Valuing guaranteed minimum death benefits in variable annuities and the option to lapse*

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Abstract

Many variable annuities provide money-back guarantees and market guarantees on invested principal. Embedded in some of these guarantees are stochastic maturity put options with adjustable strike prices. These variable annuities can be surrendered or lapsed at any time. The lapse option when exercised rationally represents an American style sell-back option that is exercised by the policyholder when the embedded put option is out-of-the-money. The death benefits we consider are only exercised involuntary, that is, upon the death of the policyholder. Critical to the valuation analysis is that the embedded put options have stochastic maturity and that the policyholder can exercise the lapse, or early-exercise, option feature to increase the value of the contract and thereby exposing the insurance company to loss of fees. We analyze specific variable annuity products by focusing on the lapse option when exercised either rationally or irrationally, taking into account the mortality risk and surrender charges.

Keywords and phrases: American option, lapse option, lookback option, stochastic maturity

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1 Introduction

The variable annuity (VA) market has shown tremendous growth in the past decade. This has largely been driven by demographic factors - i.e. the aging baby boomers and the demand for retirement savings - and the simultaneous performance of strong U.S. equity markets.

However, the risk-return profile of the VA business is highly correlated with the future evolution of U.S. equity markets. This is true for two reasons. First, the size of the lucrative asset-based fees is dependent on a strong equity market. Second, the guaranteed minimum death benefit (GMDB) - which is a ubiquitous component of every contract - exposes annuity writers to claims during prolonged periods of weak equity markets. As such, their business model is doubly exposed to equity market risks. Indeed, faced with an aggregate $1 trillion VA market - and after coming to grips with the potential magnitude of the GMDB risk - the reinsurance markets for GMDBs has virtually dried up. The few remaining players do not provide the wide range of features and capacities needed to efficiently reinsure the books of some of the largest annuity writers. At the same time, the reinsurance rate for GMDB risk also increased significantly.

Therefore, the only feasible alternative to managing the potentially hazardous risks embedded in VA policies is for a large annuity writer to undertake a *self reinsurance* plan using traded capital market instruments.

To this end, the first step is to value the GMDBs in variable annuities in the same framework that capital markets use to value derivatives. In fact, there have been studies (cf. Milevsky and Posner (2001), Grosen and Jorgensen (1977 & 1999) and references therein) on computing the *no arbitrage value* of GMDBs using the well-established derivative pricing models. The methodology traces its roots to the famous Black-Scholes option pricing formula, and has become the standard valuation technique for traded capital market instruments. The main contribution from this work is that we value the GMDBs and the option to lapse in an actual variable annuity contract and take into account the surrender charges schedule. Our results give annuity writers more realistic indications on the costs of GMDBs in their products. In this paper, we value guaranteed minimum death benefit (GMDB) options in Polaris[1] II variable annuities.

[1] Polaris II VAs are provided by SunAmerica Inc., a subsidiary of American International Group, Inc.

2 Product description

There are two types of guaranteed minimum death benefit (GMDB) options in Polaris II VA. These are: the purchase payment accumulation (PPA) option and maximum anniversary value (MAV) option. The PPA option provides interest rate guarantees as well as market guarantees. The basic PPA option guarantees a return of at least the original invested premium at time zero compounded at a 4% annual growth rate until the date of death (3% growth rate if 70 or older at the time of contract issue). Technically, the payout to the beneficiary is: $\max(e^{g\tau}V_0, V_\tau)$, g is the guaranteed instantaneous growth rate, V_0 is the invested principal and V_τ is the value of the investment at the time of death. The time of death is treated as a stochastic random variable.

The MAV option provides a money-back guarantee (without a growth rate) as well as market guarantees. The basic MAV option is based on a suitably defined highest anniversary account value. Technically, under the MAV option, the payout to the beneficiary is $\max(V_\tau, \mathrm{M}^*_\tau)$ where M^*_τ is the maximum of anniversary value of the contract prior to the policyholders' 81^{st} birthday. If the policyholder is 90 or older at the time of death, the death benefit will be equal to the value of the contract at the time of death. In option pricing theory the guarantee corresponds to having a strike price of the embedded option that *floats* and may increase to a new higher level every year. The floating piece is determined on the anniversary market value of the contract. A detailed description of the PPA and MAV options is provided in Appendix A.

The term *interest guarantee* refers to a GMDB where the original premium is guaranteed to accumulate at a fixed rate of return (for example, the PPA option). The term *market guarantee* refers to a GMDB where some degree of market returns are guaranteed in the form of anniversary resets (for example, the MAV option).

Our goal is to compute the fair values of GMDB options in Polaris II VA which are essentially derivatives that remain *alive* as long as the policyholder does not die or lapse. Rational investors will lapse the option when the embedded put options are *out-of-the-money.* However, the insurance company provides a disincentive for policyholders to leave the fund by imposing lapsation or surrender charges. Surrender charges secure sunk commissions costs, and the insurance company is also protecting itself by increasing the probability of receiving fees that are meant to pay for the death benefit option. GMDB options with a lapse option exercised rationally have an American style feature otherwise they exhibit European style features. (By European style features, we refer to options that are only exercised at maturity. Early exercisable options are said to exhibit American style features.) Unlike standard financial options, the embedded options in VA are not paid for upfront, instead they are paid for as an insurance charge deducted from the underlying fund periodically. Formal valuation methods for these GMDB options are complicated by the fact that the underlying options exhibit American-style sell-back features and they have stochastic maturity. The options are exercised involuntary since they are only triggered by death. Stochastic maturity, where the maturity is independent of the underlying asset, is eccentric to classical option pricing theory and as such it has not been dealt with sufficiently. There are some exotic stochastic maturity options, such as barrier options, where the maturity depends on the underlying asset. To the extent that the payoff structure of the death benefit options in many variable annuities are between European and American style and are triggered by death, Milevsky and Posner (cf. Milvesky and Posner (2000)) have labeled these options *Titanic options.* Related work on stochastic maturity options was studied by Carr (1998) for financial instruments he called randomized American options. While GMDB options are not freely traded between policyholders and capital markets participants, the assumption of perfect markets guides us to a relative value. In our study, we do not consider credit risk, that is, default on the side of the fund manager.

The organization of this report is as follows; we will begin by considering simplified versions of the two GMDBs in the Polaris II VA. Then, we will study the lapsation option which is the main focus of this paper. For the special cases of the product, we provide closed-form analytical solutions and the more general cases are handled by an American Monte Carlo method for the illustrative examples. We also study perpetual options to provide upper bounds for present values of MAV and PPA options. Throughout this paper, the final payoff at the date of death of the policyholder or when he/she exercises the lapse option, is assumed to be based on an initial premium of 100 units at time zero. We finish this paper with some concluding remarks.

3 The model

We will use the following nomenclature;

D_τ	$\triangleright$	death payment at time τ
P_E	$\triangleright$	GMDB (without lapsation) option payoff
P_A	$\triangleright$	GMDB (with lapsation) option payoff
V_t	$\triangleright$	market value of the investment at time t (years)
V_k^*	$\triangleright$	market value of the investment at the k^{th} anniversary date
M_τ^*	$\triangleright$	maximum anniversary value of the investment at death.
τ	$\triangleright$	stochastic death time of the policyholder
T	$\triangleright$	deterministic death time of the policyholder
g	$\triangleright$	guaranteed growth rate
α_t	$\triangleright$	lapsation (or surrender) charge at time t
δ	$\triangleright$	insurance charge and management fees
r	$\triangleright$	risk-free interest rate
σ	$\triangleright$	volatility

The maturity date refers to the termination of the accumulation phase which we may assume, for simplicity, to be set[2] at the inception of the contract. Superscripts p and m will be used to denote the PPA and the MAV death benefit options respectively.

The progression of the fund value, which we treat as a single asset is modeled as

$$dV_t = (\mu - \delta)V_t dt + \sigma V_t d\widetilde{W}_t \tag{1}$$

where $\widetilde{W}_t$ is a standard Brownian motion, μ is the drift rate and δ is a sum of management fees and insurance charges (also called mortality and expense (M&E) fees) that *pays for* the underlying option. This reflects that, unlike standard financial options, the embedded options in VA are not paid for upfront, instead they are paid for as insurance charge deducted from the underlying fund on a periodic basis. The insurance charge and the management fees are modeled as a dividend yield outflow[3] δ, since this dividend does not go to the fund-holder but to the insurance company. Thus, a continuous time payment δV_t flows to the insurer. In the examples that follow, we choose $\delta = 2\%$ and $\delta = 2.5\%$ which are consistent with the insurance charges and management asset fees in Polaris II VA.

