Sum Capacity of a Gaussian Vector Broadcast Channel *

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Abstract

This paper characterizes the sum capacity of a class of non-degraded Gaussian vector broadcast channels where a single transmitter with multiple transmit terminals sends independent information to multiple receivers. Coordination is allowed among the transmit terminals, but not among the different receivers. The sum capacity is shown to be a saddle-point of a Gaussian mutual information game, where a signal player chooses a transmit covariance matrix to maximize the mutual information, and a noise player chooses a fictitious noise correlation to minimize the mutual information. This result holds for the class of Gaussian channels whose saddle-point satisfies a full rank condition. Further, the sum capacity is achieved using a precoding method for Gaussian channels with additive side information non-causally known at the transmitter. The optimal precoding structure is shown to correspond to a decision-feedback equalizer that decomposes the broadcast channel into a series of single-user channels with interference pre-subtracted at the transmitter.

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1 Introduction

Consider a discrete-time memoryless Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{n}$, where $\mathbf{x}$ and $\mathbf{y}$ are vector-valued signals, H is a matrix channel, and $\mathbf{n}$ is a vector Gaussian random variable. The capacity of the vector channel is the maximum mutual information $I(\mathbf{X}, \mathbf{Y})$ [1]. Assuming that the input signal is Gaussian, the mutual information is evaluated as:

$$I(\mathbf{X}; \mathbf{Y}) = \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|}, \tag{1}$$

where $|\cdot|$ denotes matrix determinant, S_{xx} denotes the covariance matrix for the input $\mathbf{x}$, and S_{nn} denotes the covariance matrix for the noise $\mathbf{n}$. The above mutual information is to be maximized over all covariance matrices S_{xx} subject to some input constraint. For example, under a total power constraint P, the maximization is over all S_{xx} such that $\text{trace}(S_{xx}) \leq P$. This leads to the well-known water-filling solution based on the singular-value decomposition of H [2]. Assuming that S_{nn} is an identity matrix, the optimum S_{xx} must have its eigenvectors equal to the right singular-vectors of H, and its eigenvalues obeying the water-filling power allocation on the singular-values of H. To achieve the vector channel capacity, coordination is necessary both among the transmit terminals of $\mathbf{x}$ and among the receive terminals of $\mathbf{y}$. Transmitter coordination is necessary because the capacity-achieving transmit covariance matrix is not necessarily diagonal. The optimal transmit signals from different transmit terminals may be correlated. Producing such a correlated signal requires coordination at the transmitter. Receiver coordination is necessary because an optimal detector is required to jointly process the signals from different receive terminals. With full coordination, it is possible to choose a transmit filter to match the right singular-vectors of H, and to choose a receive filter to match the left singular-vectors of H, so that the vector Gaussian channel is diagonalized [2]. This diagonalization decomposes the Gaussian vector channel into a series of independent Gaussian scalar channels, so that single-user codes can be used on each sub-channel to collectively achieve the vector channel capacity.

When coordination is possible only among the receive terminals, but not among the transmit terminals, the vector channel becomes a Gaussian multiple access channel. Although a complete characterization of the multiuser channel capacity involves a rate region, the maximum sum capacity can still be computed in terms of the maximum mutual information $I(\mathbf{X}; \mathbf{Y})$. However, in a multiple access channel, different transmit terminals of $\mathbf{x}$ are required to be uncorrelated. So, the water-filling covariance, which is optimum for a coordinated vector channel, can no longer necessarily be synthesized. The optimum covariance matrix for the multiple access channel must be found by solving an optimization problem that restricts the off-diagonal entries of the covariance matrix to zero. This additional constraint leads to a capacity loss compared to the transmitter coordinated case. In addition, the lack of transmitter coordination makes the diagonalization of the vector channel impossible. Instead, the vector channel can only be triangularized [3] [4]. Such triangularization decomposes a vector channel into a series of single-user sub-channels each interfering with only subsequent sub-channels. The triangular structure enables a coding method based on the superposition of single-user codes and a decoding method based on successive decision-feedback to be implemented. If decisions on previous sub-channels are assumed correct, this successive decoding scheme achieves the sum capacity of a Gaussian vector multiple access channel [4]. Thus, from both the capacity and the coding points of view, the value of coordination at the transmitter is well-understood.

