The present status of Maxwell's displacement current

John Roche
Linacre College, Oxford, OX1 3JA, UK

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Abstract. Physics literature on the displacement current from Maxwell to the present day is reviewed with the intention of clarifying the concept. Two major traditions of interpretation are identified, one deriving from Maxwell which maintains that the displacement current is electromagnetically equivalent to an electric current and one deriving from Lorentz which denies that there is any displacement current. The article attempts to resolve various outstanding ambiguities and it concludes with an assessment of the present status of both traditions of interpretation. The appropriateness or otherwise of introducing the concept of a 'displacement current' to undergraduates is also discussed briefly.

1. The controversy

Maxwell's displacement current, first introduced in 1862, has been the subject of both admiration and controversy for more than a century. To-day, important questions still remain unresolved, most notably whether or not it represents an electric current in some real sense, whether it produces a magnetic field and whether or not all currents are closed—all closely related issues. Why does the controversy continue? Not, I believe, because of unrecognized physical principles, or because of insufficient conceptual effort by physicists. I believe it is largely because the competing interpretations of electromagnetism inherited from last century still cloud the issue. I also believe that there is a way forward. For more than a century many highly competent physicists have thought about this problem. By drawing together their investigations—from Maxwell to the present day—and examining them afresh, I believe that further light can be shed upon the controversy. I have used historical notation in the text and modern notation, in SI, in the boxes.

2. Maxwell's introduction of the concept of displacement current

Maxwell's writings are sometimes very difficult to understand and his texts on the displacement current from 1856 to 1873 are particularly rich in ambiguity, seeming mathematical errors and apparent incoherence. Nevertheless, in the midst of all of this, Maxwell arrived at his corrected version of 'Ampère's law' and his great discovery of the electromagnetic nature of light.

Maxwell does not tell us explicitly how he arrived at his belief in a polarization or displacement current in the ether, but he leaves us many clues. It seems to have arisen during the process of formulating what I shall call—borrowing a term from Larmor and Heaviside—Maxwell's magnetic 'vorticity' law, which is a differential form of the law of Ampère and

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Box 1. The naming of the laws of electromagnetism

There is considerable ambiguity in the naming of some of the laws of electromagnetism. I have found the name ‘Ampère’s law’, for example, given without qualification to three different laws. Also, the term ‘Biot and Savart’s law’ is frequently applied incorrectly. Is it not time for an appropriate electrical body to take a fresh look at nomenclature? My own historical investigations of these two laws have led me to the following conclusions.

1. The law of magnetic circulation: $\int \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \boldsymbol{I}$

This has a complex history. André Marie Ampère (1775–1836) introduced in 1823 the concept of the imaginary magnetic dipole shell which was magnetically equivalent to the conduction current around its boundary. Building on Ampère, Carl Friedrich Gauss (1777–1855) in the late 1830s introduced the magnetic scalar potential and discovered that the change of magnetic potential in passing once around a closed current loop is equal to $4\pi I$, where I is the current in the loop. This is equivalent to the modern circulatory law. Since the contributions of Ampère and Gauss are equally significant here, an appropriate name might be the *law of Ampère and Gauss*.

2. The magnetic intensity due to a current element, $\boldsymbol{I}\,\mathrm{d}\boldsymbol{s}$*:*

$$\mathrm{d}\boldsymbol{H} = \frac{1}{4\pi} \frac{\boldsymbol{I}\,\mathrm{d}\boldsymbol{s} \times \boldsymbol{r}}{r^3}$$

This was discovered in scalar form in December 1820 through the joint efforts of Jean Baptiste Biot (1774–1862) and Pierre Simon de Laplace (1749–1827). An appropriate name, therefore, might be the *law of Biot and Laplace*. The less general law of Biot and Savart was discovered earlier in 1820 by Biot and Félix Savart (1791–1841) and related the magnetic intensity due to an indefinitely long straight current to the magnitude of the current and the perpendicular distance to the observation point. I have used the $\boldsymbol{H}$ vector here rather than the $\boldsymbol{B}$ vector because it is historically more accurate to do so in this context.

Gauss. According to his 1856 version of this equation, the conduction current intensity is made numerically and directionally equal to the magnetic vorticity. Maxwell knew even then that this equation was invalid for open circuits since it implied that free charges never accumulate. Guided by publications of Gauss and Kirchhoff, and by an elaborate mechanical model of the ether, Maxwell over a period of nine years finally arrived in 1865 at the expression

$$r' = r + \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi}\left(\frac{\mathrm{d}\beta}{\mathrm{d}x} - \frac{\mathrm{d}\alpha}{\mathrm{d}y}\right)$$

where the z-component of the total current r' is made equal to the sum of the z-components of the conduction current intensity r and the ethereal displacement current intensity, $\mathrm{d}h/\mathrm{d}t$, and also equal to a constant factor multiplied by the z-component of the magnetic vorticity. Also,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{4\pi E^2}\frac{\mathrm{d}R}{\mathrm{d}t}$$

where a changing ‘electromotive force’ $\mathrm{d}R/\mathrm{d}t$—a changing electric field intensity in modern terms—when applied to the ether causes a change $\mathrm{d}h/\mathrm{d}t$ of polarization or displacement in the ether. He argues that the latter ‘variations constitute currents’. The divergence of Maxwell’s total current p', q', r' was zero and formed, therefore, closed loops, but the divergence of the conduction current alone can be shown to follow the law of conservation of charges and currents.

Maxwell's new magnetic vorticity equation allowed him to show easily that it predicted purely transverse waves in his electromagnetic ether, which travelled with the speed of light. This led him to interpret light as an electromagnetic disturbance and to identify the electromagnetic ether with the older optical ether in one of the great moments of theoretical unification in physics. Today it is possible to recognize that Maxwell's deduction of electromagnetic waves holds good—provided the term containing the rate of change of the electric intensity is present—whether or not he had supposed that the changing electric intensity produces an ethereal displacement current. There is little doubt, however, that such a conception was midwife to his electromagnetic theory of light and made it more acceptable to him and to his contemporaries.

Maxwell maintained that the displacement current is 'electromagnetically equivalent' to an electric current. Charges for Maxwell were not tiny physical objects or sources of attraction but seem to have been understood as strained states of the ether. With many late 19th century field theorists, Maxwell seems to blur the distinction between a material dielectric—where polarization and polarization currents were uncontroversial—and the hypothetical ether dielectric. I shall confine my attention throughout this article to Maxwell's displacement current in space, only.

Box 2. A parenthesis

One might be tempted to suppose at this point that we now have the answer to the controversy. Maxwell assumed that a displacement current was a polarization current in the ether. We now no longer believe that a material dielectric fills space and it would seem to follow logically that there is no displacement current there, either. Maxwell's $\mathrm{d}h/\mathrm{d}t$ term can be replaced by his field term

$$\frac{1}{4\pi E^2}\frac{\mathrm{d}R}{\mathrm{d}t}$$

and the equation interpreted as a relationship between conduction currents and fields—with no reference whatever to any 'displacement' current. The study of this question can surely stop right here!

This argument suffers, however, from what is known as the genetic fallacy. It does not follow that a theory which has been arrived at historically using some false assumptions must necessarily be false. The Fitzgerald contraction, for example, was also introduced in terms of an ether theory. Perhaps a different but valid justification was found subsequently for Maxwell's displacement current—or a different but valid interpretation. We must look to the later development of the theory, therefore, to see how this issue was dealt with.

3. Displacement and displacement current in the *Treatise*

In his great *Treatise on Electricity and Magnetism*, first published in 1873, Maxwell introduced the vector symbol $\boldsymbol{D}$ for displacement (as a Gothic capital) and $\boldsymbol{E}$ as the 'electromotive intensity'. For Maxwell, the electric field seems to have been primarily understood as the displacement $\boldsymbol{D}$, rather than the 'electromotive intensity' $\boldsymbol{E}$. $\boldsymbol{D}$ was measured by the charge per unit area on a real or notional test capacitor inserted in the medium (or in space) required to reproduce the given electrical condition of the medium between the plates of the capacitor. $\boldsymbol{E}$ was measured by the force on 'unity of charge'.

