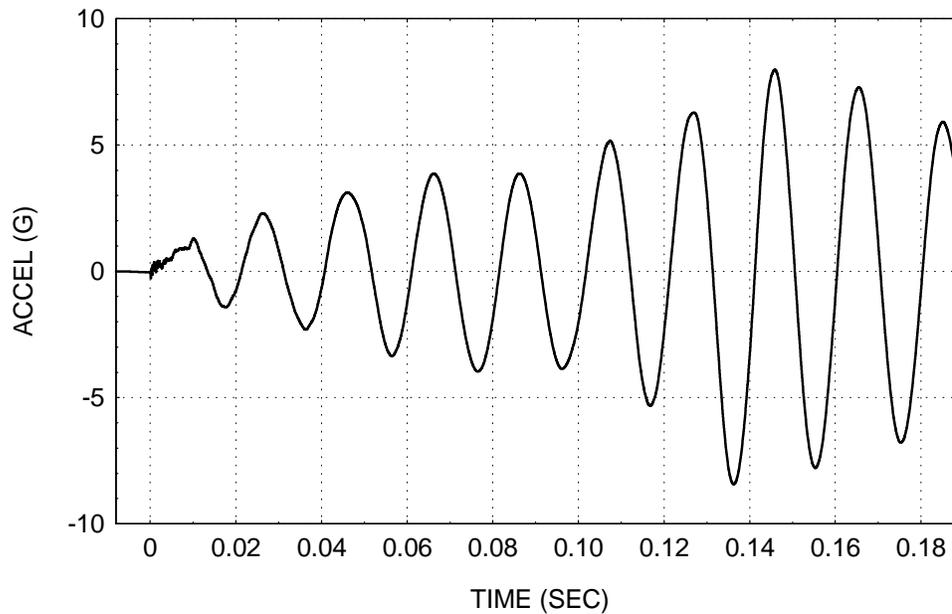
Figure 13. Acceleration Response to Pyrotechnic Pulse

SDOF VELOCITY RESPONSE (fn=50 Hz, Q=10) TO PYROTECHNIC SHOCK BASE INPUT
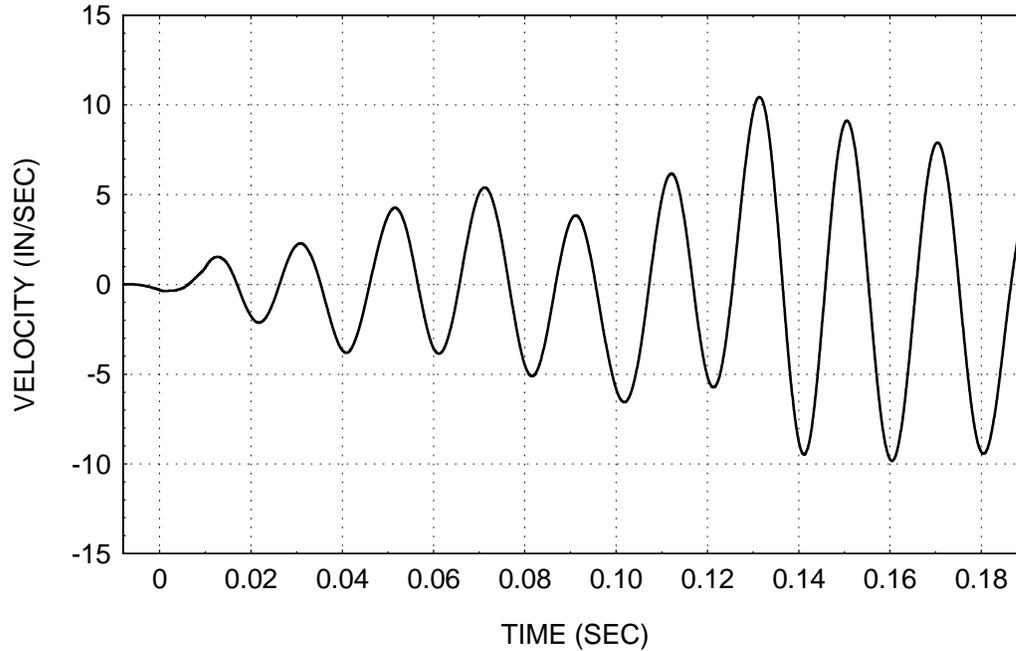
Figure 14. Velocity Response to Pyrotechnic Pulse

Shock Response Spectrum of Pyrotechnic Base Input

The pyrotechnic pulse shock response spectrum from Figure 2 is shown again in Figure 15, with an expanded range and domain. Note that the amplitude is about 8.5 G at 50 Hz for both the positive and negative curves, in agreement with Figure 13.

Two slope curves are superimposed on the plot. The overall trend is 6 dB/octave. The corresponding velocity is 11.1 in/sec, which is approximately equal to the 10.5 in/sec value in Figure 14.

The respective slopes of individual peaks, however, take on a variety of values. For example, the slope is 12 dB/octave from 10 Hz to 20 Hz. The slope in the 40 Hz to 50 Hz domain is higher yet.

SHOCK RESPONSE SPECTRUM Q=10   BASE INPUT: PYROTECHNIC SHOCK PULSE
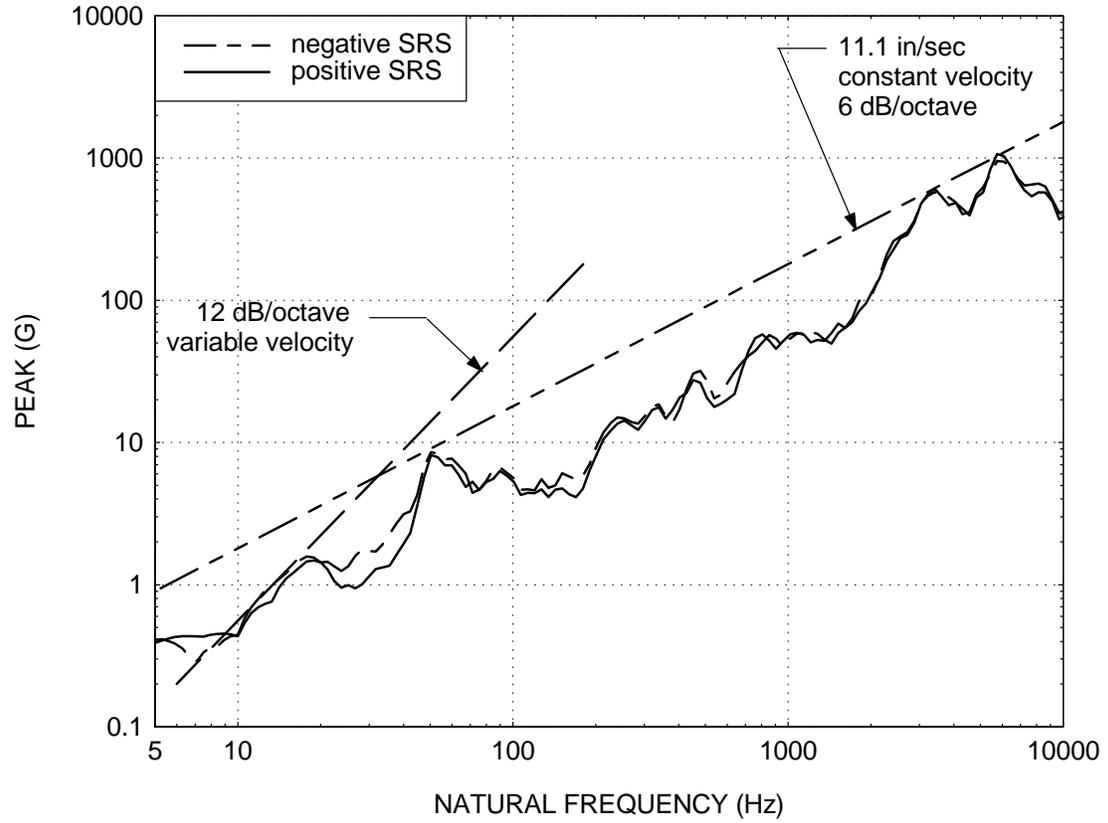
Figure 15. Shock Response Spectrum of Pyrotechnic Pulse

COMPONENT QUALIFICATION TEST METHODS

Purpose

The purpose of component qualification testing is to verify that the component can survive the shock event. Furthermore, some components are required to operate properly during the event itself. These components must be powered and monitored during the test.

Explosives

Consider that a shock response spectrum specification has been derived for a component. There are several possible methods for performing the test.

A component could be mounted to a plate. The opposite side of the plate could be excited by an explosive device or by a mechanical impact. Explosive devices may be required if the specification has a plateau above 10,000 G. The plate method often allows the specification to be satisfied simultaneously in all three orthogonal axes, regardless of excitation source.

Mechanical Impact

Mechanical impact methods are typically used if the specification has a plateau between 1000 G and 10,000 G. The impact device could be a pendulum hammer or a pneumatic piston. An example of a test set-up using a pneumatic device to excite a beam is shown in Figure 16.

Shaker Method

Certain specifications can be met on an electromagnetic shaker. Note that the main advantage of the shaker method is repeatability. The shaker method typically requires the specification level to be below 1000 G. Furthermore, the maximum frequency should be less than or equal to 2000 Hz. At higher frequencies, the shaker resonant and anti-resonant frequencies would interfere with the control algorithm.

Figure 16. Shock Test Set-up

Andy Trudgen and Rick Baird, the author's colleagues, devised this set-up. A pneumatic gun is in the foreground. The gun is powered by shop air. The black, vertical cylinder is a solenoid valve. The pressure is set at 10 to 30 psig, depending on the required shock level. The gun fires a steel rod into a plate or beam, at point blank range. In this case, the test item was a small coupler device which was mounted on top of the beam. A triaxial accelerometer was mounted on the beam adjacent to the test item.

<u>Testing Limitations</u>

Acceleration

The shock response spectrum method is an excellent method for characterizing the acceleration response of a system.

Acceleration input is easy to measure. Acceleration is an "across variable." It is analogous to voltage measurement. The voltage potential across two points of circuit can be easily measured, with negligible effect on the circuit.

Force

Force is also a significant variable, however. Force is a "through variable." Force is analogous to current. Electrical current is awkward to measure because the circuit must be broken in order to make the measurement. The current meter becomes a significant part of the circuit. Mechanical force is likewise difficult to measure because the "mechanical circuit" must be broken in order to insert the force sensor.

Dynamic force is seldom measured in component shock testing.[4]

Mechanical Impedance

Consider an avionics component mounted to a compliant bulkhead in a space vehicle. The shock level at the mounting location is measured during a ground separation test. A shock level is then derived in terms of acceleration. For the qualification test, the component is mounted on a thick metal plate. The mechanical impedance of the test plate is much greater than that of the bulkhead. The force input to the box during the component test may be entirely different than the force input during flight, even if the acceleration is similar in each case.

Agony

Authors such as Luhrs in Reference 5 have agonized over this lack of realism in testing. He claims, for example, that mechanical impact methods are overly severe for satisfying a shock response spectrum specification.

"Real-world" schedule and financial pressures seldom allow engineers the opportunity to pursue such lofty goals as impedance-matching and force measurements.

COMPONENT ISOLATION

Return to the shock response spectrum in Figure 2. Consider that it is the expected flight level for a particular component. Consider that the component has a natural frequency of 1000 Hz. Note that the peak level at 1000 Hz is about 55 G for both the positive and negative curves. The design engineer could reduce this peak level to 10 G by shifting the natural frequency to 200 Hz. One possible approach to achieve this is to mount the component with isolator bushings, made from rubber or plastic material.

