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## Abstract

The purpose of this fMRI study was to provide evidence for the mathematician's belief that mathematical thinking emerges from the interplay between symbolic and visuo-spatial systems. Twelve participants were given algebra word problems and depicted the quantitative relations on a mental number line or made parts of an equation. The regions activated in depicting the picture were also recruited to make an equation.


Mathematics is a language. Many scientists say that mathematics is a language to describe the nature of phenomena they are looking at. It is well known that Nicolas Burubaki, a group of mathematicians, stressed the crucial role of formal symbol systems in mathematics.

On the other hand, many mathematicians and physicists emphasize the role of visuo-spatial reasoning in mathematics, which recruits qualitative, language-independent representations. For example, Albert Einstein stated "Words and language, whether written or spoken, do not seem to play any part in my thought process."

As Dehaene, Spelke, Pinel, Stanescu, and Tsivkn (1999) suggested, mathematical thinking may emerge from the interplay between symbolic and visuo-spatial systems. In this fMRI study, we approach this problem and provide evidence for this kind of mathematical thinking.

Psychological studies have revealed that if a problem apparently looks like a pure symbolic task, it can require students to have some visuo-spatial representations. For example, Griffin, Case and Siegler (1994) showed that the mental "number line", a qualitative representation of the number system, is crucial readiness for early arithmetic. Lewis used a number-line-like diagram to train undergraduate students having difficulty to solve "compare" word problems (problems containing more-than or less-than relations), and succeeded in improving their performance. Paige and Simon (1966) proposed that solving word problems is not a simple translation of problem sentences into equations, as Bobrow's (1966) STUDENT did, but needs "physical cues," a visuo-spatial representation. The 6$^{th}$ grade students who used our "Picture Algebra" strategy (Koedinger & Terao, 2002) to solve the compare word problem showed relatively high performance. We expect that using this strategy may better prepare students to learn formal algebra.

Functional magnetic resonance imaging (fMRI) gives us a new source of information about the mental representations used in mathematics. Dehaene et al. (1999) showed two different mental representations are used for different tasks. Exact calculation (e.g., 4+5=9) elicited left-lateralized activation in the left inferior frontal lobe, together with left angular gyrus and left anterior cingulate. This pattern was interpreted as suggesting that the participants recruited their symbolic systems and did language dependent encoding. Approximation (e.g., 4+5 is closer to 8 than 6), on the contrary, elicited bilateral parietal lobes activation. This pattern was interpreted as suggesting that the participants recruited visuo-spatial systems and did language independent encoding. Dehaene, Piazza, Pinel and Cohen (2003) reviewed neuro-imaging and neuropsychological evidence concerning various numerical tasks and proposed a hypothesis that three parietal circuits are related to number processing. The horizontal segment of the intraparietal sulcus (HIPS) appears to be a core quantity system, analogous to a mental number line. This area seems to be supplemented by two other circuits. One is the bilateral posterior superior parietal lobule (PSPL), which is considered to be involved in attention orientation on the mental number line. The other is the left angular gyrus, which is likely to support manipulations of numbers in a symbolic form (e.g., exact calculation).

Dehaene et al. (1999, 2003) suspected that mathematical thinking may emerge from the interplay between symbolic and visuo-spatial systems but did not provide direct evidence for this idea. For example, exact calculation and approximation mainly depend on the symbolic system and the visuo-spatial system, respectively, not necessarily a collaboration between these two systems.

In this study, we try to provide direct evidence for such a collaboration. We decided to use algebra word problems for three reasons. First, previous psychological research suggests that visuo-spatial reasoning plays a crucial role in solving these problems while they explicitly require students to use symbols (i.e., equations). This kind of problem is expected to show the interplay between symbolic and visuo-spatial systems. Second, algebra word problems are widely used in school mathematics curriculum, so that we can say the observed interplay is a prevailing form of reasoning, not a special form isolated to a very specific task. Third, there are plenty of studies using algebra word problem, the accumulated findings help us in valid reasoning from our results.

If algebra word problems recruit the visuo-spatial system as well as the symbolic system, we should see activation of some visuo-spatial areas when students try to make a correct equation for a problem. To find visuo-spatial areas, we asked our participants to make a pictorial representation of the problem in one condition. This task should activate visuo-spatial areas and most of these areas should also be activated when we ask the participants to make an equation of the same problem in another condition. We especially expect that the two hypothesized parietal visuo-spatial systems, HIPS and PSPL, show activation in both conditions.

