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Risk-dominance and perfect foresight dynamics in *N*-player games

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Abstract

In *perfect foresight dynamics*, an action is *linearly stable* if expectation that people will always choose the action is self-fulfilling. A symmetric game is a *PIM game* if an opponent's particular action maximizes the incentive of an action, independently of the rest of the players. This class includes supermodular games, games with linear incentives and so forth. We show that, in PIM games, linear stability is equivalent to *u-dominance*, a generalization of risk-dominance, and that there is no path escaping a *u*-dominant equilibrium. Existing results on *N*-player coordination games, games with linear incentives and two-player games are obtained as corollaries.

JEL classification: C72; C73

Keywords: Equilibrium selection; PIM games; *u*-Dominance; Risk-dominance; Perfect foresight dynamics; Linear stability; Network games; Linear incentives; ***p***-Dominance

1. Introduction

Coordination problems appear in various areas of economics. Examples include banking [5], international finance [25], business cycles [4] and so forth. Game theoretic models of such economic situations have more than one Nash equilibria. Multiplicity of equilibria is a challenge to economic theory: economic analysis based on Nash equilibrium is silent about "the most likely outcome" when there are multiple equilibria. Thus theories of equilibrium

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selection are called for, which try to single out one particular equilibrium from among more than one equilibria.

This paper investigates equilibrium selection based on perfect foresight dynamics, as proposed by Matsui and Matsuyama [18]. Perfect foresight dynamics is a large-population dynamics with *forward-looking and rational individuals*. At each moment of time people are matched randomly and play a game. Opportunities to revise actions follow a Poisson process, which represents the inertia of action revisions. Each agent chooses an action that maximizes the expected discounted payoff given the expectation about the future path of behavior. This expectation is consistent in the sense that the actual path of behavior is the same as the expected one. Such a consistent path, called a perfect foresight path, models how "self-fulfilling expectations" govern the movement of behavior patterns in a large society.

Perfect foresight dynamics has been applied to various areas of economics such as business cycles [6], international trade [17] and development [14,19]. These models differ from Matsui and Matsuyama [18] in some respects (for example, they do not assume random matching structures), but all these related models share several crucial features, such as inertia of action revisions and perfect foresight of agents. These features are also shared by Burdzy et al. [1], who consider equilibrium selection based on perfect foresight dynamics in which payoff of the base game changes stochastically over time.

Perfect foresight dynamics is useful as a device for equilibrium selection as well as economic analysis. On the one hand, some Nash equilibria may be fragile in the sense that there exists a consistent expectation, or a perfect foresight path, such that everyone will deviate from that equilibrium in the future. Other equilibria, on the other hand, may have the property that no such expectation can be consistent. We utilize this property to select a particular equilibrium from among multiple equilibria. Matsui and Matsuyama [18] define two stability concepts, global accessibility and absorption. A state is *globally accessible* if it can be reached by a perfect foresight path from every initial state. A state is *absorbing* if no perfect foresight path can escape from it, given that the path has originated sufficiently nearby. In this paper we use the concept of *linear stability* proposed by Oyama [26], a stronger version of global accessibility, together with absorption. An action is linearly stable if for every initial state, the path in which agents choose the action at every action revision opportunity is a perfect foresight path. [1] We are especially interested in a limit case where the degree of friction approaches zero. Thus the question of equilibrium selection is formalized as follows: under what conditions is an equilibrium linearly stable (or globally accessible) and absorbing when the degree of friction is sufficiently small?

In the literature, several results are known for special classes of games. Matsui and Matsuyama [18] show that, for 2×2 games, the concept of risk-dominance [8] is a necessary and sufficient condition for global accessibility and absorption for small degrees of friction. Kim [15] generalizes this result for binary choice *N*-player coordination games (2^N coordination games). He proposes an extension of risk-dominance, and shows that his extension

[1] Perfect foresight dynamics typically exhibits multiplicity of paths, and a stationary path at a Nash equilibrium is always one of the perfect foresight paths. Thus stability concepts are called for to select an equilibrium. In contrast, the equilibrium selection of Burdzy et al. [1] does not rely on the stability concepts adopted in this paper: their method selects the risk-dominant equilibrium [8] in symmetric 2×2 games by iterated eliminations of dominated strategies.

is a necessary and sufficient condition for selection in perfect foresight dynamics. Oyama [26] investigates two-player symmetric games with more than two actions. He shows that a strict Nash equilibrium is linearly stable and absorbing for small degrees of friction if and only if it is $1/2$-dominant. Part of this result is generalized by Hofbauer and Sorger [10] to games with linear incentives. Different papers analyze different classes of games, using different concepts, and the relationship between the conditions they employ is not clear. [2] Given the wide applicability of perfect foresight dynamics to economic analysis, it is important to unify these diverse results in a more general class of games.

To address the above question, we consider perfect foresight dynamics on a class of normal-form games called PIM games. A game is said to be a *game with pairwise incentive maximizers (PIM game)* if, for every pair of players i, j and actions h and $k \neq h$, her incentive of taking h over k is uniquely maximized by an action η_{hk} with respect to actions of j, and the incentive maximizer η_{hk} is determined independently of the choice of the rest of the players. The class of PIM games includes various games of interest in economics such as generic symmetric games with linear incentives, symmetric strict supermodular or submodular games and marginal bandwagon games.

We introduce the concept of u-dominant equilibrium for PIM games. Action h is *u-dominant* if it is the unique best response to any correlated action distribution where number of opponents using the action is uniformly distributed. The notion of u-dominance coincides with risk-dominance in symmetric 2×2 games.

We compare u-dominance with an alternative extension of risk-dominance, the concept of $\boldsymbol{p}$-dominance introduced by Morris et al. [20] and Kajii and Morris [11]. Given $\boldsymbol{p} = (p_1, p_2, \ldots, p_N) \in [0, 1]^N$, a Nash equilibrium a^* is a (strict) *$\boldsymbol{p}$-dominant equilibrium* if every player i has an (strict) incentive to take a_i^* whenever others take a_{-i}^* with probability of at least p_i. In some contexts, especially when there exist strategic complementarities, u-dominance may serve as a more suitable criterion than $\boldsymbol{p}$-dominance. In choosing an operating system for a personal computer, for example, what concerns an individual is not how likely it is that *everyone* will adopt the same software. What matters more may be *how many* will use the software *with what probability*. The concept of u-dominance captures this kind of decision problem in games.