The risk neutral process for V_t follows the lognormal process;

$$dV_t = (r - \delta)V_t dt + \sigma V_t dW_t^Q \tag{2}$$

[2] In the Polaris II VA, the investor determines when the income phase begins.

[3] We distinguish these from actual dividends on the underlying assets which are assumed to be automatically reinvested in the fund.

where r is a risk-free rate, and dW_t is a Brownian motion under a new Girsanov-transformed measure Q. The stochastic differential equation (2) has a well known solution given by

$$V_t = V_0 e^{(r-\delta-\sigma^2/2)t+\sigma W_t^Q}. \tag{3}$$

In the following sections, we will consider different aspects of PPA and MAV death benefit options.

3.1 Simulations

We use a *hybrid* of Monte Carlo simulation procedures and tree methods. The Monte Carlo technique works by building simulated paths for the underlying security price which are computed recursively. Under the geometric Brownian motion assumption, the price paths are generated by solving equation (2) thus,

$$V_{i\Delta t} = V_{(i-1)\Delta t} e^{(r-\delta-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}z_i}, \qquad i = 1, 2, \cdots, n \tag{4}$$

where Δt is the length of the time interval between two successive prices and the z_i are random numbers drawn from a standard Gaussian distribution. The Monte Carlo simulation and generic tree building procedures we use are implemented in NumeriX Time Libraries provided by NumeriX, LLC (2000). The method used is similar to the stochastic mesh method (cf. Broadie and Glasserman (2000)).

3.2 Mortality

We shall use standard 1994 VA MGDB mortality data shown in Figure 1. The data shows the conditional death probability $p(x+i;1)$ for a male (and $\widetilde{p}(x+i;1)$, for a female) aged $x+i$ to die prior to age $x+i+1$ for $i = 0, 1, 2, \cdots$. For simplicity and convenience, future lifetime random variables can be expressed in continuous time by extrapolating a probability distribution function from the mortality table (cf. Milevsky and Posner (2000)).

4 Interest guarantee option

One of the death benefit options in the Polaris II VA is a *Purchase Payment Accumulation* (PPA) option. The PPA death benefit option is the greater of: the value of the contract at the time of death; or the invested premium compounded at a 4% annual growth rate until the date of death (3% growth rate if 70 or older at the time of contract issue); or the value of the contract on the

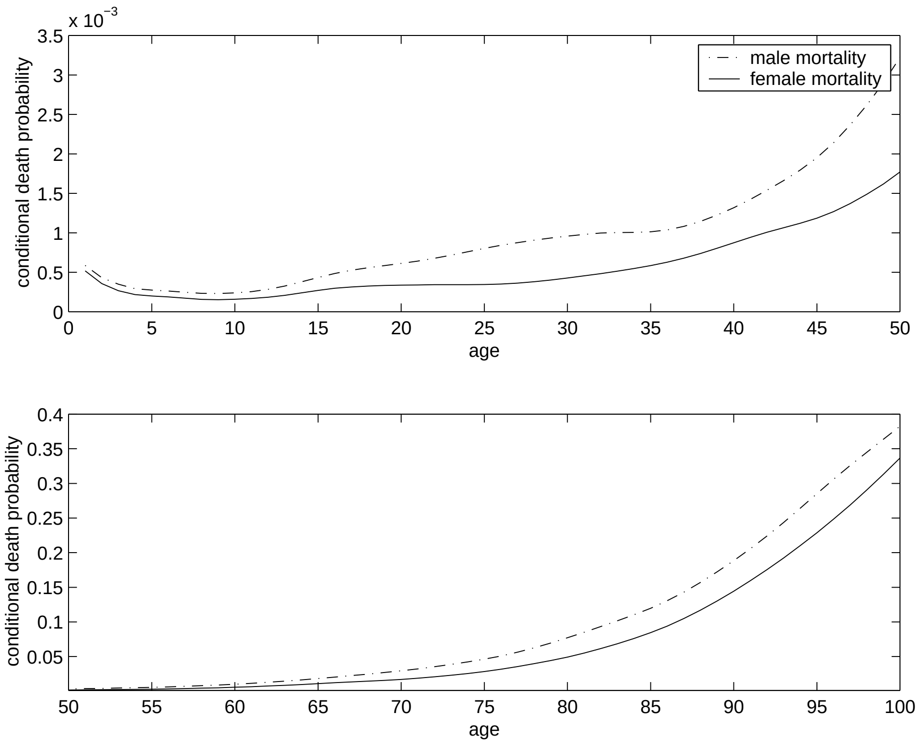

Figure 1: 1994 VA MGDB mortality data.

seventh anniversary compounded at a 4% annual growth rate until the date of death (3% growth rate if 70 or older at the time of contract issue). We can think of this option as a reset death benefit option since, as shown below, it has an embedded put option with a strike price reset option at the seventh anniversary date. However, we will call the death benefit options *interest guarantee options* because of the guranteed growth rate specified in the contract. The fund payoff of a contract containing a PPA death benefit option can be described as follows:

$$D_T^p = \begin{cases} (1-\alpha_t)V_t, & \text{for all } t, \quad \text{if lapsed} \\ \max(e^{g\tau}V_0, V_T), & t \leq 7, \quad \text{at death}\,(t=\tau) \\ \max(e^{g\tau}V_0, e^{g(\tau-7)}V_7^*, V_\tau), & t > 7, \quad \text{at death}\,(t=\tau) \end{cases} \tag{5}$$

where $\alpha_t V_t$ is the amount of the surrender charge at time t (in years). In Polaris II VA, one of the surrender charges (which we will use here) is a piecewise step (ratchet-down) function defined as

$$\alpha_t = \begin{cases} 8\% - 1\% \times \texttt{ceil}(t), & t \leq 7 \\ 0, & t > 7 \end{cases} \tag{6}$$

where $\texttt{ceil}(\cdot)$ rounds the arguments to the nearest intergers towards infinity.

4.1 PPA option with deterministic lifetime and no lapsation

If we further assume that the policyholder dies before the seventh anniversary of the contract and does not annuitize early, the fund payout is simply the death payment

$$D_T^p = \max(e^{gT}V_0, V_T) \tag{7}$$

which can conveniently be rewritten as

$$D_T^p = e^{gT}\max(V_0 - e^{-gT}V_T, 0) + V_T. \tag{8}$$

Thus,

$$D_T^p = e^{gT}P_E(T, V_0; g) + V_T \tag{9}$$

(cf. Grosen and Jorgensen (1997)). We note that $e^{gT}P_E(T, V_0; g)$ is a weighted European put option whose strike price is the initial value of the policy V_0 with an underlying asset V_t discounted by the guaranteed growth rate g. Alternatively, the exercise price of the put option increases at a rate g. Therefore, the present value D_0^p of the total claim that the investor has on the insurance company at time zero can be represented as

$$D_0^p = \mathbf{E}^Q\left\{e^{-rT}D_T^p\right\} = \mathbf{E}^Q\left\{e^{-(r-g)T}\max(V_0 - e^{-gT}V_T, 0) + e^{-rT}V_T\right\} \tag{10}$$

where $\mathbf{E}^Q\{\cdot\}$ is an expectation. Due to arbitrage considerations $0 < g < r$ at the inception of the contract. Making the classical assumption that V_t follows a geometric Brownian motion (2) and noting that

$$\mathbf{E}^Q\left\{e^{-rT}V_T\right\} = e^{-\delta T}V_0 \tag{11}$$

since

$$V_T = V_0 e^{(r-\delta-\frac{1}{2}\sigma^2)T+\sigma W}$$

we obtain a Black-Scholes type solution,

$$D_0^p = V_0\left[\mathrm{BS}(\widehat{r}, \delta, \sigma, T) + e^{-\delta T}\right] \equiv V_0([e^{-\widehat{r}T}\mathbf{N}(-d_2) - e^{-\delta T}\mathbf{N}(-d_1)] + e^{-\delta T}) \tag{12}$$

where

$$d_1 = \frac{(\widehat{r} - \delta + \sigma^2/2)\sqrt{T}}{\sigma}, \quad d_2 = d_1 - \sigma\sqrt{T} \quad \text{and} \quad \widehat{r} = r - g. \tag{13}$$