When coordination is possible only among the transmit terminals, but not among the receive terminals, the Gaussian vector channel becomes a broadcast channel. Unlike the multiple access channel, the capacity region for a broadcast channel is still not known in general [5]. The main difficulty is that a vector channel distributes information across several receive terminals, and without joint processing of the received signals, a data rate equal to $I(\mathbf{X};\mathbf{Y})$ cannot be supported. In fact, the largest-known achievable rate region for the broadcast channel involves the use of coding methods beyond that of superposition coding and successive decoding. This paper deals with a special class of non-degraded broadcast channels. We consider a Gaussian vector broadcast channel with multiple coordinated terminals at the transmitter and multiple uncoordinated terminals at the receivers. We focus on the sum capacity and ask two questions. First, what is the capacity loss when receiver terminals lack coordination? Second, what is the optimal encoding and decoding structure on such broadcast channels? These questions have been partially answered by Caire and Shamai [6] in the special case of a two-user broadcast channel with two transmit antennas and one receive antenna for each user, where they showed that a precoding strategy based on channels with transmitter side information can achieve the sum capacity. The main contribution of this paper is an extension of their result to a more general case. This paper shows that the sum capacity of a Gaussian vector broadcast channel is a saddle-point of a Gaussian mutual information game, whenever the saddle-point satisfies a full rank condition. This result holds regardless of the number of transmit antennas, the number of users, and the number of receive antennas per user. The approach in this paper further reveals that the structure of the optimal precoding strategy takes the form of a decision-feedback equalizer.

The broadcast channel problem is important in a variety of practical situations. In a cellular wireless system, the downlink direction from the base station to the subscribers can be modeled as a broadcast channel. The broadcast channel is non-degraded when the transmitter is equipped with multiple antennas [6]. Likewise, in wireline communication systems such as digital subscriber lines, because of electromagnetic coupling between lines, the downstream direction from the central office to the subscribers can be modeled as a vector broadcast channel. Further, as the inter-line coupling is usually small, the broadcast channel is typically non-degraded [7]. The solution to the sum capacity of non-degraded Gaussian vector broadcast channels gives a useful upper bound on the ultimate performance of such systems.

The rest of the paper is organized as follows. In section II, the broadcast channel is discussed in detail, and a precoding scheme based on channels with transmitter side information is described. This scheme motivates us to search for the optimal precoding structure for a Gaussian vector broadcast channel. The optimal precoding structure turns out to be closely related to the optimal coding structure for a Gaussian multiple access channel. This coding structure is based on the generalized decision-feedback equalizer, and it is the focus of section III. In section IV, an outer bound for the sum capacity of the Gaussian broadcast channel is computed, and the precoding structure is shown to achieve the outer bound, thus proving the capacity result. Finally, concluding remarks are made in section V.

The notations used in this paper are as follows. Lower case letters are used to denote scalar signals, e.g. x, y. Upper case letters are used to denote scalar random variables, e.g. X, Y, or matrices, e.g. H, where context should make the distinction clear. Bold face letters are used to denote vector signals, e.g. $\mathbf{x}$, $\mathbf{y}$, or vector random variables, e.g. $\mathbf{X}$, $\mathbf{Y}$. For matrices, $\cdot^T$ denote the transpose operation and $|\cdot|$ denotes the determinant operation. The discussions in this paper are confined to the real-valued signals. However, all results extend naturally to the complex-valued case.

2 Precoding for Broadcast Channels

In a broadcast channel, a single transmitter attempts to send information to two or more receivers at the same time. A K-user broadcast channel is specified by the conditional probability distribution $p(\mathbf{y_1}, \cdots, \mathbf{y_K}|\mathbf{x})$ where $\mathbf{x}$ is the transmit signal and $\mathbf{y_i}$, $i = 1, \cdots, K$, are the received signals for users 1 to K, all possibly vector-valued. When the conditional distribution is Gaussian, the broadcast channel is referred to as a Gaussian broadcast channel. A Gaussian broadcast channel can be represented as follows (Figure 1):

$$\begin{aligned} \mathbf{y_1} &= H_1\mathbf{x} + \mathbf{n_1} \\ \vdots \quad & \quad \vdots \\ \mathbf{y_K} &= H_K\mathbf{x} + \mathbf{n_K}, \end{aligned} \tag{2}$$

where $H_1, \cdots, H_K$ are channel matrices, and $\mathbf{n_1}, \cdots, \mathbf{n_K}$ are (possibly vector-valued) Gaussian noises independent of $\mathbf{x}$. This paper concentrates on the situation where independent information is to be transmitted to each receiver. The capacity region in this case refers to the set of rate-tuples $(R_1, \cdots, R_K)$ simultaneously achievable by users 1 to K with an arbitrarily small probability of error. Our primary interest is in the sum rate: $R_1 + \cdots + R_K$.