Our main result is as follows. In a PIM game, an action is linearly stable for sufficiently small degrees of friction if and only if it is u-dominant. Moreover, a u-dominant equilibrium is absorbing for any degree of friction. Thus we characterize stability properties under perfect foresight dynamics by the static concept of u-dominance. We may also say that u-dominant equilibrium as a criterion of equilibrium selection is justified by perfect foresight dynamics. PIM games include 2^N coordination games and symmetric games with linear incentives (and hence symmetric two-player games). Thus our result unifies the results of Kim [15] on 2^N coordination games, Hofbauer and Sorger [10] on games with linear incentives, and Oyama [26] on symmetric two-player games.

[2] In addition to games with linear incentives, Hofbauer and Sorger [9,10] investigate potential games. Their approach is generalized recently by Oyama et al. [27], who unify results of potential game and Oyama [26]'s result of $1/2$-dominance.

Here is an intuition for the important role played by u-dominance. Consider, for example, u-dominance as a sufficient condition for linear stability under a small degree of friction. Take the limit case, where friction vanishes and the initial state puts no weight on the u-dominant equilibrium h. Suppose that a linear path toward h is expected, and consider agents' incentives at $t = 0$. Zero friction amounts to no subjective discounting. In this case the discounted probability for an agent that m opponents play h is equal to the probability that m opponents will have changed action before she changes her action: in other words, it is the probability that she is the $(m + 1)$th person to obtain a revision opportunity among N players that are matched. This probability is $1/N$ by symmetry among agents. Therefore the distribution of numbers of opponents taking h converges to the uniform distribution as friction approaches zero. Here u-dominance ensures that each player has an incentive to take h, making the linear path self-fulfilling.

Carlsson and van Damme [2] propose the global games approach for equilibrium selection. This approach has been applied to a variety of economic situations in which multiple equilibria are present. Examples include international finance [21] and banking [7]. Takahashi [29] points out that perfect foresight dynamics can be seen as a model of global games with a specific signal structure, and shows that absorbing states correspond to equilibria selected in the corresponding global games in two-player supermodular games. Similarity between these two approaches are present with more than two players as well: u-dominant equilibrium is selected both by perfect foresight dynamics and by the global games method in 2^N coordination games [15]. Given that the global games approach is widely applied to economic analysis, the result of this paper could fascilitate contact between perfect foresight dynamics and other methods of equilibrium selection and also economic applications.

This paper proceeds as follows. Section 2 defines PIM games and introduces u-dominance. In Section 3, we define the setup and stability concepts of perfect foresight dynamics. In Section 4 we prove the main result. Section 5 re-examines existing results and shows that our approach unifies them. We also show generic existence of a u-dominant equilibrium in network games. Section 6 concludes.

2. Games with pairwise incentive maximizers

2.1. Games with pairwise incentive maximizers

Let $N \geqslant 2$ be an integer and consider a symmetric N-player normal-form game $G = \langle I, A, u \rangle$. $I = \{1, 2, \ldots, N\}$ is the set of players. Finite set $A = \{1, \ldots, n\}$ is the action set of each player, which is assumed to be common among players. $u : A^N \to \mathbb{R}$ is the payoff function for each player. The payoff of player i is given by $u(a_i; a_{-i})$ for action profile $a \in A^N$. We assume that payoffs are invariant to permutation of opponents' action profiles. That is, $u(a_i; a_{-i}) = u(a_i; a'_{-i})$ if a_{-i} is a permutation of a'_{-i}. We assume von Neumann–Morgenstern utility: Players are concerned about their expected payoffs.

We define the incentive function as follows. Let $a_{-1} = (a_2, \ldots, a_N) \in A^{N-1}$ be a generic action profile of $N-1$ players. Given $h, k \in A$ with $h \neq k$, the function $\Delta u_{hk}(\cdot) : A^{N-1} \to \mathbb{R}$, defined as $\Delta u_{hk}(a_{-1}) = u(h; a_{-1}) - u(k; a_{-1})$, is called the *incentive function* of action h over k. Observe that h and k are regarded as parameters. The domain of incentive functions is extended to $\Delta(A^{N-1})$ by $\Delta u_{hk}(\pi) = \sum_{a_{-1} \in A^{N-1}} \Delta u_{hk}(a_{-1})\pi(a_{-1})$ for $\pi \in \Delta(A^{N-1})$, where $\Delta(A^{N-1})$ is the set of probability distributions on A^{N-1}. Using this function, we now introduce the class of games of interest in this paper.

Definition 2.1. A symmetric normal form game G is called *a game with pairwise incentive maximizers* (PIM game for short) if the following holds. For any $i \in I$, $h, k \in A (h \neq k)$ and $j \neq i$, the incentive function $\Delta u_{hk}(\cdot)$ has the unique maximizer $\eta_{hk} \in A$ with respect to *j*'s action, and η_{hk} is independent of actions chosen by the rest of the players.

We call η_{hk} incentive maximizer of h over k. By symmetry, η_{hk} is independent of $j \in I \setminus \{i\}$. Note that the incentive maximizer of k over h, η_{kh}, is the incentive *minimizer* of h over k, since $\Delta u_{hk}(\cdot) = -\Delta u_{kh}(\cdot)$ by definition of incentive functions.

PIM games include the following games:

- Generic symmetric games with linear incentives. A game is said to have linear incentives if the incentive function is a linear function of action distribution of each opponent. Every generic symmetric game with linear incentives is a PIM game. In particular, every symmetric two-player game has linear incentives and is generically a PIM game.
- Symmetric strict supermodular games. A symmetric game is said to be strictly supermodular if $\Delta u_{hk}(\cdot)$ is strictly increasing in a_{-1} if $h > k$. Every symmetric strict supermodular game is a PIM game with $\eta_{hk} = n$ if $h > k$ and $\eta_{hk} = 1$ if $k > h$, which is independent of actions of the rest of the players.
- Symmetric strict submodular games. A symmetric game is said to be strictly submodular if $\Delta u_{hk}(\cdot)$ is strictly decreasing in a_{-1} if $h > k$. Every strict submodular game is a PIM game by the same argument as in strict supermodular games.
- Marginal bandwagon games, first defined by Kandori and Rob [13]. [3] A symmetric game is said to have the marginal bandwagon property if it is a PIM game with $\eta_{hk} = h$ for any h and $k \neq h$.

