# Post's System of Tag — A Simple Discrete Nonlinear System

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**Abstract** – We consider an instance $T_0$ of Post's system of tag for which we show that it is nonlinear. Then we discuss some computer simulations and study the periodic behavior of $T_0$.

## 1. Introduction

One of the oldest and simplest rewriting systems is Post's system of tag. Such a system $T$ with $T = (\Sigma, d, P, \omega_0)$ consists of an *alphabet* (a finite set of symbols) $\Sigma$, a natural number $d$ (the *deletion number* of $T$), a function $P : \Sigma \to \Sigma^*$ from symbols from $\Sigma$ to strings over $\Sigma$, and an *input* or *initial string* $\omega_0$ ($\omega_0 \in \Sigma^+$; $\Sigma^+$ denotes the set of all nonempty strings over $\Sigma$ and $\Sigma^* = \Sigma^+ \cup \{\lambda\}$, where $\lambda$ is the empty word, i.e., the string of length 0). Each Post's system of tag is deterministic and possesses a state space which consists of all strings over $\Sigma$. The way in which state changes take place — or, equivalently, how the string $\omega_{t+1}$ is obtained from the string $\omega_t$ ($t \geq 0$), denoted by $\omega_t \Rightarrow \omega_{t+1}$ — is as follows:

- If the left-most symbol of $\omega_t$ equals $\sigma$, then we append the string $P(\sigma)$ to the right end of $\omega_t$. We denote the string obtained in this fashion by $\omega_t'$.
- We delete the first $d$ symbols from $\omega_t'$; this yields $\omega_{t+1}$.

Post's systems of tag have been introduced in [7,8]; cf. [6] for an introduction. They are quite powerful, since they are able to simulate Turing-machine computations [5] and thus to perform every effective computation. However, in this note we restrict our attention to a single simple instance $T_0$ of such a system (§2). In §3 we show that this instance is nonlinear. Computer simulations and the periodic behavior of $T_0$ are discussed in §4 and §5, respectively. §6 consists of conclusions and an open problem.

## 2. Examples

We consider two instances of Post's system of tag and discuss their behavior. The first one happens to be linear whereas the second one turns out to be nonlinear; cf. §3.

**Example 2.1.** Let $T_1 = (\Sigma, d, P, \omega_0)$ with $\Sigma = \{a\}$, $d = 3$, $P(a) = a^2$ and $\omega_0 = a^4$. Then $aaaa \Rightarrow aaa \Rightarrow aa \Rightarrow a \Rightarrow \lambda$. Thus $\omega_0$ vanishes in four steps: there is no rule to rewrite $\lambda$.

Slightly more general is the system $T_2 = (\Sigma, d, P, \omega_0)$ with $\Sigma = \{a\}$, $P(a) = a^k$ and $\omega_0 \in \Sigma^+$. If $|\omega|$ denotes the length of $\omega$, then $T_2$ can be described by $|\omega_{t+1}| = |\omega_t| + k - d$, i.e., by a first-order linear difference equation, of which the solution results in $\omega_t = a^{(k-d)t}\omega_0$. Now for $d > k$, $\omega_t$ vanishes, i.e., $\omega_0 \Rightarrow^{\tau} \lambda$ with $\tau = \lceil |\omega_0| / (d-k) \rceil$; for $d = k$, $\omega_0$ is a fixed point: $\omega_0 \Rightarrow^t \omega_0$ ($t \geq 0$); and for $d < k$, $\omega_0$ explodes: $|\omega_t| \to \infty$ as $t \to \infty$. □

Of the simple instances of Post's system of tag the following one — already studied by Post in 1921-1922; cf. [8] — is one of the simplest nontrivial cases; cf. [1,2,3,6,7,8,9,10]. The remaining part of this note is devoted to this instance.

**Example 2.2.** Consider $T_0 = (\Sigma, d, P, \omega_0)$ with $\Sigma = \{0,1\}$, $d = 3$, $P(0) = 00$ and $P(1) = 1101$. Examples of its behavior are in Figure 1 for the input string $\omega_0$ equal to 1000 and to 1001, respectively. □

For an arbitrary Post's system of tag $T$ we can distinguish the following types of behavior:

(1) After $\tau$ steps we enter a periodic cycle with (minimal) period $\pi$: $\omega_{\tau+\pi} = \omega_\tau$. For $T_0$ with $\omega_0 = 1001$, we have $\tau = 3$ and $\pi = 2$; cf Figure 1.