## Method

### Participants

Participants were 12 right-handed, native English speakers. They were recruited by advertisement posted on an electric bulletin board in Carnegie Mellon University. Participants were provided written informed consent in accordance with the Institutional Review Boards at the University of Pittsburgh and at Carnegie Mellon University.

### Tasks and Design

There was one *representation* factor and four *problem* factors. They were all manipulated within subjects. The representation factor was the mental representation the participants made from the problems. In the *picture* condition, the participants draw a mental image describing the critical relations in the problem; in the *equation* condition, the participants were required to construct an equation to the problem.

Table 1 shows two example problems. Each problem consisted of three problem sentences and used three letters as unknown quantities. The first sentence was an assignment sentence. In the equation condition, the

Table 1: Sample of Problems.

| Consistent/more-than/intransitive problem | |
|---|---|
| Assignment | x=A |
| R1 | B is 6 more than A. |
| R2 | C is 8 more than A. |
| Inconsistent/less-than/transitive_n problem | |
| Assignment | x=A |
| R1 | A is 6 less than B. |
| R2 | B is 2 less than C. |

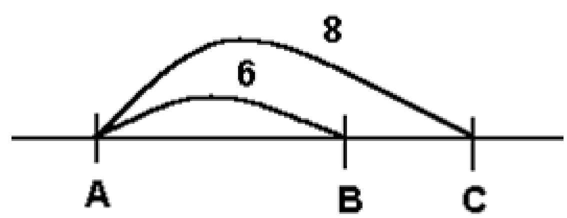



Figure 1: An example of pictures made from problems.

participants memorize what letter the "x" is assigned to. In the picture condition, the participants imagined a number line and picked locations for the letters. The second and the third sentences described a qualitative relation between two letters. Hereafter we call them R1 (meaning "Relation 1") and R2, respectively. Participants could not find a numerical solution to these problems because they did not have a sentence referring to the total amount of the three unknown quantities. For example, if the first problem in Table 1 has the sentence "The total of the three quantities is 44," we can make an equation to find the three quantities like x+(x+6)+(x+8)=44. In the equation condition, the participants were told that the total sentence would be omitted and were required to make the left side of the correct equation like x+(x+6)+(x+8). For the first problem in Table 1, the participant can make the parts (x+6) and (x+8) from R1 and R2, respectively. In the picture condition, the participants imagine a picture describing the critical relations in the problem as shown in Figure 1. For the first problem in Table 1, the participants can imagine B to the right of the location A on the mental number line and add the distance, 6, to this picture. When they read and represent R2, the whole picture can be obtained.

A first problem factor was the relation, whether R1 and R2 use *more-than* or *less-than* relations.

A second problem factor was the consistency, whether the relations used in R1 and R2 were consistent with the correct equation. A more-than problem was labeled either as a *consistent* problem if the correct equation uses "+"or labeled as an *inconsistent* problem if the correct equation uses "-". We use a similar labeling for the less-than problems.

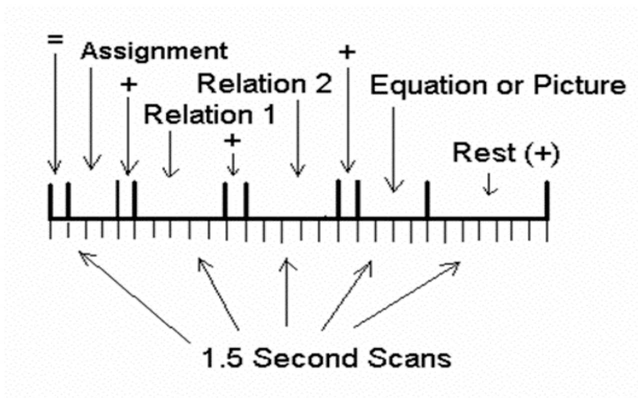



Figure 2: The 42-second structure of an fMRI trial.

A third condition was defined in accordance with the target stimuli presented at the end of each problem (trial). The problem was labeled either as a *correct* problem if the target is the correct target or as an *incorrect* problem if the target is incorrect.

