A 3-CHANNEL BIORTHOGONAL FILTER BANK CONSTRUCTION BASED ON PREDICT AND UPDATE LIFTING STEPS

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ABSTRACT

A new method for constructing M-channel biorthogonal filter banks based on predict and update lifting steps is described. In particular, a detailed treatment of the three channel case given. The filter bank construction is a generalisation of the well known two channel split – predict – update approach introduced by Sweldens [2]. The construction admits a highly efficient in-place implementation and is easily adapted to produce perfect reconstruction filter banks that map integers to integers. The filter banks exhibit excellent band pass characteristics, however they are not, in general, linear phase.

1. INTRODUCTION

The *lifting scheme* [2] is a technique for constructing a set of biorthogonal filters by modifying another existing set. Two channel biorthogonal filter banks give rise to biorthogonal wavelets via the discrete wavelet transform and it is in this context that lifting has been popularised.

Using lifting steps to construct wavelets has a number of advantages. First of all, the lifting scheme allows the construction of filter banks with very fast implementations that do not require auxiliary memory. Secondly, unlike traditional approaches to wavelet design, Fourier domain considerations are not necessary (though may be used). This allows wavelets to be designed for use in "second generation" settings - for example on bounded domains or with non-uniformly sampled data. Finally, linear lifting steps may be replaced by analogous non-linear operations allowing the construction of non-linear counterparts to the wavelet transform. In particular, invertible integer to integer transforms are possible where the lifting steps may involve rounding floating point values to integers.

There are many situations where it is advantageous to use M-channel filter banks over dyadic wavelet transforms. For example, in certain coding applications [3], and in audio signal processing [1]. There has been relatively little work applying the lifting scheme to the M-channel case. Most notable is the work of Tran [3] who constructs filter banks based on lattice structures and factors the constant matrices into lifting steps.

2. TWO CHANNEL CASE

The split – predict – update lifting step algorithm decomposes an input signal into high and low frequency components. The filters that make up the corresponding analysis and synthesis filter banks are created implicitly in this process. In the two channel setting, the input signal is first split into even and odd components. This is followed by two lifting operations. The first (*predict*), uses the even values to predict the odd values and replaces them with the difference (high frequency component). Thus:

$$HP(z) = x_o(z) - T(z)x_e(z)$$

where $T(z)$ is the predict operator and $x_e(z)$ and $x_o(z)$ are the odd and even signal components respectively. After constructing the high pass subband, the even values are *updated* to produce a coarse approximation (the low frequency component) of the original signal:

$$LP(z) = x_e(z) - S(z)HP(z)$$

where $S(z)$ is the update operator. The update operator associated with a given predict operator is usually chosen to conserve some number of moments in the low pass (coarse) sub–band.

One very appealing side effect of implementing an analysis filter bank using lifting steps is that the inverse process is immediately apparent. The even values are first reconstructed as:

$$x_e(z) = LP(z) + S(z)HP(z)$$

The odd values may then be reconstructed via:

$$x_o(z) = x_e + T(z)x_e(z)$$

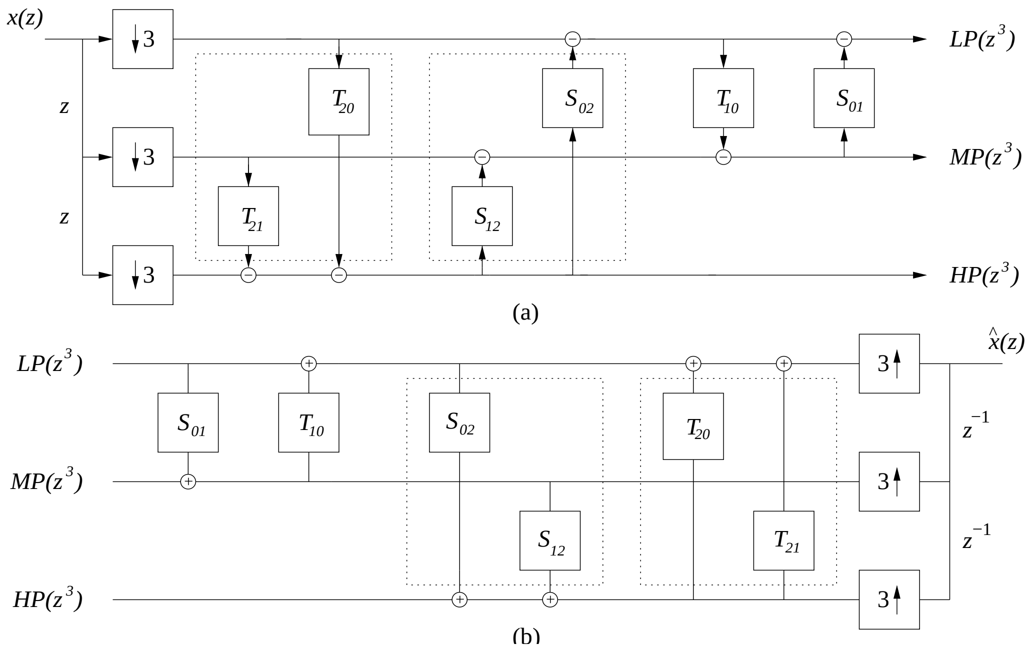

Figure 1: (a) A block diagram of the forward 3-band transform. (b) The inverse transform.

Furthermore, it is clear $T(z)$ and $S(z)$ may be replaced by non–linear operations and the inverse may be obtained just as easily.

The process of applying the predict and update operators can be written in matrix form as follows:

$$\begin{bmatrix} LP(z) \\ HP(z) \end{bmatrix} = \begin{bmatrix} 1 & -S(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -T(z) & 1 \end{bmatrix} \begin{bmatrix} x_e(z) \\ x_o(z) \end{bmatrix}$$
$$= \begin{bmatrix} 1 + S(z)T(z) & -S(z) \\ -T(z) & 1 \end{bmatrix} \begin{bmatrix} x_e(z) \\ x_o(z) \end{bmatrix}$$

Similarly the inverse process can be written as a multiplication of elementary lifting matrices and the vector containing the high and low frequency subbands. The analysis ($\widetilde{g}(z)$, $\widetilde{h}(z)$) and synthesis ($g(z)$, $h(z)$) filters can be obtained directly from the resulting polyphase matrices, noting that on the analysis side, the multiplication is by the *transpose* of the dual polyphase matrix:

$$\begin{aligned} \widetilde{g}(z^{-1}) &= z - T(z^2) \\ \widetilde{h}(z^{-1}) &= 1 + S(z^2)T(z^2) - zS(z^2) \\ g(z) &= S(z^2) + z^{-1}(1 + T(z)S(z)) \\ h(z) &= 1 + T(z^2) \end{aligned}$$

3. THREE CHANNEL CASE

In the M-channel setting, a lifting step corresponds to subtracting a filtered version of any one channel from any other. As in the two channel case, the original data may be reconstructed by applying the additive inverse operation. Filter banks with desirable properties may be constructed using a series of lifting steps. The inverse is given by un–doing each lifting step in reverse order.

The split – predict – update method can be generalised to three channels as follows. The input signal is first split into three polyphase components. The first two components are used to predict the values in the third component, which is replaced by the difference between itself and its predicted value. The resulting high pass sub-band is then used to update the first two channels such that they each constitute samples from a smoothed version of the original signal. The values in the first channel (now smoothed) are then used to predict the values in the second (also smoothed) to create the mid-pass sub-band in the second channel. Finally, the mid-pass subband is used to update the values in the first channel to produce the low-pass sub-band. This process, together with its inverse is shown diagrammatically in figure 1. It may be extended to filter banks with more than three channels, however it should be noted that increasingly lower frequency filters will have increasingly larger support, since they are constructed using more lifting steps.

The method is now described in more detail. The analysis polyphase matrix corresponding to the first (combined) predict step is given by:

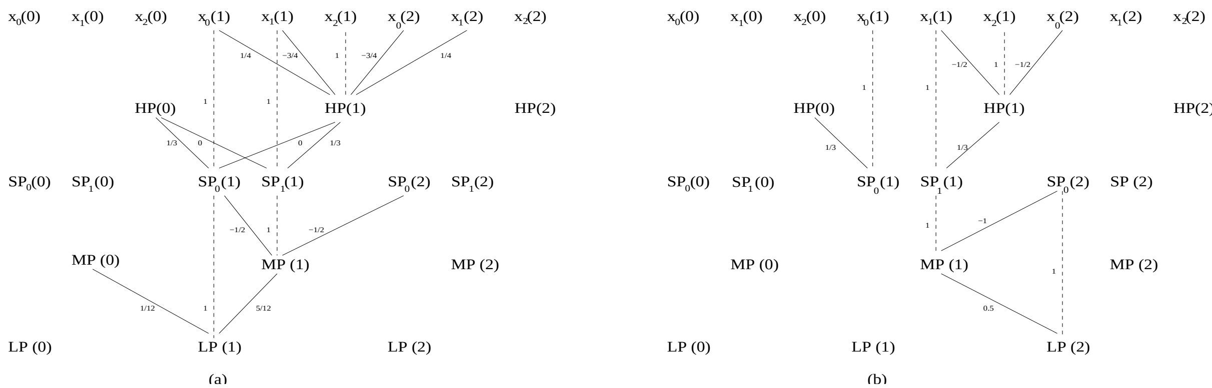

Figure 2: (a) A diagrammatic representation of the forward transform using first order lifting steps. The signal is first split into three polyphase components, $x_0(t), x_1(t)$ and $x_2(t)$. The high pass component $HP(t)$ is then created, followed by the intermediate smoothed components $SP_0(t)$ and $SP_1(t)$, followed by the mid pass sub-band $MP(t)$ followed by the low pass subband $LP(t)$. (b) Forward process using zeroth order lifting steps.

$$\widetilde{P}_0(z^{-1}) = \begin{bmatrix} 1 & 0 & -T_{20}(z) \\ 0 & 1 & -T_{21}(z) \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The analysis high pass filter $\widetilde{g}(z^{-1})$ is thus:

$$\widetilde{g}(z^{-1}) = -T_{20}(z^3) - zT_{20}(z^3) + z^2 \tag{2}$$

The corresponding polyphase matrix on the synthesis side is:

$$P_0(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_{20}(z) & T_{21}(z) & 1 \end{bmatrix} \tag{3}$$

The (smoothing) synthesis filters associated with the first two channels are thus:

$$s_0(z) = 1 + z^{-1}T_{20}(z^3) \tag{4}$$

$$s_1(z) = z^{-1} + z^{-2}T_{21}(z^3) \tag{5}$$

For the remainder of this discussion, the situation depicted in figure 2a is considered, where the predict operators are of the form:

$$T_{20}(z) = t_{20,0} + t_{20,1}z, \tag{6}$$

$$T_{21}(z) = t_{21,0} + t_{21,1}z, \tag{7}$$

There are four free parameters which allow the analysis high pass filter $\widetilde{g}$ to have up to four vanishing moments. Unfortunately, choosing the coefficients in this way yields smoothing synthesis filters $s_0(z)$ and $s_1(z)$ whose frequency response does not vanish at $\omega = \pi$. Two degrees of freedom are needed to ensure this, leaving only two degrees of freedom to enforce vanishing moments of $\widetilde{g}$. The required coefficients are:

$$(t_{20,0}, t_{21,0}, t_{20,1}, t_{21,1}) = (-\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, -\frac{1}{4}) \tag{8}$$

Thus far, the lifting filters $T_{20}(z)$ and $T_{21}(z)$ have been chosen to give high pass analysis and intermediate smoothing synthesis filters with desirable properties. They are now fixed. In the next step, the update filters $S_{02}(z)$ and $S_{12}(z)$ are chosen to give desirable characteristics to the smoothing analysis and high pass synthesis filters. The update filters corresponding to the situation depicted in figure 2a are of the form:

$$S_{02}(z) = s_{02,0}z^{-1} + s_{02,1} \tag{9}$$

$$S_{12}(z) = s_{12,0}z^{-1} + s_{12,1} \tag{10}$$

Again, there are four degrees of freedom. Two of these are used to ensure the high pass synthesis filter has two vanishing moments. The other two are used to ensure the frequency response of the analysis smoothing filters vanish at $\omega = \pi$. The required update coefficients are:

$$(s_{02,0}, s_{02,1}, s_{12,0}, s_{12,1}) = (0, -\frac{1}{3}, -\frac{1}{3}, 0); \tag{11}$$

The next lifting operator to be determined is $T_{21}(z)$. One of the degrees of freedom associated with this filter is assigned to ensuring the frequency response of the the analysis mid–pass filter vanishes at zero. The other is chosen to minimise the height of the side lobe on the frequency response of the low pass synthesis filter.