A complete characterization of the capacity region for the general broadcast channel is still an open problem. However, the capacity region is known if the broadcast channel is degraded. A two-user broadcast channel is degraded if $p(\mathbf{y_1}, \mathbf{y_2}|\mathbf{x}) = p(\mathbf{y_1}|\mathbf{x})p(\mathbf{y_2}|\mathbf{y_1})$. Intuitively, this means that one user always receives a corrupted version of the other user's received signal. For example, the Gaussian scalar broadcast is always degraded. Consider the following channel:

$$\begin{aligned} y_1 &= x + n_1 \\ y_2 &= x + n_2, \end{aligned} \tag{3}$$

where x is the scalar transmitted signal subject to a power constraint P, y_1, and y_2 are the received signals, and n_1 and n_2 are the additive white Gaussian noises with variances σ_1^2 and

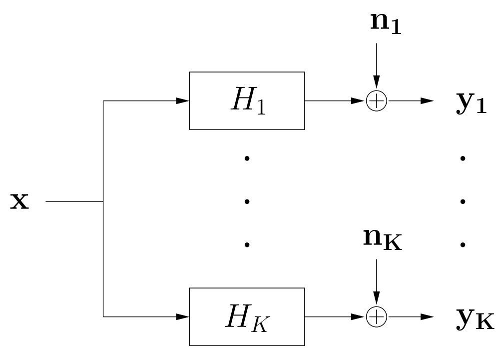

Figure 1: The vector broadcast channel

σ_2^2, respectively. Without loss of generality, assume $\sigma_1 < \sigma_2$. Then, y_2 can be regarded as a degraded version of y_1 because n_2 can be re-written as $n_2' = n_1 + n'$, where n' is independent of n_1, and $n' \sim \mathcal{N}(0, \sigma_2^2 - \sigma_1^2)$. This n_2' has the same distribution as n_2, so y_2 can now be viewed as $y_1 + n'$, a corrupted version of y_1. The capacity region for a degraded broadcast channel is achieved using a superposition coding and interference subtraction scheme due to Cover [8]. The idea is to divide the total power into $P_1 = \alpha P$ and $P_2 = (1-\alpha)P$, $(0 \le \alpha \le 1)$, and construct two independent Gaussian codebooks, one with power P_1 for the first user, and the other with power P_2 for the second user. To send two independent messages, one codeword is chosen from each codebook, and their sum is transmitted through the channel. It is not difficult to see that the following rate pair is achievable:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_1^2}\right) \tag{4}$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_2^2 + P_1}\right). \tag{5}$$

Because y_2 is a degraded version of y_1, the codeword intended for y_2 can be decoded by y_1. Thus, y_1 can subtract the codeword due to P_2, and in effect get a cleaner channel with noise σ_1^2 instead of $\sigma_1^2 + P_2$. In fact, as it was shown by Bergman [9], this superposition and interference subtraction scheme is optimum for the degraded Gaussian broadcast channel.

Unfortunately, when a Gaussian broadcast channel has multiple transmit terminals, it is no longer a degraded broadcast channel in general, and superposition coding is no longer capacity-achieving. Although the capacity region for the general non-degraded broadcast channel is still unknown, superior coding schemes beyond that of superposition do exist. The key idea is a random binning argument which was first used in [10] and subsequently allowed Marton [11] [12] to derive an enlarged achievable rate region. For a two-user channel with independent information for each user, Marton's region is as follows:

$$R_1 \le I(\mathbf{U_1}; \mathbf{Y_1}) \tag{6}$$

$$R_2 \le I(\mathbf{U_2}; \mathbf{Y_2}) \tag{7}$$

$$R_1 + R_2 \le I(\mathbf{U_1}; \mathbf{Y_1}) + I(\mathbf{U_2}; \mathbf{Y_2}) - I(\mathbf{U_1}; \mathbf{U_2}) \tag{8}$$