A forth factor was transitivity. This factor was defined by the two relational sentences, R1 and R2. Considering the picture the participants were required to make seems to be an easy way to explain this factor. For the *intransitive* problem, an arc will be drawn over another arc as shown in Figure 1. This will be the case if either the former letter or the latter letter is common in R1 and R2. For example, R1 and R2 of the first problem in Table 1 use the same latter letter (A), and this is an intransitive problem. For the *transitive_n* problem (n stands for Normal), one arc should be drawn at the next position to another arc in the correct picture. A third level of this factor is represented by another problem. This type of problem, called the *transitive_d* problem hereafter (d stands for Delay), was made by changing the R1 and R2 of the transitive_n problem. In the transitive_d problem, participants in the equation condition are not able to make a part of the equation until they read R2. They need to remember R1 and make the two parts of the equation when R2 is presented. The purpose of making this unusual problem was to find brain regions which play a role in making equations going beyond simple encoding of problem sentences. Comparing the transitive_n problem with the transitive_d problem in the time period of presenting R1 might reveal differences between memory and processing areas but we will not say much about this comparison in this paper. There may be no difference between the transitive_n and transitive_d problems in the picture condition because the participants can describe the relation when they see R1. For example, for the transitive_d problem made from the second problem in Table 1 (R1: "B is 2 less than C."), the participants can imagine the spatial relation between B and C by just ignoring A used in the assignment sentence.

Combinations of the four problem factors (2x2x2x3) yielded 24 types of problems. The participants went through all of these 24 types in both the picture condition and the equation condition in the MRI scanner, so that each participant encountered 48 problems. The 48 problems were divided into four blocks: two blocks in the picture condition and the other two blocks in the equation condition.

## Procedure

**Pre-scan Practice** Participants took about 20 minutes of pre-scan practice just before the scan. They went through one block of 12 trials (problems) in the picture condition and another block of 12 trials in the equation condition. Half of the participants started with the picture condition and the other half of the participants started with the equation condition. The time course in each trial was the same as the one in the scanner.

**Event-related fMRI scan** Event-related fMRI data were collected by using a singe-shot EPI acquisition on a Siemens 3T scanner, 1500 TR, 30 ms TE, $60^0$ flip angle, 210 mm FOV, 26 axial slices/scan with 3.2 mm thick, 64x64 matrix, and with AC-PC on the $6^{th}$ slice from the bottom. There were 28 scans (42 seconds) for each trial, 12 trials for a block and 4 blocks for each participant. Two of these 4 blocks were for the picture condition and the other two blocks were for the equation condition. Half of the participants started with the first block in the picture condition and proceeded to the second block in the equation condition, the third block in the picture condition, and the last block in the equation condition. The other half of the participants started with the first block in the equation condition, then went through picture, equation, and picture conditions in this order.

The protocol of each trial of scan is illustrated in Figure 2. The three problem sentences appeared on the screen one by one. The assignment sentence was on the screen for 3500 ms; R1 and R2 were on the screen for 7500 ms. A target equation or picture was presented after the disappearance of R2. Participants responded to this target by pressing the button. If they thought the target was correct they pressed a button with the index finger of the right hand; if they thought the target is incorrect, they pressed the other button with the middle finger of the right hand.

**fMRI data analysis** Data processing was conducted using SPM99 software (http://www.fil.ion.ucl.ac.uk/spm/). Slice timing was corrected first and images were realigned. Realigned images were normalized to Talairac coordinates. Normalized images were smoothed with an 8 mm FWHM isotropic Gaussian Kernel. Analysis was carried out using the general linear model with a box-car waveform convolved with a hemodynamic response function. Only correct trials were used for analysis.

To find brain regions of interest (ROI), a random effects model was used. At the first level, mean images for each participant were created, depicting the subtraction of BOLD response during assignment sentence from BOLD response during R1 in each condition (picture and equation). Data from transitive_d problems were excluded to do this subtraction because of the unique nature of these problems. At the second level, these mean images were combined in one-sample *t*-test. We used a height threshold of $P < 0.0005$

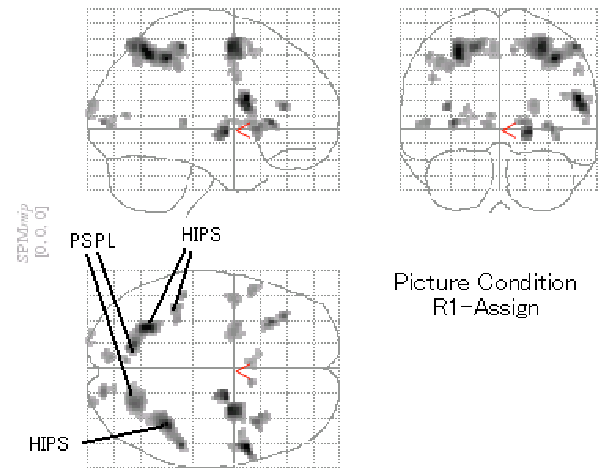



Figure 3: Regions of the brain that show activation in depicting a relation between two quantities in the picture condition as compared to the encoding of the assignment.