Finally $S_{12}(z)$ is chosen to ensure the frequency response of the synthesis mid–pass filter vanishes at zero, and

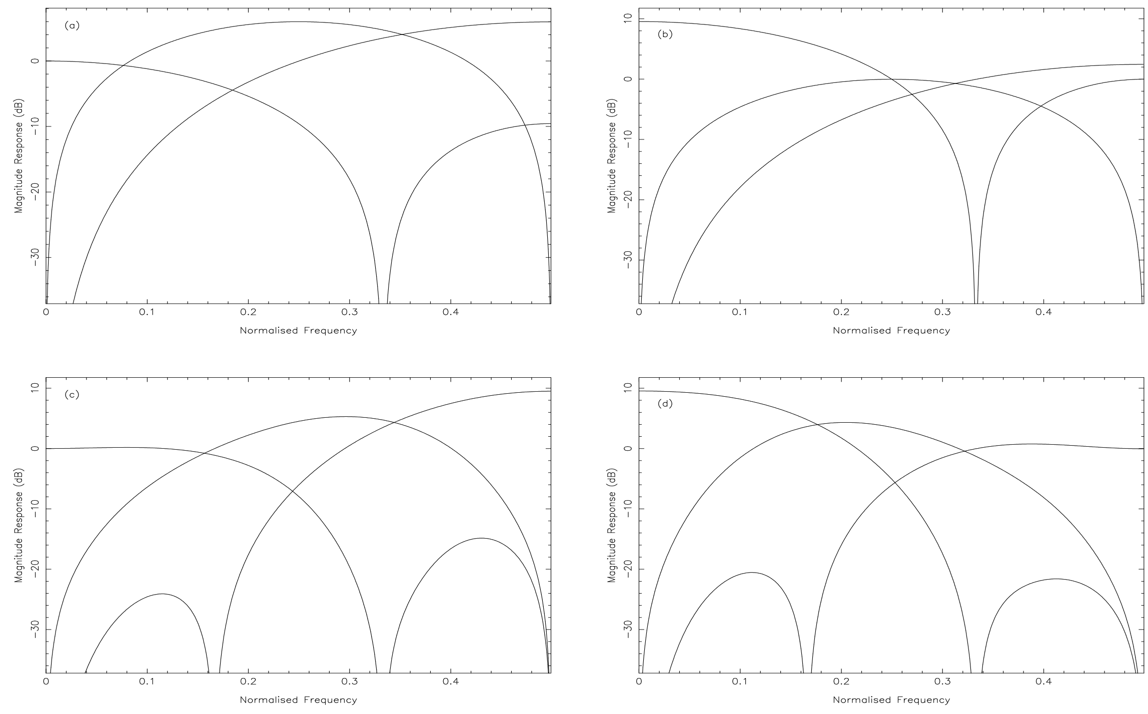

Figure 3: Top: frequency responses of the analysis (a) and synthesis (b) filter banks produced using zeroth order lifting steps. Bottom: frequency responses of the analysis (c) and synthesis (d) filter banks produced using first order lifting steps.

to minimise the height of the side lobe on the frequency response of the low pass analysis filter. The final predict and update steps are:

$$T_{10}(z) = \frac{1}{2}z + \frac{1}{2} \quad (12)$$

$$S_{01}(z) = -\frac{1}{12} - \frac{5}{12}z^{-1} \quad (13)$$

Similar reasoning can be used to derive a transform based on the same lifting step configuration but using zeroth order lifting steps (figure 2b). The lifting operators are:

$$T_{20}(z) = \frac{1}{2}z, \;\; T_{21}(z) = \frac{1}{2} \quad (14)$$

$$S_{02}(z) = \frac{1}{3}z^{-1}, \;\; S_{12}(z) = \frac{1}{3} \quad (15)$$

$$T_{10}(z) = 1, \;\; S_{01}(z) = -\frac{1}{2} \quad (16)$$

4. CONCLUSIONS

A new method for constructing three channel biorthogonal filter banks has been presented which may be extended to the general M-channel case. The construction uses just six lifting steps and thus admits a highly efficient implementation. Though not linear phase, the filter banks exhibit excellent band pass characteristics.

In contrast to the constructions of Tran [3], the method presented here was derived using reasoning based on lifting steps alone. In general, lifting steps of greater than zeroth order are used. In practice, this can result in quite involved calculations, particularly when the method is extended to more advanced cases than those presented here.

Current work is concerned with constructing highly efficient M-channel filter banks using new configurations of zeroth order lifting steps.

5. ACKNOWLEDGMENTS

M. Howlett would like to thank Richard Davis for the use of his “research tools”.

6. REFERENCES

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