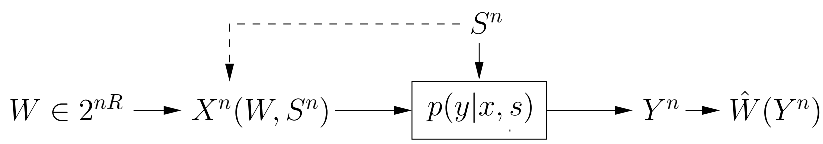

Figure 2: Channel with non-causal transmitter side information

where $(\mathbf{U_1}, \mathbf{U_2})$ is a pair of auxiliary random variables, and the mutual information is to be evaluated under a joint distribution $p(\mathbf{x}|\mathbf{u_1}, \mathbf{u_2})p(\mathbf{u_1}, \mathbf{u_2})$ such that the induced marginal distribution $p(\mathbf{x})$ satisfies the input constraint. Although the optimality of Marton's region is not known for the general broadcast channel, it is optimal for the deterministic broadcast channel [5], and by a proper choice of $(\mathbf{U_1}, \mathbf{U_2})$, it gives the capacity region of the scalar Gaussian degraded broadcast channel also. The objective of this paper is to show that a proper choice of $\mathbf{U_i}$ also gives the sum-capacity of the non-degraded Gaussian vector broadcast channel. As a first step, let's examine the degraded broadcast channel more carefully and give an interpretation of the auxiliary random variables.

The connection between the degraded broadcast channel capacity region and Marton's region lies in the study of channels with non-causal transmitter side information. A channel with transmitter side information is shown in Figure 2, where $p(y|x, s)$ is the conditional probability distribution of the channel, x is the transmit signal, y is the received signal, and s is the channel state information, whose entire sample sequence is known to the transmitter prior to transmission but not to the receiver. Gel'fand and Pinsker [13] and Heegard and El Gamal [14] characterized the capacity of such channels using an auxiliary random variable U:

$$C = \max_{p(u,x|s)} \{I(U;Y) - I(U;S)\}. \tag{9}$$

The achievability proof of this result uses a random binning argument, and it is closely connected to Marton's achievability region for the broadcast channel. Such connection was noted by Gel'fand and Pinsker in [13], and was further used by Caire and Shamai [6] for the two-by-two Gaussian broadcast channel. The following rough argument illustrates the connection. Fix a pair of auxiliary random variables (U_1, U_2) and a conditional distribution $p(x|u_1, u_2)$. Consider the effective channel $p(y_1, y_2|x)p(x|u_1, u_2)$. Construct a random-coding codebook from U_2 to Y_2 using an i.i.d. distribution according to $p(u_2)$. Evidently, a rate of $R_2 = I(U_2; Y_2)$ may be achieved. Now, since U_2 is completely known at the transmitter, the channel from U_1 to Y_1 is a channel with non-causal side information available at the transmitter. Then, Gel'fand and Pinsker's result ensures that a rate of $R_1 = I(U_1; Y_1) - I(U_2; U_1)$ is achievable. This rate-pair is precisely a corner point in Marton's achievability region for the broadcast channel. The above rough argument ignores the issue that U_1 now depends on U_2, but it turns out that for the Gaussian channel, the joint distribution is preserved, and the argument can be made rigorous.

When specialized to the Gaussian channel, the capacity of a channel with side information

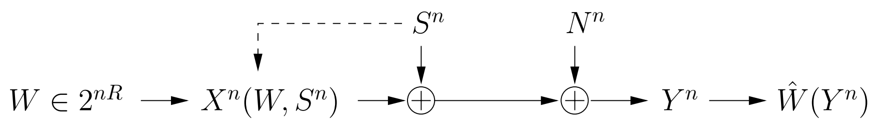

Figure 3: Gaussian channel with transmitter side information

has an interesting solution. Consider the following channel,

$$y = x + s + n, \tag{10}$$

as shown in Figure 3, where x and y are the transmitted and the received signals respectively, s is a Gaussian interfering signal whose entire non-causal realization is known to the transmitter but not to the receiver, and n is Gaussian noise independent of s. In a surprising result known as "writing-on-dirty-paper," Costa [15] showed that under a joint i.i.d. Gaussian condition on s and n, the capacity of the channel with interference s is the same as if s does not exist. In addition, the optimal transmit signal x is statistically independent of s. In effect, the optimum coding scheme can pre-subtract s at the transmitter.