$\mathbf{N}(\cdot)$ is the cumulative probability function for a random variable that is standard normally distributed. Notice that the guaranteed growth rate lowers the discounting factor, thereby increasing the value of the GMDB option, thus the embedded put option price increases at the rate of g. If

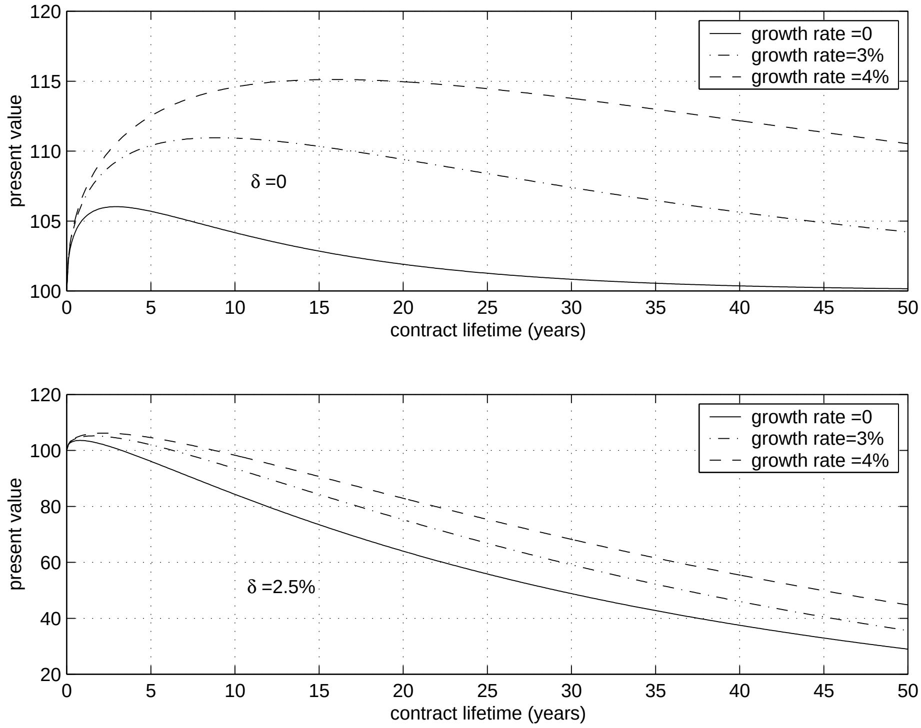

Figure 2: A comparison of a European style PPA option (without a 7^{th} year anniversary reset) and contract lifetime for $\sigma = 20\%$ and $r = 6\%$.

$g = 0$ we will call the option, one that only guarantees the invested principal, a *plain vanilla* GMDB *option*[4]. Thus a plain vanilla death benefit option has the payoff

$$D_T = \max(V_0, V_T) \tag{14}$$

and its present value is given by (12). The plain vanilla death benefit option guarantees a limited loss of $(1 - e^{-rT})V_0$ due to the time value of money. Figure 2 shows a comparison of the present value of a European style PPA option (without a 7^{th} year anniversary reset) with contract lifetime. The dividend payout δ represents payments charged to the account. These payments implicitly reduce the value of the fund under management. We note that the embedded put options actually increase in value with increasing dividends. In Appendix C, Table 3 shows (for $g = 4\%$ and $T = 3$) that the value of the put option with $\delta = 0$ is 4.1860 ($= 104.1860 - 100$) and in Table 4 (for $\delta = 2.5\%$) the value increases to 7.1776 ($= 99.9519 - e^{-0.025\times 3}100$).

Now, if we assume that the policyholder dies after the seventh anniversary, the payoff of the death

[4]The plain vanilla GMDB option defined here is not provided in Polaris II VA, but it is an artifact we use for comparison.

benefit option is

$$D_T^p = \max(e^{gT}V_0, e^{g(T-7)}V_7^*, V_T). \tag{15}$$

This death benefit option can be rewritten as

$$D_T^p = \begin{cases} \max(e^{g(T-7)}V_7^*, V_T), & V_0 < e^{-g7}V_7^* \\ \max(e^{gT}V_0, V_T), & V_0 \geq e^{-g7}V_7^*. \end{cases} \tag{16}$$

We derive a closed form solution for the present value of the option which has a *reset guarantee* by making the classical assumption that, in the risk neutral world, V_t follows the geometric Brownian motion (2). In the interest of brevity we take $\delta = g = 0$, otherwise the general derivation follows from the steps below. Therefore, we consider

$$D_T^p = V_T + \begin{cases} \max(V_7^* - V_T, 0), & V_0 < V_7^* \\ \max(V_0 - V_T, 0), & V_0 \geq V_7^* \end{cases}. \tag{17}$$

Based on the risk neutral valuation, the value at inception of the death benefit contract is given by

$$\begin{aligned} D_0^p &= V_0 \Big[\mathbf{N}(a_2(7)) \Big[e^{-\widehat{r}(T-7)}\mathbf{N}(-a_2(T-7)) - e^{-\delta(T-7)}\mathbf{N}(-a_1(T-7))\Big] \\ &+ e^{-\widehat{r}T}\mathbf{N}_2(-a_2(7), -a_2(T), \sqrt{7/T}) - \mathbf{N}_2(-a_1(7), -a_1(T), \sqrt{7/T}) + e^{-\delta T}\Big] \end{aligned} \tag{18}$$

where

$$a_1(x) = \frac{(\widehat{r} - \delta + \sigma^2/2)\sqrt{x}}{\sigma}, \quad a_2(x) = a_1(x) - \sigma\sqrt{x} \quad \text{and} \quad \widehat{r} = r - g.$$

$\mathbf{N}_2(\cdot)$ stands for the standard bivariate normal cumulative probability distribution function. Refer to the derivation in Appendix B. If $T \leq 7$, the explicit formula (18) for the death benefit reset option reduces to the explicit formula (12) for the death benefit without a reset option. Figure 3 shows a comparison of European style plain vanilla death benefit option values with the corresponding PPA and MAV options. The impact of the seventh anniversary reset option on the present value of the PPA option is large compared to the plain vanilla death benefit option. The exponential decay of the option value is due to the insurance charge and management fees. The low present values of the death benefit options suggests that for large T (i.e., young investors), policyholders *over-pay* for the embedded put options if they do not exercise the lapse option.

Example: Consider a contract with an initial value equal to 100 units of the account, that is, $V_0 = 100$ and calculate the fair value of a European style PPA option. The contract is assumed to be subject to the following market and deal parameters; $r = 6\%$, $\sigma = 10\%, 20\%$ or 30% and $\delta = 0\%$ or $\delta = 2.5\%$. Tables 3 and 4 in Appendix C show a summary of the results (written in parentheses). The results are very close to *exact values* obtained using the explicit solution (12) if the policyholder dies before the 7^{th} anniversary. For uniformity, simulation results rather than exact values are tabulated here, even for cases where closed form solutions exist, since other features of the GMDBs discussed below do not have closed form solutions.□

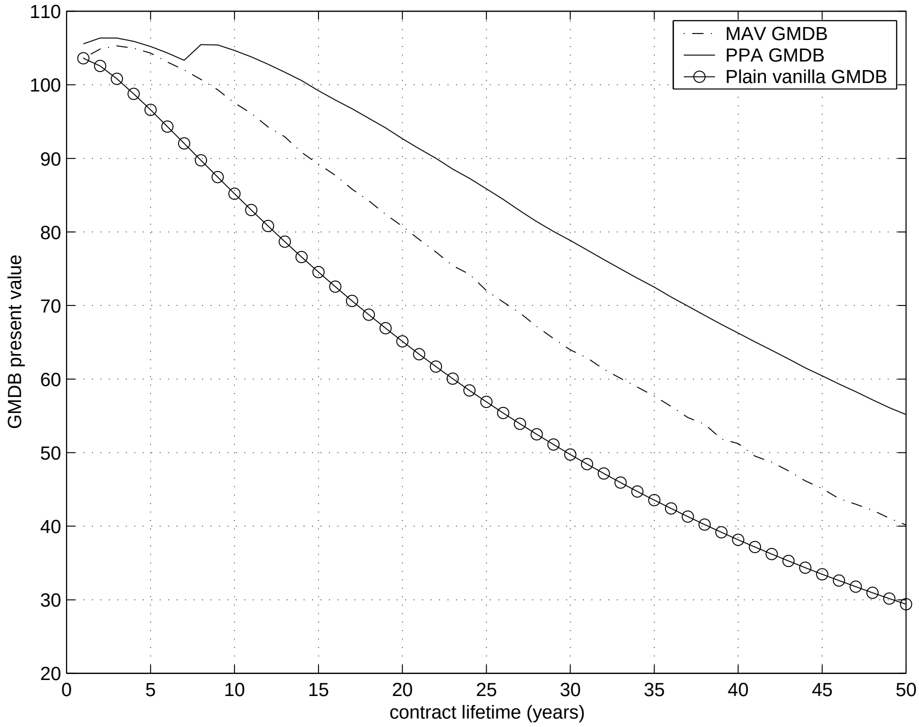

Figure 3: Present value of European style GMDB option for $\sigma = 20\%$ and $r = 6\%$.

