ON THE MATHEMATICAL SYNTHESIS OF EQUATIONAL LOGICS

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Introduction

Birkhoff [1935] initiated the general study of algebraic structure. Importantly for our concerns here, his development was from (universal) algebra to (equational) logic. Birkhoff's starting point was the informal conception of algebra based on familiar concrete examples. Abstracting from these, he introduced the concepts of signature and equational presentation, and thereby formalised what is now our notion of (abstract) algebra. Subsequently he set up the model theory of equational presentations (varieties) and analysed their structure from the standpoint of logical inference for algebraic languages. In doing so, he introduced Equational Logic as a sound and complete deductive system for reasoning about equational assertions in Algebraic Theories.

Since Birkhoff's work, our understanding of algebraic structure has deepened; having been both systematised and extended (see *e.g.* Lawvere [1963], Ehresmann [1968], Burroni [1981], Kelly and Power [1993], Power [1999]). On the other hand, the development of equational logics has remained ad hoc. The aim of the current work is to fill in this gap.

Our standpoint is that equational logics should arise from algebraic structure. In this direction, our first purpose is to provide a mathematical theory and methodology for synthesising equational logics from algebraic metatheories (see Part I). Our second purpose is to establish the practicality of the approach. In this respect, we illustrate our methodology by means of two applications: a rational reconstruction of Birkhoff's Equational Logic and a new equational logic for reasoning about algebraic structure with name-binding operators (see Part II).

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Part I. Theory

We present our mathematical framework for synthesising equational logics. For simplicity of exposition, we restrict attention to the mono-sorted context. As such, we consider algebraic metatheories given by strong monads on symmetric monoidal categories. These provide algebraic structure that allows the specification of equational presentations in the form of *Monadic Equational Systems*. Monadic Equational Systems come equipped with a canonical model theory whereby models are Eilenberg-Moore algebras satisfying the equations. An *Equational Metalogic* for reasoning about equality in such models is presented. This deductive system has been designed to guarentee sound derivations. As for completeness, a mathematical justification of the well-known use of free constructions in equational completeness proofs is given. This is backed up with an inductive method for constructing free algebras.

1. Monadic Equational Systems

Monadic Equational Systems (MESs) are defined and their model theory is explained.

1.1. Monadic Equational Systems.

The concept of MES provides a general abstract notion of equational presentation.

Definition 1.1 (Terms and equations)**.** A *term* for an endofunctor T on a category $\mathscr{C}$ of arity A and coarity C is a Kleisli map $C \to TA$ in $\mathscr{C}$. A parallel pair of terms $t \equiv t' : C \to TA$ is called an *equation*.

Definition 1.2 (Monadic Equational Systems)**.** A *MES* $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$ consists of a strong monad $\mathbf{T}$ on a symmetric monoidal closed category $\underline{\mathscr{C}}$ together with a set of equations $\mathcal{A}$.

Notation. For a strong monad $\mathbf{T}$ on a symmetric monoidal closed category $\underline{\mathscr{C}}$ we implicitly assume that the respective underlying structures are denoted by $(T, \eta, \mu, \mathsf{st})$ and $(\mathscr{C}, I, \otimes, [-,=])$.

1.2. Model theory.

Terms admit interpretations in algebras and these give a model-theoretic notion of equality.

Let (T, st) be a strong endofunctor on a symmetric monoidal closed category $\underline{\mathscr{C}}$. Every term $t : C \to TA$ admits an *interpretation*

$$[\![t]\!]_{(X,s)} : [A, X] \otimes C \to X$$

in a T-algebra $(X, s : TX \to X)$ given by the composite

$$[A, X] \otimes C \xrightarrow{[A,X]\otimes t} [A, X] \otimes TA \xrightarrow{\mathsf{st}_{[A,X],A}} T\big([A, X] \otimes A\big) \xrightarrow{T(\epsilon^A_X)} TX \xrightarrow{s} X \ .$$

We thus obtain a *satisfaction relation* between algebras and equations: for all T-algebras (X, s) and equations $u \equiv v : C \to TA$,

$$(X, s) \models u \equiv v : C \to TA \quad \text{iff} \quad [\![u]\!]_{(X,s)} = [\![v]\!]_{(X,s)} : [A, X] \otimes C \to X \ .$$

Definition 1.3 (Algebras)**.** An $\mathbb{S}$-*algebra* for a MES $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$ is an Eilenberg-Moore algebra (X, s) for the monad $\mathbf{T}$ satisfying the equations in $\mathcal{A}$; that is, such that $(X, s) \models u \equiv v : C \to TA$ for all $(u \equiv v : C \to TA) \in \mathcal{A}$.

The category $\mathbb{S}$-**Alg** is the full subcategory of the category $\mathscr{C}^{\mathbf{T}}$ (of Eilenberg-Moore algebras for the monad $\mathbf{T}$) consisting of the $\mathbb{S}$-algebras. We thus have the following situation:

$$\begin{tikzcd}
\mathbb{S}\text{-}\mathbf{Alg} \arrow[r, hook] \arrow[dr, "U_{\mathbb{S}}"'] & \mathscr{C}^{\mathbf{T}} \arrow[r, hook] \arrow[d, "U_{\mathbf{T}}"] & T\text{-}\mathbf{Alg} \arrow[dl, "U_T"] \\
& \mathscr{C} &
\end{tikzcd}$$

2. Equational Metalogic

We present a sound deductive system for reasoning about the equality of terms in MESs.

The *Equational Metalogic (EML)* associated to a MES $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$ has judgements of the form

$$\mathcal{A} \vdash u \equiv v : C \to TA \ ,$$

where u and v are terms of arity A and coarity C, and consists of the following inference rules.

- Equality rules.

$$\mathsf{Ref}\ \frac{}{\mathcal{A} \vdash u \equiv u : C \to TA} \qquad \mathsf{Sym}\ \frac{\mathcal{A} \vdash u \equiv v : C \to TA}{\mathcal{A} \vdash v \equiv u : C \to TA}$$

$$\mathsf{Trans}\ \frac{\mathcal{A} \vdash u \equiv v : C \to TA \qquad \mathcal{A} \vdash v \equiv w : C \to TA}{\mathcal{A} \vdash u \equiv w : C \to TA}$$

- Axioms.

$$\mathsf{Axiom}\ \frac{(u \equiv v : C \to TA) \in \mathcal{A}}{\mathcal{A} \vdash u \equiv v : C \to TA}$$

- Congruence of substitution.

$$\mathsf{Subst}\ \frac{\mathcal{A} \vdash u_1 \equiv v_1 : C \to TB \qquad \mathcal{A} \vdash u_2 \equiv v_2 : B \to TA}{\mathcal{A} \vdash u_1\{u_2\} \ \equiv\ v_1\{v_2\} : C \to TA}$$

  where $w_1\{w_2\}$ denotes the Kleisli composite $C \xrightarrow{w_1} TB \xrightarrow{T(w_2)} T(TA) \xrightarrow{\mu_A} TA$.

- Congruence of tensor extension.

$$\mathsf{Ext}\ \frac{\mathcal{A} \vdash u \equiv v : C \to TA}{\mathcal{A} \vdash \langle V \rangle u \ \equiv\ \langle V \rangle v : V \otimes C \to T(V \otimes A)}$$

  where $\langle V \rangle w$ denotes the composite $V \otimes C \xrightarrow{V \otimes w} V \otimes TA \xrightarrow{\mathsf{st}_{V,A}} T(V \otimes A)$.

- Local character.

$$\mathsf{Local}\ \frac{\mathcal{A} \vdash u \circ e_i \ \equiv\ v \circ e_i : C_i \to TA \quad (i \in I)}{\mathcal{A} \vdash u \equiv v : C \to TA}\ \big(\{\, e_i : C_i \to C \,\}_{i \in I} \text{ jointly epi}\big)$$

(Recall that a family of maps $\{\, e_i : C_i \to C \,\}_{i \in I}$ is said to be *jointly epimorphic* if, for any $f, g : C \to X$ such that $\forall_{i \in I}\ f \circ e_i \ = \ g \circ e_i : C_i \to X$, it follows that $f \ = \ g$.)

2.1. **Soundness.** The following result states the soundness of derivability in EML.

Theorem 2.1 (Soundness)**.** For a MES $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$,

$$\mathcal{A} \vdash u \equiv v : C \to TA \quad \text{implies} \quad (X, s) \models u \equiv v : C \to TA \text{ for all } \mathbb{S}\text{-algebras } (X, s)\ .$$

Proof. See Appendix A. □

3. Internal Completeness

In this and the following section, we build a mathematical basis for investigating completeness. Our main tools are an internal completeness result for MESs that admit free algebras together with an inductive method for constructing them.

3.1. **Internal Completeness.** Let $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$ be a MES admitting free algebras; that is, such that the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint. We denote the free $\mathbb{S}$-algebra on $X \in \mathscr{C}$ as $(T_{\mathbb{S}}X, \tau^{\mathbb{S}}_X : TT_{\mathbb{S}}X \to T_{\mathbb{S}}X)$, and the associated *free* $\mathbb{S}$*-algebra monad* as $\mathbf{T}_{\mathbb{S}} = (T_{\mathbb{S}}, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$. Then, the embedding $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \mathscr{C}^{\mathbf{T}}$ induces a strict monad morphism $\mathsf{q}^{\mathbb{S}} : \mathbf{T} \to \mathbf{T}_{\mathbb{S}}$. This has components referred to as *quotient maps* that are characterised by being the unique morphisms $\mathsf{q}^{\mathbb{S}}_X : TX \to T_{\mathbb{S}}X$ for which the diagram

$$\begin{tikzcd}
TTX \arrow[r, "T(\mathsf{q}^{\mathbb{S}}_X)"] \arrow[d, "\mu_X"] & TT_{\mathbb{S}}X \arrow[d, "\tau^{\mathbb{S}}_X"] \\
TX \arrow[r, "\exists!\, \mathsf{q}^{\mathbb{S}}_X"] & T_{\mathbb{S}}X \\
X \arrow[u, "\eta_X"] \arrow[ur, "\eta^{\mathbb{S}}_X"'] &
\end{tikzcd} \tag{3.1}$$

commutes. In this situation, we have a form of *strong completeness* stating that an equation is satisfied in all models if and only if it is satisfied in a freely generated one, if and only if it is identified by the quotient map.

Theorem 3.1 (Internal completeness)**.** For a MES $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$ admitting free algebras, the following are equivalent.

(1) $\mathbb{S}\text{-}\mathbf{Alg} \models u \equiv v : C \to TA$

(2) $(T_{\mathbb{S}}A, \tau^{\mathbb{S}}_A) \models u \equiv v : C \to TA$

(3) $\mathsf{q}^{\mathbb{S}}_A \circ u \;=\; \mathsf{q}^{\mathbb{S}}_A \circ v : C \to T_{\mathbb{S}} A$

Proof. See Appendix B. □

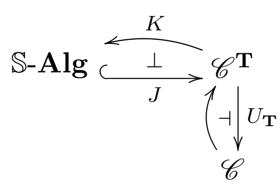
4. Free Algebras

We now give conditions under which MESs admit free algebras, and provide a construction of quotient maps (see Fiore and Hur [2009] for details).

Definition 4.1. Let $\underline{\mathscr{C}}$ be a symmetric monoidal closed category. An object A in $\mathscr{C}$ is respectively said to be *compact* and *projective* if the endofunctor $[A, -]$ on $\mathscr{C}$ respectively preserves colimits of ω-chains and epimorphisms.