The "dirty-paper" result gives us another way to derive the degraded Gaussian broadcast channel capacity. Let $x = x_1 + x_2$, where x_1 and x_2 are independent Gaussian signals with average power P_1 and P_2 respectively, where $P_1 + P_2 = P$. The message intended for y_1 is transmitted through x_1, and the message intended for y_2 is transmitted through x_2. If two independent codebooks are used for x_1 and x_2, each receiver sees the other user's signal as noise. However, the transmitter knows both messages in advance. So, the channel from x_1 to y_1 can be regarded as a Gaussian channel with non-causal side information x_2, for which Costa's result applies. Thus, a transmission rate from x_1 to y_1 that is as high as if x_2 is not present can be achieved, i.e. $R_1 = I(X_1; Y_1|X_2)$. Further, the optimal signal for x_1 is statistically independent of x_2. Thus, the channel from x_2 to y_2 still sees x_1 as independent noise, and a rate $R_2 = I(X_2; Y_2)$ is achievable. This gives an alternative derivation for the degraded Gaussian broadcast channel capacity (4)-(5). Curiously, this derivation does not use the fact that y_2 is a degraded version of y_1. In fact, y_1 and y_2 may be interchanged and the following rate pair is also achievable:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_1^2 + P_2}\right) \tag{11}$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_2^2}\right). \tag{12}$$

It can be shown that when $\sigma_1 < \sigma_2$, the above rate region is strictly smaller than the true capacity region (4)-(5).

The idea of subtracting interference at the transmitter instead of at the receiver is attractive because it is also applicable to the non-degraded broadcast channels as first shown

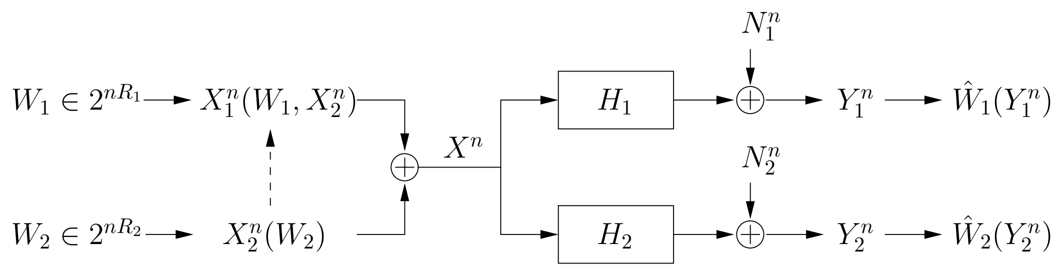

Figure 4: Coding for vector broadcast channel

in [6]. Consider the following Gaussian vector broadcast channel:

$$\begin{aligned} \mathbf{y_1} &= H_1\mathbf{x} + \mathbf{n_1} \\ \mathbf{y_2} &= H_2\mathbf{x} + \mathbf{n_2}, \end{aligned} \tag{13}$$

where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{y_1}, \mathbf{y_2} \in \mathbf{R}^m$, H_1 and H_2 are channel matrices of dimension $m \times n$, and $\mathbf{n_1}$ and $\mathbf{n_2}$ are vector Gaussian noises with covariance matrices $S_{n_1n_1}$ and $S_{n_2n_2}$ respectively. In general, H_1 and H_2 are not degraded versions of each other, and they do not necessarily have the same eigenvectors, so it is typically not possible to decompose the vector channel into independent scalar broadcast channels. (An important exception is the ISI channel which can be decomposed by a discrete Fourier transform [16].) Nevertheless, the "dirty-paper" result may be generalized to the vector case to implement interference pre-subtraction at the transmitter.

Lemma 1 ([17] [18]) *Consider a channel* $\mathbf{y} = \mathbf{x} + \mathbf{s} + \mathbf{n}$*, where* $\mathbf{s}$ *and* $\mathbf{n}$ *are independent i.i.d. vector Gaussian signals. Suppose that non-causal knowledge of* $\mathbf{s}$ *is available at the transmitter but not at the receiver. The capacity of the channel is the same as if* $\mathbf{s}$ *is not present, i.e.*

$$C = \max_{p(\mathbf{u},\mathbf{x}|\mathbf{s})} \{I(\mathbf{U};\mathbf{Y}) - I(\mathbf{U};\mathbf{S})\} = I(\mathbf{X};\mathbf{Y}|\mathbf{S}). \tag{14}$$

Further, the capacity-achieving $\mathbf{x}$ *is statistically independent of* $\mathbf{s}$*, i.e.* $p(\mathbf{u},\mathbf{x}|\mathbf{s}) = p(\mathbf{u}|\mathbf{x},\mathbf{s})p(\mathbf{x})$.