4.2 Lapse option and surrender charges

The GMDB in Polaris II VA also gives the policyholder an American style option (if the lapse option is exercised rationally) to cash out the contract from the issuing company anytime he/she likes. This feature is known in the insurance business as a surrender option. We will call this option a lapsation option. Rational investors will lapse the contract when the embedded put options are *out-of-the-money*. Lapsation will be thought of as rational if: *i*) it is immediately followed by a reestablishment of a contract with a *better* guarantee, i.e., one that has an embedded option with a higher strike price, or *ii*) the contract policy value minus surrender charges is greater than the present value of the death benefit option. In the first case, the option to lapse is really an option to raise the strike price of the embedded put option when it out-of-the-money even after the lapsation charges are applied. We will focus on the second case where we do not worry about what the policyholder does with the payoff. Clearly the option to lapse raises the value of death benefit options and as such increases the exposure of the issuing company to the loss of fees. The death benefit payout at death time τ is given by

$$D_\tau^p = P_A(\tau, V_0, V_7^*; g) + V_\tau \tag{19}$$

The American style put option in (19) is computed by using the American Monte Carlo method. We warn readers that due to the nonlinearity of the early exercise features, equation (19) cannot be solved separably (as is the case with the European style version), instead we will solve (19) as a non-separable death benefit option. In order for a cohort of policyholders to remain invested in the contract and *fully pay* for the embedded put options they are provided with, the insurance company charges for surrendering the option. Thus, the death benefit is meant to be lapse supported. The value of the death benefit depends on how rational policyholders choose to lapse in a bear or bull market. If policyholders panic, they will lapse and the company will not issue embedded GMDB option payments but it will loose fees. By definition, the insurance contract shares cost among individuals in an underwriting class.

Example Revisited Tables 3 and 4 in Appendix C show a summary of results obtained based on a step-down surrender charge function when the reset GMDB option is valued considering the lapse option. The asterisk ($*$) indicates that it is not optimal to lapse the death benefit option.$\square$

4.3 PPA option with a stochastic lifetime

When the maturity date τ is stochastic and independent of V_t, the present value of the GMDB is given by an expectation with respect to both V_t and τ. The death benefit option present value is given by

$$D_0^p = \mathbf{E}_x \left\{ \mathbf{E}^Q \left\{ e^{-r\tau} D_\tau^p \mid \tau = t \right\} \right\}. \tag{20}$$

The first expectation is taken over the random maturity while the second is taken over the future asset price at a given realization of the random maturity. The present value of the stochastic maturity European put option is then given by

$$D_0^p = V_0 [\int_0^K f_x(t) \mathrm{BS}(\hat{r}, \sigma, t) dt + 1] \tag{21}$$

where K is the maximum term of the contract and $f_x(t)$ is the probability density function of the future lifetime random variable. Equation (21) establishes a relationship between random and fixed maturity GMDBs (cf. Milevsky and Posner (2001)). Parametric mortality functions may also be used. For a given issue age, a higher value of K increases the probability that the policyholder will die and use the embedded put option.

Alternatively, the lifetime variable can also be incorporated into a binomial tree valuation scheme by assuming a time-dependent probability of death at each node. Thus, the value at each node equals the value of the contract if the policyholder does not die times his/her survival probability plus the value if he/she dies times his/her probability of dying. Using conditional probabilities of dying provided in the 1994 VA MGDB mortality table, shown in Figure 1, and the deterministic maturity death benefit option values, we obtain American style GMDB present values with a stochastic maturity given by

$$D_0^p = \sum_{j=1}^{105-x} q(x;j) D_0^p(j) \tag{22}$$

for a male aged x at the inception of the contract, where $D_0^p(j)$ is the present value (with an early exercise option) of a GMDB with maturity j. The quantity $q(x;j)$ denotes the probability that a male aged x will rationally surrender or *involuntary* exercise the death benefit option in the j^{th} year. This probability is defined by

$$q(x;j) = \prod_{i=0}^{j-2} (1 - p(x+i;1)) \cdot p(x+j-1;1) \tag{23}$$

for all $j = 2, 3, \cdots$, and

$$q(x;1) = p(x;1).$$

We note that

$$\sum_{j=1}^{105-x} q(x;j) = 1, \tag{24}$$

assuming that policyholders do not live beyond 105 years. Surrender charges are automatically incorporated into the valuation scheme. Similarly, for a female aged x, we substitute p and q in the above formulas with $\widetilde{p}$ and $\widetilde{q}$ corresponding to female conditional death probabilities. Table 1 below summarises the results for for different sexes and ages of policyholders.

4.4 Irrational lapsation

Assuming that there is a 5% irrational lapsation rate denoted by l, an investor aged x will lapse the contract in the first year with probability l or die with probability $p(x;1)$. (Lapsation statistics on variable annuities shows that between 5% and 8% of the policyholders exercise the lapse option every year.) Thus, the probability of non-optimally (or irrationally) surrendering the GMDB option or exercising it (involuntary) in the first year is $q_l(x;1) = p(x;1) + l$. In general, the probability that a male aged x will irrationally surrender or exercise the GMDB option in the j^{th} year is given by

$$q_l(x;j) = \prod_{i=0}^{j-2}(1 - p(x+i;1) - l) \cdot (p(x+j-1;1) + l) \tag{25}$$

for $j = 2, 3, \cdots$, and

$$q_l(x;1) = p(x;1) + l.$$

Thus, the GMDB present value of a male aged x assuming an annual lapsation rate of $l = 5\%$ is given by

$$D_0^p = \sum_{j=2}^{105-x} \frac{q_l(x;j)}{p(x+j-1;1)+l}\left[p(x+j-1;1)D_0^p(j) + l_{j-1}^* e^{-\delta(j-1)}V_0\right] \tag{26}$$

and

$$D_0^p = p(x;1)D_0^p(1) + e^{-\delta} l_0^* V_0$$

where

$$l_j^* = l \cdot (1 - \max(7-j, 0)\%) \tag{27}$$

has been adjusted for surrender charges in the first seven years and $D_0^p(j)$ is the present value of a European style death benefit option with maturity j. Similar results directly follow for female policyholders. Unlike in (22), the surrender charges in equation (27) are incorporated in $q_l(x+j)$ and not implemented in the valuation schemes since $D_0^p(j)$ assumes that investors die in the j^{th} year and have no early exercise feature. Table 1 shows a summary of results ⟨in triangular parenthesis⟩ considering irrational lapsation.

4.5 Perpetual plain vanilla death benefit option

The PPA option we have discussed can be generalized as providing the payoff,

$$D_T = P_A + V_T \tag{28}$$

at time T, where P_A is an American style put option that has deterministic maturity T with an appropriate strike price. It is well known that $P_A \to 0$ as $T \to \infty$ (i.e., the put option becomes worthless with increasing time to maturity). Considering dividend payouts, the value of the underlying asset $V_T \to 0$ as $T \to \infty$, since $\mathbf{E}^Q\{V_T\} = e^{-\delta T} V_0$. Therefore $D_T \to 0$, which implies that European style death benefit options are not valuable for young investors.

5 Market guarantee option

The *maximum anniversary value* (MAV) option guarantees the policyholder the greater of either: (i) the value of the contract at the time of death; or (ii) the invested premium; or the maximum anniversary value of on any contract anniversary prior to the policyholders' 81^{st} birthday. Clearly, the MAV option has a lookback feature. The fund payoff of a contract containing a MAV death benefit is given by

$$D_t^m = \begin{cases} (1-\alpha_t)V_t & \text{if lapsed} \\ \max(V_0, \mathrm{M}_\tau^*, V_\tau) & \text{at death}(t=\tau) \end{cases} \tag{29}$$

where α_t is defined by (6) and M_τ^* is the maximum of the anniversary values up to time τ.