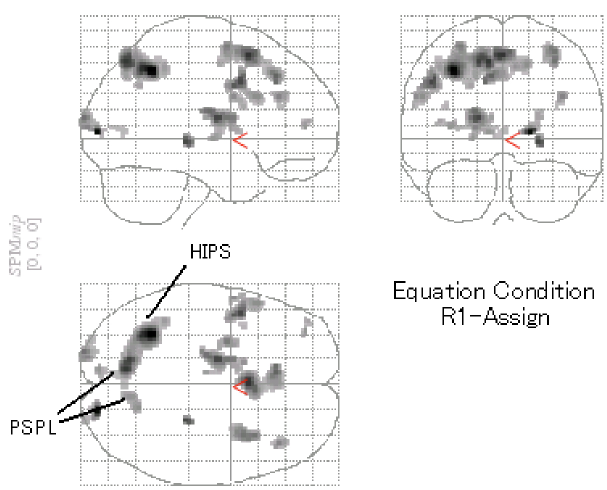



Figure 4: Regions of the brain that show activation in constructing parts of an equation in the equation condition as compared to the encoding of the assignment.

uncorrected ($t > 4.44$), with an extent threshold of eight contiguous voxels in an ROI. This analysis should show us the regions activated when constructing a number line to describe a quantitative relation or constructing parts of an equation from this relation. The subtraction between picture and equation conditions might be interesting but we did not conduct it, because this subtraction might hide visuo-spatial areas that are recruited not only in picture condition but also in equation condition.

## Results

Figure 3 shows the brain regions which show activation during the period of R1 as compared to the assignment sentence in picture condition. We can infer that these regions may be related to using a mental number line to depict a relation between two quantities.

Figure 4 shows the brain regions in the equation condition obtained by the same subtraction (R1 - assignment). These regions should be related to constructing parts of an equation from sentences.

Comparing these two figures, we can see an overlap of activation especially in the parietal lobe (HIPS and PSPL). This means that constructing an equation, which apparently is a symbolic task, recruits the visuo-spatial system.

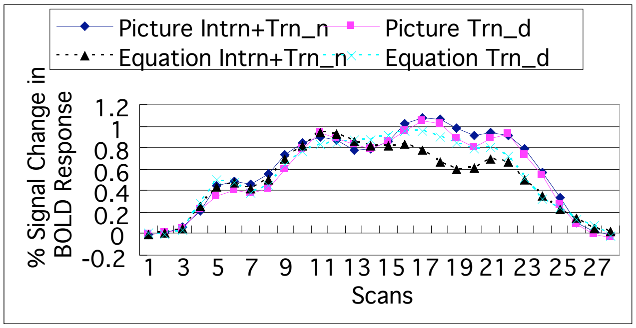
### Areas for constructing a mental number line

A pattern of bilateral activation was obtained for drawing a mental number line to represent a quantitative relation. As we had expected, the active areas in parietal lobes occupied HIPS and PSPL (Tarairach coordinates of main peaks: -40, -40, 48, Z=3.75; -28, -60, 54, Z=4.63; 38, -46, 48, Z=4.62; 24, -68, 56, Z=4.12). Activation was also found during constructing a number line from R1 in the bilateral premotor cortices (-30, 2, 52, Z=3.79; -46, 2, 34, Z=3.72; -48, 0, 50, Z=3.68; 28, -2, 56, Z=4.52; 54, -2, 40, Z=3.51), bilateral supplementary motor areas (-4, 12, 56, Z=3.59; 10, 12, 54, Z=3.73), left Broca area (-52, 12, 4, Z=3.69), right inferior frontal sulcus (54, 8, 20, Z=4.75), left and right insula (-32, 26, 4, Z=4.27; 36, 16, 0, Z=4.05), left and right corpus striatum (20, -6, -2, Z=4.49; -24, -2, 6, Z=3.51), and right DLPFC, dorsolateral prefrontal cortex (-36, 34, 16, Z=4.07). Activation found in left and right visual cortex should reflect longer exposure to the visual stimulus: The period of R1 was longer than the period of assignment sentence.