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[4] An exception is modal testing where the force input from a hammer or shaker is measured. Modal testing has a different purpose than component qualification testing, however.

Isolator mounts are used to break metal-to-metal contact between the avionics component mounting foot[5] and the vehicle mounting structure, as shown in Figure 17.
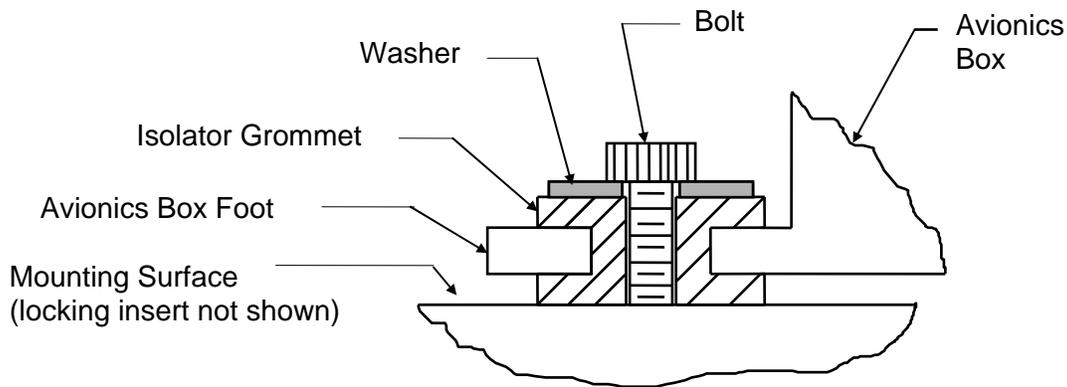

Figure 17. Typical Isolator Configuration

The bolt may be a shoulder bolt in order to control the initial compression of the isolator.

Usually, four isolator grommets are used for mounting an avionics component.   Note that small components should be mounted via an isolated base plate, where the base plate serves as ballast mass.

Isolators typically have the following effects:

1. The isolators render the avionics component as a single-degree-of-freedom-system.
2. The isolators reduce the fundamental frequency of the avionics component.
3. The isolators add damping.

The first effect tends to be the most important. A single-degree-of-freedom system attenuates energy above $\sqrt{2}$ times its natural frequency.   The isolators thus filter out high-frequency pyrotechnic energy.

The reduced natural frequency effect, which is the second effect, is almost always desirable. A reduced natural frequency improves the lowpass filtering capability of the single-degree-of-freedom system. A possible drawback is that the isolators might bring the natural frequency of the avionics component down to within one octave[6] of a bulkhead natural frequency or a forcing frequency. The isolators could thus cause some dynamic amplification at the avionics component natural frequency.   The isolator damping, however, would minimize this amplification.

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[5]   There are many varieties of isolator mounting schemes. For example, some avionics boxes are mounted on brackets which are in turn mounted via isolators to a structure. In this case, the intermediate bracket should be fairly rigid to avoid dynamic coupling effects with the isolators.

[6]   The octave rule is a rule-of-thumb. It states that the respective natural frequencies of two systems should be at least one octave apart if the systems are to be joined together. A one-octave separation means that the higher of the two frequencies is twice the value of the lower frequency.

PYROTECHNIC SHOCK PROPAGATION

Source shock data for explosive devices is given in Reference 6. The data is given in the form of shock response spectra.

Attenuation relationships are also given in this reference. These relationships give the attenuation as the shock energy propagates through material and across joints. The relationships are empirical.

APPLICATION IN OTHER INDUSTRIES

Notes

The aerospace industry is perhaps the prime user of the shock response spectrum method. Nevertheless, other industries also use this tool.

Earthquake Engineering

Equipment mounted in buildings must be designed and tested to withstand seismic excitation. This is particularly true if the building is located in California or in some other earthquake-prone area.

Seismic shock response spectra are similar to spacecraft shock response spectra except for two differences

1. Seismic shock response spectra are typically represented in terms of peak velocity rather than peak acceleration.
2. Qualitatively, the seismic spectrum is dominated by low-frequency energy, whereas the spacecraft spectrum is dominated by high-frequency energy.

Reference 7 gives examples of seismic shock response spectra.

Transportation and Shipping

The shock response spectrum method is also used to characterize certain transportation environments. MIL-STD-810E, for example, specifies a *crash hazard shock response spectrum* for certain components. This specification can be used to verify that a radio in a military vehicle would still function even after the vehicle was involved in a moderate collision.

Computer Industry

Hard disk drives are particularly susceptible to shock. The drives may receive shock during a variety of environments

1. Handling. For example, a drive may be dropped a few inches onto a worktable during the manufacturing process.
2. Chassis installation. A power tool may be used to install the mounting screws.
3. Transportation and shipping.

The computer industry is moving toward the shock response spectrum method to characterize these environments. Examples are given in Reference 8.

FORCE SHOCK

The methods in this report have been applied to an SDOF system excited by a base excitation acceleration pulse. The same methods can be adapted to a system excited by an applied force as shown in Appendix C. Note that force impulses are used to excite structures for modal testing.

SHOCK SEVERITY

Shock is usually measured in terms of acceleration because accelerometers are readily available and convenient to use. An accelerometer cannot measure velocity directly, although the resulting signal can be integrated to obtain velocity.

A laser vibrometer or a geophone is required to measure velocity directly. Laser vibrometers require a direct line-of-sight. Furthermore, lasers are expensive. Geophones are bulky and are intended for seismic measurements.

Nevertheless, some sources claim that shock should be characterized in terms of velocity rather than acceleration. Reference 9, for example, claims that velocity is a better criterion for severity.

Furthermore, an empirical rule-of-thumb in Reference 10 states that a shock response spectrum is considered severe only if one of its components exceeds the level

$$\text{Threshold} = [\ 0.8\ (\text{G/Hz}) * \text{Natural Frequency (Hz)}\ ] \tag{16}$$

For example, the severity threshold at 1000 Hz would be 800 G.

Reference 10 states that this rule is based on unpublished observations that military-quality equipment does not tend to exhibit shock failures below a shock response spectrum velocity of 100 inches/sec (254 cm/sec).

The threshold equation, however, is equivalent to 49.1 in/sec (125 cm/sec). Thus, it appears to contain a 6 dB margin. As an example, convert the coordinate (1000 Hz, 800 G) into the equivalent velocity.

$$\text{velocity} = \frac{\text{acceleration}}{2\pi f} \tag{17}$$

$$\text{velocity} = \frac{(800\ \text{G})\left(386\,\text{in}/\sec^2/\text{G}\right)}{2\pi\,(1000\,\text{Hz})} \tag{18}$$

$$\text{velocity} = 49.1\,\text{in}/\sec \tag{19}$$

Again, this rule-of-thumb is intended for military-quality equipment.

CONCLUSION

Overview

A shock response spectrum can be calculated from acceleration time history data. This spectrum shows the peak response for each natural frequency, assuming SDOF behavior.

The convolution integral method is the most efficient calculation approach. This integration can be implemented using a digital recursive filtering algorithm.

Design Considerations

Engineers can use the shock response spectrum data to design spacecraft and avionics components which can reliably withstand expected shock environments.

For example, a design engineer could make hardware modifications to reduce the peak response by shifting the natural frequency.

Test Considerations

The shock response spectrum is also expedient for testing.

Note that each time history has a unique shock response spectrum.

On the other hand, consider a component test level specified as a shock response spectrum along with its tolerance bands. This specification can be met with a reasonable variety of input pulses. This characteristic allows test lab personnel some flexibility in preparing a suitable shock test method.

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10. Environmental Test Methods and Engineering Guidelines, Military Standard MIL-STD-810E, U.S. Department of Defense, 1989.

11. Reference: L. Meirovitch, Analytical Methods in Vibration, Macmillan, New York, 1967.

ADDITIONAL RESOURCES

Additional text resources are

1. Dave Steinberg, Vibration Analysis for Electronic Equipment, Second Edition, Wiley, New York, 1988.

2. R. Kelly and G. Richman, Principles and Techniques of Shock Data Analysis, SVM-5; The Shock and Vibration Information Center, United States Department of Defense, Washingtion D.C., 1969.

3. W. Tustin and R. Mercado, Random Vibration in Perspective, Tustin Institute of Technology, Santa Barbara, 1984.

Furthermore, the Handbook for Dynamic Data Acquisition and Analysis, IEST-RP-DTE012.1 is available from the Institute of Environmental Sciences and Technology
(IEST), web site: http://www.iest.org/

The reader should also refer to the plethora of outstanding papers written by Dr. David Smallwood of Sandia National Labs. Many of his papers deal mainly with advanced concepts. He also wrote a chapter in the Shock and Vibration Handbook.

APPENDIX A

DERIVATION

The solution method proceeds by finding a solution to the *homogeneous* form of equation (5). In other words, the solution is found for $\ddot{Y}(t) = 0$.

$$\ddot{Z} + 2\xi\omega_n\dot{Z} + \omega_n^2 Z = 0 \qquad\qquad\text{(A-1)}$$

The Laplace transform method can be used as follows

$$\mathcal{L}\{\ddot{Z} + 2\xi\omega_n\dot{Z} + \omega_n^2 Z\} = \mathcal{L}\{0\} \qquad\qquad\text{(A-2)}$$

$$\mathcal{L}\{\ddot{Z}\} + 2\xi\omega_n \mathcal{L}\{\dot{Z}\} + \omega_n^2 \mathcal{L}\{Z\} = 0 \qquad\qquad\text{(A-3)}$$

$$\begin{aligned} s^2\hat{Z}(s) - sz(0) - \dot{z}(0) \\ +2\xi\omega_n s\hat{Z}(s) - 2\xi\omega_n z(0) \\ +\omega_n^2\hat{Z}(s) \qquad\qquad = 0 \end{aligned} \qquad\qquad\text{(A-4)}$$

$$[s^2 + 2\xi\omega_n s + \omega_n^2]\hat{Z}(s) + [-s - 2\xi\omega_n]z(0) - \dot{z}(0) = 0 \qquad\qquad\text{(A-5)}$$

$$[s^2 + 2\xi\omega_n s + \omega_n^2]\hat{Z}(s) = [s + 2\xi\omega_n]z(0) + \dot{z}(0) \qquad\qquad\text{(A-6)}$$

$$\hat{Z}(s) = \frac{\{[s + 2\xi\omega_n]z(0) + \dot{z}(0)\}}{\{s^2 + 2\xi\omega_n s + \omega_n^2\}} \qquad\qquad\text{(A-7)}$$

The inverse Laplace transform can be found from standard tables

$$Z(t) = \left[\exp(-\xi\omega_n t)\right]\left[Z_0\cos\omega_d t + \frac{1}{\omega_d}\left(V_0 + \xi\omega_n Z_0\right)\sin\omega_d t\right] \qquad\qquad\text{(A-8)}$$

$$\text{where } \omega_d = \omega_n\sqrt{1-\xi^2}, \qquad\qquad\text{(A-9)}$$

$Z_0 = z(0)$, initial relative displacement.

$V_0 = \dot{z}(0)$, initial relative velocity.