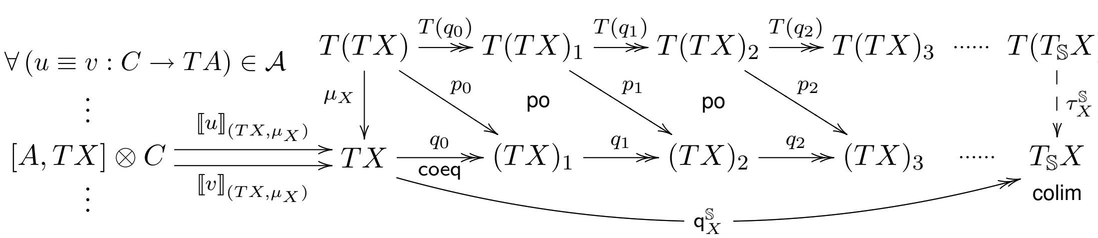
Definition 4.2. A MES $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$ is called *finitary* if the category $\mathscr{C}$ is cocomplete, the endofunctor T on $\mathscr{C}$ is ω-cocontinuous, and the arity A of each equation $u \equiv v : C \to TA$ in $\mathcal{A}$ is compact. Such a MES is called *inductive* if furthermore the endofunctor T preserves epimorphisms and the arity A of each equation $u \equiv v : C \to TA$ in $\mathcal{A}$ is projective.

For a finitary MES $\mathbb{S} = (\underline{\mathscr{C}}, \mathbf{T}, \mathcal{A})$ we have the following situation:



For each object $X \in \mathscr{C}$, since (TX, μ_X) is a free Eilenberg-Moore algebra on X, the free $\mathbb{S}$-algebra $(T_{\mathbb{S}} X, \tau^{\mathbb{S}}_X)$ on X is given by the free $\mathbb{S}$-algebra $K(TX, \mu_X)$ over the Eilenberg-Moore algebra (TX, μ_X). Satisfying the commutative diagram (3.1), the universal homomorphism $(TX, \mu_X) \to (T_{\mathbb{S}} X, \tau^{\mathbb{S}}_X)$ induced by the adjunction $K \dashv J$ yields the quotient map $\mathsf{q}^{\mathbb{S}}_X : TX \to T_{\mathbb{S}} X$.

In the case of inductive MESs, the quotient maps $\mathsf{q}^{\mathbb{S}}_X$ are constructed as follows:



(4.1)

where q_0 is the universal map that jointly coequalizes every pair $[\![u]\!]_{(TX,\mu_X)}$ and $[\![v]\!]_{(TX,\mu_X)}$ with $(u \equiv v : C \to TA) \in \mathcal{A}$.

Moreover, when the strong monad $\mathbf{T}$ arises from a left adjoint to a forgetful functor F-$\mathbf{Alg} \to \mathscr{C}$, for F a strong endofunctor that preserves colimits of ω-chains and epimorphisms, the construction of the quotient maps $\mathsf{q}^{\mathbb{S}}_X$ simplifies as follows:

$$\begin{tikzcd}
\forall\,(u \equiv v : C \to TA) \in \mathcal{A} & F(TX) \arrow[r, two heads, "F(q_0)"] \arrow[d, "\widehat{\mu}_X"'] \arrow[dr, "p_0"] & F(TX)_1 \arrow[r, two heads, "F(q_1)"] \arrow[dr, "p_1"] \arrow[d, phantom, "\mathsf{po}"] & F(TX)_2 \arrow[r, two heads, "F(q_2)"] \arrow[dr, "p_2"] \arrow[d, phantom, "\mathsf{po}"] & F(TX)_3 \arrow[r, dotted, no head] & F(T_{\mathbb{S}}X) \arrow[d, dashed, "\widehat{\tau}^{\mathbb{S}}_X"] \\
\vdots & & & & & \\
{[A,TX] \otimes C} \arrow[r, shift left, "{[\![u]\!]_{(TX,\mu_X)}}"] \arrow[r, shift right, "{[\![v]\!]_{(TX,\mu_X)}}"'] & TX \arrow[r, two heads, "q_0", "\mathsf{coeq}"'] \arrow[rrr, bend right, "\mathsf{q}^{\mathbb{S}}_X", "\mathsf{colim}"'] & (TX)_1 \arrow[r, two heads, "q_1"] & (TX)_2 \arrow[r, two heads, "q_2"] & (TX)_3 \arrow[r, dotted, no head] & T_{\mathbb{S}}X \\
\vdots & & & & &
\end{tikzcd} \tag{4.2}$$

where $(TX, \widehat{\mu}_X)$ and $(T_{\mathbb{S}}X, \widehat{\tau}^{\mathbb{S}}_X)$ are the F-algebras respectively corresponding to the Eilenberg-Moore algebras (TX, μ_X) and $(T_{\mathbb{S}}X, \tau^{\mathbb{S}}_X)$ for the monad $\mathbf{T}$.

Part II. Methodology

In view of the mathematical development of Part I, we advocate the following methodology for synthesising equational logics.

(1) *Select a symmetric monoidal closed category $\underline{\mathscr{C}}$ as universe of discourse and consider within it a syntactic notion of signature such that every signature Σ gives rise to a strong monad $\mathbf{T}_\Sigma$ on $\underline{\mathscr{C}}$.*

The universe of discourse should be carefully chosen to consist of mathematical objects with enough internal structure to allow for the algebraic realisation of the syntactic constructs that one is intending to model.

We do not insist on an a priori prescription for the definition of signature, but rather consider it as being domain specific. Of course, standard notions of signature (*e.g.* as in enriched algebraic theories—see Kelly and Power [1993], and Robinson [2002]) may be considered. However, one may need to go beyond them—see Fiore [2008] and Fiore and Hur [2010].

(2) *Select a class of coarity-arity pairs (C, A) of objects of $\mathscr{C}$ and give a syntactic description of Kleisli maps $C \to T_\Sigma A$. This yields a syntactic notion of equational presentation with an associated model theory arising from that of MESs.*

We are ultimately interested in constructing free algebras for equational presentations. In the context of finitary algebraic theories, it is thus appropriate to consider a cocomplete universe of discourse together with signatures for which the associated monad preserves colimits of ω-chains and epimorphisms, and arities that are compact and projective; so that the induced MESs are inductive.

(3) *Synthesise a deductive system for equational reasoning on syntactic terms with rules arising as syntactic counterparts of the EML rules associated to the MES.*

The analysis of the Subst rule will typically involve the consideration of a syntactic substitution operation corresponding to Kleisli composition.

(4) *In view of the internal completeness result, analyse the construction of free algebras and quotient maps so as to either establish the completeness of the synthesised equational logic, or get insight into how to extend it to make it complete.*

In practise, we have not only found that the synthesised equational logics are already complete but moreover that the method allows us to distil complete rewriting systems that provide a compuational treatment of derivability.

The resulting equational logics are thus synthesised from algebraic metatheories by means of first principles. Two sample applications of the methodology follow.

5. Algebraic Theories

5.1. **MESs for algebraic theories.** Recall that an *algebraic theory* $\mathbb{T} = (\Sigma, E)$ is given by a signature Σ, consisting of a set of operators O and an arity function $|-| : O_\Sigma \to \mathbb{N}$, together with a set of equations E. Algebraic theories may be encoded as MESs as follows.

The signature Σ induces the endofunctor $F_\Sigma(X) = \coprod_{\mathsf{o}\in O} X^{|\mathsf{o}|}$ on **Set**, for which the category of Σ-algebras, Σ-**Alg**, and the category of F_Σ-algebras, F_Σ-**Alg**, are isomorphic. The forgetful functor F_Σ-**Alg** $\to$ **Set** is monadic and the induced *term monad* $\mathbf{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma)$ is given syntactically. For a set of variables V, the set $T_\Sigma(V)$ consists of terms built up from the variables in V and the operators in O.

The endofunctor F_Σ has a canonical strength $\mathsf{st} : U \times F_\Sigma(V) \to F_\Sigma(U \times V)$ mapping a pair $(u, \iota_\mathsf{o}(v_1, \ldots, v_{|\mathsf{o}|}))$ to $\iota_\mathsf{o}((u, v_1), \ldots, (u, v_{|\mathsf{o}|}))$. The induced strength on the monad $\mathbf{T}_\Sigma$, $\widehat{\mathsf{st}} : U \times T_\Sigma(V) \to T_\Sigma(U \times V)$, maps a pair (u, t) to the term $t\{v \mapsto (u, v)\}_{v\in V}$ obtained by simultaneously substituting (u, v) for each variable $v \in V$ in the term t.

By definition, each equation $(V \vdash l \equiv r)$ in E is given by a pair of terms $l, r \in T_\Sigma(V)$, or equivalently, by a parallel pair of Kleisli maps $\underline{l}, \underline{r} : 1 \to T_\Sigma(V)$. Thus, one can encode the algebraic theory $\mathbb{T}$ as the MES $\underline{\mathbb{T}} = (\mathbf{Set}, \mathbf{T}_\Sigma, \underline{E})$ with the set of equations $\underline{E}$ given by $\{\, \underline{l} \equiv \underline{r} : 1 \to T_\Sigma V \mid (V \vdash l \equiv r) \in E \,\}$. The MES $\underline{\mathbb{T}}$ is inductive.

5.2. **Model theory.** A $\underline{\mathbb{T}}$-algebra is an Eilenberg-Moore algebra $(X, s : T_\Sigma X \to X)$ for $\mathbf{T}_\Sigma$ such that the following diagram commutes for every equation $(V \vdash t_1 \equiv t_2)$ in E:

$$X^V \times 1 \underset{X^V \times \underline{t_2}}{\overset{X^V \times \underline{t_1}}{\rightrightarrows}} X^V \times T_\Sigma V \xrightarrow{\widehat{\mathsf{st}}_{X^V, V}} T_\Sigma\big(X^V \times V\big) \xrightarrow{T_\Sigma(\epsilon^V_X)} T_\Sigma X \xrightarrow{s} X\,;$$

that is,

$$1 \underset{\underline{t_2}}{\overset{\underline{t_1}}{\rightrightarrows}} T_\Sigma V \xrightarrow{T_\Sigma(v)} T_\Sigma X \xrightarrow{s} X \quad \text{for all functions } v : V \to X\,. \tag{5.1}$$

Write $(X, [\![-]\!])$ for the Eilenberg-Moore algebra of the monad $\mathbf{T}_\Sigma$ corresponding to the Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma})$ via the isomorphism Σ-**Alg** $\cong \mathscr{C}^{\mathbf{T}_\Sigma}$. We have that the Eilenberg-Moore algebra $(X, [\![-]\!])$ satisfies (5.1) if and only if the Σ-algebra $(X, \{[\![\mathsf{o}]\!]\}_{\mathsf{o}\in\Sigma})$ satisfies the equation $(V \vdash t_1 \equiv t_2)$. It follows thus that $\underline{\mathbb{T}}$-**Alg** is isomorphic to the category $\mathbb{T}$-**Alg** of algebras for the algebraic theory $\mathbb{T}$.

5.3. **EML for algebraic theories.** The EML associated to the MES of an algebraic theory $\mathbb{T} = (\Sigma, E)$ allows the synthesis of a syntactic equational deductive system for reasoning about algebraic structure. This consists of the following rules:

$$\mathsf{Ref}\ \frac{}{V \vdash t \equiv t}\ (t \in T_\Sigma V) \qquad \mathsf{Sym}\ \frac{V \vdash t \equiv t'}{V \vdash t' \equiv t} \qquad \mathsf{Trans}\ \frac{V \vdash t \equiv t' \quad V \vdash t' \equiv t''}{V \vdash t \equiv t''}$$

$$\mathsf{Axiom}\ \frac{}{V \vdash t \equiv t'}\ (V \vdash t \equiv t') \in E \tag{5.2}$$

$$\mathsf{Subst}\ \frac{U \vdash t \equiv t' \qquad V \vdash s_u \equiv s'_u\ (u \in U)}{V \vdash t\{u \mapsto s_u\}_{u \in U}\ \equiv\ t'\{u \mapsto s'_u\}_{u \in U}}$$

In the substitution rule, $t\{u \mapsto s_u\}_{u \in U}$ denotes the term obtained by simultaneously substituting the terms s_u for the variables $u \in U$ in the term t.