In the *Treatise* Maxwell found a most significant solution of his magnetic vorticity equation. According to this solution, the vector potential function $\boldsymbol{A}$ satisfies the condition div$\boldsymbol{A} = 0$ and is expressed by

$$\boldsymbol{A} = \mu \iiint \frac{\boldsymbol{C}}{r}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

where C is the present value of Maxwell's total current.

Since $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, this meant that the magnetic 'induction' $\boldsymbol{B}$ at any point of the field is calculated by combining contributions from the present values of every conduction current element and every displacement current element. This strikingly supports Maxwell's theory that the displacement current is equivalent to the conduction current with respect to its magnetic effects. Maxwell's scalar potential function and the corresponding electric field intensity are also calculated from present values, only. This solution of Maxwell's equations is now called the 'Coulomb gauge'.

Box 3. Calculating the magnetic field strength in Maxwell's electromagnetism

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j} \times \boldsymbol{r}}{r^3}\, \mathrm{d}v + \frac{\mu_0}{4\pi} \int \frac{(\partial \boldsymbol{D}/\partial t) \times \boldsymbol{r}}{r^3}\, \mathrm{d}v$$

Present values of both the conduction current intensity and the displacement current intensity are required. Since this expression for $\boldsymbol{B}$ satisfies Maxwell's equations it applies to both open and closed circuits. It reduces to the well-established law of Biot and Laplace for steady currents in closed circuits.

4. Faraday electromagnetism and the displacement current

Maxwell associated his theory of displacement with Faraday's theory of 'lines of inductive force' by declaring $\boldsymbol{D}$ to be a measure of Faraday's state of 'static electric induction'. Although Maxwell provides a faithful mathematical rendering for many of Faraday's conceptions, his mechanical interpretation of electromagnetism belonged to a very different explanatory tradition to that of Faraday. Faraday's tubes of force had an identity, an autonomous existence and a rather esoteric character which no purely mechanical ether theory could provide.

In the late nineteenth century, Faraday's field theory steadily gathered support, especially in England. John Henry Poynting (1852–1914) and especially John Joseph Thomson (1856–1940) were in many ways the true interpreters of Faraday. They elaborated Faraday's theory of tubes of electric induction and dressed it in Maxwell's mathematics and theory of stresses. In their view the electric field was composed of Faraday induction or displacement tubes and was properly represented by the vector $\boldsymbol{D}$. Maxwell's term 'displacement' was accordingly replaced by the term 'induction' and was interpreted as an undefined 'strain'. Although 'the tubes of electric induction have their seat in the ether', for Thomson the ether seems to retreat into the background. As a result, the tubes of force became more and more autonomous and more like Faraday's later theory of pure forces stretching across space.

In 1904 Thomson wrote that 'for many years he had abandoned [Maxwell's conception] of electric displacement'. This was apparently because Poynting and Thomson had found another mechanism for the production of that part of the magnetic field which had previously been attributed to Maxwell's ethereal displacement current. According to Poynting, in 1888:

> 'The sideways propagation of electric induction is accompanied by (let us drop 'generated by') magnetic induction and equally the sideways propagation of magnetic induction is accompanied by electric induction. The two go together when a disturbance is propagated.'

J J Thomson had a slightly different theory. Where Poynting had wished simply to say that the magnetic tubes 'accompanied' the moving electric tubes, Thomson insists that they are caused by them. Also, the electric tubes are primary, the magnetic tubes secondary, for Thomson.

5. 'Electron theory' rejects the displacement current

Responses to Maxwell's displacement current in the late nineteenth century were varied, both in Britain and on the Continent, but it was given enormous importance. Many began to interpret light itself as the displacement current, a conception which was reinforced by Hertz's discovery of electromagnetic radiation in 1889. The ether continued to figure prominently. The displacement current was not universally accepted in Britain. It is well known that William Thomson (Lord Kelvin) (1824–1907) was a polite critic and we have seen that J J Thomson replaced it by a field process.

French and German physicists tended to approach Maxwell from the perspective of a theory in which electric charges attract or repel other charges. In this tradition magnetism was directly caused by charges in motion. Hermann von Helmholtz (1821–1894) from 1870 accepted the theory of a polarizable ether and interpreted electric displacement as electric polarization. Hertz's experimental discovery of electromagnetic waves in 1889 led Hertz to argue that 'the most direct conclusion [of this discovery] is the confirmation of Faraday's view according to which the electric forces are polarizations existing independently in space'. For Hertz, the ordinary electrostatic and magnetic fields are propagated from source charges at the speed of light, just like the radiation fields.

The most profound change in the interpretation of electromagnetism was brought about by the great Dutch physicist Hendrik Antoon Lorentz (1853–1928) from 1892 onwards. Experiments on gases in the 1880s had led to a revival of the atomic theory of electricity. For Lorentz, elementary charges, carried by matter, became the primary agents of electromagnetism. The growing evidence for the electron in the 1890s added strong support to his theory. For Lorentz, electric and magnetic fields were changes of state—generated by the charges—in a postulated non-mechanical ether. These disturbances travelled away from the charges at the speed of light. In the Lorentz tradition, lines or tubes of force were not the autonomous physical entities of J J Thomson: there was no tension along them or repulsion between them. They were simply a mapping of force directions in the field. Lorentz followed Maxwell in supposing that the rate of change of his displacement vector $\boldsymbol{d}$ represents an electric current.

Oliver Heaviside (1850–1925) in 1886 and Hertz in 1890 attempted to banish the vector and scalar potential functions from electromagnetism as part of their programme to emphasize what they believed to be the true physical content of Maxwell's theory. Nevertheless, the role of these functions continued to be investigated. Henri Poincaré (1854–1912) was, perhaps, the first to show that retarded potential theory questioned the physical status of the displacement current. In 1894 he wrote as follows:

> 'In calculating [the vector potential] A Maxwell takes into account the currents of conduction and those of displacement; and he supposes that the attraction takes place according to Newton's law, i.e. instantaneously. But in calculating [the retarded potential] on the contrary we take account only of conduction currents and we suppose the attraction is propagated with the velocity of light . . . It is a matter of indifference whether we make this hypothesis [of a propagation in time] and consider only the induction due to conduction currents, or . . . with Maxwell . . . take into account both currents of conduction and those of displacement.'

Box 4. The magnetic field intensity according to Lorentz theory

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(t - r/c) \times \boldsymbol{r}}{r^3} \, \mathrm{d}v$$

where

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(t - r/c)}{r} \, \mathrm{d}v$$

and $(t - r/c)$ is the retarded time.

Note that the displacement current makes no contribution to the magnetic intensity in this solution of Maxwell's equations. The solution applies to all macroscopic circuits, open or closed, and reduces to the law of Biot and Laplace for steady currents in closed circuits.

Calculationally, both approaches gave the same value for the magnetic field intensity but retarded magnetic field theory had no need of the displacement current.

Lorentz himself, who was very familiar with Poincaré's publications and who used retarded potentials from 1902 in his theory, stated pointedly in 1906 that 'an electromagnetic field in the ether is never produced by any other causes than the presence and motion of electrons'. In Lorentz theory all fields—bound electric and magnetic and free radiation fields—are generated by charges: contrary to J J Thomson, changing electric fields do not generate magnetic fields.

In 1901 Poincaré found another weakness in displacement current theory. He showed that when the displacement current is acted on by a magnetic field it 'does not experience any mechanical action according to the theory of Lorentz'. Although the displacement current neither produces a magnetic field nor is affected by a magnetic field in Lorentz electromagnetism, it remained a physical reality for both Poincaré and Lorentz as a process in the ether equivalent to light and as contributing to the closure of currents in an open circuit.

The vacuum of explanation created by the abandonment of the ether in the 1920s was gradually filled with Faraday's electric and magnetic fields, thought of as pure non-material forces stretching across the space between charges. In other ways, however, the new macroscopic field theory retained some of the properties of Lorentz's ether. The charge, not the field, was primary. Bound fields and radiation fields were propagated at the speed of light from source charges, with the radiation fields becoming autonomous upon release. Others, after the disappearance of the ether, interpreted the field vectors as mathematical artifacts, only. Indeed, a considerable variety of meanings were given to the term 'field' by physicists in the first third of the present century.