PIM games include, in particular, generic symmetric two-player games and *N*-player coordination games. Several results are known about selection in perfect foresight dynamics for these classes of games. They have also been investigated in the literature on other equilibrium selection methods. [4]

2.2. *The concept of u-dominance*

In this subsection we introduce a normal-form property of equilibrium which we call *u*-dominance. The notion of *u*-dominance plays a crucial role in the following sections. This concept is closely related to *p*-dominance and risk-dominance concepts.

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[3] Kandori and Rob [13] define the marginal bandwagon property in two-player games.

[4] Oyama [26] investigates perfect foresight dynamics for symmetric two-player games. For *N*-player coordination games, Kim [15] compares selection results by perfect foresight dynamics with those by other dynamic and static equilibrium selection methods, such as stochastic evolution by Kandori et al. [12] and global games by Carlsson and van Damme [2].

	C	D
C	a,a	b,c
D	c,b	d,d

Fig. 1. A 2×2 coordination game.

Definition 2.2. Let G be a PIM game. $h \in A$ is *u-dominant* if

$$\sum_{m=0}^{N-1} \Delta u_{hk}(\underbrace{h, \ldots, h}_{m}, \underbrace{\eta_{kh}, \ldots, \eta_{kh}}_{N-1-m}) > 0$$

for all $k \neq h$.

The concept of u-dominance requires that h is the unique best response for an action distribution where the numbers of both h and n_{kh} are uniformly distributed.[5] It is straightforward to see that if h is u-dominant, then $\boldsymbol{h} := (h, \ldots, h)$ is a strict Nash equilibrium.

Example 2.1. Consider the following three-player game $G = \langle I, A, u\rangle$, where $I = \{1, 2, 3\}$, $A = \{C, D\}$. Let u_h^m be the payoff of each player if her action is h and m opponents choose h as well. Assume that $u_C^0 = -1, u_C^1 = 0, u_C^2 = 1 + \varepsilon$ and $u_D^0 = -2, u_D^1 = 0, u_D^2 = 2$, where $\varepsilon > 0$. It is easy to see that G is a PIM game with $\eta_{CD} = C, \eta_{DC} = D$. There are two pure-strategy Nash equilibria, $\boldsymbol{C} := (C, C, C)$ and $\boldsymbol{D} := (D, D, D)$. C is u-dominant while D is not, because $\Delta u_{CD}(D, D) + \Delta u_{CD}(C, D) + \Delta u_{CD}(C, C) = \varepsilon > 0$. Note that $\boldsymbol{D}$ Pareto-dominates $\boldsymbol{C}$.

The concept of u-dominance is an extension of the risk-dominance in symmetric 2×2 coordination games to more general finite games. See the following symmetric 2×2 game given in Fig. 1.

In a symmetric 2×2 coordination game defined above, C is risk-dominant if $a + b > d + c$. The notion of u-dominance is reduced to risk-dominance in these games since C is u-dominant if and only if $\Delta u_{CD}(D) + u_{CD}(C) = (b - d) + (a - c) > 0$. More generally, in games with linear incentives, and hence in two-player games with more than two actions, an action is u-dominant if and only if it is $1/2$-dominant.[6] For games with more than three players, however, u-dominance is not equivalent to $\boldsymbol{p}$-dominance concept defined by Kajii and Morris [11].

The basic observation is the following uniqueness property of u-dominant equilibria.

Proposition 2.1. *A game has at most one u-dominant equilibrium.*

[5] "u" in u-dominance stands for uniform distribution.

[6] The definition of $1/2$-dominance and the proof of this claim are given in Section 5.

Proof. Suppose that h is u-dominant. For any $k \neq h$, we have

$$\sum_{m=0}^{N-1} \Delta u_{kh}(\underbrace{k, \ldots, k}_{m}, \underbrace{\eta_{hk}, \ldots, \eta_{hk}}_{N-1-m}) \leqslant \sum_{m=0}^{N-1} \Delta u_{kh}(\underbrace{\eta_{kh}, \ldots, \eta_{kh}}_{m}, \underbrace{h, \ldots, h}_{N-1-m})$$
$$= -\sum_{m=0}^{N-1} \Delta u_{hk}(\underbrace{h, \ldots, h}_{m}, \underbrace{\eta_{kh}, \ldots, \eta_{kh}}_{N-1-m}) < 0.$$

The last inequality results from the assumption that h is u-dominant. Therefore, k is not u-dominant. □

The existence of u-dominant equilibrium does not hold in general. In Section 5 we show that a class of special games called network games generically has a u-dominant equilibrium.

3. Perfect foresight dynamics

This section introduces perfect foresight dynamics based on Matsui and Matsuyama [18] and Oyama [26].

3.1. Perfect foresight dynamics: the model

Let $G = \langle I, A, u \rangle$ be a symmetric game. There is a population of infinitesimal and anonymous agents whose size is normalized to one. Time is continuous. At every point in time, agents are randomly matched and play the N-player normal-form game G. Agents cannot change their actions at every moment: they are committed to the same actions for a while. Chances to change actions are given to individuals by an independent Poisson process with arrival rate $\lambda > 0$. We denote the set of mixed strategies by $\Delta(A)$. A *path of behavior* is a function $\phi : [0, \infty) \to \Delta(A)$. We denote by $\phi(t)(h)$ the weight of $\phi(t)$ on action $h \in A$. For $a_{-1} \in A^{N-1}$, we define ϕ_{-1} by $\phi_{-1}(t)(a_{-1}) = \prod_{j \in \{2,\ldots,N\}} \phi(t)(a_j)$. We abuse notation to express a point mass on h by h.

We take the inertia introduced above into account and define the following concept of a feasible path.