Note that our synthesised deductive system subsumes the usual presentation of equational logic, where the substitution rule is restricted to families $s_u = s'_u$ $(u \in U)$ and a congruence rule for operators is added.

The rules Ref, Sym, Trans and Axiom directly follow from the corresponding EML rules instantiated to global Kleisli maps. The rule Subst is derived from the EML rules Local and Subst in the following way:

$$\frac{\underline{t} \equiv \underline{t'} : 1 \to T_\Sigma U \qquad \dfrac{\underline{s_u} \equiv \underline{s'_u} : 1 \to T_\Sigma V \quad (u \in U)}{[\,\underline{s_u}\,]_{u \in U}\ \equiv\ [\,\underline{s'_u}\,]_{u \in U} : U \to T_\Sigma V}\ (\text{by Local})}{\underline{t}\{\,[\,\underline{s_u}\,]_{u \in U}\,\}\ \equiv\ \underline{t'}\{\,[\,\underline{s'_u}\,]_{u \in U}\,\} : 1 \to T_\Sigma V}\ (\text{by Subst})$$

The well-known soundness of the above deductive system for algebraic theories follows from the soundness of EML.

5.4. **Completeness.** We proceed to show how the internal completeness theorem and the construction of free algebras for inductive MESs lead to semantic and rewriting completeness results.

Consider the construction (4.2) for the MES $\underline{\mathbb{T}}$. The map $q_0 : T_\Sigma X \twoheadrightarrow (T_\Sigma X)_1$ is the universal map in **Set** that coequalizes every pair $[\![t]\!], [\![t']\!] : (T_\Sigma X)^V \to T_\Sigma X$ for all $(V \vdash t \equiv t') \in E$, where $[\![t]\!]$ maps $\{\, s_v \in T_\Sigma X \,\}_{v \in V}$ to $t\{v \mapsto s_v\}_{v \in V}$. It follows that the set $(T_\Sigma X)_1$ is given as the quotient $T_\Sigma X/_{\approx_1}$ of $T_\Sigma X$ under the equivalence relation $\approx_1$ generated by the rule:

$$\frac{}{t\{v \mapsto s_v\}_{v \in V} \approx_1 t'\{v \mapsto s_v\}_{v \in V}}\ \big(\,(V \vdash t \equiv t') \in E, \{\, s_v \,\}_{v \in V} \in (T_\Sigma X)^V\,\big)$$

The map q_0 sends a term $s \in T_\Sigma X$ to its equivalence class $[s]_{\approx_1} \in T_\Sigma X/_{\approx_1}$, and the map p_0 sends $\iota_{\mathsf{o}}(s_1, \ldots, s_{|\mathsf{o}|}) \in F_\Sigma(T_\Sigma X)$ to $[\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|})]_{\approx_1} \in T_\Sigma X/_{\approx_1}$.

Recall that a pushout

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[dr, "f"'] & B \arrow[dr, "f'"] & \\ & C \arrow[r, two heads, "e'"] & C/_\approx \end{tikzcd}$$

of a surjective map $e : A \twoheadrightarrow B$ and a map $f : A \to C$ in **Set** can be constructed as the cospan $e' : C \twoheadrightarrow C/_\approx \leftarrow B : f'$, where $C/_\approx$ is the quotient of C under the equivalence

relation $\approx$ generated by setting $f(a) \approx f(a')$ for all $a, a' \in A$ such that $e(a) = e(a')$ in B; and where the surjective map $e' : C \twoheadrightarrow C/_{\approx}$ sends an element c to its equivalence class $[c]_{\approx}$ and the map $f' : B \to C/_{\approx}$ sends an element b to $e'(f(a))$ for $a \in A$ such that $e(a) = b$.

Using this construction, an inductive analysis of the maps q_n for $n \geq 1$, shows that the sets $(T_\Sigma X)_n$ for $n \geq 2$ are given as the quotients $T_\Sigma X/_{\approx_n}$ of $T_\Sigma X$ under the equivalence relations $\approx_n$ inductively generated by the following rules:

$$\frac{s \approx_{n-1} s'}{s \approx_n s'} \qquad\qquad \frac{s_1 \approx_{n-1} s'_1,\ \ldots,\ s_{|\mathsf{o}|} \approx_{n-1} s'_{|\mathsf{o}|}}{\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|}) \approx_n \mathsf{o}(s'_1, \ldots, s'_{|\mathsf{o}|})}\ (\mathsf{o} \in \Sigma)$$

The maps q_n for $n \geq 1$ send $[s]_{\approx_n} \in T_\Sigma X/_{\approx_n}$ to $[s]_{\approx_{n+1}} \in T_\Sigma X/_{\approx_{n+1}}$, and the maps p_n for $n \geq 1$ send $\iota_{\mathsf{o}}([s_1]_{\approx_n}, \ldots, [s_{|\mathsf{o}|}]_{\approx_n}) \in F_\Sigma(T_\Sigma X/_{\approx_n})$ to $[\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|})]_{\approx_{n+1}} \in T_\Sigma X/_{\approx_{n+1}}$.

By taking the colimit of the chain of quotients $\{\, q_n : T_\Sigma X/_{\approx_n} \twoheadrightarrow T_\Sigma X/_{\approx_{n+1}} \,\}_{n \geq 0}$, the set $T_{\mathbb{S}} X$ is given by the quotient $T_\Sigma X/_{\approx_E}$ of $T_\Sigma X$ under the relation $\approx_E$ generated by the following rules:

$$\mathsf{Ref}\ \frac{}{s \approx_E s}\ (s \in T_\Sigma X) \qquad \mathsf{Sym}\ \frac{s \approx_E s'}{s' \approx_E s} \qquad \mathsf{Trans}\ \frac{s \approx_E s' \quad s' \approx_E s''}{s \approx_E s''}$$

$$\mathsf{Axiom}\ \frac{}{t\{v \mapsto s_v\}_{v \in V} \approx_E t'\{v \mapsto s_v\}_{v \in V}}\ \big(\, (V \vdash t \equiv t') \in E, \{\, s_v \,\}_{v \in V} \in (T_\Sigma X)^V \,\big) \tag{5.3}$$

$$\mathsf{Cong}\ \frac{s_1 \approx_E s'_1,\ \ldots,\ s_{|\mathsf{o}|} \approx_E s'_{|\mathsf{o}|}}{\mathsf{o}(s_1, \ldots, s_{|\mathsf{o}|}) \approx_E \mathsf{o}(s'_1, \ldots, s'_{|\mathsf{o}|})}\ (\mathsf{o} \in \Sigma)$$

The quotient map $\mathsf{q}^{\mathbb{S}}_X : T_\Sigma X \to T_{\mathbb{S}} X$ sends a term s to its equivalence class $[s]_{\approx_E}$.

The rules Axiom and Cong for the relation $\approx_E$ can be merged into a single rule to yield a rewriting-style deduction system. Indeed, by an induction on the depth of proof trees, one shows that the relation $\approx_E$ on $T_\Sigma X$ coincides with the equivalence relation $\approx^{\mathsf{R}}_E$ on $T_\Sigma X$ generated by the rewriting-style rule

$$\mathbf{C}[t\{v \mapsto s_v\}_{v \in V}] \ \approx^{\mathsf{R}}_E\ \mathbf{C}[t'\{v \mapsto s_v\}_{v \in V}]$$

for $(V \vdash t \equiv t') \in E$, $\{\, s_v \,\}_{v \in V} \in (T_\Sigma X)^V$, and $\mathbf{C}[-]$ a context with one hole and possibly with variables from X.

From the internal completeness of the MES $\underline{\mathbb{T}}$, we have the soundeness and completeness of equational reasoning by bidirectional rewriting:

$$\begin{aligned}
&\mathbb{T}\text{-}\mathbf{Alg} \models (U \vdash s \equiv s') \\
&\quad\iff \underline{\mathbb{T}}\text{-}\mathbf{Alg} \models (\underline{s} \equiv \underline{s'} : 1 \to T_\Sigma U) \\
&\quad\iff \mathsf{q}^{\mathbb{S}}_U \circ \underline{s} = \mathsf{q}^{\mathbb{S}}_U \circ \underline{s'} : 1 \to T_{\mathbb{S}} U \\
&\quad\iff [s]_{\approx_E} = [s']_{\approx_E} \quad \text{in } T_\Sigma U/_{\approx_E} \\
&\quad\iff s \approx_E s' \quad \text{in } T_\Sigma U \\
&\quad\iff s \approx^{\mathsf{R}}_E s' \quad \text{in } T_\Sigma U
\end{aligned}$$

Finally, note also that the equational logic (5.2) is complete; as a proof of $s \approx_E s'$ for $s, s' \in T_\Sigma U$ constructed by the rules in (5.3) can be turned into a proof of $U \vdash s \equiv s'$ by the rules in (5.2).

6. Synthetic Nominal Equational Logic

This section provides a novel application of our theory and methodology for synthesising equational logics geared to the development of a deductive system for reasoning about algebraic structure with name-binding operators.

We consider a class of MESs, referred to as *Nominal Equational Systems (NESs)*, based on the category **Nom** of nominal sets [Gabbay and Pitts, 2001, Section 6] (or equivalently the Schanuel topos [Mac Lane and Moerdijk, 1992, Section III.9]). These we subsequently present in syntactic form to yield *Nominal Equational Presentations (NEPs)*. The model theory of NEPs is of course derived from that of NESs.

An equational logic, called *Synthetic Nominal Equational Logic (SNEL)*, for NEPs is derived from the EML associated to NESs. This is guaranteed to be sound by construction. Completeness is derived from the internal completeness theorem by an analysis of the inductive construction of free algebras in terms of equational derivability. This approach yields two completeness results: the rewriting completeness of an induced notion of *Synthetic Nominal Rewriting (SNR)* and the semantic completeness of SNEL.

A brief discussion of related work is included.

6.1. **Nominal sets.** For a fixed countably infinite set A of atoms, the group $\mathfrak{S}_0(\mathsf{A})$ of finite permutations of atoms consists of the bijections on A that fix all but finitely many elements of A. A $\mathfrak{S}_0(\mathsf{A})$-*action* $X = (\,|X|, \cdot)$ consists of a set $|X|$ equipped with a function $(-)\cdot(=) : \mathfrak{S}_0(\mathsf{A}) \times |X| \to |X|$ satisfying $\mathrm{id}_{\mathsf{A}} \cdot x = x$ and $\pi' \cdot (\pi \cdot x) = (\pi'\pi) \cdot x$ for all $x \in |X|$ and $\pi, \pi' \in \mathfrak{S}_0(\mathsf{A})$. $\mathfrak{S}_0(\mathsf{A})$-actions form a category with morphisms $X \to Y$ given by *equivariant functions*; that is, functions $f : |X| \to |Y|$ such that $f(\pi \cdot x) = \pi \cdot (fx)$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ and $x \in |X|$.