The disappearance of the ether created severe difficulties in the 1930s for Maxwell's displacement current. Most physicists and electrical engineers seem to have followed J J Thomson and accepted that the changing electric field directly produces a magnetic field, without any need for an ethereal electric current as an intermediary. $\partial \boldsymbol{D}/\partial t$, therefore, was thought of as equivalent to an electric current and it continued to be called the 'displacement current'. In Lorentz electromagnetism, however, the interpretation of $\partial \boldsymbol{D}/\partial t$ as a displacement current now lost all justification because it did not represent a disturbance of an ether, nor did it produce a magnetic field, nor was it acted on by a magnetic field. Some authors, therefore, questioned whether $\partial \boldsymbol{D}/\partial t$ in a vacuum might represent no more than the rate of change of the electric field strength there, or even an analytical device. The displacement $\boldsymbol{D}$ itself in a vacuum was sometimes considered as just another measure of the electric field strength $\boldsymbol{E}$.

Lorentz electromagnetism had made its case quite strongly by the beginning of the Second World War. Articles following the Lorentz tradition since the War, most notably in English by F W Warburton in 1954 and W G V Rosser in 1976, have argued from time to time that there is no displacement current, but they seem to have had little impact. Warburton's article of 1954, which rather immoderately describes the displacement current as a 'useless concept', illustrates the frustration of some retarded field theorists.

Clearly, many Lorentzian ideas have been absorbed into macroscopic electromagnetism. Nevertheless, its denial of the existence of the displacement current has not been generally welcomed into physics or into engineering science. Why is this so? Is it because the displacement current remains deeply rooted in electromagnetic intuition? In 1974 T R Carver and J Rajhel could still remark that '. . . the existence of electromagnetic radiation might seem to be an adequate . . . demonstration of displacement currents'. Also, many physicists appear to find satisfying the idea that all currents are closed.

With respect to the displacement current, therefore, there now appear to be at least two authoritative interpretations, that deriving from Maxwell which maintains the existence of a displacement current producing a magnetic field and securing the closure of all currents, and that deriving from Lorentz which denies both. Those philosophers of science who suppose that normal science is conducted in terms of just one theory might find this interesting. We must now examine the literature of the Maxwell tradition more closely since it has carried out its own critique of the status of the displacement current.

6. Maxwellians discover that a changing conservative electric field produces no magnetic field

Anxieties began to be expressed from the 1880s, even among those who supported Maxwell's displacement current, that it might not be capable of producing a magnetic field. In 1881 George Francis Fitzgerald (1851–1901), a strong supporter of Maxwell, wrote that:

> 'It has not, as far as I am aware, been ever absolutely demonstrated that open circuits, such as the Leyden jar discharges, produce exactly the same effects as closed circuits, and until some such effect of displacement current is observed, the whole theory will be open to question.'

Poynting, another supporter of Maxwell, in 1885 deduced from Maxwell's theory that a leaky capacitor discharging internally should produce no external magnetic field because the displacement current balances the conduction current. This result troubled Poynting. The followers of Maxwell have continued to probe these issues closely.

In 1922 Max Planck (1858–1947) accepted that $\partial \boldsymbol{D}/\partial t$ produces a magnetic field, but demonstrated that the symmetries of this function ensure that no magnetic field is actually produced when the changing electric field is a conservative field.

In a seminal paper in 1963, A P French and J R Tessman developed Planck's argument further to prove that the displacement current in a uniformly charging (or discharging) capacitor produces no magnetic field. Their result can be interpreted to mean that the magnetic field produced by the displacement current within the body of the capacitor is balanced by that produced by the fringing electric fields outside the capacitor—a rather unexpected result. They also resolved Poynting's difficulty by demonstrating that a uniformly discharging leaky capacitor does produce an external magnetic field, since the contribution from its displacement current is zero.

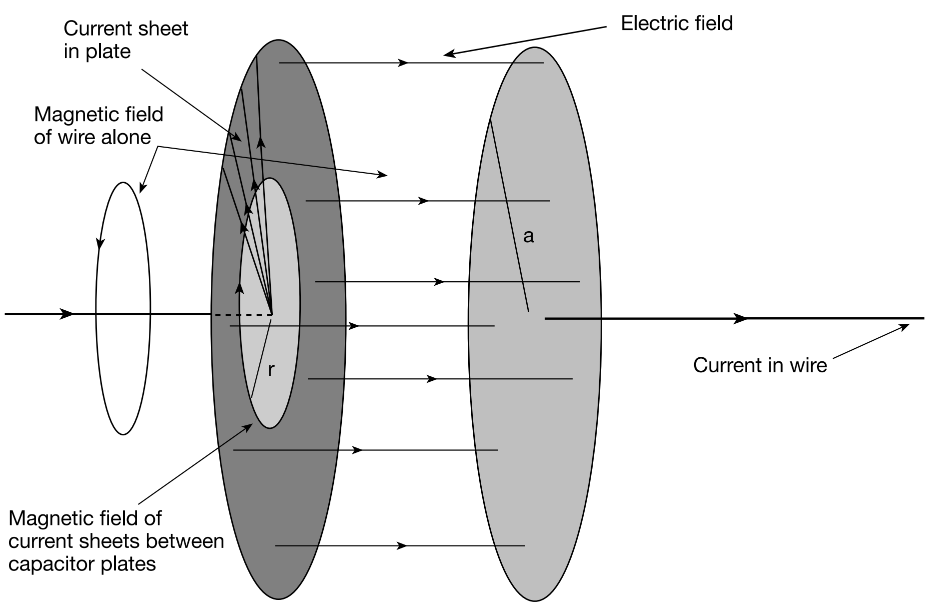
Subsequent authors have expanded on this argument. E M Purcell in 1985, with exemplary clarity, showed that the displacement currents produced by conservative fields are composed of spherically symmetric current elements whose symmetry prohibits them from generating magnetic fields. In one of the most thorough investigations to date, D F Bartlett in 1990 again proved the same result using a transformation of an integral expression for the magnetic field generated by the displacement current. 1n 1987 H Zapolsky showed, on the other hand, that it is the conservative part, only, of a displacement current which contributes to current closure in an open circuit. Purcell, Zapolsky and Bartlett's investigations make it clear that, even in Maxwell theory, only the displacement current of a rapidly changing induced (or vortex) electric field will generate a significant magnetic field. In the special case of a steadily charging capacitor the magnetic field between the capacitor plates is entirely caused by the currents in the leads and in the plates. The displacement current itself then makes no contribution.

Before this result had become well established a theory developed in elementary textbooks which assumed that the local magnetic field which appears in a uniformly charging capacitor is entirely caused by the displacement current in that capacitor. With hindsight it is now possible to wonder why the magnetic field of the conduction currents was excluded from this calculation. Also, a thin charging capacitor is an element of displacement current, only, and one would expect it to make an insignificant contribution to the magnetic field between the plates of the capacitor. Hindsight is an unfair critic, however. It is rather more surprising that the conclusions of French, Tessman, Purcell and Bartlett seem to have had very little impact on the textbook tradition.

The theory that the displacement current in a charging capacitor produces the total magnetic field there has led to numerous experiments to measure the magnetic field between the plates of a capacitor in the belief that such measurements provided a proof of the existence of the displacement current. The above results suggest that only a very small component of the magnetic field between the plates of a capacitor, if indeed any at all, will be caused by displacement currents—even if we accept Maxwell's theory.

Box 5. Calculating the magnetic field intensity in a uniformly charging circular air capacitor

In Maxwell's displacement theory there are three sources of the magnetic field between the plates: the uniform conduction current in the leads, the radial current in the plates and the displacement current generated by the changing electric field between the plates (illustrated below).



(Continued opposite)

Box 5. Continued

Each contribution at a radial distance r from the centre can be calculated directly using the Biot and Laplace law. Present values of the currents must be used. The problem will be simplified by supposing the capacitor very narrow compared with the diameter of the plates and by supposing the leads straight, semi-infinite and very thin compared with r. The H measure of the vacuum field will be used here rather than the B measure since Maxwell's magnetic vorticity equation is more naturally expressed in terms of the former. Scalar notation will be used where it is convenient.