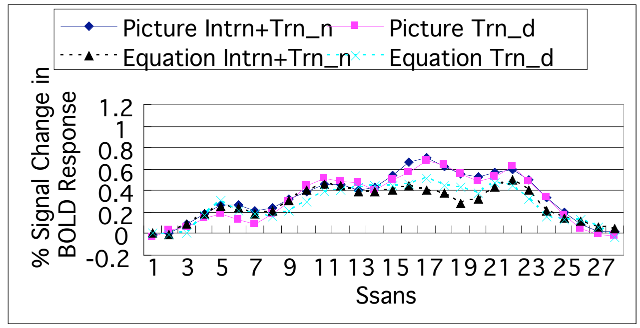
### Areas for constructing an equation

A pattern of left-lateralized activation was obtained for constructing parts of an equation from a sentence referring to a relation between two quantities. The bilateral PSPL and left HIPS activation were also found as in the picture condition (-34, -54, 46, Z=4.93; 16, -62, 50, Z=3.85). This means that the hypothesized parietal visuo-spatial system (Dehaene et al., 2003) was recruited when constructing parts of an equation from a problem sentence. Activation was also found during construction of parts of the equation from R1 in the left premotor cortex (-58, 2, 28, Z=3.87), bilateral supplementary motor areas and right Brodmann area 8 (-28, -2, 62, Z=4.00; -14, 8, 58, Z=3.79; 0, 12, 56, Z=4.53; 34, 8, 58, Z=4.06), bilateral inferior frontal sulci (-38, 8, 22, Z=3.44; -48, 6, 38, Z=4.43), left basal ganglia including thalamus and globus pallidus (-16 -10 14, Z=4.43), right parahippocampal gyrus (24, -30, -2, Z=4.38), and left and right DLPFC (40, 32 28, Z=3.95; -36, 50, 8, Z=3.77).

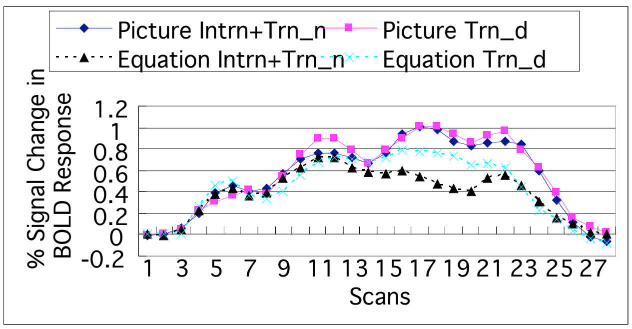
To confirm that several brain areas activated in picture condition also showed activation in equation condition, we plotted percent signal change along the time course. The base line for calculating the signal change was set by the average of first two scans. Remember that these areas were found in picture condition and no data from equation condition were used. Data from intransitive problems and transitive_n problems were combined in each condition.



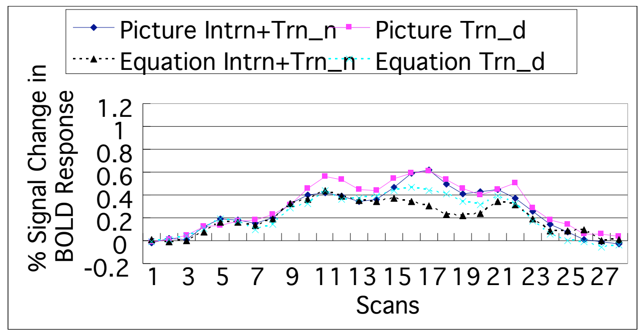

Left PSPL and HIPS




Right HIPS




Right PSPL




Right Premotor

Figure 5: Percent signal change in BOLD response in four ROIs found in picture condition.

After finding ROIs, the signal change was calculated for each of the four problem types (picture--transitive_d, picture--others, equation--transitive_d, and equation--others) and for each participant, and then averaging across the 12 participants. Among many ROIs found in picture condition, because of the limited space of this paper, we only show the percent signal change at left PSPL and HIPS (-28, -60, 54; The number of voxels is 333), right HIPS (38, -46, 48; The number of voxels is 303), right PSPL (24, -68 56; The number of voxels is 241), and right premotor cortex (28, -2, 56; The number of voxels is 220.). These are the four biggest clusters of active voxels.