For a $\mathfrak{S}_0(\mathsf{A})$-action X, a finite subset S of A is said to *support* $x \in X$ if for all atoms $a, a' \notin S$, we have that $(a\ a') \cdot x = x$, where the *transposition* $(a\ a')$ is the bijection that swaps a and a', and fixes all other atoms. A *nominal set* is a $\mathfrak{S}_0(\mathsf{A})$-action in which every element has finite support. As an example, the set of atoms A becomes the *nominal set of atoms* $\mathbb{A}$ when equipped with the evaluation action $\pi \cdot a = \pi(a)$. The category **Nom** is the full subcategory of the category of $\mathfrak{S}_0(\mathsf{A})$-actions consisting of nominal sets.

The supports of an element of a nominal set are closed under intersection, and we write $\mathsf{supp}_X(x)$, or simply $\mathsf{supp}(x)$, for the intersection of the supports of x in the nominal set X. For instance, we have that $\mathsf{supp}_{\mathbb{A}}(a) = \{\, a \,\}$. For elements x and y of two, possibly distinct, nominal sets X and Y, we write $x \# y$ whenever $\mathsf{supp}_X(x)$ and $\mathsf{supp}_Y(y)$ are disjoint. Thus, for $a \in \mathbb{A}$ and $x \in X$, $a \# x$ stands for $a \notin \mathsf{supp}_X(x)$; that is, *$a$ is fresh for x*.

The category **Nom** is complete and cocomplete. In particular, for a possibly infinite family of nominal sets $\{\, X_i \,\}_{i \in I}$, the coproduct $\coprod_{i \in I} X_i$ is given by $|\coprod_{i \in I} X_i| = \coprod_{i \in I} |X_i|$ with action $\pi \cdot \iota_i(x) = \iota_i(\pi \cdot x)$; whilst the product $\prod_{i \in I} X_i$, for a finite set I, is given by $|\prod_{i \in I} X_i| = \prod_{i \in I} |X_i|$ with action $\pi \cdot \{x_i\}_{i \in I} = \{\pi \cdot x_i\}_{i \in I}$. As usual, we write X^n for $X \times \ldots \times X$ (n times).

Further, **Nom** carries a symmetric monoidal closed structure $(1, \#, [-,=])$. The *unit* 1 is the terminal object in **Nom** (*i.e.*, the singleton set consisting of the empty tuple equipped with the unique action). The *separating tensor* $X \# Y$ is the nominal subset of $X \times Y$ with underlying set given by $\{\, (x, y) \in |X| \times |Y| \mid x \# y \,\}$. We write $X^{\#\, n}$ for $X \# \ldots \# X$ (n times). For instance, $\mathbb{A}^{\#\, n}$ consists of n-tuples of distinct atoms equipped with the pointwise action $\pi \cdot (a_1, .., a_n) = (\pi \cdot a_1, .., \pi \cdot a_n)$. Note that $X^{\#\, 0}$ is 1 for any nominal set X. Henceforth

we write $\boldsymbol{a}^n$, or simply $\boldsymbol{a}$ when n is clear from the context, as a shorthand for a tuple $a_1, \ldots, a_n$ of distinct atoms, and $\{\boldsymbol{a}^n\}$ for the set $\{a_1, \ldots, a_n\}$. For $\boldsymbol{a}^n$ and $\boldsymbol{b}^n$ such that $a_i = b_j$ iff $a_j = b_i$ for all $1 \leq i, j \leq n$, the *multi-transposition* $(\boldsymbol{a}^n\ \boldsymbol{b}^n)$ denotes the bijection on A given by $(\boldsymbol{a}^n\ \boldsymbol{b}^n)(a_i) = b_i$ and $(\boldsymbol{a}^n\ \boldsymbol{b}^n)(b_i) = a_i$ for $1 \leq i \leq n$, and $(\boldsymbol{a}^n\ \boldsymbol{b}^n)(c) = c$ for $c \notin \{\boldsymbol{a}^n\} \cup \{\boldsymbol{b}^n\}$.

The separating tensor $\#$ is closed and the associated internal-hom functor is denoted $[-,=]$. In particular, the internal homs $[\mathbb{A}^{\#\,n}, X]$, for $n \in \mathbb{N}$ and $X \in \mathbf{Nom}$, give rise to a notion of multi-atom abstraction. Indeed, the nominal set $[\mathbb{A}^{\#\,n}, X]$ has underlying set given by the quotient $|\mathbb{A}^{\#\,n} \times X|/_{\approx_\alpha}$ under the *α-equivalence relation* $\approx_\alpha$ defined as follows: $(\boldsymbol{a}, x) \approx_\alpha (\boldsymbol{b}, x')$ if and only if there exists a *fresh* $\boldsymbol{c} \in \mathbb{A}^{\#\,n}$ (*i.e.*, a tuple $\boldsymbol{c} \in \mathbb{A}^{\#\,n}$ satisfying the condition $\boldsymbol{c} \# \boldsymbol{a}, x, \boldsymbol{b}, x'$) such that $(\boldsymbol{a}\ \boldsymbol{c}) \cdot x = (\boldsymbol{b}\ \boldsymbol{c}) \cdot x'$. We write $\langle \boldsymbol{a} \rangle\, x$ for the equivalence class $[(\boldsymbol{a}, x)]_{\approx_\alpha}$. The nominal set $[\mathbb{A}^{\#\,n}, X]$ has action $\pi \cdot \langle \boldsymbol{a} \rangle\, x = \langle \pi \cdot \boldsymbol{a} \rangle\, \pi \cdot x$ Note that $\mathsf{supp}(\langle \boldsymbol{a} \rangle\, x)$ is $\mathsf{supp}(x) \backslash \{\boldsymbol{a}\}$.

6.2. **Nominal Equational Systems.** We specify a class of MESs on **Nom**, called *Nominal Equational Systems (NESs)*.

Following [Clouston and Pitts, 2007] we define a *nominal signature* Σ to be a family of nominal sets $\{\, \Sigma(n) \,\}_{n \in \mathbb{N}}$, each of which consists of the *operators* of arity n.

Example 6.1. The nominal signature Σ_λ for the untyped λ-calculus is given by the nominal sets of operators

$$\Sigma_\lambda(0) = \{\, \boldsymbol{V}_a \mid a \in \mathbb{A} \,\}\,, \quad \Sigma_\lambda(1) = \{\, \boldsymbol{L}_a \mid a \in \mathbb{A} \,\}\,, \quad \Sigma_\lambda(2) = \{\, \boldsymbol{A} \,\}$$

with action

$$\pi \cdot \boldsymbol{V}_a = \boldsymbol{V}_{\pi(a)}\ , \quad \pi \cdot \boldsymbol{L}_a = \boldsymbol{L}_{\pi(a)}\ , \quad \pi \cdot \boldsymbol{A} = \boldsymbol{A}\ .$$

To each nominal signature Σ, we associate the strong endofunctor $(F_\Sigma, \mathsf{st}^\Sigma)$ on $\underline{\mathbf{Nom}} = (\mathbf{Nom}, 1, \#, [-,=])$ as follows:

$$\begin{array}{rcccc} F_\Sigma(X) & = & \coprod_{n \in \mathbb{N}} \Sigma(n) \times X^n\ , & & \\ \mathsf{st}^\Sigma_{X,Y} & : & F_\Sigma(X) \# Y & \to & F_\Sigma(X \# Y) \\ & & \big(\iota_n(\mathsf{o}, x_1, \ldots, x_n), y\big) & \mapsto & \iota_n\big(\mathsf{o}, (x_1, y), \ldots, (x_n, y)\big) \end{array}$$

for $X, Y \in \mathbf{Nom}$ and $n \in \mathbb{N}$, $\mathsf{o} \in \Sigma(n)$, $x_1, \ldots, x_n \in X$, $y \in Y$. Since **Nom** is cocomplete and the functor F_Σ is ω-cocontinuous, free F_Σ-algebras exist. The carrier of the free F_Σ-algebra $T_\Sigma X$ on X has the following inductive syntactic description:

$$\begin{array}{rcll} t \in T_\Sigma X & ::= & x & (\, x \in X\,) \\ & \mid & \mathsf{o}\,(t_1, \ldots, t_n) & (\, \mathsf{o} \in \Sigma(n),\ t_1, .., t_n \in T_\Sigma X\,) \end{array}$$

with action given by $\pi \cdot x = \pi \cdot_X x$ and $\pi \cdot \mathsf{o}(t_1, \ldots, t_n) = (\pi \cdot \mathsf{o})(\pi \cdot t_1, \ldots, \pi \cdot t_n)$. The associated *term monad* $\mathbf{T}_\Sigma = (T_\Sigma, \eta^\Sigma, \mu^\Sigma)$ is strong, with strength $\widehat{\mathsf{st}}^\Sigma$ given as follows:

$$\begin{array}{rcccc} \widehat{\mathsf{st}}^\Sigma_{X,Y} & : & T_\Sigma(X) \# Y & \to & T_\Sigma(X \# Y) \\ & : & (t, y) & \mapsto & t\{x \mapsto (x, y)\}_{x \in X} \end{array}$$

where $t\{x \mapsto (x, y)\}_{x \in X}$ denotes the term obtained by simultaneously substituting (x, y) for each x in the term t.

A NES is a MES $(\underline{\mathbf{Nom}}, \mathbf{T}_\Sigma, \mathcal{A})$ for Σ a nominal signature and $\mathcal{A}$ a set of equations with arities $\coprod_{i=1}^{\ell} \mathbb{A}^{\#\,n_i}$ for $\ell, n_1, \ldots, n_\ell \in \mathbb{N}$ and coarities $\mathbb{A}^{\#\,n}$ for $n \in \mathbb{N}$.

6.3. **Nominal Equational Presentations.** We introduce *Nominal Equational Presentations (NEPs)* as syntactic counterparts of NESs.

We define a *variable context* as a finite set of variables $|V|$ together with a function $V : |V| \to \mathbb{N}$ assigning a *valence* to each variable. We write $x_1 : n_1, \ldots, x_\ell : n_\ell$ for the variable context with variables $x_1, \ldots, x_\ell$ respectively of valence $n_1, \ldots, n_\ell$. Variable contexts are syntax for arities, and every such V determines the arity

$$\underline{V} = \coprod_{x \in |V|} \mathbb{A}^{\#\, V(x)} \,.$$

We write $x(\boldsymbol{a})$ for the element $\iota_x(\boldsymbol{a})$ of $\underline{V}$ and, when convenient, further abbreviate $x()$ as x.