Definition 3.1. A path of behavior $\phi : [0, \infty) \to \Delta(A)$ is *feasible* if it is Lipschitz continuous with Lipschitz constant λ and for almost all $t \in [0, \infty)$ there exists $\alpha(t) \in \Delta(A)$ such that

$$\dot{\phi}(t) = \lambda(\alpha(t) - \phi(t)).$$

By definition, $\alpha(t)(h) \geqslant 0$ for every t and h. The equality $\alpha(t)(h) = 0$ implies that agents with only measure zero choose action h at the action revision opportunity at time t. The strict inequality $\alpha(t)(h) > 0$ implies, on the other hand, that a positive fraction of the population chooses action h.

We assume that an agent, when given an opportunity, changes her action to maximize the expected value of her discounted payoff along the expected path ϕ^e. Let $\theta > 0$ be the discount rate, which is assumed to be the same across all agents.

Definition 3.2. Given $t \in [0, \infty)$, a feasible path $\phi^e : [0, \infty) \to \Delta(A)$ and an action $h \in A$, the *expected payoff* at period t along ϕ^e when she chooses action h is[7]

$$V(h, t; \phi^e) = (\lambda + \theta) \int_{s=0}^{\infty} \sum_{a_{-1} \in A^{N-1}} u(h; a_{-1}) \phi^e_{-1}(t+s)(a_{-1}) e^{-(\lambda+\theta)s} \, ds.$$

In addition we define the *time-averaged action distribution* $\pi(a_{-1}, t; \phi^e)$ for i as

$$\pi(a_{-1}, t; \phi^e) = (\lambda + \theta) \int_{s=0}^{\infty} \phi^e_{-1}(t+s)(a_{-1}) e^{-(\lambda+\theta)s} \, ds.$$

By construction, $\sum_{a_{-1} \in A^{N-1}} \pi(a_{-1}, t; \phi^e) = 1$. Therefore $\pi(\cdot, t; \phi^e) \in \Delta(A^{N-1})$.[8] With this notation, the expected payoff of an agent h at period t is expressed by

$$V(h, t; \phi^e) = \sum_{a_{-1} \in A^{N-1}} \pi(a_{-1}, t; \phi^e) u(h; a_{-1}).$$

Recall that $\pi(\cdot, t; \phi^e)$ is a probability distribution on A^{N-1}. So $V(h, t; \phi^e)$ can be interpreted as the expected utility of taking h when opponents follow a correlated action $\pi(\cdot, t; \phi^e)$.

Definition 3.3. A feasible path $\phi : [0, \infty) \to \Delta(A)$ is a *perfect foresight path* if for almost all $t > 0$,

$$\dot{\phi}(t) = \lambda(\alpha(t) - \phi(t))$$

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[7] Rationale for defining the expected payoff in this manner is as follows. Consider an agent who is committed to action h for a given time of length $s \in [0, \infty)$. Given this commitment, the expected discounted payoff is calculated as

$$\int_{z=0}^{s} \sum_{a_{-1} \in A^{N-1}} u(h; a_{-1}) \phi^e_{-1}(t+z)(a_{-1}) e^{-\theta z} \, dz.$$

So the expected payoff is obtained by taking the expected value of this from $s = 0$ to ∞.

$$\int_{s=0}^{\infty} \left(\int_{z=0}^{s} \sum_{a_{-1} \in A^{N-1}} u(h; a_{-1}) \phi^e_{-1}(t+z)(a_{-1}) e^{-\theta z} \, dz \right) \lambda e^{-\lambda s} \, ds$$
$$= \int_{s=0}^{\infty} \sum_{a_{-1} \in A^{N-1}} u(h; a_{-1}) \phi^e_{-1}(t+s)(a_{-1}) e^{-(\lambda+\theta)s} \, ds$$

by partial integration. We multiply this by $(\lambda + \theta)$, a normalization constant. Then the expression in Definition 3.2 is obtained.

[8] A time-averaged distribution is correlated. Intuitively speaking, this is because time t works as a public signal that coordinates agents' action revisions.

for some $\alpha(t) \in \Delta(A)$ such that

$$\alpha(t)(h) > 0 \Rightarrow V(h, t; \phi^{\mathrm{e}}) \geqslant V(k, t; \phi^{\mathrm{e}}) \quad \text{for } \forall k \in A$$

holds, and $\phi^{\mathrm{e}} = \phi$.

Interpretation of the concept of a perfect foresight path is as follows: the first condition implies that if a positive fraction of people chooses h, then it must be a best response to the expected path ϕ^{e}. In other words, an agent acts to maximize her expected payoff along the imagined path ϕ^{e}. The second condition, $\phi^{\mathrm{e}} = \phi$, is a condition that people are endowed with perfect foresight: the expected path ϕ^{e} turns out to be the actual path. In this way, the notion of perfect foresight path captures a mechanism where a "self-fulfilling expectation" governs the movement of behavior patterns in a large society.

We define the *degree of friction* by $\delta = \theta/\lambda$. It is straightforward to see that the qualitative nature of the dynamics is invariant for different values of θ and λ as long as its ratio δ is kept unchanged. We set $\lambda = 1$ for the rest of this paper.

3.2. Stability concepts in perfect foresight dynamics

The following property of perfect foresight paths enables us to use it as a device for equilibrium selection: given $\delta > 0$, it is possible that there are perfect foresight paths which escape from some Nash equilibria, while other Nash equilibria do not allow such deviating paths. This subsection defines stability concepts we adopt in perfect foresight dynamics.

Definition 3.4. Action $h \in A$ is *linearly stable* if for every initial state $x \in \Delta(A)$ the following path

$$\phi(t) = xe^{-t} + h(1 - e^{-t})$$

is a perfect foresight path. A feasible path which satisfies the above equation is called *a linear path*.

Definition 3.5. A state $x^* \in \Delta(A)$ is *absorbing* if there exists a neighborhood of x^* such that every perfect foresight path originating at a state within this neighborhood converges to x^*.