The initial relative displacement is taken as zero. The homogeneous relative displacement thus becomes:

$$Z(t) = \frac{V_0}{\omega_d}\left[\exp(-\xi\,\omega_n t)\right]\left[\sin\omega_d t\right] \tag{A-10}$$

A unit impulse response function h(t) may be defined for this homogeneous case as:

$$h(t) = \frac{1}{\omega_d}\left[\exp(-\xi\,\omega_n t)\right]\left[\sin\omega_d t\right] \tag{A-11a}$$

Thus,

$$Z(t) \;=\; V_0 h(t) \tag{A-11b}$$

The *particular* solution is not solved in a direct manner. Rather, the arbitrary excitation is represented by the initial relative velocity term, which is considered as an impulse. This approach assumes a sudden change in velocity without an appreciable change in displacement. Consider the base input time history shown in Figure A-1. For convenience the sample input in Figure A-1 is shown having uniform amplitude polarity, but an input may oscillate about the zero baseline. The response of an SDOF system to a particular impulse from this example is also given in this figure.
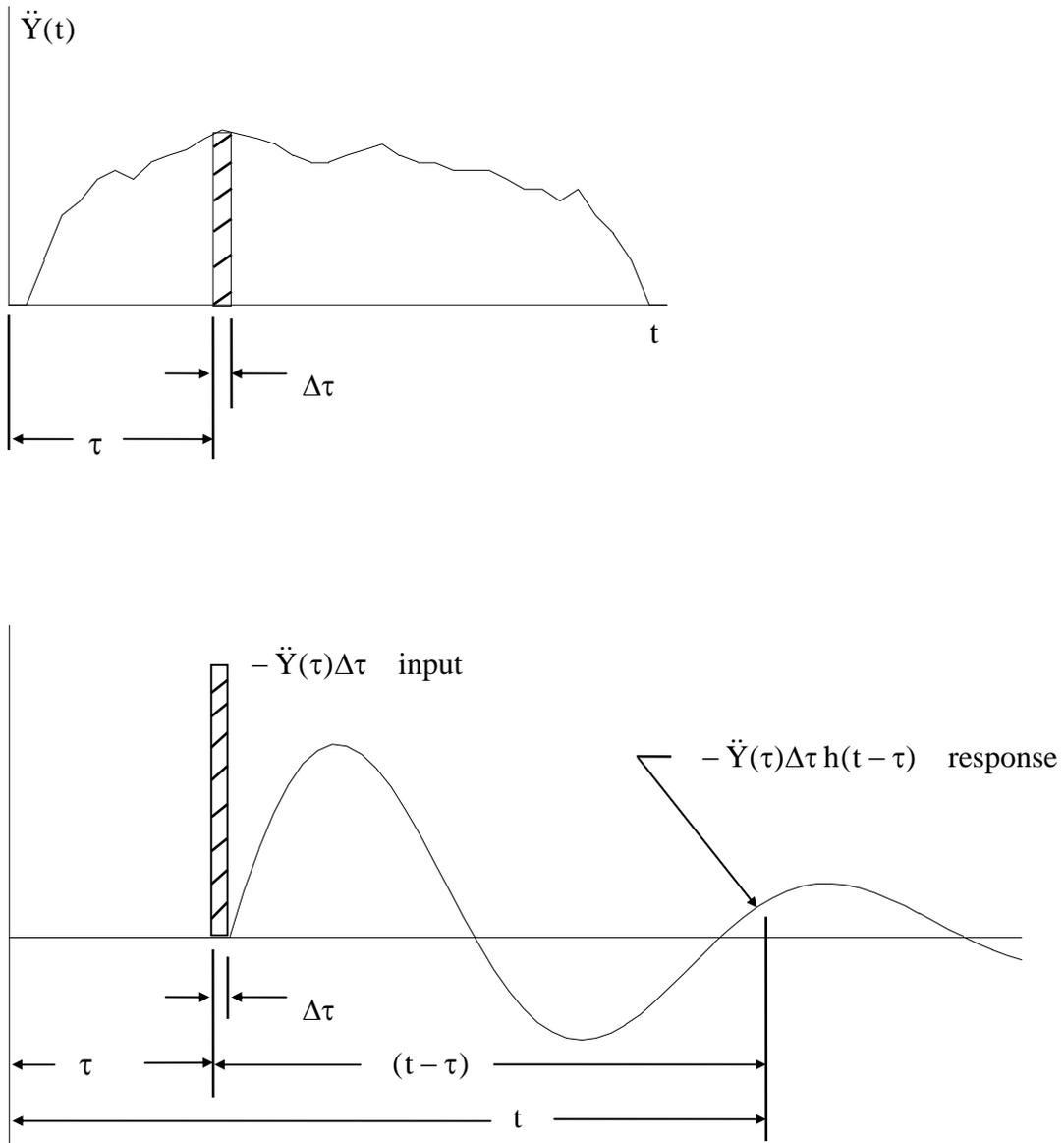
Figure A-1. Arbitrary Base Input and Response

Note that the relative velocity is

$$V_0 = -\ddot{Y}(\tau) \, \Delta\tau \tag{A-12}$$

The total relative displacement is found by summing the response to each of the velocity impulses. Note that the principle of superposition holds because the system is linear.

$$Z(t_n) = - \sum_{i=1}^{n} \ddot{Y}(\tau_i) \Delta \tau_i \, h(t_n - \tau_i), \qquad t_n > \tau_i \tag{A-13}$$

Substituting for the impulse response function,

$$Z(t_n) = - \frac{1}{\omega_d} \sum_{i=1}^{n} \ddot{Y}(\tau_i) \left[\exp\{-\xi\omega_n(t_n - \tau_i)\} \right] \left[\, \sin\omega_d(t_n - \tau_i) \right] \Delta\tau_i \tag{A-14}$$

The series form changes to an integral form as an infinitesimally small velocity interval is taken.

$$Z(t) = - \frac{1}{\omega_d} \int_0^t \ddot{Y}(\tau) \left[\exp\{-\xi\omega_n(t - \tau)\} \right] \left[\sin\omega_d(t - \tau) \right] d\tau \tag{A-15}$$

The Leibnitz rule for differentiation under the integral sign is

$$\frac{d}{dt} \int_0^t f(\tau) g(t - \tau) d\tau = f(t)g(0) + \int_0^t f(\tau) g'(t - \tau) d\tau \tag{A-16}$$

where the prime indicates differentiation with respect to $t - \tau$. Equation (A-16) is taken from Meirovitch, page 16, Reference 11.

The relative velocity is

$$\dot{Z}(t) = \frac{d}{dt} \left\{ - \frac{1}{\omega_d} \int_0^t \ddot{Y}(\tau) \left[\exp(-\xi\omega_n(t - \tau)) \right] \left[\sin \omega_d(t - \tau) \right] d\tau \right\} \tag{A-17}$$

$$\dot{Z}(t) = \left[- \frac{1}{\omega_d} \right] \ddot{Y}(\tau) \left[\exp(0) \right] \left[\, \sin 0 \, \right] - \frac{1}{\omega_d} \int_0^t \ddot{Y}(\tau) \left\{ \left[\exp(-\xi\omega_n(t - \tau)) \right] \left[\sin\omega_d(t - \tau) \right] \right\}' d\tau \tag{A-18}$$

$$\dot{Z}(t) = - \frac{1}{\omega_d} \int_0^t \ddot{Y}(\tau) \left\{ \left[\exp(-\xi\omega_n(t - \tau)) \right] \left[\sin\omega_d(t - \tau) \right] \right\}' d\tau \tag{A-19}$$

Finally, the relative velocity is

$$\dot{Z}(t) = \int_0^t \ddot{Y}(\tau) \left[\exp(-\xi\omega_n(t-\tau)) \right] \left[\left\{ \frac{\xi\omega_n}{\omega_d} \right\} \sin\omega_d(t-\tau) - \cos\omega_d(t-\tau) \right] d\tau$$

(A-20)

The relative acceleration is

$$\ddot{Z}(t) = \frac{d}{dt} \left\{ \int_0^t \ddot{Y}(\tau) \left[\exp(-\xi\omega_n(t-\tau)) \right] \left[\left\{ \frac{\xi\omega_n}{\omega_d} \right\} \sin\omega_d(t-\tau) - \cos\omega_d(t-\tau) \right] d\tau \right\}$$

(A-21)

$$\ddot{Z}(t) = -\ddot{Y}(\tau) + \int_0^t \ddot{Y}(\tau) \left\{ \left[\exp(-\xi\omega_n(t-\tau)) \right] \left[\left\{ \frac{\xi\omega_n}{\omega_d} \right\} \sin\omega_d(t-\tau) - \cos\omega_d(t-\tau) \right] \right\}' d\tau$$

(A-22)

$$\ddot{Z}(t) = \left\{ -\xi\omega_n \int_0^t \ddot{Y}(\tau) \left[\exp(-\xi\omega_n(t-\tau)) \right] \left[\left\{ \frac{\xi\omega_n}{\omega_d} \right\} \sin\omega_d(t-\tau) - \cos\omega_d(t-\tau) \right] d\tau \right\}$$
$$+ \left\{ \int_0^t \ddot{Y}(\tau) \left[\exp(-\xi\omega_n(t-\tau)) \right] \left[\xi\omega_n \cos\omega_d(t-\tau) + \omega_d \sin\omega_d(t-\tau) \right] d\tau \right\}$$
$$- \ddot{Y}(t)$$

(A-23)

$$\ddot{Z}(t) =$$
$$-\ddot{Y}(t) + \frac{1}{\omega_d} \int_0^t \ddot{Y}(\tau) \left[\exp(-\xi\omega_n(t-\tau)) \right] \left[(\omega_d^2 - \xi^2\omega_n^2)\sin\omega_d(t-\tau) + 2\xi\omega_n\omega_d\cos\omega_d(t-\tau) \right] d\tau$$

(A-24)

Recall

$$\ddot{Z} = \ddot{X} - \ddot{Y}$$

(A-25)

$$\ddot{X} = \ddot{Z} + \ddot{Y}$$

(A-26)

The absolute response acceleration becomes

$$\ddot{X}(t) = \frac{1}{\omega_d} \int_0^t \ddot{Y}(\tau) \left[\exp\{-\xi\omega_n(t-\tau)\} \right] \left[(\omega_d^2 - \xi^2\omega_n^2)\sin\omega_d(t-\tau) + 2\xi\omega_n\omega_d\cos\omega_d(t-\tau) \right] d\tau$$

(A-27)

Note that there is a unit impulse function $\hat{h}(t)$ in equation (A-27), which is

$$\hat{h}(t) = \frac{1}{\omega_d}\left[\exp\{-\xi\omega_n t\}\right]\left[(\omega_d{}^2 - \xi^2\omega_n{}^2)\sin\omega_d t + 2\xi\omega_n\omega_d\cos\omega_d t\right] \tag{A-28}$$

The discrete form assuming a constant time step is

$$\ddot{X}(t_n) =$$

$$\frac{\Delta t}{\omega_d}\sum_{i=0}^{n}\ddot{Y}(\tau_i)\left[\exp\{-\xi\omega_n(t_n - \tau_i)\}\right]\left[(\omega_d{}^2 - \xi^2\omega_n{}^2)\sin\omega_d(t_n - \tau_i) + 2\xi\omega_n\omega_d\cos\omega_d(t_n - \tau_i)\right]$$

$$\tag{A-29}$$

Thomson gives details about these steps in Reference 1, although he assumes zero damping.

Again, equations (A-27) and (A-29) are based on the principle that the response at any time depends in part on the responses at all previous times. Note that the integration is carried out from $\tau = 0$ to $\tau = t$ in equation (A-27).