For $n \in \mathbb{N}$ and a variable context V, the following bijection

$$\begin{aligned} &\{\, t : \mathbb{A}^{\#\, n} \to T_\Sigma \underline{V} \,\} \\ &\quad \cong \{\, t : 1 \to [\mathbb{A}^{\#\, n}, T_\Sigma \underline{V}] \,\} \\ &\quad \cong \{\, t \in [\mathbb{A}^{\#\, n}, T_\Sigma \underline{V}] \mid \mathsf{supp}(t) = \emptyset \,\} \\ &\quad = \{\, \langle \boldsymbol{a} \rangle\, t \in [\mathbb{A}^{\#\, n}, T_\Sigma \underline{V}] \mid \mathsf{supp}(t) \subseteq \{\boldsymbol{a}\} \,\} \end{aligned}$$

shows that a Kleisli map $\mathbb{A}^{\#\, n} \to T_\Sigma \underline{V}$ is determined by an α-equivalence class $\langle \boldsymbol{a} \rangle\, t$ for $\boldsymbol{a} \in \mathbb{A}^{\#\, n}$ and $t \in (T_\Sigma \underline{V})_{\boldsymbol{a}} = \{\, t \in T_\Sigma \underline{V} \mid \mathsf{supp}(t) \subseteq \{\boldsymbol{a}\} \,\}$. The set $(T_\Sigma \underline{V})_{\boldsymbol{a}}$ has the following inductive syntactic description:

$$\begin{array}{rcll} t \in (T_\Sigma \underline{V})_{\boldsymbol{a}} & ::= & x(\boldsymbol{a'}) & (\, x(\boldsymbol{a'}) \in \underline{V} \text{ such that } \{\boldsymbol{a'}\} \subseteq \{\boldsymbol{a}\} \,) \\ & \mid & \mathsf{o}(t_1, \ldots, t_n) & (\, \mathsf{o} \in \Sigma(n) \text{ such that } \mathsf{supp}(\mathsf{o}) \subseteq \{\boldsymbol{a}\}, \\ & & & \quad \text{and } t_1, \ldots, t_n \in (T_\Sigma \underline{V})_{\boldsymbol{a}} \,) \,. \end{array}$$

Directly motivated by this analysis, we define the notion of *Nominal Equational Presentation (NEP)* as follows. A *nominal context* $[\boldsymbol{a}]V$ consists of an atom context $\boldsymbol{a} \in \mathbb{A}^{\#\, n}$, for $n \in \mathbb{N}$, and a variable context V. A *nominal term* t, for a nominal signature Σ, in a nominal context $[\boldsymbol{a}]V$, denoted $[\boldsymbol{a}]V \vdash t$, is given by a term $t \in (T_\Sigma \underline{V})_{\boldsymbol{a}}$; a *nominal equation* $[\boldsymbol{a}]V \vdash t \equiv t'$ is given by a pair of nominal terms t and t' in the same nominal context $[\boldsymbol{a}]V$. A NEP $\mathbb{T} = (\Sigma, \mathcal{A})$ consists of a nominal signature Σ and a set of nominal equations $\mathcal{A}$.

By construction, thus, NEPs represent NESs. Indeed, a NEP $\mathbb{T} = (\Sigma, \mathcal{A})$ determines the NES $\underline{\mathbb{T}} = (\underline{\mathbf{Nom}}, \mathbf{T}_\Sigma, \underline{\mathcal{A}})$ with the set of equations $\underline{\mathcal{A}}$ given by

$$\{\, \underline{[\boldsymbol{a}]V \vdash t} \equiv \underline{[\boldsymbol{a}]V \vdash t'} : \mathbb{A}^{\#\, n} \to T_\Sigma \underline{V} \,\}_{(\, [\boldsymbol{a}]V \vdash t \equiv t' \,) \,\in\, \mathcal{A}}$$

where $\underline{[\boldsymbol{a}]V \vdash t}$ is the Kleisli map corresponding to $\langle \boldsymbol{a} \rangle\, t$ via the above bijection.

Example 6.2 (continued from Example 6.1, *cf.* [Gabbay and Mathijssen, 2007] and [Clouston and Pitts, 2007])**.** The NEP $\mathbb{T}_\lambda = (\Sigma_\lambda, \mathcal{A}_\lambda)$ for $\alpha\beta\eta$-equivalence of untyped λ-terms has the following equations:

$$
\begin{array}{lrcl}
(\alpha) & [a,b]\; x:1 & \vdash & \boldsymbol{L}_a.\, x(a) \;\equiv\; \boldsymbol{L}_b.\, x(b) \\
(\beta_\kappa) & [a]\; x:0, y:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, x\,,\, y(a)\big) \;\equiv\; x \\
(\beta_{\boldsymbol{V}}) & [a]\; x:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, \boldsymbol{V}_a\,,\, x(a)\big) \;\equiv\; x(a) \\
(\beta_{\boldsymbol{L}}) & [a,b]\; x:2, y:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, \boldsymbol{L}_b.\, x(a,b)\,,\, y(a)\big) \;\equiv\; \boldsymbol{L}_b.\, \boldsymbol{A}\big(\boldsymbol{L}_a.\, x(a,b)\,,\, y(a)\big) \\
(\beta_{\boldsymbol{A}}) & [a]\; x:1, y:1, z:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, \boldsymbol{A}(x(a), y(a))\,,\, z(a)\big) \\
 & & & \quad \equiv\; \boldsymbol{A}\big(\boldsymbol{A}\big(\boldsymbol{L}_a.\, x(a), z(a)\big)\,,\, \boldsymbol{A}\big(\boldsymbol{L}_a.\, y(a), z(a)\big)\big) \\
(\beta_\varepsilon) & [a,b]\; x:1 & \vdash & \boldsymbol{A}\big(\boldsymbol{L}_a.\, x(a), \boldsymbol{V}_b\big) \;\equiv\; x(b) \\
(\eta) & [a]\; x:0 & \vdash & \boldsymbol{L}_a.\, \boldsymbol{A}(x, \boldsymbol{V}_a) \;\equiv\; x
\end{array}
$$

where we write $\boldsymbol{L}_a.\, t$ for $\boldsymbol{L}_a(t)$.

6.4. **Model theory.** The model theory of a NEP $\mathbb{T} = (\Sigma, \mathcal{A})$ is derived from that of the NES $\underline{\mathbb{T}}$. This we now spell out in elementary terms.

A $\mathbb{T}$-*algebra* is an F_Σ-algebra $(M, \mathsf{e} : F_\Sigma M \to M)$ such that

$$[\![\, [\boldsymbol{a}]V \vdash t_1 \,]\!]_{(M,\mathsf{e})} = [\![\, [\boldsymbol{a}]V \vdash t_2 \,]\!]_{(M,\mathsf{e})} \;\; : [\![[\boldsymbol{a}]V]\!](M) \to M$$

for all nominal equations $[\boldsymbol{a}]V \vdash t_1 \equiv t_2$ in $\mathcal{A}$, where

$$[\![[\boldsymbol{a}]V]\!](M) \;=\; \big(\textstyle\prod_{x \in |V|} [\mathbb{A}^{\#\, V(x)}, M]\big) \# \mathbb{A}^{\#\, n}$$

and where $[\![\, [\boldsymbol{a}]V \vdash t \,]\!]_{(M,\mathsf{e})}$ is inductively defined as follows:

- $[\![\, [\boldsymbol{a}]V \vdash x(\boldsymbol{a'}) \,]\!]_{(M,\mathsf{e})}\big(\{\, \langle \boldsymbol{d}_x \rangle\, m_x \,\}_{x \in |V|},\, \boldsymbol{b}\big) = (\boldsymbol{d}_x\; \boldsymbol{b'}) \cdot m_x$ for $\boldsymbol{b'} = (\boldsymbol{a}\; \boldsymbol{b}) \cdot \boldsymbol{a'}$,
- $[\![\, [\boldsymbol{a}]V \vdash \mathsf{o}(t_1, .., t_k) \,]\!]_{(M,\mathsf{e})}\big(\{\, \langle \boldsymbol{d}_x \rangle\, m_x \,\}_{x \in |V|},\, \boldsymbol{b}\big) = \mathsf{e}_k(\mathsf{o}', t'_1, .., t'_k)$ for $\mathsf{e}_k : \Sigma(k) \times M^k \to M$ the k-component of the structure map e, and

$$\mathsf{o}' = (\boldsymbol{a}\; \boldsymbol{b}) \cdot \mathsf{o} \;\;, \quad t'_i = [\![\, [\boldsymbol{a}]V \vdash t_i \,]\!]_{(M,\mathsf{e})}\big(\{\, \langle \boldsymbol{d}_x \rangle\, m_x \,\}_{x \in |V|},\, \boldsymbol{b}\big) \;\;.$$

The category $\mathbb{T}$-**Alg** is the full subcategory of F_Σ-**Alg** consisting of $\mathbb{T}$-algebras. $\mathbb{T}$-**Alg** and $\underline{\mathbb{T}}$-**Alg** are isomorphic by construction.

Example 6.3 (continued from Example 6.2)**.** A $\mathbb{T}_\lambda$-algebra has a carrier $M \in \mathbf{Nom}$ with structure maps

$$[\![\boldsymbol{V}]\!] : \mathbb{A} \to M \;\;, \quad [\![\boldsymbol{L}]\!] : \mathbb{A} \times M \to M \;\;, \quad [\![\boldsymbol{A}]\!] : M^2 \to M$$

satisfying the equations of the theory. For instance, according to the equation (α), we have that

$$[\![\boldsymbol{L}]\!]\big(a, (c\; a) \cdot m\big) = [\![\boldsymbol{L}]\!]\big(b, (c\; b) \cdot m\big) \quad \text{for all } (\langle c \rangle\, m, (a,b)) \in [\mathbb{A}, M] \# \mathbb{A}^{\#\, 2}$$

and, according to the equation (η), we have that

$$[\![\boldsymbol{L}]\!]\big(a, [\![\boldsymbol{A}]\!](m, [\![\boldsymbol{V}]\!](a))\big) = m \quad \text{for all } (m, a) \in M \# \mathbb{A}\,.$$

$$\mathsf{Eqvar}\ \frac{[\boldsymbol{a}^n]V \vdash t \equiv t'}{[\boldsymbol{b}^n]V \vdash (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t \equiv (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t'}\ (\forall_{1\leq i,j\leq n}\ a_i = b_j \Leftrightarrow a_j = b_i)$$

$$\mathsf{Ref}\ \frac{}{[\boldsymbol{a}]V \vdash t \equiv t}\ [\boldsymbol{a}]V \vdash t \quad \mathsf{Sym}\ \frac{[\boldsymbol{a}]V \vdash t \equiv t'}{[\boldsymbol{a}]V \vdash t' \equiv t} \quad \mathsf{Trans}\ \frac{[\boldsymbol{a}]V \vdash t \equiv t' \quad [\boldsymbol{a}]V \vdash t' \equiv t''}{[\boldsymbol{a}]V \vdash t \equiv t''}$$

$$\mathsf{Axiom}\ \frac{}{[\boldsymbol{a}]V \vdash t \equiv t'}\ \big([\boldsymbol{a}]V \vdash t \equiv t'\big) \in \mathcal{A} \qquad \mathsf{Elim}\ \frac{[\boldsymbol{a}^n, \boldsymbol{b}^m]V \vdash t \equiv t'}{[\boldsymbol{a}^n]V \vdash t \equiv t'}\ \big(\boldsymbol{b} \# \boldsymbol{a}, t, t'\big)$$

$$\mathsf{Intro}\ \frac{[\boldsymbol{a}^n]V \vdash t \equiv t'}{[\boldsymbol{a}^n, \boldsymbol{b}^m]V^{\langle \boldsymbol{b}\rangle} \ \vdash\ t\{x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})\}_{x\in|V|} \ \equiv\ t'\{x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})\}_{x\in|V|}}\ (\boldsymbol{b} \# \boldsymbol{a})$$

$$\text{where } |V^{\langle \boldsymbol{b}\rangle}| \ =\ |V| \text{ and } \forall_{x\in|V|}\ V^{\langle \boldsymbol{b}\rangle}(x) \ =\ V(x) + m$$

$$\mathsf{Subst}\ \frac{[\boldsymbol{a}^n]U \vdash t \equiv t' \qquad \{\ [\boldsymbol{b}_x{}^{U(x)}]V \vdash s_x \equiv s'_x\ \}_{x\in|U|}}{[\boldsymbol{a}^n]V\ \vdash\ t\{x(\boldsymbol{b}_x) \mapsto s_x\}_{x\in|U|}\ \equiv\ t'\{x(\boldsymbol{b}_x) \mapsto s_x\}_{x\in|U|}}$$

Figure 1: Rules of SNEL.