5.1 MAV option with deterministic lifetime and no lapsation

If we further assume that $V_0 < \mathrm{M}_T^*$, which is a reasonable assumption for sufficiently large T, or define $\mathrm{M}_0^* = V_0$, then

$$D_T^m = \max(\mathrm{M}_T^* - V_T, 0) + V_T. \tag{30}$$

The first term on the right side of the above equation is a *floating strike European lookback put option* whose discretely-sampled maximum coincides with the policy anniversary dates. If we further assume, the more valuable and computationally less challenging, continuously sampled maximum, then

$$D_T^m = \mathrm{M}_T. \tag{31}$$

We define the *maximum* stochastic process

$$\mathrm{M} = \{\mathrm{M}_t,\, t \geq 0\} = \max_{0 \leq s \leq t} V_s \tag{32}$$

where $\mathrm{M}_0 = V_0$ is the starting value for the maximum process. If the policyholder chooses to exercise the option at time $\hat{t} \geq 0$, he receives the payoff $\mathrm{M}_{\hat{t}}$. We observe that no exercise price is required. The embedded put option in (31), without dividends, has the closed form solution

$$\mathrm{GSG}(\mathrm{M}, V_0, r, \sigma, T) = \mathrm{M}e^{-rT}[\mathbf{N}(b_1) - \frac{\sigma^2}{2r}e^{\eta}\mathbf{N}(-b_3)] + V_0[\frac{\sigma^2}{2r}\mathbf{N}(-b_2) - \mathbf{N}(b_2)] \tag{33}$$

where

$$
\begin{aligned}
b_1 &= \frac{\ln(\mathrm{M}/V_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\
b_2 &= b_1 - \sigma\sqrt{T} \\
b_3 &= \frac{\ln(\mathrm{M}/V_0) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \\
\eta &= \frac{2(r - \sigma^2/2)\ln(\mathrm{M}/V_0)}{\sigma^2}
\end{aligned}
$$

and M is the maximum investment value achieved to date (cf. Goldman, Sosin and Gatto (1979)).

As for the MAV death benefit option, under the continuously sampled maximum assumption, we obtain the present value of the death benefit option as

$$D_0^m = \mathrm{GSG}(\mathrm{M}, V_0, r, \sigma, T) + V_0 \tag{34}$$

where $\delta = 0$. The present value of the death benefit option payoff for the more practical discretely-sampled maximum can be approximated by simulation and tree methods or by continuous adjustments of the discretely-sampled maximum (cf. Broadie *et al* (1998)).

Example Revisited: Tables 5 and 6 in Appendix B show a summary of European style MAV present values (shown in parentheses) considering step-down surrender charges. The asterisk ($*$) indicates that it is not optimal to lapse the GMDB option.$\Box$

5.2 Lapse option and surrender charges

Example Revisited: Tables 5 and 6 in Appendix B show a summary of a American style MAV present values considering step-down surrender charges. The asterisk ($*$) indicates that it is not optimal to lapse the GMDB option.$\Box$

Figure 4 shows a comparison of American style GMDBs. The present values approach a limit of approximately 93 for increasing contract years. This limit is related to surrender charges.

5.3 MAV option with a stochastic lifetime

When τ is stochastic and independent of V_t, the present value of the death benefit option is given by an expectation with respect to both V_t and τ. The death benefit option present value is given by

$$D_0^p = \mathbf{E}_x\left\{\mathbf{E}^Q\left\{e^{-r\tau}D_\tau^p \mid \tau = t\right\}\right\}$$

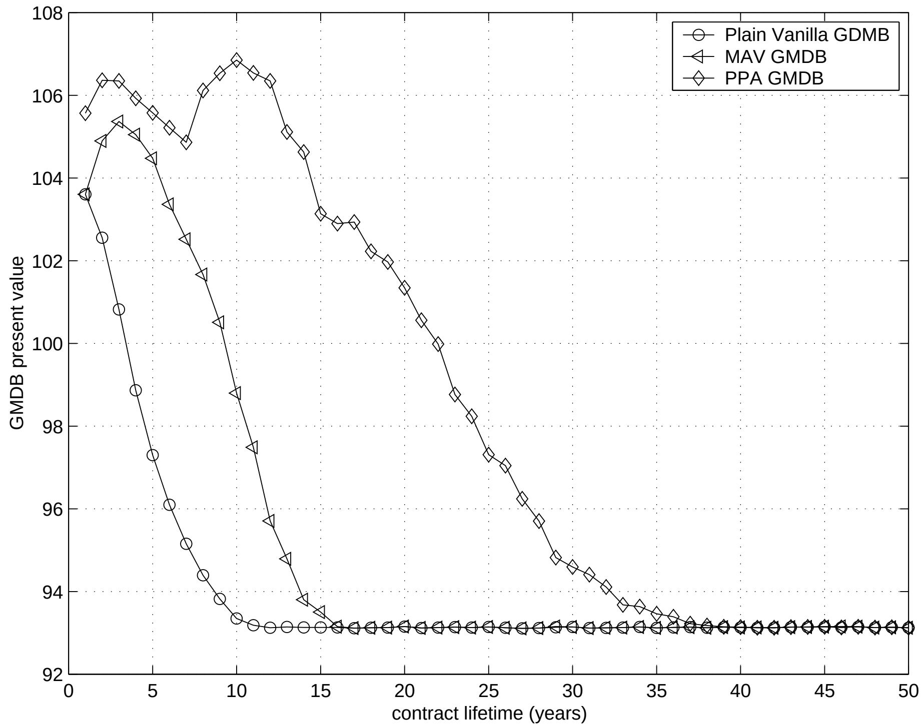

Figure 4: The American style death benefit option with insurance charges and management fees for $\sigma = 20\%$, $\delta = 2.5\%$ and $r = 6\%$.

The present value of the stochastic maturity European type MAV option, with continuously sampled maximum, is then given by

$$D_0^m = \int_0^K f_x(t)[\mathrm{GSG}(\mathrm{M}, V_0, r, \sigma, t) + V_0]dt \tag{35}$$

where K is the maximum term of the contract and $f_x(t)$ is the probability density function of the future lifetime random variable. Using the mortality data shown in Figure 1 and already computed MAV options with deterministic maturity, we obtain present values for different sexes and ages shown in Table 2 below. We assume that the policyholder annuitizes at their 90^{th} birthday since after that the MAV option does not have a death benefit option.

5.4 Perpetual MAV option

The perpetual American style MAV option is related to a Russian option which at any time chosen by the holder as the stopping time, pays out the maximum (defined by equation (32)) realised asset price M up to that date. Thus, the Russian option value provides an upper bound for the value of the perpetual MAV option. When the dividend yield is zero, it is never optimal to hold a Russian option. If the holder of the option chooses to exercise the option at $\tau^* \geq 0$, he/she receives the payoff M^*_τ. The fair price for the Russian option, which provides an upper bound for the perpetual MAV, is given by

$$D_0 = \frac{V_0}{\gamma_1 - \gamma_2}\left[\gamma_2 \alpha^{\gamma_1} - \gamma_1 \alpha^{\gamma_2}\right] \tag{36}$$

where

$$\alpha = \left(\frac{\gamma_2(\gamma_1 - 1)}{\gamma_1(\gamma_2 - 1)}\right)^{\frac{1}{\gamma_2 - \gamma_1}} \tag{37}$$

and

$$\gamma_{1,2} = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \mp \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{38}$$

with $\gamma_1 < 0 < 1 < \gamma_2$, and the optimal stopping time t^* is the first time t for which

$$t^* = \inf\{t \geq 0 : \alpha V_t \leq \mathrm{M}_t\} \tag{39}$$

(cf. Basso and Pianca (2001)).

The present values of perpetual MAV option, assuming continuously sampled maximum, are summarized in Table 6. Clearly the results show that continuous sampling makes the MAV option very valuable. When the dividend yield is zero i.e., $\delta = 0$, it is clearly never optimal to lapse the MAV option.

6 Concluding remarks

In this paper we have proposed a methodology for pricing GMDBs using the well-developed capital markets valuation framework by taking into account mortality risk, lapse option and surrender charges. The mortality risk is implemented discretely using the 1994 VA GMDB mortality data. Our results indicate that the lapse option significantly increases the GMDB option value. Both death benefit options are much more valuable for middle aged to senior investors compared to younger investors because of M&E fees. Younger investors are less likely going to die (and exercise the embedded option) when the GMDBs are most valuable. Our results show that the Polaris II VAs are very sensitive to M&E charges. The valuation strategy we have presented could be used to value many insurance products such as equity-linked life insurance policies (ELLIPs) with interests rate guarantees, guaranteed investment contracts (GICs), guaranteed minimum accumulation benefits (GMABs) and guaranteed minimum income benefits (GIMBs). The valuation strategy can also be used to influence insurance product design.