Equation (A-27) is very inefficient to solve, even using a computer. It may be simplified via a *digital recursive filtering relationship.* The term *recursive* refers to the fact that the response at any time depends in part on the responses at all previous times. The term *filtering* refers to the fact that the SDOF system itself behaves as a simple lowpass filter. In addition, the integration process itself is a filtering operation.

The derivation of the filtering relationship may be performed by an extensive number of steps in the time domain. It may also be performed using frequency domain transformation methods, particularly the Z-transform method.

Z-TRANSFORM APPROACH

The author expresses gratitude to Dr. Dennis B. Nelson, Sandia National Laboratories, Livermore, California; for this approach.

The Z-transform is related to the Laplace transform and the Fourier transform. The Z-transform operates on digital data only.

The one-sided Z-transform is

$$X(z) = \sum_{k=0}^{\infty} x_k\, z^{-k} \tag{A-30}$$

Consider the unit impulse response function $\hat{h}(t)$, equation (A-28).

$$\hat{h}(t) = \frac{1}{\omega_d} \left[\exp\{-\xi\omega_n t\} \right] \left[(\omega_d^2 - \xi^2\omega_n^2)\sin\omega_d t + 2\xi\omega_n\omega_d \cos\omega_d t \right] \tag{A-31}$$

$$\hat{h}(t) = \left[\exp\{-\xi\omega_n t\} \right] \left[\alpha\sin\omega_d t + \beta\cos\omega_d t \right] \tag{A-32}$$

where

$$\alpha = \frac{1}{\omega_d} \left[\omega_d^2 - \xi^2\omega_n^2 \right]$$

$$\beta = \frac{1}{\omega_d} \left[2\xi\omega_n\omega_d \right]$$

Let $\ddot{x}$ be the absolute acceleration response of the mass. Let $\ddot{y}$ be the base input acceleration. Consider the following recursive formula as a proposed solution

$$\ddot{x}(kT) = A\,\ddot{x}(kT - T) + B\,\ddot{x}(kT - 2T) + C\,\ddot{y}(kT) + D\,\ddot{y}(kT - T) + E\,\ddot{y}(kT - 2T) \tag{A-33}$$

       where T is the time increment

       and $k = 0, 1, 2, 3,......$

Take the Z-transform of equation (A-33). Omit the double-dot notation for brevity.

$$X(z) = A\,z^{-1}X(z) + Bz^{-2}X(z) + C\,Y(z) + D\,z^{-1}Y(z) + Ez^{-2}Y(z) \tag{A-34}$$

Now that z^{-1} is a unit delay in the Z-transform approach.

Collecting terms,

$$\left[1 - A\,z^{-1} - Bz^{-2} \right]X(z) = \left[C + D\,z^{-1} + E\,z^{-2} \right]Y(z) \tag{A-35}$$

$$X(z) = \frac{\left[C + D\,z^{-1} + E\,z^{-2} \right]}{\left[1 - A\,z^{-1} - Bz^{-2} \right]}\,Y(z) \tag{A-36}$$

Multiplying through by z^2,

$$X(z) = \frac{\left[C z^2 + D z + E\right]}{\left[z^2 - A z - B\right]} Y(z) \qquad \text{(A-37)}$$

Let y(kT) be a discrete unit impulse function.

$$y(kT) = \begin{cases} \dfrac{1}{T} & \text{for } k = 0 \\[2mm] 0 & \text{for } k \neq 0 \end{cases} \qquad \text{(A-38)}$$

The Z-transform of a constant is equal to that constant. Thus, the transform is

$$Y(kT) = \begin{cases} \dfrac{1}{T} & \text{for } k = 0 \\[2mm] 0 & \text{for } k \neq 0 \end{cases} \qquad \text{(A-39)}$$

By substitution,

$$X(z) = \frac{1}{T} \frac{\left[C z^2 + D z + E\right]}{\left[z^2 - A z - B\right]} \qquad \text{(A-40)}$$

Note that X(z) is equal to H(z) for the discrete impulse input.

$$H(z) = \frac{1}{T} \frac{\left[C z^2 + D z + E\right]}{\left[z^2 - A z - B\right]} \qquad \text{(A-41)}$$

H(z) is the Z-transform of the impulse response of the digital filter.

$$Z\{h(kT)\} = H(z) \qquad \text{(A-42)}$$

Now take the Z-transform of equation (A-32).

$$Z\{\hat{h}(t)\} = Z\{[\exp\{-\xi\omega_n t\}][\alpha\sin\omega_d t + \beta\cos\omega_d t]\} \tag{A-43}$$

Separating terms,

$$Z\{\hat{h}(t)\} = Z\{[\exp\{-\xi\omega_n t\}][\alpha\sin\omega_d t]\} + Z\{[\exp\{-\xi\omega_n t\}][\beta\cos\omega_d t]\} \tag{A-44}$$

From standard Z-transform tables,

$$Z\{\hat{h}(t)\} = \frac{\alpha z\{\exp[-\xi\omega_n T]\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}}$$

$$+ \frac{\beta z\{z - \exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}} \tag{A-45}$$

$$Z\{\hat{h}(t)\} = \frac{z^2\beta + z\{\exp[-\xi\omega_n T]\}\{\alpha\sin[\omega_d T] - \beta\cos[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}} \tag{A-49}$$

Equations (A-36) through (A-38) lead to the following,

$$\left[\frac{1}{T}\right]\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z^2\beta + z\{\exp[-\xi\omega_n T]\}\{\alpha\sin[\omega_d T] - \beta\cos[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}}$$

$$\tag{A-53}$$

Multiply through by T.

$$\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z^2 \beta T + zT\{\exp[-\xi\omega_n T]\}\{\alpha\sin[\omega_d T] - \beta\cos[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}}$$

(A-48)

Equation (A-48) is satisfied if

$$A = 2\exp[-\xi\omega_n T]\cos[\omega_d T]$$

(A-49)

$$B = -\exp[-2\xi\omega_n T]$$

(A-50)

$$C = \beta T$$

(A-51)

$$D = T\exp[-\xi\omega_n T]\{\alpha\sin[\omega_d T] - \beta\cos[\omega_d T]\}$$

(A-52)

$$E = 0$$

(A-53)

By substitution, the C and D coefficients become

$$C = 2\xi\omega_n T$$

(A-54)

$$D = \frac{T}{\omega_d}\exp[-\xi\omega_n T]\left\{\left[\omega_d^2 - \xi^2\omega_n^2\right]\sin[\omega_d T] - \left[2\xi\omega_n\omega_d\right]\cos[\omega_d T]\right\}$$

(A-55)

$$D = \frac{T}{\omega_d}\exp[-\xi\omega_n T]\left\{\left[\omega_n^2\left(1-\xi^2\right) - \xi^2\omega_n^2\right]\sin[\omega_d T] - \left[2\xi\omega_n\omega_d\right]\cos[\omega_d T]\right\}$$

(A-56)

$$D = \frac{\omega_n T}{\omega_d}\exp[-\xi\omega_n T]\left\{\left[\omega_n\left(1-\xi^2\right) - \xi^2\omega_n\right]\sin[\omega_d T] - \left[2\xi\omega_d\right]\cos[\omega_d T]\right\}$$

(A-57)

$$D = \frac{\omega_n T}{\omega_d} \exp\left[-\xi\omega_n T\right] \left\{ \left[\omega_n\left(1-2\xi^2\right)\right]\sin\left[\omega_d T\right] - \left[2\xi\omega_d\right]\cos\left[\omega_d T\right] \right\}$$

$$\text{(A-58)}$$

$$D = \omega_n T \exp\left[-\xi\omega_n T\right] \left\{ \left[\frac{\omega_n}{\omega_d}\left(1-2\xi^2\right)\right]\sin\left[\omega_d T\right] - 2\xi\cos\left[\omega_d T\right] \right\}$$

$$\text{(A-59)}$$

Now replace T with Δt.

$$A = 2\exp\left[-\xi\omega_n\Delta t\right]\cos\left[\omega_d\Delta t\right] \tag{A-60}$$

$$B = -\exp\left[-2\xi\omega_n\Delta t\right] \tag{A-61}$$

$$C = 2\xi\omega_n\Delta t \tag{A-62}$$

$$D = \omega_n\Delta t \exp\left[-\xi\omega_n\Delta t\right] \left\{ \left[\frac{\omega_n}{\omega_d}\left(1-2\xi^2\right)\right]\sin\left[\omega_d\Delta t\right] - 2\xi\cos\left[\omega_d\Delta t\right] \right\} \tag{A-63}$$

$$E = 0 \tag{A-64}$$

Now substitute the coefficients into equation (A-33).

$$\ddot{x}(k\Delta t) =$$

$$+ 2\exp\left[-\xi\omega_n\Delta t\right]\cos\left[\omega_d\Delta t\right]\ddot{x}(k\Delta t - \Delta t)$$

$$- \exp\left[-2\xi\omega_n\Delta t\right]\ddot{x}(k\Delta t - 2\Delta t)$$

$$+ 2\xi\omega_n\Delta t\,\ddot{y}(k\Delta t)$$

$$+ \omega_n\Delta t \exp\left[-\xi\omega_n\Delta t\right] \left\{ \left[\frac{\omega_n}{\omega_d}\left(1-2\xi^2\right)\right]\sin\left[\omega_d\Delta t\right] - 2\xi\cos\left[\omega_d\Delta t\right] \right\}\ddot{y}(k\Delta t - \Delta t)$$

$$\text{(A-65)}$$

Now change to index notation,

$$\ddot{x}_i =$$

$$+ 2\exp[-\xi\omega_n\Delta t]\cos[\omega_d\Delta t]\ddot{x}_{i-1}$$

$$- \exp[-2\xi\omega_n\Delta t]\ddot{x}_{i-2}$$

$$+ 2\xi\omega_n\Delta t\, \ddot{y}_i$$

$$+ \omega_n\Delta t\exp[-\xi\omega_n\Delta t]\left\{\left[\frac{\omega_n}{\omega_d}\left(1-2\xi^2\right)\right]\sin[\omega_d\Delta t] - 2\xi\cos[\omega_d\Delta t]\right\}\ddot{y}_{i-1}$$

$$(A\text{-}66)$$

Note that Z-transform theory is described in many signal analysis and control systems textbooks. An example is:

R. Dorf, Modern Control Systems, Addison-Wesley, Reading, Massachusetts, 1980.

Equation (A-66) can be readily calculated via a computer program, written in FORTRAN, C/C++, or Matlab. Again, the damping value is typically fixed, whereas the natural frequency is an independent variable. A sample program is given in Appendix E.

The derivation is repeated in Appendix E for relative displacement. Note that relative displacement might be of interest to civil engineers who must calculate the stress in a building foundation resulting from an earthquake.

APPENDIX B

NOTES

The most efficient method for calculating the shock response spectrum is the time domain method given in the main body of the text.

As an alternative, the shock response spectrum can be calculated in the frequency domain.

FREQUENCY DOMAIN CALCULATION MODEL

An equivalency exists between the convolution in the time domain and Fourier transform multiplication in the frequency domain.

If $F(\omega)$ and $G(\omega)$ are the complex Fourier transforms of $f(x)$ and $g(x)$, respectively, the inverse transformation of the product of $F(\omega)$ and $G(\omega)$ is the convolution integral.