6.5. **Synthetic Nominal Equational Logic.** For a NEP $\mathbb{T} = (\Sigma, \mathcal{A})$, we consider the EML associated to the NES $\underline{\mathbb{T}}$ in syntactic form and thereby obtain a deductive system for deriving valid nominal equations in $\mathbb{T}$-algebras. The resulting *Synthetic Nominal Equational Logic (SNEL)* has the inference rules given in Figure 1. The *substitution operation* in the Intro and Subst rules maps

$$t \in T_\Sigma \underline{U}\ ,\quad \{\ \langle \boldsymbol{c}_x\rangle\, m_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma(M)]\ \}_{x\in|U|}$$

to the nominal term

$$t\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|}\ \in\ T_\Sigma(M)$$

defined by structural induction on t as follows:

$$\begin{aligned} x(\boldsymbol{a})\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|} &= (\boldsymbol{c}_x\ \boldsymbol{a})\cdot m_x \\ \mathsf{o}(t_1, .., t_k)\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|} &= \mathsf{o}\big(t_1\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|}, \ldots, t_k\{x(\boldsymbol{c}_x) \mapsto m_x\}_{x\in|U|}\big)\,. \end{aligned}$$

By construction, if a nominal equation $[\boldsymbol{a}]V \vdash t \equiv t'$ is derivable in SNEL, then the equation $\underline{\mathcal{A}} \vdash \underline{[\boldsymbol{a}]V \vdash t} \equiv \underline{[\boldsymbol{a}]V \vdash t'}$ is derivable in EML. We now explain why this is so for each rule.

- The SNEL rule Eqvar arises from the fact that $\underline{[\boldsymbol{a}^n]V \vdash t} = \underline{[\boldsymbol{b}^n]V \vdash (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t} : \mathsf{A}^{\#n} \to T_\Sigma(\underline{V})$ for all nominal terms $[\boldsymbol{a}^n]V \vdash t$ and $[\boldsymbol{b}^n]V \vdash (\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t$.
- The SNEL rules Ref, Sym, Trans and Axiom directly mimic the corresponding EML rules.
- The SNEL rule Elim arises from the EML rule Local with respect to the epimorphic projection map $\mathbb{A}^{\#(n+m)} \twoheadrightarrow \mathbb{A}^{\#\,n}$ sending $(\boldsymbol{a}^n, \boldsymbol{b}^m)$ to $(\boldsymbol{a}^n)$.
- The SNEL rule Intro arises from the EML rule Ext extended with the nominal set $\mathbb{A}^{\#\,m}$.

Note that $\underline{[\boldsymbol{a}^n, \boldsymbol{b}^m]V^{\langle \boldsymbol{b}\rangle} \vdash t\{x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})\}_{x\in|V|}}$ amounts to the composite

$$\mathbb{A}^{\#(n+m)} \cong \mathbb{A}^{\#\,m} \# \mathbb{A}^{\#\,n} \xrightarrow{\langle \mathbb{A}^{\#\,m}\rangle\, \underline{[\boldsymbol{a}]V\vdash t}} T_\Sigma\big(\mathbb{A}^{\#\,m} \# \underline{V}\big) \cong T_\Sigma\big(\textstyle\coprod_{x\in|V|} \mathbb{A}^{\#(V(x)+m)}\big)\ .$$

- The SNEL rule Subst arises from the EML rule Subst together with the rule Local as follows:

$$\dfrac{\underline{[\boldsymbol{a}^n]U \vdash t} \equiv \underline{[\boldsymbol{a}^n]U \vdash t'} \qquad \dfrac{\{\,\underline{[\boldsymbol{b}_x{}^{U(x)}]V \vdash s_x} \equiv \underline{[\boldsymbol{b}_x{}^{U(x)}]V \vdash s'_x}\,\}_{x\in|U|}}{\big[\underline{[\boldsymbol{b}_x]V \vdash s_x}\big]_{x\in|U|} \;\equiv\; \big[\underline{[\boldsymbol{b}_x]V \vdash s'_x}\big]_{x\in|U|}}\ \text{(by Local)}}{\underline{[\boldsymbol{a}]U \vdash t}\Big\{\big[\underline{[\boldsymbol{b}_x]V \vdash s_x}\big]_{x\in|U|}\Big\} \;\equiv\; \underline{[\boldsymbol{a}]U \vdash t'}\Big\{\big[\underline{[\boldsymbol{b}_x]V \vdash s'_x}\big]_{x\in|U|}\Big\}}\ \text{(by Subst)}$$

  where the rule Local applies with respect to the jointly epimorphic family of maps

$$\Big\{\iota_x : \mathbb{A}^{\#\,U(x)} \to \Big(\coprod_{x\in|U|} \mathbb{A}^{\#\,U(x)}\Big) = \underline{U}\Big\}_{x\in|U|}\ .$$

  Note that $\underline{[\boldsymbol{a}]U \vdash t}\Big\{\big[\underline{[\boldsymbol{b}_x]V \vdash s_x}\big]_{x\in|U|}\Big\} \;=\; \underline{[\boldsymbol{a}]V \vdash t\{x(\boldsymbol{b}_x) \mapsto s_x\}_{x\in|U|}}$.

Thus, the soundness of SNEL follows from that of EML.

Example 6.4 (continued from Example 6.3)**.** We give a derivation of

$$[a]\ x:1, y:0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_a.\,x(a), y) \;\equiv\; \boldsymbol{L}_a.\,x(a)$$

in the SNEL of $\mathbb{T}_\lambda$.

$$\mathbf{A}:\ \dfrac{\begin{array}{rll} & [a,b]\ x:1 \vdash \boldsymbol{L}_a.\,x(a) \;\equiv\; \boldsymbol{L}_b.\,x(b) & \text{by Axiom } (\alpha) \\ x \mapsto & [c]\ x:1, y:0 \vdash x(c) \;\equiv\; x(c) & \text{by Ref} \end{array}}{[a,b]\ x:1, y:0 \vdash \boldsymbol{L}_a.\,x(a) \;\equiv\; \boldsymbol{L}_b.\,x(b)}\ \text{by Subst}$$

$$\mathbf{B}:\ \dfrac{\begin{array}{rll} & [a,b]\ z:2, w:0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.z(a,b), w) \;\equiv\; \boldsymbol{A}(\boldsymbol{L}_a.z(a,b), w) & \text{by Ref} \\ z \mapsto & [a,b]\ x:1, y:0 \vdash \boldsymbol{L}_a.\,x(a) \;\equiv\; \boldsymbol{L}_b.\,x(b) & \text{by } \mathbf{A} \\ w \mapsto & [\ ]\ x:1, y:0 \vdash y \;\equiv\; y & \text{by Ref} \end{array}}{[a,b]\ x:1, y:0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_a.\,x(a), y) \;\equiv\; \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_b.\,x(b), y)}\ \text{by Subst}$$

$$\mathbf{C}:\ \dfrac{[a]\ x:0, y:1 \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\,x\,,\ y(a)\big) \;\equiv\; x \quad \text{by Axiom } (\beta_\kappa)}{[a,b]\ x:1, y:2 \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\,x(b)\,,\ y(a,b)\big) \;\equiv\; x(b)}\ \text{by Intro}$$

$$\mathbf{D}:\ \dfrac{\begin{array}{rll} & [a,b]\ x:1, y:2 \vdash \boldsymbol{A}\big(\boldsymbol{L}_a.\,x(b)\,,\ y(a,b)\big) \;\equiv\; x(b) & \text{by } \mathbf{C} \\ x \mapsto & [b]\ x:1, y:0 \vdash \boldsymbol{L}_b.\,x(b) \;\equiv\; \boldsymbol{L}_b.\,x(b) & \text{by Ref} \\ y \mapsto & [a,b]\ x:1, y:0 \vdash y \;\equiv\; y & \text{by Ref} \end{array}}{[a,b]\ x:1, y:0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_b.\,x(b), y) \;\equiv\; \boldsymbol{L}_b.\,x(b)}\ \text{by Subst}$$

$$\dfrac{[a,b]\ x:1, y:0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_a.\,x(a), y) \equiv \boldsymbol{L}_a.\,x(a) \quad \text{by Trans(Trans}(\mathbf{B},\mathbf{D}),\text{Sym}(\mathbf{A}))}{[a]\ x:1, y:0 \vdash \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_a.\,x(a), y) \equiv \boldsymbol{L}_a.\,x(a)}\ \text{by Elim}$$

$$\mathsf{Ref}\ \frac{}{t \approx_E t}\ t \in |T_\Sigma X| \qquad \mathsf{Sym}\ \frac{t \approx_E t'}{t' \approx_E t} \qquad \mathsf{Trans}\ \frac{t \approx_E t' \quad t' \approx_E t''}{t \approx_E t''}$$

$$\mathsf{Axiom}\ \frac{}{\big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t\big)\{x(c_x) \mapsto s_x\}_{x\in|U|} \approx_E \big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t'\big)\{x(c_x) \mapsto s_x\}_{x\in|U|}}\ \big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in E$$

$$\boldsymbol{b}^n \in \mathbb{A}^{\#\,n}\,,\ \{\,\langle \boldsymbol{c}_x\rangle\, s_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma X]\,\}_{x\in|U|}\ \text{ such that}$$
$$\forall_{1\le i,j\le n}\ a_i = b_j \Leftrightarrow a_j = b_i \ \text{ and }\ \forall_{x\in|U|}\ \boldsymbol{b}^n \# \langle \boldsymbol{c}_x\rangle\, s_x$$

$$\mathsf{Cong}\ \frac{s_1 \approx_E s'_1,\ \ldots,\ s_k \approx_E s'_k}{\mathsf{o}(s_1,..,s_k) \approx_E \mathsf{o}(s'_1,..,s'_k)}\ \big(\mathsf{o} \in \Sigma(k)\big)$$

Figure 2: Rules for $\approx_E$.

6.6. **Completeness.** We provide a sound and complete rewriting-style deduction system for NEPs, referred to as *Synthetic Nominal Rewriting (SNR)*.

For every NEP $\mathbb{T}$, the associated NES $\underline{\mathbb{T}}$ is inductive. Indeed, using that in **Nom** finite limits commute with filtered colimits and equivariant functions are epimorphic iff their underlying function is surjective, one sees that the endofunctor F_Σ associated to a nominal signature Σ preserves colimits of ω-chains and epimorphisms. Moreover, since for every $n \in \mathbb{N}$, the right adjoint $[\mathsf{A}^{\#n}, -]$ is also a left adjoint, it follows that, for every variable context V, the nominal set $\underline{V}$ is compact and projective.

For a NEP $\mathbb{T} = (\Sigma, \mathcal{A})$, we consider the construction (4.2) for the associated NES $\underline{\mathbb{T}}$. Since the forgetful functor $|-| : \mathbf{Nom} \to \mathbf{Set}$ creates colimits, we have the following explicit description. For a nominal set X, the nominal set $(T_\Sigma X)_1$ has as underlying set the quotient $|T_\Sigma X|/_{\approx_1}$ under the equivalence relation $\approx_1$ on $|T_\Sigma X|$ generated by the following rule:

$$\frac{}{\big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t\big)\{x(c_x) \mapsto s_x\}_{x\in|U|} \approx_1 \big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t'\big)\{x(c_x) \mapsto s_x\}_{x\in|U|}}\ \big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in \mathcal{A}$$

for $\boldsymbol{b}^n \in \mathbb{A}^{\#\,n}$, $\{\,\langle \boldsymbol{c}_x\rangle\, s_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma X]\,\}_{x\in|U|}$ such that $\forall_{1\le i,j\le n}\ a_i = b_j \Leftrightarrow a_j = b_i$ and $\forall_{x\in|U|}\ \boldsymbol{b}^n \# \langle \boldsymbol{c}_x\rangle\, s_x$. The action of $(T_\Sigma X)_1$ is given by $\pi \cdot [t]_{\approx_1} = [\pi \cdot t]_{\approx_1}$. The equivariant function $q_0 : T_\Sigma X \twoheadrightarrow (T_\Sigma X)_1$ maps a nominal term t to its equivalence class $[t]_{\approx_1}$.