Due to the conduction current I in the leads the contribution is

$$H_1 = I/2\pi r$$

provided we neglect the small reduction due to the missing current element in the capacitor region.

Due to the radial current sheets in the plates the total contribution is $H_2 = J_s$, where J_s is the radial current crossing unit length of the plates. It is easy to show that

$$J_s = \frac{I}{2\pi r}\left(1 - \frac{r^2}{a^2}\right) \qquad \text{and} \qquad H_2 = \frac{I}{2\pi r}\left(\frac{r^2}{a^2} - 1\right)$$

choosing the direction of H_1 as positive.

The contribution from the displacement current is

$$\boldsymbol{H}_3 = \frac{1}{4\pi}\int \frac{(\partial \boldsymbol{D}/\partial t) \times \boldsymbol{r}}{r^3}\,\mathrm{d}v = -\frac{\varepsilon_o}{4\pi}\frac{\partial}{\partial t}\int \boldsymbol{E} \times \nabla\frac{1}{r}\,\mathrm{d}v.$$

Using the vector identity

$$\boldsymbol{E} \times \nabla\frac{1}{r} = -\nabla \times \frac{\boldsymbol{E}}{r} + \frac{\nabla \times \boldsymbol{E}}{r}$$

we obtain

$$\boldsymbol{H}_3 = \frac{\varepsilon_o}{4\pi}\frac{\partial}{\partial t}\int \nabla \times \frac{\boldsymbol{E}}{r}\,\mathrm{d}v - \frac{\varepsilon_o}{4\pi}\frac{\partial}{\partial t}\int \frac{\nabla \times \boldsymbol{E}}{r}\,\mathrm{d}v$$

Using the further identity

$$\int_v \nabla \times \frac{\boldsymbol{E}}{r}\,\mathrm{d}v = \int_s \frac{\mathrm{d}\boldsymbol{S} \times \boldsymbol{E}}{r}$$

the first term on the right of $\boldsymbol{H}_3$ vanishes on the surface of the capacitor and its integral vanishes at infinity since $\boldsymbol{E}$ is a dipole field. The second term on the right vanishes in the case considered because $\boldsymbol{E} = \nabla\phi$ and $\nabla \times \nabla\phi = 0$. The resultant magnetic intensity between the plates is, therefore,

$$H = H_1 + H_2 = \frac{Ir}{2\pi a^2}.$$

In this case, therefore, both the electric and magnetic fields between the plates are caused by the conduction currents alone. The quantitative association between these two fields is expressed by Maxwell's magnetic vorticity equation $\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$. The integral form of this equation,

$$\int \nabla \times \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{S} = \int \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}$$

leads to exactly the same expression for H.

7. Cullwick's paradox

E G Cullwick in 1939 may have been the first to recognize that the use of Maxwell's magnetic vorticity equation to calculate the magnetic field of a displacement current seems to lead to a physical contradiction. He calculated the magnetic field due to a uniformly moving charge using the law of Biot and Laplace, and then the magnetic field due to the accompanying displacement current using Maxwell's magnetic vorticity equation. When he added the two contributions together he obtained twice the established value of the local magnetic field. Cullwick's response was to say that the two methods of calculation are alternatives and must not be used together. The difficulty with this strategy is that Maxwell's displacement theory demands that we must add together the magnetic contributions of both the conduction and displacement currents—and Maxwell's theory is, of course, consistent with Maxwell's equations.

It seems to me that the paradox arises from assuming that the magnetic vorticity $\nabla \times \boldsymbol{H}$ in the equation

$$\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$$

is caused by the displacement current. However, the careful investigations described above have shown that the displacement current produces no magnetic field in this case. The magnetic field associated with $\partial \boldsymbol{D}/\partial t$ above, and that deduced from the motion of the charge, are one and the same field—looked at from different points of view. Had Cullwick applied the law of Biot and Laplace to calculate the magnetic field of the displacement current he would have found that the result—in this case—was zero.

8. Fine-tuning the arguments

To conclude this review, I shall now attempt to evaluate the various debates and to examine their bearing on the present status of Maxwell's displacement current. Retarded field theory, which has denied the displacement current, seems to have been almost silenced during the second part of this century by Maxwell theory, which asserts it. I have offered some suggestions above for this. I also believe that both traditions suffer somewhat from the abstract influence of mathematics, which does not always encourage a clear physical interpretation of mathematical expressions. The problem of the displacement current is now primarily one of interpretation, not one of mathematics or experiment.

I believe that some retarded field theorists undermine their case from the outset by employing the vector potential. Rightly or wrongly, most physicists believe, with Hertz and Heaviside, that the vector potential is an analytical device, only, and that the use of the Lorentz or Coulomb gauge has no physical relevance. Zapolsky, for example, is not convinced by the argument that the absence of the displacement current from the calculation of the Lorentz vector potential means that the displacement current does not exist. He adopts a middle ground and argues, developing Poincaré, that both viewpoints can be useful and sees the debate as a matter of semantics.

I believe that the retarded field argument would be far more effective if the Lorentz expression for the magnetic field intensity (Box 4) was defended in its own right independently of the historical fact that it was deduced from a vector potential function. The Lorentz intensity satisfies Maxwell's equations and it reduces to the well established Biot and Laplace law for steady currents in closed circuits. It has been used successfully for more than a century. Of course, exactly the same can be said for Maxwell's expression for the magnetic field intensity (Box 3). However, if we now accept that conduction currents are a source of the magnetic field, then we will view the two expressions quite differently. The Maxwell expression asserts that these fields are transmitted instantaneously to the observation point while the Lorentz expression implies that the transmission is retarded at the speed of light. It is hardly necessary to point out the very serious difficulties involved in the first interpretation. Most physicists already seem to accept the second, which is strongly supported by special relativity and by relativistic quantum electrodynamics.

It would seem to follow from these arguments that the Lorentz expression is a better physical description of the relationship between the magnetic field and its sources. Significantly, the Lorentz expression does not include the displacement current in its calculation of the magnetic field. If it did it would violate Maxwell's equations. Retarded field theory, therefore, implicitly rejects the displacement current as a source of magnetic field. It also rejects the closure of all currents: what is closed is not the currents but the geometrical tube which is defined by the conduction current streamlines in the material part of the circuit and by the lines of $\partial \boldsymbol{D}/\partial t$ in the open part of the circuit.

A second argument in favour of retarded field theory here is that it offers one source, only, of magnetic fields, namely moving electric charges. Maxwell's displacement theory on the other hand makes both charges and electric fields the source of magnetic fields. The latter view might appear to some to be at variance with their intuitive belief in the principle of economy in nature.

What then of the Maxwell expression for $\boldsymbol{B}$? It is quite common in physics for the mathematics which is set up to describe a physical process to have more degrees of freedom than the physical process itself: we constantly discard unphysical solutions of the equations of physics. Perhaps Maxwell's integral for $\boldsymbol{B}$, although perfectly sound mathematically, does not correctly describe the relationship between a magnetic field and its physical sources.

If these arguments are valid it would seem to follow that—in a vacuum—$\partial \boldsymbol{D}/\partial t$ (called by Rosser the 'Maxwell term') is no more and no less than what it appears to be: the rate of change of the electric field strength. Enormously important though it is, in no sense can it be described as an electric current. Does this mean that the concept, and even the term, 'displacement current' should disappear from physics? I will address this question first from a pedagogical point of view and then in terms of mathematical convenience. Undergraduates already find the concept of a displacement current in a vacuum puzzling since they think of a current as a transport of charges and there is no such transport here. The above analysis suggests that there is no theoretical foundation, either, for such a concept. It could, however, be argued that, since the law of Biot and Laplace is used with present values of the conduction currents in elementary calculations of the magnetic fields, one is then logically obliged to interpret $\partial \boldsymbol{D}/\partial t$ as equivalent to an (imaginary) current. I have been persuaded by the arguments of a referee, however, that the tiny gain in calculational accuracy achieved by this would be more than offset by the perplexity caused to students. Perhaps, therefore, the term 'displacement current' should not even be mentioned to undergraduates?