This concept is shown in Figure B-1, taken from Reference 11.

input y(t) | Linear System   output x(t)

$Y(\omega)$ h(t) impulse response function

                          $X(\omega)$

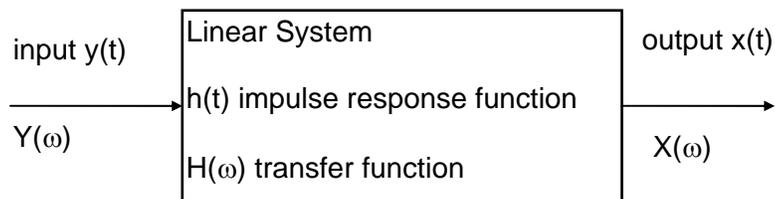 $H(\omega)$ transfer function

Figure B-1. Linear System

The theorem suggests another approach to shock response spectrum calculation, as shown in Table B-1.

Table B-1. Frequency Domain Approach		
Step	Description	Math
1	Take the Fourier transform of the input.	Convert y(t) to Y(ω)
2	Multiply this transform by the appropriate transfer function for each SDOF system.	X(ω)=H(ω)Y(ω)
3	The response time history is simply equal to the inverse transform of the product.	Convert X(ω) to x(t)
4	Search the response time history for the extreme amplitude values.	Find extremes in x(t)

Note that the forward Fourier transform of a discrete series is

$$F(k) = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ x(n) \exp\left(-j \frac{2\pi}{N} nk \right) \right\}, \qquad \text{for } k = 0, 1, ..., N-1 \qquad (B-1)$$

The inverse transform is

$$x(n) = \sum_{k=0}^{N-1} \left\{ F(k) \exp\left(+j \frac{2\pi}{N} nk \right) \right\}, \qquad \text{for } n = 0, 1, ..., N-1 \qquad (B-2)$$

Recall the governing equation of motion,

$$\ddot{Z}(t) + 2\xi\omega_n \dot{Z}(t) + \omega_n^2 Z(t) = -\ddot{Y}(t) \qquad (B-3)$$

Take the transform of both sides of (B-3),

$$\int_{-\infty}^{\infty} \left\{ \ddot{Z}(t) + 2\xi\omega_n \dot{Z}(t) + \omega_n^2 Z(t) \right\} \{ \exp[-j\omega t] \} dt = -\int_{-\infty}^{\infty} \ddot{Y}(t) \exp[-j\omega t] dt \qquad (B-4)$$

$$\int_{-\infty}^{\infty} \ddot{Z}(t) \exp[-j\omega t] dt + 2\xi\omega_n \int_{-\infty}^{\infty} \dot{Z}(t) \exp[-j\omega t] dt + \omega_n^2 \int_{-\infty}^{\infty} Z(t) \exp[-j\omega t] dt$$
$$= -\int_{-\infty}^{\infty} \ddot{Y}(t) \exp[-j\omega t] dt \qquad (B-5)$$

$$-\omega^2 \int_{-\infty}^{\infty} Z(t)\exp[-j\omega t]dt + j2\xi\omega\omega_n \int_{-\infty}^{\infty} Z(t)\exp[-j\omega t]dt + \omega_n^2 \int_{-\infty}^{\infty} Z(t)\exp[-j\omega t]dt$$

$$= -\int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt \tag{B-6}$$

$$\{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\}\int_{-\infty}^{\infty} Z(t)\exp[-j\omega t]dt = -\int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt \tag{B-7}$$

Note that

$$\int_{-\infty}^{\infty} Z(t)\exp[-j\omega t]dt = -\left[\frac{1}{\omega^2}\right]\int_{-\infty}^{\infty} \ddot{Z}(t)\exp[-j\omega t]dt \tag{B-8}$$

$$\frac{\{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n\}}{\omega^2}\int_{-\infty}^{\infty} \ddot{Z}(t)\exp[-j\omega t]dt = \int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt \tag{B-9}$$

$$\int_{-\infty}^{\infty} \ddot{Z}(t)\exp[-j\omega t]dt = \left\{\frac{\omega^2}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n}\right\}\int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt \tag{B-10}$$

Recall that,

$$\ddot{Z} = \ddot{X} - \ddot{Y} \tag{B-11}$$

Taking Fourier transforms,

$$\int_{-\infty}^{\infty} \ddot{Z}(t)\exp(-j\omega t)dt = \int_{-\infty}^{\infty} \ddot{X}(t)\exp(-j\omega t)dt - \int_{-\infty}^{\infty} \ddot{Y}(t)\exp(-j\omega t)dt \tag{B-12}$$

Substituting (B-12) into (B-10),

$$\int_{-\infty}^{\infty} \ddot{X}(t)\exp[-j\omega t]dt - \int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt = \left\{\frac{\omega^2}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n}\right\}\int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt \tag{B-13}$$

$$\int_{-\infty}^{\infty} \ddot{X}(t)\exp[-j\omega t]dt = \left\{1 + \frac{\omega^2}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n}\right\}\int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt \tag{B-14}$$

$$\int_{-\infty}^{\infty} \ddot{X}(t)\exp[-j\omega t]dt = \left\{\hat{H}(\omega)\right\}\int_{-\infty}^{\infty} \ddot{Y}(t)\exp[-j\omega t]dt \tag{B-15}$$

where

$$\hat{H}(\omega) = \left\{ \frac{\omega_n^2 + j2\xi\omega\omega_n}{(\omega_n^2 - \omega^2) + j2\xi\omega\omega_n} \right\}$$

(B-16)

$\hat{H}(\omega)$ Is thus the transfer function which is multiplied by the input Fourier transform to yield the output Fourier transform. Again, the response time history is calculated by taking the inverse of the output Fourier transform.

Note that the transfer function can also be derived from the Fourier transform of the unit impulse function $\hat{h}(t)$, which was given in equation (A-27).

APPENDIX C

FORCE SHOCK

<u>Derivation of Impulse Response Functions</u>

Consider a single-degree-of-freedom system.
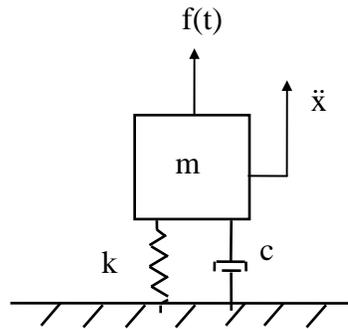
where

        m   is the mass

        c   is the viscous damping coefficient

        k   is the stiffness

        x   is the absolute displacement of the mass

      f(t)   is the applied force

Note that the double-dot denotes acceleration.

The free-body diagram is
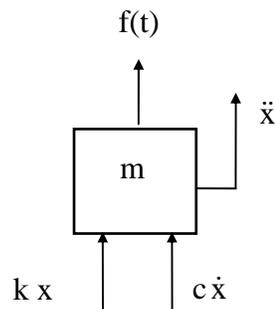
Summation of forces in the vertical direction

$$\sum F = \quad m\ddot{x} \qquad\qquad\qquad (C\text{-}1)$$

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$$m\ddot{x} = -c\dot{x} - kx + f(t) \tag{C-2}$$

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{C-3}$$

Divide through by m,

$$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \left(\frac{1}{m}\right)f(t) \tag{C-4}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = \omega_n^2$$

where

ω_n is the natural frequency in (radians/sec)

ξ is the damping ratio.

By substitution,

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}f(t) \tag{C-5a}$$

Equation (C-5) does not have a closed-form solution for the general case in which f(t) is an arbitrary function. A convolution integral approach must be used to solve the equation.

The solution method proceeds by finding a solution to the *homogeneous* form of equation (C-5). In other words, the solution is found for f(t)=0.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \tag{C-5b}$$

Equation (C-6) is essentially the same as equation (A-1), except that equation (C-6) is expressed in terms of absolute displacement. Also note that

$$V_0 = \frac{f(\tau)}{m}\Delta\tau \tag{C-6}$$

Some of the intermediate derivation steps can thus be skipped.

The displacement equation becomes.

$$x(t) = \frac{1}{m\omega_d} \int_0^t f(\tau)\left[\exp\{-\xi\omega_n(t-\tau)\}\right]\left[\sin\omega_d(t-\tau)\right]d\tau \tag{C-7}$$

The corresponding impulse response function for the displacement is

$$\hat{h}_d(t) = \frac{1}{m\omega_d}\left[\exp\{-\xi\omega_n t\}\right]\left[\sin\omega_d t\right] \tag{C-8}$$

The Leibnitz rule for differentiation under the integral sign is

$$\frac{d}{dt}\int_0^t f(\tau)g(t-\tau)d\tau = f(t)g(0) + \int_0^t f(\tau)g'(t-\tau)d\tau \tag{C-9}$$

where the prime indicates differentiation with respect to $t-\tau$. Equation (C-9) is taken from Meirovitch, page 16, Reference 11.

The velocity is

$$\dot{x}(t) = \frac{d}{dt}\left\{ \frac{1}{m\omega_d}\int_0^t f(\tau)\left[\exp(-\xi\omega_n(t-\tau))\right]\left[\sin\omega_d(t-\tau)\right]d\tau \right\} \tag{C-10}$$

$$\dot{x}(t) = \frac{1}{m\omega_d}f(\tau)\left[\exp(0)\right]\left[\sin 0\right] + \frac{1}{m\omega_d}\int_0^t f(\tau)\left\{ \left[\exp(-\xi\omega_n(t-\tau))\right]\left[\sin\omega_d(t-\tau)\right]\right\}' d\tau \tag{C-11}$$

$$\dot{x}(t) = \frac{1}{m\omega_d}\int_0^t f(\tau)\left\{ \left[\exp(-\xi\omega_n(t-\tau))\right]\left[\sin\omega_d(t-\tau)\right]\right\}' d\tau \tag{C-12}$$

Finally, the velocity is

$$\dot{x}(t) = \frac{1}{m}\int_0^t f(\tau)\left[\exp(-\xi\omega_n(t-\tau))\right]\left[\left\{\frac{-\xi\omega_n}{\omega_d}\right\}\sin\omega_d(t-\tau) + \cos\omega_d(t-\tau)\right]d\tau \tag{C-13}$$

The corresponding impulse response function for the velocity is

$$\hat{h}_v(t) = \frac{1}{m}\left[\exp(-\xi\omega_n t)\right]\left[\left\{\frac{-\xi\omega_n}{\omega_d}\right\}\sin\omega_d t + \cos\omega_d t\right] \tag{C-14}$$

The acceleration is

$$\ddot{x}(t) = \frac{d}{dt}\left\{\left[\frac{1}{m}\right]\int_0^t f(\tau)\left[\exp(-\xi\omega_n(t-\tau))\right]\left[\left\{\frac{-\xi\omega_n}{\omega_d}\right\}\sin\omega_d(t-\tau)+\cos\omega_d(t-\tau)\right]d\tau\right\}$$

(C-15)

$$\ddot{x}(t) = \left[\frac{f(t)}{m}\right]+\left[\frac{1}{m}\right]\int_0^t f(\tau)\left\{\left[\exp(-\xi\omega_n(t-\tau))\right]\left[\left\{\frac{-\xi\omega_n}{\omega_d}\right\}\sin\omega_d(t-\tau)+\cos\omega_d(t-\tau)\right]\right\}' d\tau$$

(C-16)

$$\ddot{x}(t) = +\left[\frac{f(t)}{m}\right]+\left[\frac{1}{m}\right]\left\{-\xi\omega_n\int_0^t f(\tau)\left[\exp(-\xi\omega_n(t-\tau))\right]\left[\left\{\frac{-\xi\omega_n}{\omega_d}\right\}\sin\omega_d(t-\tau)+\cos\omega_d(t-\tau)\right]d\tau\right\}$$

$$+\left[\frac{1}{m}\right]\left\{\int_0^t f(\tau)\left[\exp(-\xi\omega_n(t-\tau))\right]\left[-\xi\omega_n\cos\omega_d(t-\tau)-\omega_d\sin\omega_d(t-\tau)\right]d\tau\right\}$$