(2) After $\tau$ steps we reach a fixed point ($\pi = 1$). $T_0$ has no fixed points other than $\lambda$.

(3) After $\tau$ steps the string vanishes ($\pi = 0$). For $T_0$ strings $0^n$ vanish with $\tau = n$. See also Figure 1.

(4) $\omega_0$ explodes, i.e., $|\omega_t| \to \infty$ as $t \to \infty$. This is the only form of chaotic behavior for $T$ ($\tau + \pi = \infty$). For $T_0$ no examples of this behavior are known up to now.

| $t$ | $\omega_t$ with $\omega_0 = 1000$ | $\omega_t$ with $\omega_0 = 1001$ |
|---|---|---|
| 0 | 1000 | 1001 |
| 1 | 01101 | 11101 |
| 2 | 0100 | 011101 |
| 3 | 000 | 10100 |
| 4 | 00 | 001101 |
| 5 | 0 | 10100 |
| 6 | | 001101 |
| 7 | | 10100 |
| 8 | | . . . |

**Figure 1.**

## 3. Nonlinearity of $T_0$

If $\Sigma$ consists of a single symbol, then $T$ is equivalent to a linear difference equation (Example 2.1). But when $\Sigma$ contains more than one symbol, $T$ becomes nonlinear. For $T_0$ we have the following result.

**Proposition 3.1.** *$T_0$ is equivalent to the following nonlinear difference equation*

$$u_{t+1} = \begin{cases} \lfloor (26 \cdot 2^{\lg(u_t)} + u_t - 7)/8 \rfloor & \text{if } u_t \text{ is even,} \\ \lfloor (3 \cdot 2^{\lg(u_t)} + u_t - 7)/8 \rfloor & \text{if } u_t \text{ is odd,} \end{cases}$$

*where* $\lg(u_t)$ *is the length of the 2-adic representation of* $u_t$*, i.e., if*

$$u_t = \sum_{i=0}^{n} d_i \cdot 2^i$$

*with* $d_i \in \{1,2\}$ $(0 \le i \le n)$*, then* $\lg(u_t) = n$.

*Proof (sketch):* Starting from $T_0$ we first change each 1 into 2 and each 0 into 1. In this way we obtain $T_1$. Next we interchange "left" and "right", which results in $T_2$. The effect of these manipulations on $P$ and on a string $\omega_t$ are shown in the following table.

| | $T_0$ | $T_1$ | $T_2$ |
|---|---|---|---|
| $P$ | $P(0) = 00,\ \ P(1) = 1101$ | $P(1) = 11,\ \ P(2) = 2212$ | $P(1) = 11,\ \ P(2) = 2122$ |
| $\omega_t$ | 001101 | 112212 | 212211 |

Finally, we interpret all strings over $\{1,2\}$ as 2-adic representations, keeping in mind that 3 and 26 in decimal notation correspond to 11 and 2122, respectively. □

In essence $T_0$ has the same structure as a younger but more famous simple nonlinear system, viz.

**The 3x + 1 problem.** (Also: Ulam's problem, Collatz's problem, the Syracuse problem, etc. [4]):

$$v_{t+1} = \begin{cases} v_t/2 & \text{if } v_t \text{ is even,} \\ 3 \cdot v_t + 1 & \text{if } v_t \text{ is odd.} \end{cases}$$ □

**The 3x + 1 conjecture.** For each initial value $v_0$ $(v_0 \in \mathbb{N})$ the ultimate behavior of $v_t$ is the 3-cycle $\cdots \Rightarrow 4 \Rightarrow 2 \Rightarrow 1 \Rightarrow 4 \cdots$ $(*)$. □

But comparing $T_0$ and the 3x + 1 problem yields the following two differences. (i) The periodic behavior of $T_0$ is much richer than the single 3-cycle $(*)$; cf. §4 and §5. (ii) Each state of the 3x + 1 problem has a predecessor: for each $n$ in $\mathbb{N}$ there exists a number $m$ in $\mathbb{N}$ (viz. $m = 2 \cdot n$) such that $m \Rightarrow n$. On the other hand $T_0$ possesses so-called *Garden-of-Eden states*, i.e., states that can only exist at time $t = 0$ (viz. all strings ending in 11, 10, 001 or 0101).