If action h is linearly stable, the expectation that the society will head for h is always consistent with incentives of agents. And if h is also absorbing, people cannot escape from h once they get sufficiently close to it. Thus action h with both linear stability and absorption is considered to have stability in this dynamic context. Furthermore, the following properties are clear by definition: if h is absorbing, then no other action is linearly stable. If h is linearly stable, then no other state is absorbing. For each $\delta > 0$, therefore, there is at most one action h which is both linearly stable and absorbing. This property gives a rationale for utilizing linear stability and absorption to select certain equilibria. The existence of such an action profile is not guaranteed in general, however. It turns out that in a network game, whose definition will be given in Section 5, there generically exists an equilibrium which is linearly stable and absorbing given that the degree of friction is sufficiently small.

Remark. The current dynamics is based on a single population setting. An alternative setting of the dynamics is a multi-population setting, where N agents are randomly chosen from M different populations ($1 < M \leqslant N$) and play a game, which is not necessarily symmetric. The concepts of perfect foresight paths, linear stability and absorption are defined in a manner similar to those in single population cases. Because of the differences in the setting, selection results which hold in one setting do not necessarily imply a counterpart in the other and vice versa. It is straightforward, however, to show that Theorem 4.1, and hence results such as Theorem 5.1 derived from Theorem 4.1, have counterparts in multi-population cases. In the following, we consider only single population cases.

4. Main result

Now we present the main theorem. This theorem characterizes linear stability of an equilibrium by u-dominance. It also shows that any u-dominant equilibrium is absorbing for any degree of friction.

Theorem 4.1. *Let G be a PIM game.*

1. *Action h is linearly stable for every sufficiently small friction $\delta > 0$ if and only if it is u-dominant.*
2. *If h is u-dominant, then it is absorbing for every friction $\delta > 0$.*

Proof. See the appendix.

The intuition for linear stability is as follows (intuition behind absorption is analogous). Let us focus on the limit case as friction vanishes and consider an agent who expects the linear path to h with no initial weight on h. The discounted probability for her that m opponents play h equals the probability that exactly m player will have changed action before she obtains action revision opportunity, as there is no subjective discounting: in other words, it is the probability that she is the $(m + 1)$th person to obtain a revision opportunity among N players that are matched. This probability is $1/N$ since every agent faces the same process of action revision. Therefore, under zero friction, an agent at $t = 0$ expects to face a discounted action distribution where numbers of opponents taking h is uniformly distributed. Since h is u-dominant, the agent has a strict incentive to take h under this condition. For $t > 0$, the incentive to take h is shown not to become small enough to reverse the incentive to take h, so every agent has an incentive to take h along the linear path. A similar argument also goes through when there is positive initial weight on h.

The concept of u-dominance can be defined in a general (not necessarily PIM or not even symmetric) finite game, and a similar result can be obtained, except for the necessity of the u-dominance for linear stability.

u-Dominant equilibrium also plays a role in the literature of global games [2]. Carlsson and van Damme [3] and Kim [15] showed that a u-dominant equilibrium is played in a global game in binary choice N-player coordination games (2^N coordination games). The next example is a simplified version of global games studied by Kim [15].

Example 4.1. Consider the following class of symmetric 2^N coordination games on $A_i = \{D, C\}$. Assume that fundamentals t is a random variable. For given t, assume $u_C^m + t$ is a payoff of each player if her action is C and m other players choose C as well, whereas u_D^m is the payoff of taking D against m players with D independently of t. Assume that u_h^m is strictly increasing in m for any t. Define $\underline{t}$, $\bar{t}$ as $\underline{t} = u_D^0 - u_C^{N-1}$ and $\bar{t} = u_D^{N-1} - u_C^0$ so that D (resp. C) is a dominant action if $t < \underline{t}$ (resp. $t > \bar{t}$). Suppose that

$$\sum_{m=0}^{N-1} u_C^m = \sum_{m=0}^{N-1} u_D^m,$$

thus C is u-dominant if $t > 0$ and D is u-dominant if $t < 0$. Suppose that t is uniformly distributed on $\mathbb{R}$. Players observe t with idiosyncratic noises: player i observes $t_i = t + \varepsilon e_i$, where $\varepsilon > 0$ is a constant and e_i is a random variable. i observes neither t nor e_i directly. Suppose that $(e_i)_{i\in I}$ are i.i.d. variables with continuous density with support $[-1, 1]$. $(e_i)_{i\in I}$ are assumed to be independent of t and to have zero mean. This signal structure is common knowledge.

We argue that it is an equilibrium for every player who receives a signal t_i higher than 0 to take C and every player with a signal lower than 0 to take D, i.e. for every player to choose an action which comprises a u-dominant equilibrium under her own signal. [9] To see this, consider a player at the cutoff value, $t_i = 0$. The probability that m of i's opponents take C turns out to be $1/N$ for any $m = 0, 1, \ldots, N-1$, i.e. i forms a uniform distribution. Therefore the player at the cutoff is indeed indifferent between C and D, and by monotonicity every player with a higher (resp. lower) signal takes C (resp. D). In particular, if the realized signal is $t > \varepsilon$ (resp. $t < -\varepsilon$), then every player takes C (resp. D) in equilibrium.

The reason that the player at the cutoff value puts uniform probability to the number of opponents taking C is as follows. By assumption, the number of players taking C is the same as the number of players with signal $t_j > 0 = t_i$. What information does t_i tell player i about the number of players taking C? Since the density of t is uniform, the probability that m players obtain $t_j > t_i$ given t_i is independent of t_i. Thus this probability is equal to the unconditional probability that m players obtain $e_j > e_i$. It is independent of m, and thus equal to $1/N$ since $(e_i)_{i\in I}$ are assumed to be i.i.d.

As the above example illustrates, the uniform distribution arises naturally in global games as well as in perfect foresight dynamics. A uniform distribution in global games comes from a uniform distribution of fundamentals and symmetric idiosyncratic signals, whereas that in perfect foresight dynamics comes from symmetric belief on arrival of revision opportunity. We note that a continuous analog of u-dominant equilibrium is played also in binary choice games with continuum of players. See [22,23].