(C-17)

$$\ddot{x}(t) =$$
$$\frac{f(t)}{m}+\frac{1}{m\omega_d}\int_0^t f(\tau)\left[\exp(-\xi\omega_n(t-\tau))\right]\left[(-\omega_d^2+\xi^2\omega_n^2)\sin\omega_d(t-\tau)-2\xi\omega_n\omega_d\cos\omega_d(t-\tau)\right]d\tau$$

(C-18a)

$$\left\{\ddot{x}(t)-\frac{f(t)}{m}\right\} =$$

$$+\frac{1}{m\omega_d}\int_0^t f(\tau)\left[\exp(-\xi\omega_n(t-\tau))\right]\left[(-\omega_d^2+\xi^2\omega_n^2)\sin\omega_d(t-\tau)-2\xi\omega_n\omega_d\cos\omega_d(t-\tau)\right]d\tau$$

(C-18b)

The unit impulse function for acceleration is

$$\hat{h}_a(t) = \frac{1}{m\omega_d}\left[\exp\{-\xi\omega_n t\}\right]\left[(-\omega_d{}^2 + \xi^2\omega_n{}^2)\sin\omega_d t - 2\xi\omega_n\omega_d\cos\omega_d t\right] \tag{C-19}$$

Dispalcement Z-transform

Now take the Z-transform of equation (C-8).

$$Z\{\hat{h}_d(t)\} = Z\left\{\left[\frac{1}{m\omega_d}\right]\left[\exp\{-\xi\omega_n t\}\right]\left[\sin\omega_d t\right]\right\} \tag{C-20}$$

From standard Z-transform tables,

$$Z\{\hat{h}_d(t)\} = \frac{z\left[\dfrac{1}{m\omega_d}\right]\{\exp[-\xi\omega_n T]\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}} \tag{C-21}$$

The following relationship results,

$$\left[\frac{1}{T}\right]\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z\left\{\dfrac{1}{m\omega_d}\right\}\{\exp[-\xi\omega_n T]\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}} \tag{C-22}$$

Multiply through by T,

$$\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z\left\{\dfrac{T}{m\omega_d}\right\}\{\exp[-\xi\omega_n T]\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}}$$

$$\tag{C-23}$$

Equation (C-23) is satisfied if

$$A = 2\exp\left[-\xi\omega_n T\right]\cos\left[\omega_d T\right] \tag{C-24}$$

$$B = -\exp\left[-2\xi\omega_n T\right] \tag{C-25}$$

$$C = 0 \tag{C-26}$$

$$D = \left\{\frac{T}{m\omega_d}\right\}\exp[-\xi\omega_n T]\{\sin[\omega_d T]\} \tag{C-27}$$

$$E = 0 \tag{C-28}$$

Now replace T with Δt.

$$A = 2\exp[-\xi\omega_n \Delta t]\cos[\omega_d \Delta t] \tag{C-29}$$

$$B = -\exp[-2\xi\omega_n \Delta t] \tag{C-30}$$

$$C = 0 \tag{C-31}$$

$$D = \left\{\frac{\Delta t}{m\omega_d}\right\}\exp[-\xi\omega_n \Delta t]\{\sin[\omega_d \Delta t]\} \tag{C-32}$$

$$E = 0 \tag{C-33}$$

Now substitute the coefficients into the recursive equation.

$$x(kT) = A\, x(kT - T) + Bx(kT - 2T) + Cf(kT) + D\, f(kT - T) + E\, f(kT - 2T) \tag{C-34}$$

$$x(kT) =$$

$$+ 2\exp[-\xi\omega_n \Delta t]\cos[\omega_d \Delta t]x(kT - T)$$

$$- \exp[-2\xi\omega_n \Delta t]x(kT - 2T)$$

$$+ \left\{\frac{\Delta t}{m\omega_d}\right\}\exp[-\xi\omega_n \Delta t]\{\sin[\omega_d \Delta t]\}f(kT - T)$$

$$\tag{C-35}$$

Change to index notation.

$$x_i =$$

$$+ 2\exp[-\xi\omega_n\Delta t]\cos[\omega_d\Delta t]x_{i-1}$$

$$- \exp[-2\xi\omega_n\Delta t]x_{i-2}$$

$$+ \left\{\frac{\Delta t}{m\omega_d}\right\}\exp[-\xi\omega_n\Delta t]\{\sin[\omega_d\Delta t]\}f_{i-1}$$

(C-36)

Equation (C-36) is the digital recursive filtering relationship for the displacement response to an arbitrary force shock.

Velocity Z-transform

Take the Z-transform of equation (C-14).

$$Z\{\hat{h}_v(t)\} = Z\left\{\frac{1}{m}[\exp(-\xi\omega_n t)]\left[\left\{\frac{-\xi\omega_n}{\omega_d}\right\}\sin\omega_d t + \cos\omega_d t\right]\right\}$$

(C-37)

$$Z\{\hat{h}(t)\} = Z\{[\exp\{-\xi\omega_n t\}][\alpha\sin\omega_d t + \beta\cos\omega_d t]\}$$

(C-38)

where

$$\alpha = \frac{-\xi\omega_n}{m\omega_d}$$

(C-39)

$$\beta = \frac{1}{m}$$

(C-40)

Use the method derived in Appendix A.

$$\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z^2\beta T + zT\{\exp[-\xi\omega_n T]\}\{\alpha\sin[\omega_d T] - \beta\cos[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}}$$

(C-41)

Equation (C-41) is satisfied if

$$A = 2\exp\left[-\xi\omega_n T\right]\cos\left[\omega_d T\right] \tag{C-42}$$

$$B = -\exp\left[-2\xi\omega_n T\right] \tag{C-43}$$

$$C = \beta T \tag{C-44}$$

$$D = T\exp\left[-\xi\omega_n T\right]\left\{\alpha\sin\left[\omega_d T\right] - \beta\cos\left[\omega_d T\right]\right\} \tag{C-45}$$

$$E = 0 \tag{C-46}$$

By substitution,

$$A = 2\exp\left[-\xi\omega_n T\right]\cos\left[\omega_d T\right] \tag{C-47}$$

$$B = -\exp\left[-2\xi\omega_n T\right] \tag{C-48}$$

$$C = \left[\frac{1}{m}\right]T \tag{C-49}$$

$$D = T\exp\left[-\xi\omega_n T\right]\left\{\left[\frac{-\xi\omega_n}{m\omega_d}\right]\sin\left[\omega_d T\right] - \left[\frac{1}{m}\right]\cos\left[\omega_d T\right]\right\} \tag{C-50}$$

$$E = 0 \tag{C-51}$$

Now replace T with Δt.

$$A = 2\exp\left[-\xi\omega_n\Delta t\right]\cos\left[\omega_d\Delta t\right] \tag{C-52}$$

$$B = -\exp\left[-2\xi\omega_n\Delta t\right] \tag{C-53}$$

$$C = \left[\frac{1}{m}\right]\Delta t \tag{C-54}$$

$$D = \Delta t\exp\left[-\xi\omega_n\Delta t\right]\left\{\left[\frac{-\xi\omega_n}{m\omega_d}\right]\sin\left[\omega_d\Delta t\right] - \left[\frac{1}{m}\right]\cos\left[\omega_d\Delta t\right]\right\} \tag{C-55}$$

$$E = 0 \tag{C-56}$$

Now substitute the coefficients into the recursive equation.

$$x(kT) = A\,x(kT-T) + Bx(kT-2T) + Cf(kT) + Df(kT-T) + Ef(kT-2T) \tag{C-57}$$

$\dot{x}(kT) =$

$+ 2\exp[-\xi\omega_n \Delta t]\cos[\omega_d \Delta t]\dot{x}(kT - T)$

$- \exp[-2\xi\omega_n \Delta t]\dot{x}(kT - 2T)$

$+ \left[\dfrac{1}{m}\right]\Delta t\, f(kT)$

$+ \Delta t\, \exp[-\xi\omega_n \Delta t]\left\{\left[\dfrac{-\xi\omega_n}{m\omega_d}\right]\sin[\omega_d \Delta t] - \left[\dfrac{1}{m}\right]\cos[\omega_d \Delta t]\right\}f(kT - T)$

$$(C-58)$$

Change to index notation.

$\dot{x}_i =$

$+ 2\exp[-\xi\omega_n \Delta t]\cos[\omega_d \Delta t]\dot{x}_{i-1}$

$- \exp[-2\xi\omega_n \Delta t]\dot{x}_{i-2}$

$+ \left[\dfrac{1}{m}\right]\Delta t\, f_i$

$+ \Delta t\, \exp[-\xi\omega_n \Delta t]\left\{\left[\dfrac{-\xi\omega_n}{m\omega_d}\right]\sin[\omega_d \Delta t] - \left[\dfrac{1}{m}\right]\cos[\omega_d \Delta t]\right\}f_{i-1}$

$$(C-59)$$

Acceleration Z-transform

Take the Z-transform of equation (C-19).

$$Z\{\hat{h}_a(t)\} = Z\left\{\frac{1}{m\omega_d}[\exp\{-\xi\omega_n t\}]\left[(-\omega_d{}^2 + \xi^2\omega_n{}^2)\sin\omega_d t - 2\xi\omega_n\omega_d\cos\omega_d t\right]\right\} \qquad (C-60)$$

$$Z\{\hat{h}_a(t)\} = Z\{[\exp\{-\xi\omega_n t\}][\alpha\sin\omega_d t + \beta\cos\omega_d t]\} \qquad (C-61)$$

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where

$$\alpha = \frac{(-\omega_d^2 + \xi^2 \omega_n^2)}{m \omega_d} \tag{C-62}$$

$$\beta = \frac{-2\xi \omega_n \omega_d}{m \omega_d} \tag{C-63a}$$

$$\beta = \frac{-2\xi \omega_n}{m} \tag{C-63b}$$

Use the method derived in Appendix A.

$$\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z^2 \beta T + zT\{\exp[-\xi \omega_n T]\}\{\alpha \sin[\omega_d T] - \beta \cos[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi \omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi \omega_n T]\}} \tag{C-64}$$

Equation (C-64) is satisfied if

$$A = 2\exp[-\xi \omega_n T]\cos[\omega_d T] \tag{C-65}$$

$$B = -\exp[-2\xi \omega_n T] \tag{C-66}$$

$$C = \frac{-2\xi \omega_n}{m} T \tag{C-67}$$

$$D = T\exp[-\xi \omega_n T]\left\{\alpha \sin[\omega_d T] - \left[\frac{-2\xi \omega_n}{m}\right]\cos[\omega_d T]\right\} \tag{C-68}$$

$$E = 0 \tag{C-69}$$

By substitution,

$$A = 2\exp[-\xi \omega_n T]\cos[\omega_d T] \tag{C-70}$$

$$B = -\exp[-2\xi \omega_n T] \tag{C-71}$$

$$C = \left[\frac{-2\xi \omega_n}{m}\right]T \tag{C-72}$$

$$D = T\exp\left[-\xi\omega_n T\right]\left\{\left[\frac{(-\omega_d{}^2+\xi^2\omega_n{}^2)}{m\omega_d}\right]\sin[\omega_d T]-\left[\frac{-2\xi\omega_n}{m}\right]\cos[\omega_d T]\right\} \qquad \text{(C-73)}$$

$$E = 0 \qquad \text{(C-74)}$$

Now replace T with Δt.