## 4. Computer Simulations of $T_0$

We performed two series of computer simulations of $T_0$'s behavior. Both series were obtained by varying the initial string $\omega_0$. First we consider small inputs, i.e., $|\omega_0| \le 15$. There is no need to consider all these small strings, since many of them show the same ultimate behavior. We introduce the following obvious equivalence relation.

**Definition 4.1.** Let $d$ be a natural number. A position $i$ in a string $\omega$ is called *critical* if $i \bmod d = 1$. Two strings $\omega$ and $\omega'$ are *equivalent* — notation: $\omega \equiv \omega'$ — if $|\omega| = |\omega'|$, and they are equal at their critical positions. □

**Observation 4.2.** Each equivalence class $[\omega]$ with respect to $\equiv$ contains a minimal element $\rho$ with respect to lexicographical order. This representative $\rho$ is obtained from $\omega$ by changing the 1's at the non-critical positions in $\omega$ into 0's. □

**Example 4.3.** Let $d = 3$ and $\omega = 1010111$. Then $\omega \equiv 1110111$ and $\omega \equiv 1100011$, but $\omega \not\equiv 1000000$. The representative $\rho$ for $[\omega]$ is 1000001. □

In the first series of simulations (small inputs) we take all nonempty strings over $\{0,1\}$ of length at most 15. Next we determine the 186 equivalence classes with respect to $\equiv$ and the corresponding representatives $\rho(n)$ $(1 \le n \le 186)$. E.g. $\rho(1) = 0$, $\rho(2) = 1$, $\rho(3) = 00$, $\rho(4) = 10$, $\rho(5) = 000$, . . . . and $\rho(186) = 100100100100100$. The corresponding instances $T_0(n) = (\Sigma, d, P, \omega_0(n))$ with $\omega_0(n) = \rho(n)$ of $T_0$ are indexed by $n$ with $1 \le n \le 186$.

The second series of simulations (worst case inputs) is easier to describe. The instances are $T_0(m) = (\Sigma, d, P, \omega_0(m))$ with $\omega_0(m) = (100)^m$ and will be indexed by $m$ with $1 \le m \le 200$. The fact that (at least) the first $m+1$ steps are in the direction of an explosion accounts for the "worst case" character. Note that the intersection of these two series consists of $m = 1, 2, 3, 4, 5$ or, equivalently, $n = 6, 18, 43, 90, 186$ respectively.

Due to space limitations we only give some examples of the results of these simulations; a more complete overview will appear in [3]. Generally spoken, little can be said with respect to the initial behavior $(0 \le t \le \tau)$; regular patterns only turn up in the periodic behavior $(t \ge \tau)$. See also §5.

| $n$ | $\rho(n)$ | $\tau(n)$ | $M(n)$ | $T(n)$ | $n$ | $\rho(n)$ | $\tau(n)$ | $M(n)$ | $T(n)$ |
|---|---|---|---|---|---|---|---|---|---|
| 130 | 00000010010010 | 422 | 56 | 108 | 159 | 000000100000000 | 17 | 15 | 0 |
| 131 | 00010000000000 | 16 | 14 | 0 | 160 | 000000100000100 | 19 | 15 | 0 |
| 133 | 00010000010000 | 18 | 14 | 0 | 164 | 000100000000100 | 21 | 15 | 0 |
| 136 | 00010010000010 | 24 | 15 | 3 | 165 | 000100000100000 | 23 | 15 | 0 |
| 141 | 10000000010000 | 20 | 15 | 1 | 166 | 000100000100100 | 419 | 56 | 105 |
| 143 | 10000010000000 | 22 | 15 | 1 | 168 | 000100100000100 | 417 | 56 | 103 |
| 144 | 10000010000010 | 420 | 56 | 106 | 173 | 100000000100000 | 421 | 56 | 107 |
| 145 | 10000010010000 | 418 | 56 | 104 | 176 | 100000100000100 | 415 | 56 | 101 |
| 149 | 10010000010000 | 416 | 56 | 102 | 178 | 100000100100100 | 413 | 56 | 99 |
| 150 | 10010000010010 | 414 | 56 | 100 | 181 | 100100000100000 | 27 | 18 | 7 |
| 154 | 10010010010010 | 412 | 56 | 98 | 186 | 100100100100100 | 411 | 56 | 97 |

**Table 1.** $\omega_0(n) = \rho(n)$ $(130 \le n \le 186)$

In §2 we already mentioned that all inputs of the form $0^n$ vanish in $n$ steps. But there are many more inputs that vanish ultimately; see Tables 1 and 2 where $M$ is the maximum string length encountered on the interval $[0,\tau]$ and $T$ is the first moment at which this maximum length occurs. Some of these inputs ($m = 106$, 160 and 195) need a considerable time to disappear.