Example 4.2. Consider the game we discussed in Example 2.1: $I = \{1, 2, 3\}$, $A = \{C, D\}$, $u_C^0 = -1$, $u_C^1 = 0$, $u_C^2 = 1 + \varepsilon$ and $u_D^0 = -2$, $u_D^1 = 0$, $u_D^2 = 2$ with $\varepsilon > 0$. Recall that the payoff of a player depends only on the number of the opponents who take the same action in

[9] One can show that the above strategy profile is the unique equilibrium. Also, the result is generalized for a more general payoff and signal structure.

this game. As we have shown in Example 2.1, C is u-dominant, whereas D is not. Applying Theorem 4.1, we conclude that C is absorbing for every degree of friction and linearly stable for every degree of friction sufficiently close to zero. Note that $\boldsymbol{D} := (D, D, D)$ is Pareto dominant. Perfect foresight dynamics selects a u-dominant equilibrium, rather than a Pareto-dominant equilibrium in this example.

The following corollary gives a relationship between linear stability and absorption. Generalizing Oyama's [26] observation in symmetric two-player games, it states that linear stability, which is defined as a global property, implies local stability as well.

Corollary 4.1. *In a PIM game, if action h is linearly stable for sufficiently small* $\delta > 0$, *then it is absorbing for every* $\delta > 0$.

Proof. Suppose that h is linearly stable for sufficiently small degrees of friction. Then it is u-dominant by the first part of Theorem 4.1. Then we apply the second part of Theorem 4.1 to complete the proof. □

Matsui and Matsuyama [18] define an alternative stability concept.

Definition 4.1. A state $x^* \in \Delta(A)$ is *accessible from* $x \in \Delta(A)$ if there is a perfect foresight path that originates at x, that is, $\phi(0) = x$, and $\phi(t) = x^*$ for some t or $\lim_{t\to\infty} \phi(t) = x^*$. A state x^* is *globally accessible* if it is accessible from every initial state $x \in \Delta(A)$.

If a state x^* is globally accessible, there exists a consistent expectation that the society will evolve into x^* in the future, no matter the initial state x. A linearly stable action profile is globally accessible by definition, but the converse is not true in general. Thus linear stability is a stronger concept than global accessibility. Therefore we have an immediate corollary of Theorem 4.1.

Corollary 4.2. *If h is a u-dominant equilibrium, then there exists* $\bar{\delta} > 0$ *such that h is globally accessible for every* $\delta < \bar{\delta}$ *and it is absorbing for every* δ.

However, u-dominance is not necessary for global accessibility. Consider the following 3×3 game by Young [31].

	C	D	E
C	6,6	0,5	0,0
D	5,0	7,7	5,5
E	0,0	5,5	8,8

As a generic two-player game, this game has the PIM property. Symmetry is satisfied. (C, C), (D, D) and (E, E) are symmetric Nash equilibria. It is easy to see that none of them is u-dominant. Therefore this game has no equilibrium which is linearly stable for small δ. Oyama [26] shows, on the other hand, that E is globally accessible for any small degree of friction and absorbing for every degree of friction. See [26] for detailed discussion.

5. Unification of existing results

This section revisits several known results of equilibrium selection in perfect foresight dynamics. We show results for 2^N coordination games by Kim [15], games with linear incentives by Hofbauer and Sorger [10], and symmetric two-player games by Oyama [26] as corollaries of Theorem 4.1. We also give a subclass of games, network games, where there generically exists an equilibrium that is u-dominant, hence linearly stable and absorbing in the face of small degrees of friction.

5.1. Network games and 2^N coordination games

We define network games as follows. We assume that the payoff of each player depends only on the number of the opponents who take the same action as her own action. Let u_h^m be a payoff of each player if her action is h and m other players choose h as well. Assume that u_h^m is strictly increasing in m. A network game is a PIM game with $\eta_{hk} = h$. Consider generic cases (by "generic" we mean that $\sum_{m=0}^{N-1} u_h^m$ has unique maximum at some h). In network games, u-dominance is equivalent to the unique maximization of the sum $\sum_{m=0}^{N-1} u_h^m$. Thus we have the following proposition.

Proposition 5.1. *In a generic network game, there exists exactly one u-dominant equilibrium.*

By Theorem 4.1, we obtain the following theorem, which establishes generic existence of an equilibrium selected in perfect foresight dynamics.

Theorem 5.1. *In a generic network game, there exists a Nash equilibrium which is linearly stable for every small degree of friction $\delta > 0$ and absorbing for all $\delta > 0$. h is linearly stable and absorbing for every small degree of friction $\delta > 0$ and absorbing for all $\delta > 0$ if and only if*

$$\sum_{m=0}^{N-1} u_h^m > \sum_{m=0}^{N-1} u_k^m$$

for all $k \neq h$.

Therefore, we conclude that a generic network game has an equilibrium with stability properties in perfect foresight dynamics.

We consider, in particular, the following 2^N coordination games. A 2^N coordination game is an N-player network game where each player has only two actions, $A = \{0, 1\}$. The following corollary of Theorem 5.1 is a selection result first proved by Kim [15].

Corollary 5.1 (*Kim [15]*). *In 2^N coordination games, 0 is linearly stable for every small $\delta > 0$ and absorbing for every $\delta > 0$ if and only if $\sum_{m=0}^{N-1} u_0^m > \sum_{m=0}^{N-1} u_1^m$.*

Remark. A symmetric game is called a *potential game* if there exists a function $p : A^N \to \mathbb{R}$ such that $\Delta u_{hk}(a_{-i}) = p(h, a_{-i}) - p(k, a_{-i})$ for every i, h, k and a_{-i}. Hofbauer and Sorger [10] prove that in potential games, if a unique maximizer of the potential function exists, then it is globally accessible for small $\delta > 0$ and absorbing for any $\delta > 0$. A network game is a potential game with a potential function defined by $p(a) = \sum_h \sum_{m=0}^{\#\{i \in I : a_i = h\}-1} u_h^m$. If h maximizes $\sum_{m=0}^{N-1} u_h^m$, then $\boldsymbol{h}$ is a unique maximizer of p. Therefore global accessibility and absorption follows from Hofbauer and Sorger [10]. Theorem 5.1 is different from their result in that it guarantees the existence of an equilibrium with linear stability, rather than global accessibility, for small $\delta > 0$.