Further research will expand on the methodology outlined in this paper to deal with more complex realistic market parameters such as stochastic interest rate and volatility. Complex lapsation issues also need to be addressed. Finally, alternative hedging strategies for the death benefits will be investigated.

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Appendix A: Product Description

(a) *Purchase Payment Accumulation Option* (PPA) *Option*:

The death benefit is the greater of:

1. the value of the contract at the time the policyholders' death; or
2. total payment less any withdrawals (and any fees or charges applicable to such withdrawals), compound at a 4% annual growth rate until the date of the policyholders' death (3% growth rate if 70 or older at the time of contract issue plus any Purchase Payment less withdrawals recorded after the policyholders' death (and any fees and charges applicable to such withdrawals); or
3. the value of the contract on the seventh contract anniversary, plus any Purchase Payments and less any withdrawals (and any fees or charges applicable to such withdrawals), since the seventh contract anniversary, all compounded at a 4% annual growth rate until the time of policyholders' death (3% growth rate if 70 or older at the time of contract issue) plus any Purchase Payments less withdrawals recorded after the date of death (and any fees or charges applicable to such withdrawals).

(b) *Maximum Anniversary Value* (MAV) *Option*:

The death benefit is the greater of:

1. the value of the contract at the time the policyholders' death; or
2. total payment less any withdrawals (and any fees or charges applicable to such withdrawals); or
3. the maximum anniversary value on any contract anniversary prior to policyholders' 81^{st} birthday. The anniversary value equals the value of the contract on its anniversary plus any Purchase Payments and less any withdrawals (and any fees or charges applicable to such withdrawals), since that anniversary.

If the policyholder is aged 90 or older at the time of death, the MAV death benefit option will be equal to value of the contract at the time of the policyholders' death. Accordingly, the policyholders do not get the advantage of the MAV option if:

- they are over age 80 at the time of contract issue, or
- they are aged 90 or older at the time of their death.

Fees and charges:

1. An annual insurance charge of 1.52% applies for the value of contract invested in Variable Portfolios.
2. A withdrawal charges applies against each Purchase Payment put into the contract. The withdrawal charge percentage declines (by a percentage point) each year a Purchase Payment is in the contract. After a Purchase Payment has been in the contract for 7 complete years (or 9 years if the policyholder elects to participate in the Principal Rewards Program) no withdrawal charge applies.
3. A maintenance fee is subtracted from the policyholders' account once per year. A $35 contract maintenance fee ($30 in North Dakota) from the policyholders' account value on the contract anniversary. If the policyholder withdraws their entire contract value, a maintenance fee is deducted from that withdrawal. If the contract value is $50,000 or more on the contract anniversary date, the charge will be waived.

Appendix B: Derivation of the PPA Solution

Based on the risk neutral valuation, the value at inception of the death benefit contract is given by;

$$\begin{aligned} D_0^p &= e^{-rT}\mathbf{E}^Q\left\{D_T^p\right\} \\ &= e^{-rT}\mathbf{E}^Q\left\{(V_7^* - V_T)\mathcal{I}_A\right\} + e^{-rT}\mathbf{E}^Q\left\{(V_0 - V_T)\mathcal{I}_B\right\} + e^{-rT}\mathbf{E}^Q\left\{V_T\right\} \\ &\equiv D_0^{pA} + D_0^{pB} + V_0 \end{aligned} \tag{40}$$

where D_T^p is given by (), $\mathcal{I}_{(\cdot)}$ is an indicator function and A and B are the events;

$$A \Leftrightarrow \{V_7^* > V_0\,;\, V_T < V_7^*\}$$

and

$$B \Leftrightarrow \{V_7^* \le V_0\,;\, V_T < V_0\}\,.$$

The first expectation can easily be calculated by observing that given the value of the death benefit option at the seventh anniversary V_7^*, the expected value of the option when the event A occurs at the seventh anniversary is given by

$$\begin{aligned} D_7^{pA} &= e^{-r(T-7)}\mathbf{E}^Q\left\{\max\left(V_7^* - V_T, 0\right)\right\} \\ &= V_7^*\left[e^{-r(T-7)}\mathbf{N}(-d_2(T-7)) - \mathbf{N}(-d_1(T-7))\right] \end{aligned} \tag{41}$$

where

$$d_1(x) = \frac{(r+\sigma^2/2)\sqrt{x}}{\sigma} \quad \text{and} \quad d_2(x) = d_1(x) - \sigma\sqrt{x}.$$

Since D_7^{pA} is independent of V_7^*, we compute the final expectation for the event A by multiplying by the probability $Pr(V_0 < V_7^*)$ which implies

$$Pr(V_0 < V_0 e^{\mu 7 + \sigma W_7}) = Pr(-\sigma W_7 < \mu 7) = \mathbf{N}(d_2(7)) \tag{42}$$

where $\mu = r - \sigma^2/2$. Therefore,

$$\begin{aligned} D_0^{pA} &= e^{-r7}\mathbf{E}^Q\left\{\mathbf{N}(d_2(7))D_7^{pA}\right\} \\ &= V_0\,\mathbf{N}(d_2(7))\left[\mathbf{N}(d_1(T-7)) - e^{-r(T-7)}\mathbf{N}(d_2(T-7))\right] \end{aligned} \tag{43}$$

since $e^{-r7}\mathbf{E}^Q\{V_7^*\} = V_0$. For the event B, we have

$$\begin{aligned} \mathbf{E}^Q\left\{\mathcal{I}_B\right\} &= Pr(V_7^* \le V_0\,,\, V_T < V_0) \\ &= Pr(V_0 e^{\mu 7 + \sigma W_7} \le V_0\,,\, V_0 e^{\mu T + \sigma W_T} < V_0). \end{aligned} \tag{44}$$

Thus,

$$Pr(\sigma W_7 \le -\mu 7\,,\, \sigma W_T < -\mu T) = \mathbf{N}_2(-d_2(7), -d_2(T), \sqrt{7/T}) \tag{45}$$

where $\mathbf{N}_2(\cdot)$ stands for the standard bivariate normal cumulative probability distribution function,

$$\mathbf{N}_2(\xi, \eta, \rho) \equiv \frac{1}{2\pi(1-\rho^2)} \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} e^{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}} dxdy \tag{46}$$

where ρ is the correlation coefficient. So,

$$\mathbf{E}^Q \left\{ V_T \mathcal{I}_B \right\} = \mathbf{E}^Q \left\{ V_0 e^{\mu T + \sigma W_T} \mathcal{I}_B \right\} \tag{47}$$

We define $\mathcal{Q}$ by setting

$$\frac{d\mathcal{Q}}{dQ} = e^{\sigma W_T - \frac{1}{2}\sigma^2 T} \tag{48}$$

Girsanov's theorem implies that $\widehat{W}_t = W_t - \sigma t$ follows a standard Brownian motion under the probability measure $\mathcal{Q}$, therefore,

$$e^{rT} \mathbf{E}^Q \left\{ e^{\sigma W_T - \frac{1}{2}\sigma^2 T} \mathcal{I}_B \right\} \equiv e^{rT} \mathbf{E}^{\mathcal{Q}} \left\{ \mathcal{I}_B \right\}. \tag{49}$$

Following the previous steps, we obtain

$$\begin{aligned} &\mathbf{E}^{\mathcal{Q}} \left\{ V_0 e^{\mu 7 + \sigma \widehat{W}_7 + \sigma^2 7} \le V_0\,;\, V_0 e^{\mu T + \sigma \widehat{W}_T + \sigma^2 T} < V_0 \right\} \\ &= \mathbf{N}_2(-d_1(7), -d_1(T), \sqrt{7/T}) \end{aligned} \tag{50}$$

Thus,

$$\begin{aligned} D_0^{p_B} &= e^{-rT} \mathbf{E}^Q \left\{ (V_0 - V_T) \mathcal{I}_B \right\} \qquad (51) \\ &= V_0 [e^{-rT} \mathbf{N}_2(-d_2(7), -d_2(T), \sqrt{7/T}) - \mathbf{N}_2(-d_1(7), -d_1(T), \sqrt{7/T})]. \end{aligned}$$