Next we turn to periodic cycles; see also §5. In the sequel we use $A = P(0) = 00$, $B = P(1) = 1101$, $x = AB$ and $y = BBAA$ as abbreviations. Examples of periodic cycles are in Table 3 of which the last column contains a string over $\{A,B,x,y\}$ preceded by its position in the cycle. Clearly, this string completely determines the cycle. So for $m = 24$, 33, 109 and 167 we have the same cycle. The same cycle may also be entered at different points; see $m = 7$ and 16.

| $m$ | $\tau(m)$ | $M(m)$ | $T(m)$ | $m$ | $\tau(m)$ | $M(m)$ | $T(m)$ | $m$ | $\tau(m)$ | $M(m)$ | $T(m)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 411 | 56 | 97 | 70 | 14196 | 456 | 1750 | 144 | 395712 | 2076 | 115972 |
| 13 | 703 | 68 | 51 | 74 | 18338 | 548 | 7098 | 148 | 836540 | 1836 | 465486 |
| 14 | 37912 | 768 | 18168 | 78 | 8676 | 592 | 2062 | 160 | 1433762 | 7130 | 1009788 |
| 22 | 1778 | 172 | 1068 | 80 | 44316 | 636 | 15212 | 166 | 56908 | 1264 | 8054 |
| 25 | 4129 | 206 | 2949 | 91 | 11347 | 748 | 5081 | 169 | 380399 | 3092 | 156625 |
| 46 | 6062 | 362 | 1508 | 93 | 609693 | 3146 | 281347 | 190 | 35920 | 1128 | 3080 |
| 47 | 3781 | 278 | 395 | 106 | 1262756 | 3518 | 1093810 | 195 | 59966591 | 20716 | 7024103 |
| 54 | 33530 | 794 | 16618 | 117 | 29611 | 734 | 1683 | 199 | 375599 | 2120 | 111757 |
| 63 | 30461 | 722 | 20689 | 118 | 18312 | 760 | 3654 | | | | |
| 65 | 13523 | 572 | 6309 | 128 | 20518 | 894 | 4074 | | | | |

**Table 2.** $\omega_0(m) = (100)^m \qquad (1 \le m \le 200)$

| $m$ | $\tau(m)$ | $\pi(m)$ | $\lvert\omega_{\tau(m)}\rvert$ | $M(m)$ | $T(m)$ | |
|---|---|---|---|---|---|---|
| 7 | 2128 | 28 | 85 | 176 | 1293 | 2 xyyxyxyxy |
| 16 | 127 | 28 | 85 | 88 | 78 | 24 xyyxyxyxy |
| 24 | 4346269 | 6 | 37 | 4432 | 935110 | 2 xyxy |
| 27 | 7018 | 6 | 73 | 378 | 2781 | 2 xyxyxyxy |
| 28 | 3885 | 6 | 163 | 206 | 2874 | 2 xyxyxyxyxyxyxyxyxy |
| 29 | 14632 | 6 | 55 | 432 | 9883 | 2 xyxyxy |
| 30 | 7019 | 6 | 19 | 380 | 3186 | 2 xy |
| 32 | 4277 | 52 | 157 | 290 | 996 | 2 yxyxyxyxyxyxyxyxy |
| 33 | 147688 | 6 | 37 | 1336 | 109875 | 2 xyxy |
| 75 | 16844 | 6 | 163 | 560 | 3973 | 2 xyxyxyxyxyxyxyxyxy |
| 76 | 5787 | 6 | 33 | 450 | 780 | 4 BAAxyx |
| 92 | 20985 | 10 | 31 | 534 | 1206 | 2 yxy |
| 94 | 716549 | 6 | 33 | 1684 | 496266 | 4 BAAxyx |
| 104 | 102805 | 36 | 109 | 946 | 82736 | 2 xxxxyxyxyxyxy |
| 105 | 27782 | 22 | 67 | 694 | 8313 | 2 yxyxyxy |
| 107 | 22894 | 6 | 69 | 822 | 15945 | 4 BAAxyxyxyx |
| 108 | 358281 | 6 | 55 | 2598 | 184550 | 2 xyxyxy |
| 109 | 9110 | 6 | 37 | 620 | 931 | 2 xyxy |
| 138 | 432155 | 40 | 121 | 2016 | 136792 | 2 xyxyxyxyxyyxy |
| 167 | 2885053985 | 6 | 37 | 171768 | 1487843937 | 2 xyxy |