5.2. Games with linear incentives

Next we consider games with linear incentives defined by Selten [28]. A game has linear incentives if, for each player, the incentive function is a linear function of an action distribution of each opponent. A symmetric game has linear incentives if and only if, for any h and $k \neq h$, there exists a function $\Delta v_{hk} : A^2 \to \mathbb{R}$ satisfying

$$\Delta u_{hk}(a_{-1}) = \sum_{j \in \{2,\dots,N\}} \Delta v_{hk}(a_j)$$

for any $a_{-1} = (a_2, \dots, a_N)$.

Action h is 1/2-dominant if for every mixed action distribution $\pi \in \Delta(A)^{N-1}$ with $\pi_j(h) \geqslant 1/2$ for all j, h is the unique best response for each player.

Proposition 5.2. *Let G be a generic symmetric game with linear incentives. $h \in A$ is 1/2-dominant if and only if it is u-dominant.*

Proof. In a generic symmetric game with linear incentives, it is clear that h is 1/2-dominant if and only if, for any $k \neq h$,

$$\Delta u_{hk}(\pi) > 0$$

for $\pi \in \Delta(A^{N-1})$ satisfying $\pi_j(h) = \pi_j(\eta_{kh}) = 1/2$ for any $j \in \{2, \dots, N\}$. Since

$$\begin{aligned}\sum_{m=0}^{N-1} \Delta u_{hk}(\underbrace{h, \dots, h}_{m}, \underbrace{\eta_{kh}, \dots, \eta_{kh}}_{N-1-m}) &= \sum_{m=0}^{N-1} \big(m\Delta v_{hk}(h) + (N-1-m)v_{hk}(\eta_{hk})\big) \\ &= \frac{N(N-1)}{2}\big(\Delta v_{hk}(h) + \Delta v_{hk}(\eta_{kh})\big) \\ &= N\Delta u_{hk}(\pi),\end{aligned}$$

the above inequality holds for any $k \neq h$ if and only if h is u-dominant. □

By Theorem 4.1 and Proposition 5.2, we obtain the following result.

Corollary 5.2. *In generic symmetric games with linear incentives, action h is linearly stable for any small $\delta > 0$ if and only if it is $1/2$-dominant. If h is $1/2$-dominant, then it is absorbing for all $\delta > 0$.*

Corollary 5.2 extends Hofbauer and Sorger [10], who show sufficiency of 1/2-dominance for linear stability and absorption. It also generalizes Oyama [26], who shows the above result for symmetric two-player games. Note, however, that Hofbauer and Sorger [10] prove their result without the symmetry or genericity assumption, and Oyama [26] shows his result without the genericity assumption.

6. Concluding remarks

We defined the class of PIM games and the static concept of u-dominance, and studied the role this concept plays in perfect foresight dynamics. Theorem 4.1 shows that an equilibrium in a PIM game is u-dominant if and only if it is linearly stable for sufficiently small degrees of friction. Moreover, a u-dominant equilibrium is also absorbing independently of friction. Then we defined network games and showed that there generically exists an equilibrium that is linearly stable and absorbing for small degrees of friction in these games. Finally, our approach unified several existing results, namely generic symmetric games with linear incentives, 2^N coordination games and two-player symmetric games, using the concept of u-dominance.

Here we discuss other results and questions we did not address in this paper. Matsui and Matsuyama [18] consider asymmetric 2×2-games. They adopt the concept of global accessibility as a global stability property and show that the asymmetric version of risk-dominance is a necessary and sufficient condition for global accessibility and absorption for small $\delta > 0$. Hofbauer and Sorger [9,10] consider potential games and show that the unique global maximizer of the potential function is globally accessible and absorbing for small degrees of friction. Based on Morris and Ui [24], Oyama, Takahashi and Hofbauer [27] show that any (strict) monotone-potential maximizer in a (strict) monotone-potential game is globally accessible and absorbing. Since the concept of monotone-potential maximizers is a generalization of both $\boldsymbol{p}$-dominant equilibria with $\sum_{i=1}^{N} p_i < 1$ and potential-maximizers in potential games, $\boldsymbol{p}$-dominant equilibrium with $\sum_{i=1}^{N} p_i < 1$ is globally accessible and absorbing for small degrees of friction. Kojima and Takahashi [16] also show this result on $\boldsymbol{p}$-dominance as well. Their result includes Matsui and Matsuyama [18] for asymmetric 2×2 games, since strict $\boldsymbol{p}$-dominance with $\sum_{i \in I} p_i < 1$ is reduced to risk-dominance in asymmetric 2×2 games. These equilibria are known to be selected by other methods, such as the global games approach by Carlsson and van Damme [2] and the robustness to incomplete information due to Kajii and Morris [11]. Such coincidences suggest some connection between perfect foresight dynamics and other methods. These results are beyond the scope of our paper. It is an open question whether these remaining divergent methods will eventually be unified or whether a fundamental difference lies between them. Another direction of research in perfect foresight dynamics is a set-valued extension by Tercieux [30]. He introduces a set-valued extension of p-dominance, p-best response set, and shows that the minimal $1/2$-best response set always exists and is unique. He shows that the minimal

1/2-best response set generically coincides with a linearly stable set and an absorbing set for any small degree of friction.

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Appendix. Proof of Theorem 4.1

We first introduce distributions μ_δ and v_δ with $\delta > 0$ as a parameter. For given $\delta > 0$ define $\mu_\delta : \{0, 1, \dots, N-1\} \to [0, 1]$ by

$$\mu_\delta(m) = \binom{N-1}{m} B(N - m + \delta, m + 1),$$

where $\binom{k}{l} := k!/[l!(k-l)!]$ is the binomial coefficient and $B : (0, \infty)^2 \to \mathbb{R}$ is the well-known *beta function* defined by

$$\begin{aligned} B(x, y) &:= \int_0^1 r^{x-1}(1-r)^{y-1}\, dr \\ &= \int_0^\infty e^{-xt}(1-e^{-t})^{y-1}\, dt. \end{aligned}$$

Note that the second expression of the beta function results from transformation of variable, $r = e^{-t}$.