Figure 5 shows the percent signal change in each of these four ROIs. We can see that these areas found in picture condition also played a role, more or less, in equation condition. Other areas found in picture condition also showed a similar pattern of activation.

## Discussion

The results of this experiment indicate that mathematical thinking emerges from the interplay between symbolic and visuo-spatial systems. The hypothesized two parietal visuo-spatial regions (Dehaene et al., 2003) showed activation not only when participants imagined a picture from a problem sentence but also when participants constructed parts of an equation from the sentence. We cannot deny a possibility that these parietal regions are involved in non-visuospatial mathematical reasoning. But it is reasonable we consider them as picture regions until some evidence is fond for this possibility in brain imaging research.

We might expect language areas to show greater activation on these symbolic tasks than in the more visuo-spatial picture task. However, while we found activation in language areas, particularly Broca's and the inferior frontal sulcus, we did not find clear differences in those areas between the two conditions. Perhaps the language areas are doing different kinds of computations in the two conditions, but we found no evidence to support this claim.

Our results seem to be consistent with the recent version of the ACT-R theory (Anderson et al., submitted). The ACT-R theory hypothesizes several buffers and their locations in the brain. For example, the goal buffer is supposed to be in DLPFC and the imaginal buffer in the parietal lobe. The theory also hypothesizes that production rules are stored in the corpus striatum. We found activation in these regions in this experiment. The recent version of ACT-R theory can predict the percent signal change in BOLD response based on the task analysis. We did a task analysis before conducting this experiment. Constructing an ACT-R model could provide us an explanation about the signal change in this experiment.

Readers might suspect that the parietal activation in PSPL and HIPS only reflects longer exposure to the relational sentence R1 (7500 ms) than the assignment sentence (4500 ms). This effect might be the case but it cannot explain the different pattern of activation between the picture and equation conditions (see Figures 3 and 4) because it should have the same effect in both conditions.

Readers might also suspect that activation of visuo-spatial areas in the equation condition was an effect of use of the mental number line in the picture blocks spilling over to the equation blocks. Because we used a within-subject design, we cannot deny this possibility. Even if it is true, however, it is the effect that we expect in educational settings. Those participants who used visuo-spatial systems on equations may have enhanced their performance. Using a framework of production systems, we can write production rules that represent a purely symbolic processing of a problem sentence. For example, we can think of the following production rule to process R1:

```
IF x is bind to $A
   and R1 says “$B is $N more than $A
THEN represent $B as x+$N,
```

where the letter with $ means a variable. Visuo-spatial systems are not necessary if using this kind of production rule but it appears participants still used visuo-spatial reasoning to construct an equation. The following alternative set of production rules illustrates how use of the visuo-spatial systems may make this task easy:

```
IFR1 says “$B is $N more than $A
THEN
   $B is to the right of $A on the number line

IF x is bind to $A
   and $B is to the right of $A on the number line
THEN represent $B using a plus sign, as x+$N,
```

If the first production is already exists prior to algebra instruction, the second rule is easier to learn, perhaps, then the one above.

It has been shown that using a pictorial representation helps students solve algebra word problems (e.g., Koedinger & Terao, 2002; Lewis, 1989). These results can be interpreted that the students who learned to use visuo-spatial systems improved their ability to solve algebra word problems. Even if the results of this study only show that students can use visuo-spatial systems to solve algebra word problems only after trained with a pictorial representation, this encourages the educational practice of using a pictorial representation as a scaffold of learning.

There is an interesting episode in the pre-scan, practice session. A few participants showed bad performance in the equation condition. We asked them what they did to make equations. They revealed that they used a “direct translation” strategy which was similar to the strategy Bobrow’s (1966) STUDENT used. For example, if R1 says “B is 6 more than A,” they translated this sentence into the form of “B=6+A” before substituting “x” for a quantity. This strategy does not seem to need any visuo-spatial reasoning. The fact that the poor performer first used this kind of strategy suggests that students having difficulty with word problems may not learn to make use of visuo-spatial systems.

This study is still in progress and we only have scratched the surface in dada analysis. Further data analysis (e.g., statistical comparisons of the two conditions) should be done and it will provide insight into mathematical thinking.

## Conclusion

Mathematical thinking emerges from the interplay between symbolic and visuo-spatial systems. Algebra word problems, which are widely used in current school curriculum, are not a pure language processing task. They appear to depend on the use of visuo-spatial systems.

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## Acknowledgments

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