**Table 3.** Some periodic cycles.

## 5. Periodic Behavior of $T_0$

Periodicity in the behavior of $T_0$ has been investigated in [10]. However, a better starting point is the following "folklore theorem".

**Theorem 5.1.** *Let $p,q,r \in \Sigma^*$ with $p \ne \lambda$, $r \ne \lambda$, and $pq = qr$. Then either $q = \lambda$ and $p = r$, or there exist strings $u,v \in \Sigma^*$ and a number $j$ ($j \in \mathbb{N}$) such that $p = uv$, $r = vu$, and $q = (uv)^j u = u(vu)^j$.* □

We distinguish the following two cases. By the definition of $P$ each period must be even.

*Case 1:* The periodic cycle contains a string of which the length is a multiple of three. For periodic strings of minimal length we have $q = \lambda$, $p = \omega_t = \omega_{t+\pi} = r$ ($t \ge \tau$) and $|p| = |r| = 3 \cdot \pi$. Simple combinatorial arguments yield, e.g.,

- $\pi = 2, \quad p = r = AB = x, \quad q = \lambda, \quad |p| = |AB| = 6,$
- $\pi = 4, \quad p = r = BBAA = y, \quad q = \lambda, \quad |p| = |BBAA| = 12.$

Once we have some periodic strings it is easy to proceed since if $p_i \in \Sigma^*$ ($i = 1,2$) are cyclic with period

$\pi_i$ and $|p_i| = 0 \bmod 3$, then $p_1 p_2$ is periodic with period $\pi$ and $\pi \le \pi_1 + \pi_2$. But sometimes we have $\pi = \pi_1 + \pi_2$, e.g., for $xy$ or $yx$ we have $\pi = 6$.

**Proposition 5.2.** *For each k* $(k \ge 1)$, *there exists a string that generates a cycle of period* $2 \cdot k$.

*Proof:* For $k = 1, 2, 3$ we take $x$, $y$ and $xy$, respectively. Otherwise in case $2 \cdot k = 6 \cdot i + 2$, $2 \cdot k = 6 \cdot i + 4$ or $2 \cdot k = 6 \cdot i + 6$ we take $x(xy)^i$, $(xy)^i y$ and $x(xy)^i y$, respectively. E.g., for $\pi = 8$ we have

$$xxy \Rightarrow^2 yxx \Rightarrow^2 yxx \Rightarrow^4 xxy \Rightarrow \cdots \qquad \square$$

**Observation 5.3.** *If a string p of which the length is a multiple of three is periodic with period* $\pi$, *then each string* $p^n$ $(n \ge 1)$ *is also periodic with period* $\pi$. $\square$

*Case 2:* The periodic cycle does not contain a string of which the length is a multiple of three. Now we only have a single period.

**Proposition 5.4.** *For each n* $(n \ge 0)$, *the string* $BAAx(yx)^n$ *is periodic with* $\pi = 6$.

*Proof:* For $n \ge 1$, $BAAx(yx)^n = BAAxyx(xy)^{n-1} \Rightarrow^2 Axyx(yx)^{n-1}BB = AABBBAAx(yx)^{n-1}BB \Rightarrow^4 BAAx(yx)^{n-1}BBAAAB = BAAx(yx)^n$ and for $n = 0$, $BAAx \Rightarrow^2 AxBB = AABBB \Rightarrow^4 BAAAB = BAAx$. $\square$

Comparing this proof and Theorem 5.1 yields $p = BAAxB$, $q = BAAx(yx)^{n-1}$, $r = yx$, $u = BAAx$, $v = B$ for $n \ge 1$, and $p = BAAxB$, $q = B^{-1}$, $r = yx$, $u = BAAx$ and $v = B$ for $n = 0$. The "strange" value for $q$ in case $n = 0$ corresponds to a substring $B$ that is both generated and consumed during a single period. For Case 2 we have no analogue of 5.3.