$$A = 2\exp\left[-\xi\omega_n\Delta t\right]\cos[\omega_d\Delta t] \qquad \text{(C-75)}$$

$$B = -\exp\left[-2\xi\omega_n\Delta t\right] \qquad \text{(C-76)}$$

$$C = \left[\frac{-2\xi\omega_n}{m}\right]\Delta t \qquad \text{(C-77)}$$

$$D = \Delta t\exp\left[-\xi\omega_n\Delta t\right]\left\{\left[\frac{(-\omega_d{}^2+\xi^2\omega_n{}^2)}{m\omega_d}\right]\sin[\omega_d\Delta t]+\left[\frac{2\xi\omega_n}{m}\right]\cos[\omega_d\Delta t]\right\} \qquad \text{(C-78)}$$

$$E = 0 \qquad \text{(C-79)}$$

Now substitute the coefficients into the recursive equation.

$$x(kT) = A\,x(kT-T)+Bx(kT-2T)+Cf(kT)+Df(kT-T)+Ef(kT-2T) \qquad \text{(C-80)}$$

Let

$$\ddot{w}_i = \ddot{x}_i - [f_i/m] \qquad \text{(C-81)}$$

$$\ddot{w}(kT) =$$

$$+2\exp\left[-\xi\omega_n\Delta t\right]\cos[\omega_d\Delta t]\ddot{w}(kT-T)$$

$$-\exp\left[-2\xi\omega_n\Delta t\right]\ddot{w}(kT-2T)$$

$$+\left\{\left[\frac{-2\xi\omega_n}{m}\right]\Delta t\right\}f(kT)$$

$$+\Delta t\exp\left[-\xi\omega_n\Delta t\right]\left\{\left[\frac{(-\omega_d{}^2+\xi^2\omega_n{}^2)}{m\omega_d}\right]\sin[\omega_d\Delta t]+\left[\frac{2\xi\omega_n}{m}\right]\cos[\omega_d\Delta t]\right\}f(kT-T)$$

$$\text{(C-82)}$$

Change to index notation.

$$\ddot{w}_i =$$

$$+\,2\exp\!\left[-\,\xi\omega_n\Delta t\right]\cos\!\left[\omega_d\Delta t\right]\ddot{w}_{i-1}$$

$$-\exp\!\left[-\,2\xi\omega_n\Delta t\right]\ddot{w}_{i-2}$$

$$+\left\{\left[\frac{-2\xi\omega_n}{m}\right]\Delta t\ \right\}f_i$$

$$+\,\Delta t\,\exp\!\left[-\,\xi\omega_n\Delta t\right]\left\{\left[\frac{(-\omega_d{}^2+\xi^2\omega_n{}^2)}{m\omega_d}\right]\sin\!\left[\omega_d\Delta t\right]+\left[\frac{2\xi\omega_n}{m}\right]\cos\!\left[\omega_d\Delta t\right]\right\}f_{i-1}$$

$$(C\text{-}83)$$

$$\ddot{w}_i =$$

$$+\,2\exp\!\left[-\,\xi\omega_n\Delta t\right]\cos\!\left[\omega_d\Delta t\right]\ddot{w}_{i-1}$$

$$-\exp\!\left[-\,2\xi\omega_n\Delta t\right]\ddot{w}_{i-2}$$

$$+\left\{\left[\frac{-2\xi\omega_n}{m}\right]\Delta t\ \right\}f_i$$

$$+\,\Delta t\,\exp\!\left[-\,\xi\omega_n\Delta t\right]\left\{\left[\frac{\left(-\omega_n{}^2\left(1-\xi^2\right)+\xi^2\omega_n{}^2\right)}{m\omega_d}\right]\sin\!\left[\omega_d\Delta t\right]+\left[\frac{2\xi\omega_n}{m}\right]\cos\!\left[\omega_d\Delta t\right]\right\}f_{i-1}$$

$$(C\text{-}84)$$

$$\ddot{w}_i =$$

$$+ 2\exp[-\xi\omega_n\Delta t]\cos[\omega_d\Delta t]\ddot{w}_{i-1}$$

$$- \exp[-2\xi\omega_n\Delta t]\ddot{w}_{i-2}$$

$$+ \left\{ \left[\frac{-2\xi\omega_n}{m} \right]\Delta t \right\} f_i$$

$$+ \Delta t \exp[-\xi\omega_n\Delta t]\left\{ \left[\frac{(-\omega_n{}^2 + 2\xi^2\omega_n{}^2)}{m\omega_d} \right]\sin[\omega_d\Delta t] + \left[\frac{2\xi\omega_n}{m} \right]\cos[\omega_d\Delta t] \right\} f_{i-1}$$

$$\text{(C-85)}$$

$$\ddot{w}_i =$$

$$+ 2\exp[-\xi\omega_n\Delta t]\cos[\omega_d\Delta t]\ddot{w}_{i-1}$$

$$- \exp[-2\xi\omega_n\Delta t]\ddot{w}_{i-2}$$

$$+ \left\{ \left[\frac{-2\xi\omega_n}{m} \right]\Delta t \right\} f_i$$

$$+ \Delta t \exp[-\xi\omega_n\Delta t]\left\{ \left[\frac{\omega_n{}^2(-1 + 2\xi^2)}{m\omega_d} \right]\sin[\omega_d\Delta t] + \left[\frac{2\xi\omega_n}{m} \right]\cos[\omega_d\Delta t] \right\} f_{i-1}$$

$$\text{(C-86)}$$

The absolute acceleration of the mass is

$$\ddot{x}_i = \ddot{w}_i + [f_i / m] \tag{C-87}$$

APPENDIX D

RELATIVE DISPLACEMENT

Consider the unit impulse response function $h(t)$, equation (A-11).

$$h(t) = \frac{-1}{\omega_d}[\exp(-\xi\,\omega_n t)][\sin\,\omega_d t] \tag{D-1}$$

Note that the minus sign comes from equation (A-13).

Let r be the relative displacement of the mass. Let $\ddot{y}$ be the base input acceleration.

Consider the following recursive formula as a proposed solution

$$r(kT) = A\,r(kT\text{-}T) + B\,r(kT\text{-}2T) + C\,\ddot{y}(kT) + D\,\ddot{y}(kT\text{-}T) + E\,\ddot{y}(kT\text{-}2T)$$

$$\tag{D-2}$$

      where T is the time increment

      and k = 0, 1, 2, 3,......

Take the Z-transform of equation (D-2).

$$R(z) = A\,z^{-1}R(z) + B\,z^{-2}R(z) + C\,Y(z) + D\,z^{-1}Y(z) + E\,z^{-2}Y(z)$$

$$\tag{D-3}$$

Now that z^{-1} is a unit delay in the Z-transform approach.

Collecting terms,

$$\left[1\text{-}A\quad z^{-1} - B\quad z^{-2}\right]R(z) = \left[C + D\,z^{-1} + E\,z^{-2}\right]Y(z) \tag{D-4}$$

$$R(z) = \frac{\left[C + D\,z^{-1} + E\,z^{-2}\right]}{\left[1\text{-}A\quad z^{-1} - B\quad z^{-2}\right]}\,Y(z) \tag{D-5}$$

Multiplying through by z^2,

$$R(z) = \frac{\left[Cz^2 + Dz + E\right]}{\left[z^2 - A\ z - B\right]} Y(z) \qquad \text{(D-6)}$$

Let y(kT) be a discrete unit impulse function.

$$y(kT) = \begin{cases} \dfrac{1}{T} & \text{for } k = 0 \\[2em] 0 & \text{for } k \neq 0 \end{cases} \qquad \text{(D-7)}$$

The Z-transform of a constant is equal to that constant. Thus, the transform is

$$Y(kT) = \begin{cases} \dfrac{1}{T} & \text{for } k = 0 \\[2em] 0 & \text{for } k \neq 0 \end{cases} \qquad \text{(D-8)}$$

By substitution,

$$R(z) = \frac{1}{T} \frac{\left[Cz^2 + Dz + E\right]}{\left[z^2 - A\ z - B\right]} \qquad \text{(D-9)}$$

Note that R(z) is equal to H(z) for the discrete impulse input.

$$H(z) = \frac{1}{T} \frac{\left[Cz^2 + Dz + E\right]}{\left[z^2 - A\ z - B\right]} \qquad \text{(D-10)}$$

H(z) is the Z-transform of the impulse response of the digital filter.

$$Z\{h(kT)\} = H(z) \qquad \text{(D-11)}$$

Now take the Z-transform of equation (D-1).

$$Z\{h(t)\} = Z\left\{\frac{-1}{\omega_d}[\exp\{-\xi\omega_n t\}][\sin\omega_d t]\right\} \tag{D-12}$$

From standard Z-transform tables,

$$Z\{h(t)\} = \frac{z\left\{\frac{-1}{\omega_d}\exp[-\xi\omega_n T]\right\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}} \tag{D-13}$$

Equations (D-10) through (D-13) lead to the following,

$$\left[\frac{1}{T}\right]\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z\left\{\frac{-1}{\omega_d}\exp[-\xi\omega_n T]\right\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}} \tag{D-14}$$

Multiply through by T.

$$\frac{Cz^2 + Dz + E}{z^2 - Az - B} = \frac{z\left\{\frac{-T}{\omega_d}\exp[-\xi\omega_n T]\right\}\{\sin[\omega_d T]\}}{z^2 - 2z\{\exp[-\xi\omega_n T]\}\{\cos[\omega_d T]\} + \{\exp[-2\xi\omega_n T]\}} \tag{D-15}$$

Equation (D-15) is satisfied if

$$A = 2\exp[-\xi\omega_n T]\cos[\omega_d T] \tag{D-16}$$

$$B = -\exp[-2\xi\omega_n T] \tag{D-17}$$

$$C = 0 \tag{D-18}$$

$$D = \frac{-T}{\omega_d}\exp[-\xi\omega_n T]\{\sin[\omega_d T]\} \tag{D-19}$$

$$E = 0 \tag{D-20}$$

Now replace T with Δt.

$$A = 2\exp[-\xi\omega_n \Delta t]\cos[\omega_d \Delta t] \tag{D-21}$$

$$B = -\exp[-2\xi\omega_n \Delta t] \tag{D-22}$$

$$C = 0 \tag{D-23}$$

$$D = \frac{-\Delta t}{\omega_d}\exp[-\xi\omega_n \Delta t]\{\sin[\omega_d \Delta t]\} \tag{D-24}$$

$$E = 0 \tag{D-25}$$

Now substitute the coefficients into equation (D-2).