This result has been noted by several authors [17] [18] under different conditions. A direct proof can be found in [18], where it is shown that the capacity-achieving $p(\mathbf{u},\mathbf{x}|\mathbf{s})$ is such that $\mathbf{x}$ and $\mathbf{s}$ are independent, and $\mathbf{u}$ takes the form of $\mathbf{u} = \mathbf{x} + F\mathbf{s}$, where F is a fixed matrix determined by the covariance matrices of $\mathbf{s}$ and $\mathbf{n}$. Lemma 1 suggests a coding scheme for the broadcast channel as shown in Figure 4. The following theorem formalizes this idea.

Theorem 1 *Consider the vector Gaussian broadcast channel* $\mathbf{y}_i = H_i\mathbf{x} + \mathbf{n}_i, i = 1, \cdots, K$*, under a power constraint* P*. The following rate region is achievable:*

$$\left\{ (R_1, \cdots, R_K) : R_i \le \frac{1}{2} \log \frac{\left| \sum_{k=i}^K H_i S_k H_i^T + S_{n_in_i} \right|}{\left| \sum_{k=i+1}^K H_i S_k H_i^T + S_{n_in_i} \right|} \right\} \tag{15}$$

where $S_{n_i n_1}$'s are covariance matrices for $\mathbf{n}_i$, and S_i's are positive semi-definite matrices satisfying the constraint: $\sum_{i=1}^{K} \text{trace}(S_i) \leq P$.

Proof: For simplicity, consider only the case for $K = 2$. The extension to the general case is straightforward. Let $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$, where $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent vector Gaussian signals with covariance matrices S_1 and S_2, respectively, such that trace $(S_1 + S_2) \leq P$. Fix $\mathbf{U_2} = \mathbf{x_2}$. Now, choose the conditional distribution $p(\mathbf{u_1}|\mathbf{u_2}, \mathbf{x_1})$ to be such that it maximizes $I(\mathbf{U_1}; \mathbf{Y_1}) - I(\mathbf{U_1}; \mathbf{U_2})$. By Lemma 1, the maximizing distribution is such that $\mathbf{x_1}$ and $\mathbf{U_2}$ are independent. So, assuming that $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent *a priori* is without loss of generality. Further, by (14), the maximizing distribution gives $I(\mathbf{U_1}; \mathbf{Y_1}) - I(\mathbf{U_1}; \mathbf{U_2}) = I(\mathbf{X_1}; \mathbf{Y_1}|\mathbf{U_2})$. Using this choice of $(\mathbf{U_1}, \mathbf{U_2})$ in Marton's region (6)-(8), the following rates are obtained: $R_1 = I(\mathbf{X_1}; \mathbf{Y_1}|\mathbf{X_2})$, $R_2 = I(\mathbf{X_2}; \mathbf{Y_2})$. The mutual information can be evaluated as:

$$R_1 = \frac{1}{2} \log \frac{|H_1 S_1 H_1^T + H_1 S_2 H_1^T + S_{n_1 n_1}|}{|H_1 S_2 H_1^T + S_{n_1 n_1}|} \tag{16}$$

$$R_2 = \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_{n_2 n_2}|}{|S_{n_2 n_2}|}, \tag{17}$$

which is the desired result. □

This theorem is a generalization of an earlier result used by Caire and Shamai [6] in their derivation of the two-by-two broadcast channel sum capacity, where essentially rank-one S_i's are considered. Theorem 1 computes Marton's region by identifying $(\mathbf{U_1}, \mathbf{U_2})$ to be of a special form. Thus, finding the appropriate $(\mathbf{U_1}, \mathbf{U_2})$ is now reduced to finding the appropriate (S_1, S_2). However, such specialization may be capacity-lossy, and even if it is capacity lossless, finding the optimal set of S_i's can still be difficult. For example, the order of interference pre-subtraction is arbitrary, and it is possible to split the transmitting covariance matrix into more than K users and achieve the rate-splitting points. However Caire and Shamai [6] partially circumvented the difficulty by deriving an outer bound for the broadcast channel sum capacity. Then, they assumed a particular precoding order and essentially searched through the subset of all rank-one S_i's numerically for the two-user broadcast channel. Their channel model assumes that the transmitter has two terminals, and each receiver has one terminal. They proved that for this special two-by-two case, Marton's region indeed coincides with the outer bound. Unfortunately, this numerical procedure does not generalize easily, and it does not reveal the structure of the optimal S_i.