I believe that the situation is rather different in advanced electromagnetism. There the Coulomb gauge is frequently used for calculational convenience. It is sometimes considerably easier to work with present values of currents than with retarded values where time—and hence the current—changes with distance. In the Coulomb gauge $\partial \boldsymbol{D}/\partial t$ is, of course, treated as mathematically equivalent to an electric current.

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LETTERS AND COMMENTS

On Maxwell's displacement current

Oleg D Jefimenko

Physics Department, West Virginia University, PO Box 6315, Morgantown, WV 26506-6315, USA

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Abstract. A recent article in this journal (Roche J 1998 *Eur. J. Phys.* **19** 155–66) discusses Maxwell's displacement current and its use for calculating magnetic fields. Some of the equations appearing in that article need clarification.

In an interesting recent article, Roche [1] discusses the physical significance of Maxwell's displacement current and its use for magnetic field calculations. I should like to comment on two equations that Roche mentions in his article.

The first equation is the equation 'for calculating the magnetic field strength in Maxwell's electromagnetism' in Box 3 of Roche's article:

$$\begin{aligned}\boldsymbol{B} &= \nabla\times\boldsymbol{A}\\ &= \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}\times\boldsymbol{r}}{r^3}\,\mathrm{d}v\\ &\quad+\frac{\mu_0}{4\pi}\int\frac{(\partial\boldsymbol{D}/\partial t)\times\boldsymbol{r}}{r^3}\,\mathrm{d}v.\end{aligned}\qquad(1)$$

Although this equation (which is correct) appears to be useful for calculating $\boldsymbol{B}$, in reality it can only be used for calculating time-independent fields, where $\partial\boldsymbol{D}/\partial t = 0$. In order to see why equation (1) cannot be used for calculating time-dependent fields, we need to look at the complete set of field and potential equations in the Coulomb gauge. This set is (see e.g. [2, p 524])

$$\boldsymbol{E} = -\nabla\varphi - \partial\boldsymbol{A}/\partial t \qquad(2)$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \qquad(3)$$

$$\varphi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho}{r}\,\mathrm{d}v \qquad(4)$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}+\partial\boldsymbol{D}/\partial t}{r}\,\mathrm{d}v. \qquad(5)$$

As one can see, in order to calculate $\boldsymbol{B}$ by means of equation (1), we need to calculate $\boldsymbol{A}$ by using equation (5). In order to use equation (5), we need to know $\boldsymbol{D}$, or, since $\boldsymbol{D} = \epsilon_0\boldsymbol{E}$, we need to know $\boldsymbol{E}$. In order to find $\boldsymbol{E}$, we need to use equation (2). In order to use equation (2) we need to know $\boldsymbol{A}$. But the only way to find $\boldsymbol{A}$ without using equation (5) is to use equation (3) and to express $\boldsymbol{A}$ in terms of $\boldsymbol{B}$. Thus, for calculating $\boldsymbol{B}$ in time-dependent systems by means of equation (1), we need to know $\boldsymbol{B}$ in the first place! (In principle, equation (1) could be used for calculating $\boldsymbol{B}$ for time-dependent fields if $\partial\boldsymbol{D}/\partial t$ were available from measurements, but this is hardly practical, since it is much easier to measure $\boldsymbol{B}$ directly.)

Detailed investigations of causal relations in electromagnetic fields have shown [3] that the only viable way to compute electric and magnetic fields in terms of their causative sources is to use retarded field integrals (or retarded potentials). This brings us to Roche's equation 'the magnetic field intensity according to Lorentz theory' in Box 4 of his article:

$$\begin{aligned}\boldsymbol{B} &= \nabla\times\boldsymbol{A}\\ &= \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}(t-r/c)\times\boldsymbol{r}}{r^3}\,\mathrm{d}v.\end{aligned}\qquad(6)$$

Unfortunately, equation (6), expressing $\boldsymbol{B}$ in terms of a retarded integral, is not quite correct; an important term is missing.

The complete equations expressing the electric and magnetic fields in terms of retarded integrals are

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\int\left\{\frac{[\rho]}{r^3} + \frac{1}{r^2 c}\left[\frac{\partial \rho}{\partial t}\right]\right\}\boldsymbol{r}\,\mathrm{d}\nu - \frac{1}{4\pi\epsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial \boldsymbol{j}}{\partial t}\right]\mathrm{d}\nu \qquad (7)$$

and

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int\left\{\frac{[\boldsymbol{j}]}{r^3} + \frac{1}{r^2 c}\left[\frac{\partial \boldsymbol{j}}{\partial t}\right]\right\}\times\boldsymbol{r}\,\mathrm{d}\nu \qquad (8)$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are determined for the instant t, and the square brackets denote the 'retardation symbol' indicating that the quantities in the square brackets are taken at the retarded time $t' = t - r/c$. These equations constitute solutions of Maxwell's equations for fields in a vacuum. They can be derived from Maxwell's equations via the inhomogeneous wave equations (see [2, pp 46–52, 514–7] and [4]) or via retarded potentials [5–7]. As one can see from equations (7) and (8), the causative sources of the electric field are the electric charges (both steady and time-variable) as well as the time-variable true electric current. The causative sources of the magnetic field are the true electric currents (both steady and time-variable), but not the displacement current.

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LETTERS AND COMMENTS

Reply to Comment 'On Maxwell's displacement current'

John Roche
Linacre College, Oxford, OX1 3JA, UK

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Abstract. This brief note replies to two comments made by Professor Oleg Jefimenko on my recent article (March 1998) on Maxwell's displacement current.

I am grateful to Professor Oleg Jefimenko [1] for pointing out a missing term

$$\left(\frac{1}{r^2 c}\left[\frac{\partial \boldsymbol{J}}{\partial t}\right] \times \boldsymbol{r}\right)$$

in my retarded version of the Biot and Laplace law [2, page 160, Box 4]. I have not had the opportunity to examine fully the literature since Maxwell on this point but Professor Jefimenko may have been the first to draw attention to this term, at least in the second half of the present century. It has been corroborated by other investigators. Happily, the omission does not seem to affect my argument.

Professor Jefimenko introduces an intriguing logical circle in the interpretation of the corresponding Maxwell or Coulomb gauge version of the Biot and Laplace law: he argues that in order to calculate $\boldsymbol{B}$ from this expression we need to know $\boldsymbol{B}$! However, he provides us with an escape from the circle by conceding that $\boldsymbol{B}$ can be calculated *in principle* from the predicate of the equation—provided each term in the latter can be measured. Unless I am missing the point, the difficulty with using the equation would seem to be practical, therefore, and not logical.

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Maxwell's displacement current revisited

J D Jackson
University of California, Berkeley, and Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

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Abstract. A careful analysis is made of the example of a charging capacitor to show the meaning and significance ('reality') of Maxwell's displacement current.

1. Introduction

In a recent paper in this journal, Roche [1] presents a historical survey of the 'controversy' over the reality of Maxwell's displacement current, with Maxwell's view that it is 'electromagnetically equivalent' to a conduction current contrasted with the majority view that the sources of the electromagnetic field are solely the charge and current densities. While Roche's paper contains some useful historical information, it is marred by imprecision, sloppy notation, and downright mistakes. In particular, his important Box 5, 'Calculating the magnetic field intensity in a uniformly charging circular air capacitor', is incorrect in detail and logically contradictory in its conclusion. Roche purports to show that the displacement current does not contribute to the magnetic field within the capacitor, but then concludes with a statement of the integral form of Ampère's law with the integral of the displacement current as source term on the right-hand side.

I begin with a brief discussion of my explanation of the logic of Maxwell's insistence on the 'reality' of the displacement current (although 'electromagnetically equivalent' is not as extreme a position as some would believe). I then carefully analyse the charging capacitor example to show in what way the displacement current enters: in one method of solution, the whole source of the magnetic field, and in another, a contribution, together with the conduction current, that removes a logical inconsistency. For simplicity, I treat the fields in vacuum, with $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ and $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$.