$r(k\Delta t) =$

$+ 2\exp[-\xi\omega_n \Delta t]\cos[\omega_d \Delta t]r(k\Delta t - \Delta t)$

$- \exp[-2\xi\omega_n \Delta t]r(k\Delta t - 2\Delta t)$

$- \frac{\Delta t}{\omega_d}\exp[-\xi\omega_n \Delta t]\{\sin[\omega_d \Delta t]\}\ddot{y}(k\Delta t - \Delta t)$

$$\tag{D-26}$$

Now change to index notation. The relative displacement equation becomes

$$r_i =$$

$$+ 2\exp[-\xi\omega_n\Delta t]\cos[\omega_d\Delta t]r_{i-1}$$

$$- \exp[-2\xi\omega_n\Delta t]r_{i-2}$$

$$- \frac{\Delta t}{\omega_d}\exp[-\xi\omega_n\Delta t]\{\sin[\omega_d\Delta t]\}\ddot{y}_{i-1}$$

$$(D-27)$$

<u>Matlab Script</u>

```
disp(' ')
disp(' srs.m   ver 2.9   March 18, 2009')
disp(' by Tom Irvine   Email: tomirvine@aol.com')
disp(' ')
disp(' This program calculates the shock response spectrum')
disp(' of an acceleration time history, which is pre-loaded into
Matlab.')
disp(' The time history must have two columns: time(sec) &
acceleration')
disp(' ')
%
clear t;
clear y;
clear yy;
clear n;
clear fn;
clear a1;
clear a2
clear b1;
clear b2;
clear jnum;
clear THM;
clear resp;
clear x_pos;
clear x_neg;
%
iunit=input(' Enter acceleration unit:   1= G   2= m/sec^2  ');
%
disp(' ')
disp(' Select file input method ');
disp('   1=external ASCII file ');
disp('   2=file preloaded into Matlab ');
disp('   3=Excel file');
file_choice = input('');
%
if(file_choice==1)
        [filename, pathname] = uigetfile('*.*');
        filename = fullfile(pathname, filename);
        fid = fopen(filename,'r');
        THM = fscanf(fid,'%g %g',[2 inf]);
        fclose(fid);
        THM=THM';
end
if(file_choice==2)
        THM = input(' Enter the matrix name:  ');
end
if(file_choice==3)
        [filename, pathname] = uigetfile('*.*');
```

```matlab
        xfile = fullfile(pathname, filename);
%
        THM = xlsread(xfile);
%
end
%
t=double(THM(:,1));
y=double(THM(:,2));
%
tmx=max(t);
tmi=min(t);
nnn = size(y);
n=max(nnn);
%
out1 = sprintf('\n  %d samples ',n);
disp(out1)
%
dt=(tmx-tmi)/(n-1);
sr=1./dt;
%
disp(' ')
disp(' Time Step ');
dtmin=min(diff(t));
dtmax=max(diff(t));
%
out4 = sprintf(' dtmin  = %8.4g sec  ',dtmin);
out5 = sprintf(' dt     = %8.4g sec  ',dt);
out6 = sprintf(' dtmax  = %8.4g sec  ',dtmax);
disp(out4)
disp(out5)
disp(out6)
%
disp(' ')
disp(' Sample Rate ');
out4 = sprintf(' srmin  = %8.4g samples/sec  ',1/dtmax);
out5 = sprintf(' sr     = %8.4g samples/sec  ',1/dt);
out6 = sprintf(' srmax  = %8.4g samples/sec  \n',1/dtmin);
disp(out4)
disp(out5)
disp(out6)
%
ncontinue=1;
if(((dtmax-dtmin)/dt)>0.01)
    disp(' ')
    disp(' Warning:  time step is not constant.  Continue calculation?
1=yes 2=no ')
    ncontinue=input(' ')
end
if(ncontinue==1)
%
    fn(1)=input(' Enter the starting frequency (Hz)  ');
    if fn(1)>sr/30.
        fn(1)=sr/30.;
    end
```

```matlab
%
    idamp=input(' Enter damping format:  1= damping ratio   2= Q  ');
%
    disp(' ')
    if(idamp==1)
        damp=input(' Enter damping ratio (typically 0.05) ');
    else
        Q=input(' Enter the amplification factor (typically Q=10) ');
        damp=1./(2.*Q);
    end
%
    disp(' ')
    disp(' Select algorithm: ')
    disp(' 1=Kelly-Richman  2=Smallwood ');
    ialgorithm=input(' ');
%
    tmax=(tmx-tmi) + 1./fn(1);
    limit = round( tmax/dt );
    n=limit;
    yy=zeros(1,limit);
    for i=1:max(nnn)
        yy(i)=y(i);
    end
%
    disp(' ')
    disp(' Calculating response..... ')
%
%  SRS engine
%
    for j=1:1000
%
        omega=2.*pi*fn(j);
        omegad=omega*sqrt(1.-(damp^2));
        cosd=cos(omegad*dt);
        sind=sin(omegad*dt);
        domegadt=damp*omega*dt;
%
        if(ialgorithm==1)
            a1(j)=2.*exp(-domegadt)*cosd;
            a2(j)=-exp(-2.*domegadt);
            b1(j)=2.*domegadt;
            b2(j)=omega*dt*exp(-domegadt);
            b2(j)=b2(j)*( (omega/omegad)*(1.-2.*(damp^2))*sind -
2.*damp*cosd );
            b3(j)=0;
%
        else
              E=exp(-damp*omega*dt);
               K=omegad*dt;
               C=E*cos(K);
               S=E*sin(K);
               Sp=S/K;
%
               a1(j)=2*C;
```

```matlab
                a2(j)=-E^2;
                b1(j)=1.-Sp;
                b2(j)=2.*(Sp-C);
                b3(j)=E^2-Sp;
            end
        forward=[ b1(j),  b2(j),  b3(j) ];
        back   =[      1, -a1(j), -a2(j) ];
%
        resp=filter(forward,back,yy);
%
        x_pos(j)= max(resp);
        x_neg(j)= min(resp);
%
        jnum=j;
        if  fn(j) > sr/8.
            break
        end
        fn(j+1)=fn(1)*(2. ^ (j*(1./12.)));
    end
%
    maximaxSRS=max(x_pos,abs(x_neg));
%
    fmax=0;
    zmax=0;
    for(i=1:max(size(fn)))
        if(x_pos(i)>zmax)
            zmax=x_pos(i);
            fmax=fn(i);
        end
        if(abs(x_neg(i))>zmax)
            zmax=abs(x_neg(i));
            fmax=fn(i);
        end
    end
%
    if(iunit==1)
        out5 = sprintf('\n Absolute Peak is %10.5g G at %10.5g Hz
',zmax,fmax);
    else
        out5 = sprintf('\n Absolute Peak is %10.5g m/sec^2 at %10.5g Hz
',zmax,fmax);
    end
    disp(out5)
%
%  Output options
%
    disp(' ')
    disp(' Select output option ');
    choice=input(' 1=plot only   2=plot & output text file  ' );
    disp(' ')
%
    if choice == 2
%%
```

```
        [writefname, writepname] = uiputfile('*','Save positive &
negative SRS data as');
          writepfname = fullfile(writepname, writefname);
          writedata = [fn' x_pos' (abs(x_neg))' ];
          fid = fopen(writepfname,'w');
          fprintf(fid,'  %g \t %g \t %g\n',writedata');
          fclose(fid);
%%
        [writefname, writepname] = uiputfile('*','Save maximax SRS data
as');
          writepfname = fullfile(writepname, writefname);
          writedata = [fn' maximaxSRS' ];
          fid = fopen(writepfname,'w');
          fprintf(fid,'  %g \t %g \n',writedata');
          fclose(fid);
%%
%   disp(' Enter output filename ');
%   SRS_filename = input(' ','s');
%
%   fid = fopen(SRS_filename,'w');
%   for j=1:jnum
%        fprintf(fid,'%7.2f %10.3f %10.3f
\n',fn(j),x_pos(j),abs(x_neg(j)));
%   end
%   fclose(fid);
    end
%
%  Plot SRS
%
    disp(' ')
    disp(' Plotting output..... ')
%
%  Find limits for plot
%
    srs_max = max(x_pos);
    if max( abs(x_neg) ) > srs_max
        srs_max = max( abs(x_neg ));
    end
    srs_min = min(x_pos);
    if min( abs(x_neg) ) < srs_min
        srs_min = min( abs(x_neg ));
    end
%
    disp(' select plot type:  1=positive & negative   2=maximax ');
    ipt=input(' ');
%
    clear figure(1);
    figure(1);
%
    if(ipt==1)
        plot(fn,x_pos,fn,abs(x_neg),'-.');
        legend ('positive','negative',2);
    else
        plot(fn,maximaxSRS);
```

```matlab
    end
%
    SRS_pn=[fn',x_pos',abs(x_neg)'];
    SRS_max=[fn',x_pos',maximaxSRS'];
%
    if iunit==1
        ylabel('Peak Accel (G)');
    else
        ylabel('Peak Accel (m/sec^2)');
    end
    xlabel('Natural Frequency (Hz)');
    Q=1./(2.*damp);
    out5 = sprintf(' Acceleration Shock Response Spectrum Q=%g ',Q);
    title(out5);
    grid;

set(gca,'MinorGridLineStyle',':','GridLineStyle',':','XScale','log','YS
cale','log');
%
    ymax= 10^(round(log10(srs_max)+0.8));
    ymin= 10^(round(log10(srs_min)-0.6));
%
    fmax=max(fn);
    fmin=fmax/10.;
%
    fmax= 10^(round(log10(fmax)+0.5));
%
    if  fn(1) >= 0.1
        fmin=0.1;
    end
    if  fn(1) >= 1
        fmin=1;
    end
    if  fn(1) >= 10
        fmin=10;
    end
    if  fn(1) >= 100
        fmin=100;
    end
    axis([fmin,fmax,ymin,ymax]);
%
    disp(' Matlab matrices: ')
    disp('      SRS_pn  - Acceleration SRS positive & negative ')
    disp('      SRS_max - Acceleration SRS maximax              ')
%
    disp(' ')
    disp(' Plot pseudo velocity? ');
    vchoice=input(' 1=yes   2=no  ' );
    if(vchoice==1)
        disp(' select plot type:  1=positive & negative   2=maximax ');
        ipt=input(' ');
%
%   Convert to pseudo velocity
%
```

```
        for j=1:jnum
            if iunit==1
                x_pos(j)=386.*x_pos(j)/(2.*pi*fn(j));
                x_neg(j)=386.*x_neg(j)/(2.*pi*fn(j));
            else
                x_pos(j)=x_pos(j)/(2.*pi*fn(j));
                x_neg(j)=x_neg(j)/(2.*pi*fn(j));
            end
        end
%
        srs_max = max(x_pos);
        if max( abs(x_neg) ) > srs_max
            srs_max = max( abs(x_neg ));
        end
        srs_min = min(x_pos);
        if min( abs(x_neg) ) < srs_min
            srs_min = min( abs(x_neg ));
        end
%
        clear figure(2);
        figure(2);
        clear xpn;
        xpn=max(x_pos,abs(x_neg));
        figure(2);
        if(ipt==1)
            plot(fn,x_pos,fn,abs(x_neg),'-.');
            legend ('positive','negative',2);
        else
            plot(fn,xpn);
        end
        pseudo_velocity_pn=[fn',x_pos',abs(x_neg)'];
        pseudo_velocity_max=[fn',xpn'];
%
        if iunit==1
            ylabel('Velocity (in/sec)');
        else
            ylabel('Velocity (m/sec)');
        end
        xlabel('Natural Frequency (Hz)');
        Q=1./(2.*damp);
        out5 = sprintf(' Pseudo Velocity Shock Response Spectrum Q=%g
',Q);
        title(out5);
        grid;

set(gca,'MinorGridLineStyle',':','GridLineStyle',':','XScale','log','YS
cale','log');
%
        ymax= 10^(round(log10(srs_max)+0.8));
        ymin= 10^(round(log10(srs_min)-0.6));
%
        axis([fmin,fmax,ymin,ymax]);
    end
%
```

```
%
    disp(' Matlab matrices: ')
    disp('        pseudo_velocity_pn  - pseudo_velocity SRS positive &
negative ')
    disp('        pseudo_velocity_max - pseudo_velocity SRS maximax
')
%
end
```