Finally, putting all the above results together, we obtain the present value of the seventh anniversary reset death benefit option

$$\begin{aligned} D_0^p &= V_0 \Big[\mathbf{N}(d_2(7)) \Big[e^{-r(T-7)} \mathbf{N}(-d_2(T-7)) - \mathbf{N}(-d_1(T-7)) \Big] \qquad (52) \\ &+ e^{-rT} \mathbf{N}_2(-d_2(7), -d_2(T), \sqrt{7/T}) - \mathbf{N}_2(-d_1(7), -d_1(T), \sqrt{7/T}) + 1 \Big]. \end{aligned}$$

Similarly, with dividend payments and a growth rate, we obtain

$$\begin{aligned} D_0^p &= V_0 \Big[\mathbf{N}(a_2(7)) \Big[e^{-\widehat{r}(T-7)} \mathbf{N}(-a_2(T-7)) - e^{-\delta(T-7)} \mathbf{N}(-a_1(T-7)) \Big] \qquad (53) \\ &+ e^{-\widehat{r}T} \mathbf{N}_2(-a_2(7), -a_2(T), \sqrt{7/T}) - \mathbf{N}_2(-a_1(7), -a_1(T), \sqrt{7/T}) + e^{-\delta T} \Big] \end{aligned}$$

where

$$a_1(x) = \frac{(\widehat{r} - \delta + \sigma^2/2)\sqrt{x}}{\sigma} \quad \text{and} \quad a_2(x) = a_1(x) - \sigma\sqrt{x}.$$

Appendix C: Summary of Results

age $\triangleright$			55	60	65	70	75
Male	$\delta = 2.5\%$	$\sigma = 20\%$	98.7030	100.5875	102.4030	99.3962	100.9581
			⟨83.6815⟩	⟨87.3824⟩	⟨91.1713⟩	⟨85.5780⟩	⟨87.1286⟩
			(86.7559)	(91.8422)	(96.2315)	(91.2610)	(92.6950)
		$\sigma = 30\%$	117.5104	125.5134	127.0612	113.7303	113.8410
			⟨88.0719⟩	⟨92.5118⟩	⟨96.9490⟩	⟨83.5354⟩	⟨86.9441⟩
			(99.9563)	(104.9284)	(108.8903)	(93.8603)	(97.6883)
Female	$\delta = 2.5\%$	$\sigma = 20\%$	96.9686	98.7374	100.7219	97.8598	99.6249
			⟨80.8301⟩	⟨84.4413⟩	⟨88.4218⟩	⟨85.7375⟩	⟨87.2058⟩
			(81.7166)	(87.2940)	(92.4041)	(90.2503)	(91.7956)
		$\sigma = 30\%$	114.6783	118.0423	119.9269	113.4936	113.7261
			⟨84.5957⟩	⟨89.0189⟩	⟨93.8335⟩	⟨79.9668⟩	⟨84.3720⟩
			(94.7843)	(100.5412)	(105.5652)	(89.8503)	(94.7959)
Male	$\delta = 2\%$	$\sigma = 20\%$	102.1590	104.0559	105.6319	101.5943	102.9152
			⟨87.1905⟩	⟨90.5553⟩	⟨93.9724⟩	⟨91.1800⟩	⟨94.2820⟩
			(91.2817)	(95.8665)	(99.7200)	(97.0480)	(100.2615)
		$\sigma = 30\%$	119.9274	123.9425	129.2617	118.3564	119.6957
			⟨91.0424⟩	⟨95.4810⟩	⟨99.9738⟩	⟨96.5472⟩	⟨98.5230⟩
			(101.0197)	(107.5791)	(112.3303)	(108.4844)	(110.7069)
Female	$\delta = 2\%$	$\sigma = 20\%$	100.2258	102.2774	104.2520	100.8547	101.8547
			⟨84.6010⟩	⟨87.9164⟩	⟨91.5565⟩	⟨88.9010⟩	⟨91.0670⟩
			(86.6530)	(91.7826)	(96.3800)	(93.5823)	(95.8629)
		$\sigma = 30\%$	113.4389	121.5093	124.8407	115.5179	117.4975
			⟨87.3867⟩	⟨91.8681⟩	⟨97.2086⟩	⟨94.0552⟩	⟨97.1880⟩
			(94.0599)	(101.9184)	(108.6804)	(105.6845)	(109.2304)

Table 1: Present values of PPA with stochastic maturity for American type death benefit option with $r = 6\%$. (European type death benefit option present values are in parentheses.) ⟨ Irrational lapsation present values are in triangular parentheses. ⟩

age ▷			55	60	65	70	75
Male	$\delta = 2.5\%$	$\sigma = 20\%$	94.3208 ⟨80.2057⟩ (75.2928)	95.1073 ⟨83.7263⟩ (81.7271)	96.1560 ⟨87.5147⟩ (87.8268)	97.5618 ⟨91.4164⟩ (93.2336)	100.4126 ⟨95.3998⟩ (97.8262)
		$\sigma = 30\%$	105.2010 ⟨86.1462⟩ (93.2848)	107.8749 ⟨90.6629⟩ (99.4495)	110.0316 ⟨95.3435⟩ (105.0306)	112.7163 ⟨99.9007⟩ (109.3480)	114.2086 ⟨104.2567⟩ (112.4550)
Female	$\delta = 2.5\%$	$\sigma = 20\%$	93.8251 ⟨77.5434⟩ (70.1107)	94.3420 ⟨80.7965⟩ (76.7889)	95.0534 ⟨84.5693⟩ (83.5855)	96.1521 ⟨88.8374⟩ (90.0170)	98.9025 ⟨93.5456⟩ (95.7427)
		$\sigma = 30\%$	101.9377 ⟨82.6483⟩ (88.1776)	106.2933 ⟨86.9690⟩ (94.8304)	108.0558 ⟨91.8770⟩ (101.5862)	111.6858 ⟨97.1318⟩ (107.2200)	113.7806 ⟨102.6067⟩ (111.6295)
Male	$\delta = 2\%$	$\sigma = 20\%$	95.1989 ⟨84.2647⟩ (81.8253)	96.3358 ⟨87.4466⟩ (87.3982)	98.6930 ⟨90.8334⟩ (92.6051)	100.5343 ⟨94.2846⟩ (97.1222)	102.7832 ⟨97.7485⟩ (100.8019)
		$\sigma = 30\%$	110.9807 ⟨90.4765⟩ (100.8718)	112.7460 ⟨94.6336⟩ (105.9145)	115.5282 ⟨98.8918⟩ (110.4263)	117.1695 ⟨102.9621⟩ (113.6741)	117.6693 ⟨106.7489⟩ (115.7018)
Female	$\delta = 2\%$	$\sigma = 20\%$	94.4344 ⟨81.8626⟩ (77.3077)	95.2884 ⟨84.8421⟩ (83.1610)	98.2280 ⟨88.2684⟩ (89.0671)	99.9031 ⟨92.1039⟩ (94.5423)	102.4440 ⟨96.2493⟩ (99.2105)
		$\sigma = 30\%$	109.3570 ⟨87.2536⟩ (96.6330)	110.5135 ⟨91.2829⟩ (102.1421)	114.4235 ⟨95.8380⟩ (107.8274)	116.9644 ⟨100.6296⟩ (112.2999)	117.5102 ⟨105.4854⟩ (115.4369)

Table 2: The present values of MAV with a stochastic maturity for American type death benefit option for $r = 6\%$. (European type death benefit option present values are in parentheses.) ⟨ Irrational lapsation present values are in triangular parentheses. ⟩

T	$\sigma = 10\%$	$\sigma = 20\%$	$\sigma = 30\%$
1	103.0370*	106.9370*	110.8450*
	(103.0370)	(106.9370)	(110.8450)
2	103.7860*	109.1770*	114.5910*
	(103.7860)	(109.1770)	(114.5910)
3	104.1860*	110.6410*	117.1380
	(104.1860)	(110.6410)	(117.1370)
4	104.4180*	111.7050*	119.0690
	(104.4180)	(111.7050)	(119.0540)
5	104.5510*	112.5410	120.7740
	(104.5510)	(112.5160)	(120.5660)
6	104.6250	113.2360	122.6360
	(104.6190)	(113.1490)	(121.7900)
7	104.6440	113.7990	123.5310
	(104.6410)	(113.6500)	(122.7970)
8	106.9640	118.9840	132.0190
	(106.6390)	(118.2390)	(130.3450)
9	107.5840	121.0110	136.2050
	(107.0370)	(119.7880)	(133.3310)
10	107.8530	122.5170	140.7310
	(107.0920)	(120.5650)	(134.9580)
20	108.1910	130.7540	166.1810
	(105.9940)	(122.0570)	(139.9540)
30	107.9880	132.0070	169.3547
	(104.4530)	(120.1400)	(138.1540)
40	108.5080	129.2340	163.l670
	(103.1860)	(117.9890)	(135.3170)
∞	108.1920†	125.0000†	141.0011†
	(100.0000)	(100.0000)	(100.0000)

Table 3: The present values of PPA option without insurance charges and management fees. (European type GMDB present values are in parentheses.) The dagger (†) shows results obtained without considering the 7^{th} year anniversary reset or surrender charges.