2. Maxwell's choice of gauge and the 'reality' of the displacement current

Although I know that Maxwell had very explicit mechanical models for electromagnetic fields as disturbances in the elastic aether, for me a key to understanding Maxwell's insistence on treating the displacement current on the same footing as the conduction current is his constant choice of the Coulomb gauge for the potentials. The Maxwell equations in vacuum are

$$\begin{aligned} \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= 0 \qquad & \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{H} &= \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad & \nabla \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0}. \end{aligned} \tag{1}$$

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Introducting the scalar and vector potentials Φ and $\boldsymbol{A}$ according to

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial A}{\partial t} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{2}$$

the inhomogeneous Maxwell equations become

$$-\nabla^2\Phi - \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) = \frac{\rho}{\varepsilon_0} \qquad -\nabla^2\boldsymbol{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) = \mu_0\left[\boldsymbol{J} + \frac{\partial D}{\partial t}\right]. \tag{3}$$

Note that in the second equation, in order to exhibit the displacement current explicitly, we have not eliminated $\boldsymbol{D}$ in favour of $\boldsymbol{A}$ and Φ. Following Maxwell, we exploit the freedom of gauges to choose the Coulomb gauge, namely, $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. Then the equations for the potentials are

$$-\nabla^2\Phi = \frac{\rho}{\varepsilon_0} \qquad -\nabla^2\boldsymbol{A} = \mu_0\left[\boldsymbol{J} + \frac{\partial D}{\partial t}\right]. \tag{4}$$

The first equation is just the Poisson equation, which yields the so-called instantaneous scalar potential. The second equation has the appearance of a Poisson equation for the vector potential, with a source term that is the *sum of the conduction current density and the displacement current*. From this point of view, the displacement current is clearly 'electromagnetically equivalent' to the conduction current. I believe it is in this sense, and within the framework of the Coulomb gauge, that Maxwell could insist on the reality of the displacement current as a contribution to the total (effective) current.

There are, of course, flaws in such an interpretation. Because $\boldsymbol{D}$ is the sum of terms involving Φ and $\boldsymbol{A}$, $\partial D/\partial t$ is not really a source term. The two equations in (4) are coupled partial differential equations for Φ and $\boldsymbol{A}$ with ρ and $\boldsymbol{J}$ the true external sources. Issues of causality and the finite speed of propagation of electromagnetic disturbances are obscured by the choice of the Coulomb gauge: the potentials are manifestly not causal, but the fields can be shown to be [2], [3, p 291, problem 6.20]. Nevertheless, for quasi-static fields, equations (4) can serve as a basis for successive approximations. The instantaneous (electrostatic) solution for the scalar potential yields at lowest order the instantaneous electric field. This field can be used in the displacement current as a source term, along with the conduction current, to find the first-order vector potential and hence the first-order magnetic field, and so on. We apply this approach below to the Maxwell equations (1) directly.

Incidentally, the expression for $\boldsymbol{B}$ in Roche's Box 3, called Maxwell's expression for the magnetic field intensity, follows from the second Poisson equation in (4) if the right-hand side is treated as a known source. This instantaneous expression for $\boldsymbol{B}$ is not a solution, of course, but only a true formal integral relation between the fields and the current source.

In his Box 4, Roche states the familiar 'Lorentz' (retarded) expressions for $\boldsymbol{A}$ and $\boldsymbol{B}$ that emerge from choosing the gauge condition, $\boldsymbol{\nabla}\cdot\boldsymbol{A} + \partial\Phi/\partial t = 0$. His expression for $\boldsymbol{A}$ is correct, but his result for $\boldsymbol{B}$ is not [3, p 247, equation (6.56)].

3. Quasi-static magnetic field within a charging capacitor, direct perturbative approach

As did Roche, we consider a parallel-plate capacitor consisting of two thin, flat, circular, conducting plates of radius a, separated by a distance d, and positioned such as to be parallel to the xy plane and have their centres on the z-axis at $z = 0$ and $z = d$. Very long straight and thin wires lie along the z-axis and terminate on the outer sides of the two plates. A current $I(t)$ flows in the negative z-direction, bringing a total charge $Q(t)$ slowly to the top plate and removing an equal amount of charge from the bottom plate. The charge and current are related according to $I(t) = \mathrm{d}Q(t)/\mathrm{d}t$. We assume that $d \ll a$, so that fringing fields can be neglected. In the static limit, charge $Q(t)$ is therefore uniformly distributed over the inner side of the top plate, so that the surface charge density there is $\sigma(t) = Q(t)/\pi a^2$. The electric displacement has only a z-component, uniform throughout the volume between the plates and equal to $D_z = -\sigma(t) = -Q(t)/\pi a^2$. The lowest-order displacement current is therefore

$$\frac{\partial \boldsymbol{D}^{(0)}}{\partial t} = -\hat{\boldsymbol{z}}\frac{I(t)}{\pi a^2}. \tag{5}$$

With this approximation for the displacement current inserted into the Ampère–Maxwell equation, we have a differential equation for the first-order magnetic field:

$$\boldsymbol{\nabla} \times \boldsymbol{H}^{(1)} = -\hat{\boldsymbol{z}}\frac{I(t)}{\pi a^2}.$$

The displacement current source term is equivalent to a uniform current density in the negative z-direction throughout a circular wire of radius a. As is well known, such a current causes an azimuthal magnetic field whose magnitude can be found by applying Stokes's theorem to a circular path of radius ρ and fixed z centred on the z-axis. The elementary result for the magnetic field for $0 < \rho < a$ and $0 < z < d$ is

$$H_\phi^{(1)} = -\frac{I(t)\rho}{2\pi a^2}. \tag{6}$$

In the region between the plates, where no conduction current exists, this first-order result for the magnetic field depends solely on the existence of Maxwell's 'displacement current.' Of course, in modern parlance we do not stress the reality of the electric displacement current $\partial \boldsymbol{D}/\partial t$ as a source term, any more than we speak of the 'magnetic displacement current' $-\partial \boldsymbol{B}/\partial t$ as a source term in Faraday's law. The Maxwell equations involve the *spatial* and *temporal* derivatives of the fields, coupled to the charge and current sources. The displacement current emerged as an effective source term above because we utilized an iterative approximation scheme whereby the lowest-order electric field gave a known quasi-static displacement current of the same order as the conduction current in the wires outside the capacitor.

4. Roche's Box 5 example, via linear superposition

The geometry of the parallel plate capacitor and the associated current flows is sketched in figure 1. Roche begins by saying that he will ignore 'the small reduction due to the missing current element in the capacitor region.' He thus assumes that the long-wire result, $H_\phi = -I(t)/2\pi\rho$, holds between the plates of the capacitor as well as outside. As we show below, this neglect cannot be made. He then makes the further (and compensating) mistake of identifying the radial surface current density on the top plate with the magnetic field on the under side of the top plate, rather than with the discontinuity of field across that plate. He adds these two contributions to obtain the correct result (6), claiming that the displacement current does not contribute. He concludes with the contradictory statement that the integral form of Ampère's law with the displacement current as 'source' yields the same result.

We solve the problem of finding the magnetic field within the capacitor by linear superposition. The simple field of the continuous straight wire carrying the current $I(t)$ must be augmented by the field caused by the current flow pattern sketched on the right in figure 1. With the assumption of uniform charge density at all times on the inner surfaces of the two capacitor plates, the continuity equation for charge and current densities yields the radial surface current density stated by Roche:

$$K_\rho(\rho, t) = \frac{I(t)}{2\pi\rho}\left(1 - \frac{\rho^2}{a^2}\right) \tag{7}$$

where equation (7) applies for the upper plate and its negative for the lower plate. The current $I(t)$ flows from the lower to the upper plate along the z-axis, as indicated in figure 1. The integral form of the Ampère–Maxwell equation is

$$\oint_C \boldsymbol{H} \cdot \mathrm{d}\ell = \int_S \left[\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right] \cdot \boldsymbol{n}\, \mathrm{d}A \tag{8}$$

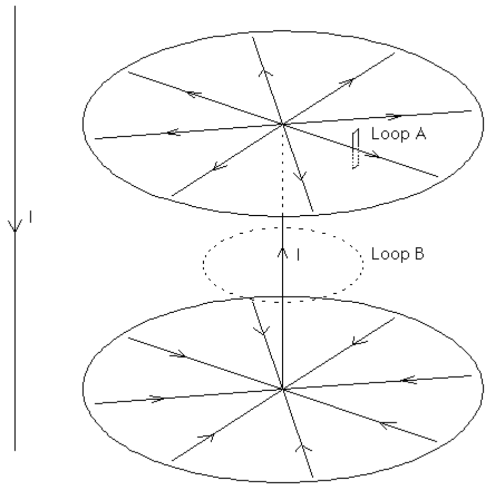

Figure 1. Charging capacitor. The magnetic field produced by the current pattern shown on the right must be added to the field of the long, continuous straight wire to give the magnetic field within the capacitor and outside it. The spacing d between the plates is greatly exaggerated compared with the radius a for clarity.

where the integral on the left is around a closed circuit C, such as loop A or loop B in figure 1, and the integral on the right is over the open surface S spanning the circuit.