The nominal set $(T_\Sigma X)_n$, for $n \ge 2$, has as underlying set the quotient $|T_\Sigma X|/_{\approx_n}$ under the equivalence relation $\approx_n$ on $|T_\Sigma X|$ generated by the following rules:

$$\frac{s \approx_{n-1} s'}{s \approx_n s'} \qquad \frac{s_1 \approx_{n-1} s'_1,\ \ldots,\ s_k \approx_{n-1} s'_k}{\mathsf{o}(s_1,..,s_k) \approx_n \mathsf{o}(s'_1,..,s'_k)}\ \big(\mathsf{o} \in \Sigma(k)\big)$$

The action of $(T_\Sigma X)_n$ is given by $\pi \cdot [t]_{\approx_n} = [\pi \cdot t]_{\approx_n}$. The equivariant function $q_{n-1} : (T_\Sigma X)_{n-1} \twoheadrightarrow (T_\Sigma X)_n$ maps $[t]_{\approx_{n-1}}$ to $[t]_{\approx_n}$.

The nominal set $T_{\underline{\mathbb{T}}} X$, being the colimit of the chain $\{\, q_n : (T_\Sigma X)_n \twoheadrightarrow (T_\Sigma X)_{n+1}\,\}_{n\ge 0}$, is given by $|T_{\underline{\mathbb{T}}} X| = |T_\Sigma X|/_{\approx_E}$ with action $\pi \cdot [t]_{\approx_E} = [\pi \cdot t]_{\approx_E}$, for $\approx_E$ the equivalence relation on $|T_\Sigma X|$ specified by the rules of Figure 2. The quotient map $\mathsf{q}^{\mathbb{T}}_X : T_\Sigma X \to T_{\underline{\mathbb{T}}} X$ sends a nominal term t to its equivalence class $[t]_{\approx_E}$.

The rules Axiom and Cong of Figure 2 can be merged into a single one to yield a rewriting-style deduction system. Indeed, by an induction on the depth of proof trees, one shows that the relation $\approx_E$ coincides with the equivalence relation $\approx^{\mathsf{R}}_E$ generated by the

rule

$$\mathbf{C}\big[\big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t\big)\{x(c_x)\mapsto s_x\}_{x\in|U|}\big]\ \approx^{\mathsf{R}}_E\ \mathbf{C}\big[\big((\boldsymbol{a}^n\ \boldsymbol{b}^n)\cdot t'\big)\{x(c_x)\mapsto s_x\}_{x\in|U|}\big] \tag{6.1}$$

for $\big([\boldsymbol{a}^n]U\vdash t\equiv t'\big)\in\mathcal{A}$ and $\mathbf{C}[-]$ a context with one hole and possibly with elements from X, and for $\boldsymbol{b}^n\in\mathbb{A}^{\#\,n}$, $\{\,\langle\boldsymbol{c}_x\rangle\, s_x\in[\mathbb{A}^{\#\,U(x)},T_\Sigma X]\,\}_{x\in|U|}$ such that $\forall_{1\le i,j\le n}\ a_i=b_j\Leftrightarrow a_j=b_i$ and $\forall_{x\in|U|}\ \boldsymbol{b}^n\,\#\,\langle\boldsymbol{c}_x\rangle\, s_x$. Henceforth, the rewriting of nominal terms by the rule (6.1) is referred to as *Synthetic Nominal Rewriting (SNR)*.

The soundness and completeness of SNR is established by means of the internal completeness of the NES $\underline{\mathbb{T}}$:

$$\begin{aligned}
&[\boldsymbol{a}^n]V\vdash s\equiv s' \text{ is satisfied by all } \mathbb{T}\text{-algebras}\\
&\iff\ \underline{\mathbb{T}}\text{-}\mathbf{Alg}\ \models\ \underline{[\boldsymbol{a}^n]V\vdash s}\equiv\underline{[\boldsymbol{a}^n]V\vdash s'}\ :\ \mathbb{A}^{\#\,n}\to T_\Sigma\underline{V}\\
&\iff\ \mathsf{q}^{\mathbb{T}}_{\underline{V}}\circ\underline{[\boldsymbol{a}^n]V\vdash s}\ =\ \mathsf{q}^{\mathbb{T}}_{\underline{V}}\circ\underline{[\boldsymbol{a}^n]V\vdash s'}\ :\ \mathbb{A}^{\#\,n}\to T_{\underline{\mathbb{T}}}\underline{V}\\
&\iff\ \mathsf{q}^{\mathbb{T}}_{\underline{V}}\big(\underline{[\boldsymbol{a}^n]V\vdash s}(\boldsymbol{a}^n)\big)\ =\ \mathsf{q}^{\mathbb{T}}_{\underline{V}}\big(\underline{[\boldsymbol{a}^n]V\vdash s'}(\boldsymbol{a}^n)\big)\quad\text{in } |T_{\underline{\mathbb{T}}}\underline{V}|/_{\approx_E}\\
&\iff\ [s]_{\approx_E}=[s']_{\approx_E}\quad\text{in } |T_{\underline{\mathbb{T}}}\underline{V}|/_{\approx_E}\\
&\iff\ s\approx_E s'\quad\text{in } |T_{\underline{\mathbb{T}}}\underline{V}|\\
&\iff\ s\approx^{\mathsf{R}}_E s'\quad\text{in } |T_{\underline{\mathbb{T}}}\underline{V}|\ .
\end{aligned}$$

The deductive completeness of SNEL follows, as every proof of $s\approx_E s'$, for $s,s'\in(T_\Sigma\underline{V})_{\boldsymbol{a}}$ can be readily turned into a proof of $[\boldsymbol{a}]V\vdash s\equiv s'$ in SNEL.

Example 6.5 (continued from Example 6.4)**.** We give a derivation of

$$[a]\ x:1,y:0\vdash \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_a.\,x(a),y)\ \equiv\ \boldsymbol{L}_a.\,x(a)$$

in SNR (*cf.* the proof given in Example 6.4):

$$\begin{aligned}
&\boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_a.\,x(a),y)\\
&\qquad\text{by }(\alpha):\ \approx^{\mathsf{R}}_E\ \begin{array}{l}\boldsymbol{A}(\boldsymbol{L}_a.\big[\quad \boldsymbol{L}_a.\,x(a)\{x(c)\mapsto x(c)\}\quad\big],y)\\ \boldsymbol{A}(\boldsymbol{L}_a.\big[\quad \boldsymbol{L}_b.\,x(b)\{x(c)\mapsto x(c)\}\quad\big],y)\end{array}\\
&\approx^{\mathsf{R}}_E\quad \boldsymbol{A}(\boldsymbol{L}_a.\,\boldsymbol{L}_b.\,x(b),y)\\
&\qquad\text{by }(\beta_\kappa):\ \approx^{\mathsf{R}}_E\ \begin{array}{l}\big[\quad \boldsymbol{A}(\boldsymbol{L}_a.\,x,y(a))\{x\mapsto \boldsymbol{L}_b.\,x(b);y(a)\mapsto y\}\quad\big]\\ \big[\qquad\qquad\qquad x\{x\mapsto \boldsymbol{L}_b.\,x(b);y(a)\mapsto y\}\quad\big]\end{array}\\
&\approx^{\mathsf{R}}_E\quad \boldsymbol{L}_b.\,x(b)\\
&\qquad\text{by }(\alpha):\ \approx^{\mathsf{R}}_E\ \begin{array}{l}\big[\quad \boldsymbol{L}_b.\,x(b)\{x(c)\mapsto x(c)\}\quad\big]\\ \big[\quad \boldsymbol{L}_a.\,x(a)\{x(c)\mapsto x(c)\}\quad\big]\end{array}\\
&\approx^{\mathsf{R}}_E\quad \boldsymbol{L}_a.\,x(a)\ .
\end{aligned}$$

6.7. **Related work.** Algebraic structure and rewriting in a nominal setting have already been considered in the literature. Gabbay and Mathijssen [2006, 2007] and Clouston and Pitts [2007] introduced an essentially equivalent notion of nominal algebra and provided sound and complete equational logics for them, whilst Fernández et al. [2004] introduced nominal rewriting.

Our SNEL and the *Nominal Equational Logic (NEL)* of Clouston and Pitts [2007] are equivalent. Indeed, see Hur [2010, Section 8.2.6] for a translation between the equality judgements of SNEL and NEL that respects the corresponding satisfaction relations. Thus, by virtue of the associated completeness theorems, SNEL and NEL establish the same theorems under different syntactic presentations.

The *Nominal Rewriting (NR)* of [Fernández and Gabbay, 2007] appears to be a term-rewriting version of NEL. However, it has the shortcoming of not being complete for nominal equational reasoning (see [Hur, 2010, Section 8.2.7]).

Finally, our approach allows us to also put the *Equational Logic for Binding Terms (ELBT)* of Hamana [2003] in the nominal context. Whereas SNEL arises from an EML on $\mathbf{Nom}$, ELBT arises from a related EML on the super-category $\mathbf{Set}^{\mathbb{I}}$, for $\mathbb{I}$ the category of finite sets and injections. Crucially, however, the embedding $\mathbf{Nom} \hookrightarrow \mathbf{Set}^{\mathbb{I}}$ does not preserve the epimorphic projection map $p_{n,m} : \mathbb{A}^{\#(n+m)} \to \mathbb{A}^{\#n}$ $(n \geq 0, m > 0)$. Thus, the only essential difference between SNEL and ELBT, is that the latter lacks the rule Elim (which arises from the EML rule Local with respect to $p_{n,m}$) as it is unsound.

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Appendix A. Proof of Theorem 2.1

Notation. For $f : V \otimes A \to B$, we write $\overline{f} : A \to [V, B]$ for the transpose of f with respect to the adjunction $V \otimes - \dashv [V, -]$ and $\widehat{f} : V \to [A, B]$ for the transpose of f with respect to the adjuction $- \otimes A \dashv [A, -]$.

Proof of 2.1. We establish the soundness of each EML rule; *i.e.*, that every $\mathbb{S}$-algebra satisfying the premises of a rule also satisfies the conclusion. The soundness of the rules Ref, Sym, Trans, and Axiom is trivial. To show the soundness of the rule Subst, one uses that, for all $u_1 : C \to TB$ and $u_2 : B \to TA$,

$$[\![u_1\{u_2\}]\!]_{(X,s)} = [\![u_1]\!]_{(X,s)} \circ \big(\overline{[\![u_2]\!]_{(X,s)}} \otimes C\big) : [A, X] \otimes C \to X\,.$$

To show the soundness of the rule Ext, one uses that, for all $u : C \to TA$,

$$[\![\langle V\rangle u]\!]_{(X,s)} = [\![u]\!]_{(X,s)} \circ (\widehat{\mathbf{p}} \otimes C) \circ \widetilde{\alpha}^{-1}_{[V\otimes A,X],V,C} : [V \otimes A, X] \otimes (V \otimes C) \to X\,,$$

where

$$\mathbf{p} = \Big(\ \big([V \otimes A, X] \otimes V\big) \otimes A \xrightarrow{\widetilde{\alpha}_{[V\otimes A,X],V,A}} [V \otimes A, X] \otimes (V \otimes A) \xrightarrow{\epsilon_X^{V\otimes A}} X\ \Big)\,.$$

Finally, to show the soundness of the rule Local, one uses that, for all $u : C \to TA$ and $e : C' \to C$,

$$[\![u \circ e]\!]_{(X,s)} = [\![u]\!]_{(X,s)} \circ ([A, X] \otimes e) : [A, X] \otimes C' \to X$$

and that, for every jointly epimorphic family $\{\, e_i : C_i \to C \,\}_{i\in I}$, the family $\{\, [A, X] \otimes e_i : [A, X] \otimes C_i \to [A, X] \otimes C \,\}_{i\in I}$ is also jointly epimorphic. $\square$

Appendix B. Proof of Theorem 3.1

We introduce several lemmas before proceeding to prove the theorem.