T	$\sigma = 10\%$		$\sigma = 20\%$		$\sigma = 30\%$	
	$\delta = 2\%$	$\delta = 2.5\%$	$\delta = 2\%$	$\delta = 2.5\%$	$\delta = 2\%$	$\delta = 2.5\%$
1	101.9290* (101.9290)	101.6800* (101.6800)	105.8290* (105.8290)	105.5680* (105.5680)	109.7110* (109.7110)	109.4400* (109.4400)
2	101.4960* (101.4960)	101.0040* (101.0040)	106.8880* (106.8880)	106.3630* (106.3630)	112.3110 (112.2300)	111.6780 (111.6760)
3	100.6770* (100.6770)	99.9519* (99.9519)	107.1330* (107.1330)	106.3470* (106.3470)	113.5010 (113.4980)	112.6660 (112.6610)
4	99.6681 (99.6663)	98.7270 (98.7160)	106.9590 (106.9530)	105.9270 (105.9120)	114.1310 (114.1060)	113.0440 (112.9980)
5	98.5905 (98.5405)	97.6807 (97.3736)	106.6530 (106.5040)	105.5750 (105.2140)	114.9230 (114.2870)	113.7450 (112.8930)
6	97.7571 (97.3397)	96.8409 (95.9650)	106.4150 (105.8680)	105.2190 (104.3340)	115.4180 (114.1670)	114.0260 (112.4990)
7	97.1182 (96.0879)	96.1457 (94.5136)	106.1390 (105.0920)	104.8620 (103.3230)	115.5340 (113.8180)	114.0890 (111.8840)
8	96.9510 (96.3365)	95.7543 (94.4467)	108.3240 (107.6560)	106.1190 (105.4610)	120.3200 (118.9130)	117.8330 (116.4660)
9	96.3852 (95.5157)	95.1032 (93.4236)	109.0250 (107.8540)	106.5330 (105.4120)	122.7190 (120.3410)	119.8200 (117.5560)
10	95.7821 (94.4145)	94.1032 (92.1424)	109.5560 (107.3600)	106.8520 (104.6830)	125.2730 (120.4980)	121.9480 (117.4530)
20	93.0628 (81.7387)	93.0553 (78.2166)	105.4890 (96.9936)	101.3460 (92.6714)	131.1010 (112.3970)	124.0430 (107.2890)
30	93.0573 (69.3919)	93.0501 (65.1646)	98.3791 (84.2423)	94.5952 (78.8493)	124.1020 (97.8854)	115.9490 (92.3701)
40	93.0534 (58.4667)	93.0478 (53.9250)	95.6209 (72.1701)	93.1396 (66.2560)	112.8765 (85.0332)	99.7819 (78.2069)

Table 4: The present values of PPA option with insurance charges and management fees. (European type GMDB present values are in parentheses.)

T	$\sigma = 10\%$	$\sigma = 20\%$	$\sigma = 30\%$
1	101.6360* (101.6360)	105.1670* (105.1670)	108.8960* (108.8960)
2	102.3580* (102.3580)	108.1890 (108.1710)	114.6200 (114.5640)
3	102.8240* (102.8240)	110.4920 (110.4470)	119.3190 (119.1460)
4	103.0580 (103.0510)	112.0580 (112.0040)	122.6960 (122.5670)
5	103.2430 (103.2310)	113.4310 (113.3220)	125.9760 (125.5850)
6	103.2890 (103.2600)	114.3680 (114.1500)	128.2460 (127.6160)
7	103.4820 (103.4150)	115.5540 (115.1450)	130.8020 (129.7370)
8	103.8450 (103.4050)	116.6510 (115.7830)	133.4200 (131.8400)
9	104.1750 (103.6060)	117.6050 (116.4800)	135.5790 (133.6290)
10	103.9690 (103.4430)	117.7520 (116.5760)	137.5030 (134.4850)
20	104.3950 (103.6090)	121.762 (119.6320)	155.1910 (135.8010)
30	104.7450 (103.6680)	121.7940 (119.5610)	155.5520 (138.0750)
40	105.1030 (103.6140)	128.3660 (120.5730)	207.5310 (144.9700)
∞	$\infty^\dagger$ $(\infty^\dagger)$	$\infty^\dagger$ $(\infty^\dagger)$	$\infty^\dagger$ $(\infty^\dagger)$

Table 5: The present values of MAV option without insurance charges and management fees. (European type MAV option present values are in parentheses.) The dagger (†) indicates that we are assuming that the strike price of the embedded put option of the GMDB option is reset to maximum anniversary value every year forever.

T	$\sigma = 10\%$		$\sigma = 20\%$		$\sigma = 30\%$	
	$\delta = 2\%$	$\delta = 2.5\%$	$\delta = 2\%$	$\delta = 2.5\%$	$\delta = 2\%$	$\delta = 2.5\%$
1	100.2320 (100.2320)	99.9075* (99.9075)	103.6060* (103.9060)	103.9060 (103.6060)	107.6610 (107.6610)	107.3630* (107.3630)
2	99.4133 (99.4133)	98.7399 (98.7392)	105.5240 (105.4970)	104.8990 (104.8690)	111.9890 (111.9150)	111.3670 (111.2910)
3	98.2553 (98.2473)	97.2278 (97.2147)	106.3240 (106.2520)	105.3680 (105.2870)	115.1510 (114.9430)	114.1730 (113.9550)
4	96.8382 (96.8278)	95.4478 (95.4312)	106.3490 (106.2740)	105.0500 (104.9720)	116.9310 (116.7100)	115.5940 (115.3570)
5	95.3552 (95.2992)	93.6467 (93.5527)	106.105 (105.9550)	104.4770 (104.3040)	118.4630 (118.0120)	116.7380 (116.2640)
6	93.7693 (93.6275)	93.0428 (91.5228)	105.3570 (105.1060)	103.3620 (103.0960)	119.0430 (118.3540)	116.9490 (116.2570)
7	93.0538 (92.1028)	93.0451 (89.6656)	104.8590 (104.3810)	102.5200 (102.0120)	119.5050 (118.5310)	116.9770 (116.0270)
8	93.0561 (90.3373)	93.0492 (87.5753)	104.3380 (103.4150)	101.6680 (100.7210)	120.2160 (118.7950)	117.2790 (115.9060)
9	93.0570 (88.8244)	93.0492 (85.6925)	103.5110 (102.3440)	100.5100 (99.3252)	120.4960 (118.7150)	117.1980 (115.4550)
10	93.0495 (86.9779)	93.0431 (83.5594)	102.0680 (100.8680)	98.8000 (97.5416)	120.5410 (117.7830)	116.8520 (114.1320)
20	93.0624 (71.7295)	93.0553 (65.7316)	93.8390 (86.9904)	93.1502 (80.7760)	115.9720 (107.1790)	108.3680 (100.0750)
30	93.0573 (58.7546)	93.0501 (51.2586)	93.6297 (71.9736)	93.1440 (63.9861)	103.5720 (91.1920)	97.4791 (81.9129)
40	93.0534 (48.1048)	93.0473 (39.9570)	93.4043 (60.2419)	93.1328 (51.2291)	112.3250 (79.8788)	95.2270 (68.6733)
∞	130.2390† (00.0000)	107.2584† (00.0000)	134.5724† (00.0000)	130.2390† (00.0000)	183.4495† (00.0000)	172.2528† (00.0000)

Table 6: The present values of MAV option with insurance charges and management fees. (European type MAV option present values are in parentheses.) The dagger (†) indicates values computed by assuming continuously sampled maximum without surrender charges.