We note that the symmetry of the problem assures that the magnetic field has only an azimuthal component, independent of angle, both within and outside the capacitor. We also note that the displacement current, derived in section 3, has only a z-component and is constant within the capacitor. As a consequence, if we apply equation (8) to loop A, with vertical sides Δz and horizontal sides $\rho\Delta\phi$ and $\boldsymbol{n}$ in the radial direction, only the surface conduction current contributes. We find

$$\rho\Delta\phi(\Delta H_\mathrm{i} - \Delta H_\mathrm{t}) = \rho\Delta\phi\, K_\rho(\rho, t). \tag{9}$$

Here ΔH_i and ΔH_t are the azimuthal magnetic fields just below and just above the top plate caused by the radial current flow. Lowering loop A to be totally within the capacitor leads to the conclusion that ΔH_i is independent of z, and lowering it to intersect the bottom plate leads to the relation

$$\rho\Delta\phi(\Delta H_\mathrm{i} - \Delta H_\mathrm{b}) = \rho\Delta\phi\, K_\rho(\rho, t). \tag{10}$$

Here ΔH_b is the magnetic field just below the bottom plate. The difference of equations (9) and (10) yields $\Delta H_\mathrm{t} = \Delta H_\mathrm{b}$, a result equally well obtained by extending loop A vertically to go above the top plate and below the bottom. Since ΔH_t and ΔH_b are independent of z, they must both vanish, because remote from the capacitor the total field is given by the expression for the long straight wire on axis.

The total magnetic field within the capacitor is the sum of the field of the long straight wire and ΔH_i:

$$H_\phi = -\frac{I(t)}{2\pi\rho} + K_\rho(\rho, t) = -\frac{I(t)\rho}{2\pi a^2} \tag{11}$$

in agreement with equation (6). Now, it appears that we agree with Roche's first conclusion, that the displacement current does not enter the determination of the magnetic field within the charging capacitor. However, before leaving the example, consider loop B of radius ρ in figure 1. Now the normal $\boldsymbol{n}$ points in the positive z-direction; the displacement current in equation (8) contributes to the integral. Explicitly, we have

$$2\pi\rho\,\Delta H_{\mathrm{i}} = I(t) + \pi\rho^2\hat{\boldsymbol{z}}\cdot\frac{\partial \boldsymbol{D}^{(0)}}{\partial t} \tag{12}$$

where the displacement current contribution is given by equation (5). Adding the straight-wire contribution (which cancels the $I(t)$ on the right-hand side of (12)), we obtain the total magnetic field in the capacitor as

$$H_\phi = \frac{\rho}{2}\hat{\boldsymbol{z}}\cdot\frac{\partial \boldsymbol{D}^{(0)}}{\partial t}. \tag{13}$$

Just as in section 3, we find from loop B that the whole field comes from the displacement current as 'source'. Without the displacement current, the answer for ΔH_{i} from loop B would be wrong and inconsistent with the result from loop A.

5. Summary and conclusions

We can understand Maxwell's insistence on the reality of the displacement current, on an equal footing with the conduction current, because of his use of the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ for the potentials. In this gauge one has 'instantaneous' Poisson equations for both the scalar and vector potentials, with the 'source' term, $\boldsymbol{J} + \partial\boldsymbol{D}/\partial t$, for the latter. In this sense, the displacement current is on a par with the conduction current (although really not an external source). The implementation of displacement current as a source term is explored through a perturbation approach, with lowest-order (instantaneous electrostatic) electric fields providing the displacement-current source term in Ampère's law for the first-order magnetic fields, and so on. The example of a charging parallel plane capacitor is treated in section 3 with the displacement current to the fore, and in section 4 with the conduction current dominating, but with the displacement current crucial in removing an inconsistency. Along the way, Roche's confusing and sometimes erroneous discussion is corrected.

It is clear from section 3 and equations (11) and (13) that it is a matter of procedure and taste whether or not the displacement current enters explicitly in a calculation. Maxwell is wrong if he asserts that the displacement current is a real external current density on a par with the conduction current density, but he is right if he says that it is electromagnetically equivalent (in the sense that it can appear on the right-hand side of the Ampère–Maxwell equation together with the conduction current density). Implementation of the displacement current as an 'external source' depends, however, on a perturbative approach, starting with electrostatics.

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LETTERS AND COMMENTS

Reply to J D Jackson's 'Maxwell's displacement current revisited'

J Roche
Linacre College, Oxford, OX1 3JA, UK

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Abstract. The points of difference between J D Jackson's 'Maxwell's displacement current revisited' (1999 *Eur. J. Phys.* **20** 495–9) and J Roche's earlier article 'The present status of Maxwell's displacement current' (1998 *Eur. J. Phys.* **19** 155–66) are carefully examined with the intention of further clarifying the issues involved.

I am delighted that someone of Professor Jackson's distinction is taking an interest in the debate over the displacement current. I believe the differences between his article [1] and my earlier article [2] are slight. Nevertheless, I would like to explore these differences.

(1) I think he may have misunderstood me when he says [1, page 497] that I conclude my argument

> '. . . with the contradictory statement that the integral form of Ampère's law, with the displacement current as "source'', yields the same result'.

Here are my words [2, page 163]:

> 'In this case, therefore, both the electric and magnetic fields between the plates are caused by the conduction currents alone. The quantitative association between these two fields is expressed by Maxwell's magnetic vorticity equation $\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$'

When phrasing these sentences I took particular care to use the word 'association' and *not* to say that $\partial \boldsymbol{D}/\partial t$ is a source term. I make the same point again on page 164. On no occasion in the text do I describe $\partial \boldsymbol{D}/\partial t$ as a 'source'. One of the most important themes of my article was to demonstrate that the Maxwell term $\partial \boldsymbol{D}/\partial t$ is not a source term, and that the equation $\nabla \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$ is an association or correlation between field properties, each of which derives from electric sources elsewhere, and is not an effect–cause relation.

(2) I think Professor Jackson may have misunderstood me again when he writes [1, page 497] that I make the

> '. . . mistake of identifying the radial surface current density on the top plate with the magnetic field on the under side of the top plate, rather than with the discontinuity of field across that plate.'

Here is my text [2, page 163]:

> 'Due to the radial currents in the sheets the total contribution is $H_2 = J_s$.

What I mean here is that the total contribution from the current sheets in each capacitor plate adds up to $H_2 = J_s$. I did not mean that H_2 is the total field in the region between the plates. Indeed, I use H_1 for the field between the plates due to the broken current and H_3 for that due to the displacement current. Furthermore, Professor Jackson's expression for this field discontinuity is exactly the same as mine. I do accept, however, that I might have been more explicit.

(3) In his calculation of the magnetic field between the plates of the capacitor, Professor Jackson omits the contribution of the fictitious reversed conduction current element that he inserts between the plates [1, pages 497–8]. This, of course, is perfectly justified in

a first-order calculation. But he appears to do exactly what he criticizes me for doing—he ignores the magnetic field dip caused by the interruption in the long current due to the presence of capacitor plates. He also asserts that I only obtain the correct result because my error in (2) compensates that in (3). However, these two 'errors' are of different orders of magnitude and cannot compensate each other. In fact, again, Professor Jackson obtains exactly the same expression as I do for the contribution of the broken current to the magnetic field intensity between the capacitor plates, and I can find no errors either in his derivation, or in mine.