The function μ_δ can be interpreted as follows. Consider a linear path from initial distribution x to h, where $x(h) = 0$. At time t, the probability that an agent faces the situation that exactly m opponents take h is $\binom{N-1}{m}(1-e^{-t})^m e^{-(N-1-m)t}$. Therefore the discounted probability for her to face exactly m players who take h is

$$\begin{aligned} &\int_0^\infty \binom{N-1}{m}(1-e^{-t})^m e^{-(N-1-m)t} e^{-(1+\delta)t}\, dt \\ &= \binom{N-1}{m} B(N - m + \delta, m + 1), \end{aligned}$$

which is equal to $\mu_\delta(m)$ by definition. Thus $\mu_\delta(m)$ is the discounted probability, as of $t = 0$, that for each player m opponents take h under the linear path whose initial distribution poses no weight on h.

We define $v_\delta : \{0, 1, \ldots, N-1\} \to [0, 1]$ by

$$v_\delta(m) = \binom{N-1}{m} B(m+1+\delta, N-m).$$

Distribution v_δ can be interpreted as follows. Consider a feasible path such that the initial state is h and every agent chooses some $k \neq h$ at every opportunity. Then at time t, the probability that an agent faces exactly m opponents with action h is $\binom{N-1}{m} e^{-mt}(1 - e^{-t})^{N-1-m}$. Therefore the discounted probability of facing exactly m opponents with h is

$$\int_0^\infty \binom{N-1}{m} e^{-mt}(1-e^{-t})^{N-1-m} e^{-(1+\delta)t}\, dt$$
$$= \binom{N-1}{m} B(m+1+\delta, N-m),$$

which is $v_\delta(m)$ by definition. Thus $v_\delta(m)$ is the discounted probability, at $t = 0$, that m opponents take h along a path starting at and escaping from h. Note that both μ_δ and v_δ are distribution functions on $\{0, 1, 2, \ldots, N-1\}$ by construction.

Lemma. *For any* $m \in \{0, \ldots, N-1\}$,

$$\lim_{\delta\to 0} \mu_\delta(m) = \lim_{\delta\to 0} v_\delta(m) = \frac{1}{N}.$$

Proof. We show $\lim_{\delta\to 0} \mu_\delta(m) = 1/N$.

$$\lim_{\delta\to 0} \mu_\delta(m) = \lim_{\delta\to 0} \binom{N-1}{m} B(m+1+\delta, N-m)$$
$$= \binom{N-1}{m} B(m+1, N-m)$$
$$= \frac{1}{N}.$$

The last equality results from the well-know equality $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\Gamma(x) = (x-1)!$ for positive integer x, where $\Gamma(\cdot)$ is the gamma function.
$\lim_{\delta\to 0} v_\delta(m) = 1/N$ is shown in a similar way. □

Given $\pi \in \Delta(A^{N-1})$ and $h \in A$ we define $f_\pi(\cdot|h) : \{0, 1, \ldots, N-1\} \to [0, 1]$ with respect to π and h as

$$f_\pi(m|h) = \pi(\{a_{-1} \in A^{N-1} : \#\{j \in \{2, \ldots, N\} | a_j = h\} = m\}).$$

Therefore $f_\pi(m|h)$ is the discounted probability that m players out of $(N-1)$ play h under π. Note that $f_\pi(\cdot|h)$ is a distribution function, since $f_\pi(m|h) \geqslant 0$ for any m and $\sum_{m=0}^{N-1} f_\pi(m|h) = 1$ by construction.

We introduce a partial order relation on the set of distributions: given two distribution functions f and $g : \{0, 1, \ldots, N-1\} \to [0, 1]$, we say that f *stochastically dominates* g (denoted $f \succcurlyeq g$) if $\sum_{k=m}^{N-1} f(k) \geqslant \sum_{k=m}^{N-1} g(k)$ for all $m \in \{0, 1, \ldots, N-1\}$.

Define $u : \{0, \ldots, N-1\} \to \mathbb{R}$ by $u(m) = 1/N$ for any $m \in \{0, \ldots, N-1\}$.

Proof of Theorem 4.1. (1) The 'if' part. Let $\phi(t) = e^{-t}x + (1 - e^{-t})h$. By the above Lemma and the assumption of u-dominance, there exists $\bar{\delta} > 0$ such that

$$\Delta u_{hk}(\pi) > 0$$

for any $\pi \in \Delta(A^{N-1})$ with $f_\pi(\cdot|h) \succsim \mu_{\bar{\delta}}$. For any $\delta > 0$ and $t > 0$, we have $f_\pi(\cdot, t; \phi) \succsim \mu_\delta$. Therefore, for any $\delta < \bar{\delta}$, we have $f_\pi(\cdot, t; \phi) \succsim \mu_\delta \succsim \mu_{\bar{\delta}}$ and hence h is the best response for an agent at t. This shows that ϕ is a perfect foresight path and h is linearly stable.

The 'only if' part. Assume that h is not u-dominant. Then for some $k \neq h$, we have the following inequality:

$$\sum_{m=0}^{N-1} \frac{1}{N} \Delta u_{hk}(\underbrace{h, \ldots, h}_{m}, \underbrace{\eta_{kh}, \ldots, \eta_{kh}}_{N-1-m}) \leqslant 0.$$

Let $x = \eta_{kh}$. Then the distribution $\pi \in \Delta(A^{N-1})$ induced by the linear path from x to h satisfies $f_\pi(\cdot|h) \prec u$. Since every action that appears in an action profile with nonzero weight is h and η_{kh}, from a well-known relation on stochastic dominance we obtain

$$\Delta u_{hk}(\pi) < \sum_{m=0}^{N-1} \frac{1}{N} \Delta u_{hk}(\underbrace{h, \ldots, h}_{m}, \underbrace{\eta_{kh}, \ldots, \eta_{kh}}_{N-1-m})$$

for every $\delta > 0$ by definition of PIM and η_{kh}. Therefore we have $\Delta u_{hk}(\pi) < 0$, which implies that action h is not a best response at time zero. This implies that the linear path is not a perfect foresight path and h is not linearly stable.

(2) For any $\delta > 0$, $v_\delta \succsim u$. Therefore, by u-dominance, there exists a neighborhood U of h such that, for any feasible path ϕ with initial state $x \in U$, h is the unique best response at $t = 0$. This implies that h is absorbing. □

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