Notation. For $f : V \otimes A \to B$, we write $\overline{f} : A \to [V, B]$ for the transpose of f with respect to the adjunction $V \otimes - \dashv [V, -]$ and $\widehat{f} : V \to [A, B]$ for the transpose of f with respect to the adjuction $- \otimes A \dashv [A, -]$.

Lemma B.1. Let $\mathbb{S} = (\mathscr{C}, \mathbf{T}, \mathcal{A})$ be a MES. For every $\mathbb{S}$-algebra $(X, s : TX \to X)$, the T-algebra $\big([V, X], s^V : T[V, X] \to [V, X]\big)$, where s^V is the transpose of

$$\underline{s^V} = \big(V \otimes T[V, X] \xrightarrow{\mathsf{st}_{V,[V,X]}} T(V \otimes [V, X]) \xrightarrow{T(\epsilon^V_X)} TX \xrightarrow{s} X \big),$$

is an $\mathbb{S}$-algebra.

Proof. That $([V, X], s^V)$ is an Eilenberg-Moore algebra follows from transposing the following identities:

(1) $\underline{s^V} \circ (V \otimes \eta_{[V,X]}) \;=\; \epsilon^V_X \;:\; V \otimes [V, X] \to X,$

(2) $\underline{s^V} \circ (V \otimes \mu_{[V,X]}) \;=\; \underline{s^V} \circ (V \otimes T(\underline{s^V})) \;:\; V \otimes TT[V, X] \to X.$

To show that every equation in $\mathcal{A}$ is satisfied in $([V, X], s^V)$, one uses that, for all $w : C \to TA$,

$$[\![w]\!]_{([V,X],s^V)} \;=\; \overline{[\![w]\!]_{(X,s)} \circ (\widehat{\mathbf{p}} \otimes C) \circ \alpha^{-1}_{V,[A,[V,X]],C}} \;:\; [A, [V, X]] \otimes C \to [V, X]$$

where

$$\mathbf{p} = \big((V \otimes [A, [V, X]]) \otimes A \xrightarrow{\alpha_{V,[A,[V,X]],A}} V \otimes ([A, [V, X]] \otimes A) \xrightarrow{V \otimes \epsilon^A_{[V,X]}} V \otimes [V, X] \xrightarrow{\epsilon^V_X} X \big).$$

□

Lemma B.2. For $\mathbb{S} = (\mathscr{C}, \mathbf{T}, \mathcal{A})$ a MES admitting free algebras, the free $\mathbb{S}$-algebra monad $\mathbf{T}_{\mathbb{S}}$ on $\mathscr{C}$ is strong. The components of the strength $\mathsf{st}^{\mathbb{S}}$ are given by the unique maps such that the following diagram commutes:

$$\begin{tikzcd}
V \otimes TT_{\mathbb{S}}X \arrow[r, "\mathsf{st}_{V,T_{\mathbb{S}}X}"] \arrow[d, "V \otimes \tau^{\mathbb{S}}_X"'] & T(V \otimes T_{\mathbb{S}}X) \arrow[r, "T(\mathsf{st}^{\mathbb{S}}_{V,X})"] & TT_{\mathbb{S}}(V \otimes X) \arrow[d, "\tau^{\mathbb{S}}_{V \otimes X}"] \\
V \otimes T_{\mathbb{S}}X \arrow[rr, dashed, "\exists!\, \mathsf{st}^{\mathbb{S}}_{V,X}"] & & T_{\mathbb{S}}(V \otimes X) \\
V \otimes X \arrow[u, "V \otimes \eta^{\mathbb{S}}_X"] \arrow[urr, "\eta^{\mathbb{S}}_{V \otimes X}"'] & &
\end{tikzcd}$$

Proof. First note that $\overline{\mathsf{st}^{\mathbb{S}}_{V,X}} : T_{\mathbb{S}}X \to [V, T_{\mathbb{S}}(V \otimes X)]$ is the unique homomorphic extension of $\overline{\eta^{\mathbb{S}}_{V \otimes X}} : X \to [V, T_{\mathbb{S}}(V \otimes X)]$ with respect to the $\mathbb{S}$-algebra $\big([V, T_{\mathbb{S}}(V \otimes X)], (\tau^{\mathbb{S}}_{V \otimes X})^V\big)$.

The naturality of $\mathsf{st}^{\mathbb{S}}$ follows from the fact that, for all $f : V \to V'$ and $g : C \to C'$, the maps

$$\overline{T_{\mathbb{S}}(f \otimes g) \circ \mathsf{st}^{\mathbb{S}}_{V,X}} \;,\; \overline{\mathsf{st}^{\mathbb{S}}_{V',X'} \circ (f \otimes T_{\mathbb{S}}(g))} \;:\; T_{\mathbb{S}}X \to [V, T_{\mathbb{S}}(V' \otimes X')]$$

are both an homomorphic extension of $\overline{\eta^{\mathbb{S}}_{V' \otimes X'} \circ (f \otimes g)} : X \to [V, T_{\mathbb{S}}(V' \otimes X')]$ with respect to the $\mathbb{S}$-algebra $\big([V, T_{\mathbb{S}}(V' \otimes X')], (\tau^{\mathbb{S}}_{V' \otimes X'})^V\big)$.

Three of the four coherence conditions for strength follow from the fact that the maps

$$\begin{array}{rcl} \overline{T_{\mathbb{S}}(\lambda_X) \circ \mathsf{st}^{\mathbb{S}}_{I,X}} \ , \ \overline{\lambda_{T_{\mathbb{S}}X}} & : & T_{\mathbb{S}}X \to [I, T_{\mathbb{S}}X] \\ \overline{T_{\mathbb{S}}(\alpha_{U,V,X}) \circ \mathsf{st}^{\mathbb{S}}_{U.V,X}} \ , \ \overline{\mathsf{st}^{\mathbb{S}}_{U,V\otimes X} \circ (U \otimes \mathsf{st}^{\mathbb{S}}_{V,X}) \circ \alpha_{U,V,T_{\mathbb{S}}X}} & : & T_{\mathbb{S}}X \to [U \otimes V, T_{\mathbb{S}}(U \otimes (V \otimes X))] \\ \overline{\mathsf{st}^{\mathbb{S}}_{V,X} \circ (V \otimes \mu^{\mathbb{S}}_X)} \ , \ \overline{\mu^{\mathbb{S}}_{V\otimes X} \circ T_{\mathbb{S}}(\mathsf{st}^{\mathbb{S}}_{V,X}) \circ \mathsf{st}^{\mathbb{S}}_{V,T_{\mathbb{S}}X}} & : & T_{\mathbb{S}}T_{\mathbb{S}}X \to [V, T_{\mathbb{S}}(V \otimes X)] \end{array}$$

are respectively homomorphic extensions of

$$\begin{array}{rcl} \overline{\eta^{\mathbb{S}}_X \circ \lambda_X} & : & X \to [I, T_{\mathbb{S}}X] \\ \overline{\eta^{\mathbb{S}}_{U\otimes(V\otimes X)} \circ \alpha_{U,V,X}} & : & X \to [U \otimes V, T_{\mathbb{S}}(U \otimes (V \otimes X))] \\ \overline{\mathsf{st}^{\mathbb{S}}_X} & : & T_{\mathbb{S}}X \to [V, T_{\mathbb{S}}(V \otimes X)] \end{array}$$

with respect to the $\mathbb{S}$-algebras $\big([I, T_{\mathbb{S}}X], (\tau^{\mathbb{S}}_X)^I\big)$, $\big([U\otimes V, T_{\mathbb{S}}(U\otimes(V\otimes X))], (\tau^{\mathbb{S}}_{U\otimes(V\otimes X)})^{(U\otimes V)}\big)$, $\big([V, T_{\mathbb{S}}(V\otimes X)], (\tau^{\mathbb{S}}_{V\otimes X})^V\big)$. The remaining coherence condition is the triangle in the diagram above. □

Lemma B.3. Let $\mathbb{S} = (\mathscr{C}, \mathbf{T}, \mathcal{A})$ be a MES admitting free algebras. Then, the quotient natural transformation $\mathsf{q}^{\mathbb{S}} : T \dot{\to} T_{\mathbb{S}}$ is a strong functor morphism between the strong monads $\mathbf{T}$ and $\mathbf{T}_{\mathbb{S}}$. That is, the following diagram commutes:

$$\begin{array}{ccc} V \otimes TX & \xrightarrow{V\otimes \mathsf{q}^{\mathbb{S}}_X} & V \otimes T_{\mathbb{S}}X \\ {\scriptstyle \mathsf{st}_{V,X}}\downarrow & & \downarrow{\scriptstyle \mathsf{st}^{\mathbb{S}}_{V,X}} \\ T(V \otimes X) & \xrightarrow[\mathsf{q}^{\mathbb{S}}_{V\otimes X}]{} & T_{\mathbb{S}}(V \otimes X) \end{array}$$

Proof. The commutativity of the diagram follows from the fact that both

$$\overline{\mathsf{st}^{\mathbb{S}}_{V,X} \circ (V \otimes \mathsf{q}^{\mathbb{S}}_X)} \ , \ \overline{\mathsf{q}^{\mathbb{S}}_{V\otimes X} \circ \mathsf{st}_{V,X}} \ : \ TX \to [V, T_{\mathbb{S}}(V \otimes X)]$$

are an homomorphic extension of $\overline{\eta^{\mathbb{S}}_{V\otimes X}} : X \to [V, T_{\mathbb{S}}(V\otimes X)]$ with respect to the Eilenberg-Moore algebra $([V, T_{\mathbb{S}}(V \otimes X)], (\tau^{\mathbb{S}}_{V\otimes X})^V)$ for the monad $\mathbf{T}$. □

We are now ready to prove the internal completeness theorem.

Proof of Theorem 3.1. We show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

1 ⇒ 2. Holds vacuously.

2 ⇒ 3. Because, for all $w : C \to TA$, the map $\mathsf{q}^{\mathbb{S}}_A \circ w : C \to T_{\mathbb{S}}A$ factors as the composite

$$[\![w]\!]_{(T_{\mathbb{S}}A, \tau^{\mathbb{S}}_A)} \circ (\widehat{\mathbf{p}} \otimes C) \circ \lambda_C^{-1}$$

for $\mathbf{p} = (I \otimes A \xrightarrow{\lambda_A} A \xrightarrow{\eta^{\mathbb{S}}_A} T_{\mathbb{S}}A)$.

3 ⇒ 1. Because for every $(X, s : TX \to X) \in \mathbb{S}$-**Alg** and $w : C \to TA$, the interpretation map $[\![w]\!]_{(X,s)} : [A, X] \otimes C \to X$ factors as the composite

$$s^* \circ T_{\mathbb{S}}(\epsilon^A_X) \circ \mathsf{st}^{\mathbb{S}}_{[A,X],A} \circ \big([A, X] \otimes (\mathsf{q}^{\mathbb{S}}_A \circ w)\big)$$

where $s^* : T_{\mathbb{S}}X \to X$ is the unique homomorphic extension of the identiy map on X with respect to the $\mathbb{S}$-algebra (X, s). □