In [10] it is claimed that there are no other periodic cycles than those mentioned above.

## 6. Conclusions and Open Problem

Simple discrete nonlinear systems like $T_0$ and the 3x + 1 problem are still as intractable and intriguing as ever. We showed that the periodic behavior of $T_0$ is more complicated than the simple 3-cycle in the 3x + 1 conjecture; cf. §4 and §5. For $T_0$ we have, apart from vanishing inputs, that for each $k$ $(k \ge 1)$, there exist infinitely many periodic cycles with period $2 \cdot k$. And there is still the possibility of "exploding" input strings.

**Figure 2.** **Figure 3.**

A suitable candidate for this last possibility is the case $m = 110$ for which the computer simulation did not yet produce a final answer. After $6864.41 * 10^9 = 6.86441 * 10^{12}$ simulation steps we only know that $M(110) \ge 17\,144\,686$ and $T(110) \ge 5\,395\,443\,171\,968$. See Figures 2-4 where we plotted the string length as function of the number of steps (in $10^9$'s). This is very long computation indeed, viz. 12.4 MIPSyear. (A *MIPSyear* is the number of instructions performed on a 1 MIPS machine during a

**Figure 4.** **Figure 5.**

year, i.e., 1 MIPSyear $= 3.1536 * 10^{13}$ instructions). So on a DEC μVAX-2000 (1 MIPS machine), a SUN 3/60 (2.5 MIPS), a SUN SPARC (12 MIPS) and a SUN SPARC+ (16 MIPS) it takes 12.4, 5.0, 1.0 and 0.8 year, respectively. Per hour we simulate on these machines $0.09 * 10^9$, $0.22 * 10^9$, $1.10 * 10^9$ and $1.35 * 10^9$ steps, respectively. In case $T_0$ will explode for $m = 110$, then it is a very slow (or "soft") explosion since it takes 2000 to 3000 steps in the average to increase the string length by 1. Cf. Figure 5 where we show $M(t)/t$ as function of $t$ ; $M(t)$ being the maximum string length reached on the interval $[0,t]$.

## References


1. P.R.J. Asveld: On a Post's system of tag, *Bull. Europ. Assoc. Theor. Comp. Sci.* No. 36 (1988) 96-102.
2. P.R.J. Asveld: A simple system with chaotic behavior, *Memoranda Informatica* 88-51 (Dept. of Comp. Sci., Twente University of Technology). Also in: D. Kleima et al (Eds.): *Liber Amicorum for E.W. Gröneveld* (1988), Dept. of Elec. Eng., Twente University of Technology, Enschede, The Netherlands.
3. P.R.J. Asveld: On Post's system of tag with $0 \to 00$ and $1 \to 1101$ as productions, *Memoranda Informatica* (in preparation), (Dept. of Comp. Sci., Twente University of Technology).
4. J.C. Lagarias: The $3x+1$ problem and its generalizations, *Amer. Math. Monthly* **92** (1985) 3-23.
5. M.L. Minsky: Recursive unsolvability of Post's problem of "tag" and other topics in the theory of Turing machines, *Annals of Math.* **74** (1961) 437-455.
6. M.L. Minsky: *Computation — Finite and Infinite Machines* (1972), Prentice-Hall, Englewood Cliffs, N.J.
7. E.L. Post: Formal reductions of the general combinatorial decision problem, *Amer. J. Math.* **65** (1943) 197-215.
8. E.L. Post: Absolutely unsolvable problems and relatively undecidable propositions — account of an anticipation, pp. 340-433 in: M. Davis: *The Undecidable — Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions* (1965), Raven, New York.
9. H. Wang: Tag systems and lag systems, *Math. Ann.* **152** (1963) 65-74.
10. S. Watanabe: Periodicity of Post's normal process of tag, Proc. of the Symp. on Math. Theory of Automata 1962, Polytechnic Press, Brooklyn, N.Y. (1963) pp. 83-99.