(4) On page 497, Professor Jackson writes:

> 'The displacement current source term is equivalent to a uniform current density in the negative z-direction throughout a circular wire of radius a. As is well known, such a current causes an azimuthal magnetic field whose magnitude can be found by applying Stokes's theorem to a circular path of radius ρ and fixed z centred on the z-axis.'

On page 499 of his text he argues in a similar manner that:

> 'the whole field [of the capacitor] comes from the displacement current as "source".'

Is this a valid argument?

Consider the corresponding equation for a pure conduction current, $\nabla \times \boldsymbol{H} = \boldsymbol{j}$. Although we say correctly that currents cause magnetic fields, we do not say that the local conduction current density $\boldsymbol{j}$ *causes* the local magnetic vorticity $\nabla \times \boldsymbol{H}$. The local magnetic field is, of course, caused by neighbouring currents both near and far. The above equation does not, therefore, relate the local source of a magnetic field to the local magnetic field caused by that source. What the above equation establishes is a quantitative *correlation* between the local magnetic vorticity and the local current density. Surely that is all it establishes, both in this case and in the case of the displacement current density.

Furthermore, the full displacement current is a current element, only, in this case—since we are assuming a very thin capacitor. Even if we accept that the displacement current generated by the capacitor is a possible source of magnetic field, it will, therefore, only make a second-order contribution to the total capacitor field, if it makes any contribution at all. Indeed, if the displacement current running between the plates of the capacitor were the full source of the magnetic field there, then halving the distance between the plates would halve the magnetic field intensity between the plates. In fact, in our capacitor, such an operation would have no first-order effect on the magnetic field. Also, when the magnetic field due to the capacitor displacement current is rigorously calculated from Maxwell's Coulomb gauge field integral, it is found to be zero [2, page 163]. Finally, if the magnetic field were caused in full or even in part by the changing bound electric field between the plates, placing a current loop perpendicular to the magnetic field between the capacitor plates would produce a torque on the loop and a reaction torque on the changing electric field (the 'current'), which, of course, is impossible. The reaction torque will be experienced only by the conduction currents in the circuit (c.f. my discussion of Poincaré's work in [2, page 160].)

(5) It seems to me that some readers may be puzzled by a seeming conflict between Professor Jackson's statement on page 498 where he writes:

> '... it appears that we agree with Roche's first conclusion, that the displacement current does not enter the determination of the magnetic field within the charging capacitor'

and his statement on page 499 that:

> 'the whole field comes from the displacement current as "source".'

I think this does need to be clarified. My own view is that, in the Coulomb gauge, the displacement current can, in general, be considered to be equivalent to an electric current but that in the special case of the uniformly charging capacitor it does not contribute at all to the magnetic field there. But the Coulomb gauge is not a physical gauge and the equivalence of $\partial \boldsymbol{D}/\partial t$ to a current is always purely fictional.

References

[1] Jackson J D 1999 Maxwell's displacement current revisited *Eur. J. Phys.* **20** 495–9

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LETTERS AND COMMENTS

Reply to Comment by J Roche on 'Maxwell's displacement current revisited'

J D Jackson
University of California, Berkeley, and Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

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Abstract. I reply to a Comment by J Roche (2000 *Eur. J. Phys.* **21** L27-8) on my paper 'Maxwell's displacement current revisited' (1999 *Eur. J. Phys.* **20** 495-9), which contrasted commonly accepted views with those expressed by Roche in his earlier paper 'The present status of Maxwell's displacement current' (1998 *Eur. J. Phys.* **19** 155–66).

In his original paper [1], John Roche argues vigorously against the interpretation of the term $\partial \boldsymbol{D}/\partial t$ in the Ampère–Maxwell equation as a 'displacement current'. My own paper [2] was intended (i) to clarify what I believe is the commonly accepted view of the 'displacement current', (ii) to illustrate how the displacement current can serve as an effective source for the magnetic field in quasi-static situations, and (iii) to correct some errors and obscurities in Roche's discussion.

Now Roche's reply [3] takes issue with me, while stating that our differences are slight. It is a fact that both of us and the vast majority of physicists and engineers living today do not think that the unfortunately named 'displacement current' is a true current. I believe that it is also widely understood how Maxwell, by choosing to work in the radiation or Coulomb gauge ($\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$), was led to speak of the total current as the sum of the conduction current $\boldsymbol{J}$ and the displacement current $\partial \boldsymbol{D}/\partial t$. Roche and I agree that the Ampère–Maxwell equation is better written as $\boldsymbol{\nabla} \times \boldsymbol{H} - \partial \boldsymbol{D}/\partial t = \boldsymbol{J}$, not as $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{total}} = \boldsymbol{J} + \partial \boldsymbol{D}/\partial t$. The external charge and current densities are the true sources for the fields.

An extensive discussion of semantics is not useful, but I remark briefly on Roche's items, in his order of presentation.

(1) Roche takes umbrage at what he interprets as my attribution to him of the use of the word 'source' to describe the role of the displacement current in Maxwell's equations. Apparently my use of the word 'source' in quotation marks was not sufficient to indicate that I meant effective source. His quotation from his paper in his defence is incomplete. I was referring to the last sentence on his page 163, which continues his partial quotation:

> 'The integral form of this equation,
>
> $$\int \boldsymbol{\nabla} \times \boldsymbol{H} \cdot \mathrm{d}S = \int \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}S$$
>
> leads to exactly the same expression for $\boldsymbol{H}$.'

(2) No comment seems necessary. I stand by what I said. The next item is relevant.

(3) Roche is wrong to say that I ignored the contribution from the upward central current element between the plates (see my figure 1). As my discussion at the bottom of page 497 and on page 498 makes clear, the application of the integral form of the Ampère–Maxwell equation, my equation (8), to the loops A and B is completely general. When the loop A straddles the top sheet, the discontinuity in the azimuthal magnetic field (the only component in the problem) is determined by the radial current in the sheet and has nothing to do with the remote central current flow. When the loop is lowered, to the region between the plates, there is no conduction or displacement

current through the loop. This establishes that the azimuthal magnetic field does not depend on z. I need not continue the argument—it is all there on my page 498. The lack of z dependence is a consequence of the cancelling z-dependences of the contributions from the current flow in the central wire segment and the radial currents in the two plates. A tedious direct calculation of the magnetic field in the limit $d/a \ll 1$ verifies the behaviour found in the simple fashion of my paper.

(4) Here the disagreements are semantic. It is common usage to call the charge and current densities the sources of the electromagnetic fields [4–6]. A standard theorem [7], loosely stated, is: if the divergence and the curl of a vector field are given through all space and they vanish at infinity, then the vector field can be determined everywhere. It is in this sense that monograph and textbook writers use the term 'source'. Roche's discussion here fails to recognize the iterative process involved in the quasi-static treatment of the charging capacitor. In the zeroth approximation, all is electrostatics. The time derivative of the zeroth-order electric field can serve as an effective source term in the Ampère equation to determine the first-order magnetic field. Continued iteration generates the proper Bessel function series for the electric and magnetic fields inside a cylindrical resonant cavity.

(5a) Roche seems not to have understood my treatment of section 3 and my use of the two loops, A and B, in my figure 1. In my section 3, the symmetry of the problem is used in the context of the quasi-static approach to derive the first-order magnetic field from the zeroth-order displacement current as an effective source in Ampère's equation (from the Ampère–Maxwell equation with no current density). With the 'loop A' approach, all is due to the conduction current. With loop B, one only gets consistency with the combination of conduction current and displacement current. None of this is surprising. If the language is a problem, use other terminology.

(5b) Roche's last sentence is objectionable on two counts. Firstly, he says,

> 'But the Coulomb gauge is not a physical gauge. . . .'

The choice of gauge is purely a matter of convenience. The Coulomb gauge is no more or less physical than any other. It is convenient for some problems, inconvenient for others. The fields are the reality. The Coulomb gauge has potentials with peculiar ('unphysical') relativistic properties, but the fields derived from them are the same as the fields derived from the potentials in any other gauge, causal and with finite speed of propagation [8, 9]. My second objection is to the last part of the sentence:

> 'and the equivalence of $\partial \boldsymbol{D}/\partial t$ to a current is always purely fictional.'

Roche believes our differences are slight. I find his statement too doctrinaire.

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