In an independent effort, [7] demonstrated a precoding technique for a broadcast channel with a transmitter having N terminals, and N receivers each having a single terminal. The channel is modeled as $\mathbf{y} = H\mathbf{x} + \mathbf{n}$, where $\mathbf{y}$ is an $N \times 1$ vector with each component as a receiver. The choice for S_i, $i = 1, \cdots, N$ is made as follows. Let $H = RQ$ be a QR-decomposition, where Q is orthogonal, and R is triangular. The transmit direction for each user is chosen to be the row vectors of Q. More precisely, let $\mathbf{x} = Q^T\mathbf{u}$, where $\mathbf{u}$ is a vector with each component as a data stream for each receiver. Then, the broadcast channel is decomposed into N parallel sub-channels. Because R is triangular, the first sub-channel is a usual Gaussian channel; the second sub-channel has non-causal side information from the first sub-channel; the third sub-channel has non-causal side information from the first two sub-channels, etc. Thus, an interference pre-subtracting scheme can be used on each sub-channel to eliminate the interference from previous sub-channels completely. In fact, this QR-type decomposition was also independently considered by Caire and Shamai [6], who proved that the QR method is rate-sum optimal in both low and high SNR regions, although sub-optimal in general. The rest of this paper is devoted to the identification of the optimal precoding strategy. Interestingly, as will be shown, the optimal precoder has the structure of a decision feedback equalizer.

3 Decision-feedback Precoding

3.1 GDFE

Decision-feedback equalization (DFE) is a technique used to compensate intersymbol interference (ISI) in linear dispersive channels. In a channel with ISI, each transmitted symbol produces a sequence of time-delayed samples at the channel output. So, each received sample contains contribution from the current symbol as well as from the previous symbols. The idea of a decision-feedback equalizer is to untangle the interference by subtracting the effect of each symbol after it is decoded. The decision-feedback equalizer is usually analyzed under the assumption that all previous symbols are decoded correctly so that error propagation does not occur. Under this assumption, it can be shown that a minimum mean-square error decision-feedback equalizer (MMSE-DFE) achieves the channel capacity of a Gaussian linear dispersive channel [19].

The study of the decision-feedback equalizer is related to the study of the multiple access channel. Each transmitted symbol in an ISI channel can be regarded as a separate user. Suppose that the transmitted symbols are independent, then there is no transmitter coordination. The decision-feedback equalizer is then equivalent to a successive interference subtraction scheme. This connection can be formalized by considering a decision-feedback structure that operates on a block-by-block basis. This finite block-length version, introduced in [3] as the Generalized Decision Feedback Equalizer (GDFE) for the block-processing of ISI channels, was also developed independently for the multiple access channel [4]. As this paper will eventually show, the GDFE structure is also applicable to the broadcast channel problem. Toward this end, an information theoretical derivation of the generalized decision-feedback equalizer is first presented. This derivation is largely based on [3].

Consider a vector Gaussian channel $\mathbf{y} = H\mathbf{x} + \mathbf{n}$, where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{n}$ are vector Gaussian signals. Let $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, and without loss of generality, assume $\mathbf{n} \sim \mathcal{N}(0, I)$. The Shannon capacity of this channel is $I(\mathbf{X}; \mathbf{Y}) = \frac{1}{2} \log |HS_{xx}H^T + I|$. The capacity can be achieved with a random Gaussian vector codebook, where each codeword consists of vector-valued symbols generated from an i.i.d. distribution $\mathcal{N}(0, S_{xx})$. Evidently, sending a message using such a vector codebook requires the joint-processing of components of $\mathbf{x}$ at the encoder. Now, write $\mathbf{x}$ as $\mathbf{x}^T = [\mathbf{x_1}^T \mathbf{x_2}^T]$, and suppose further that $\mathbf{x_1}$ and $\mathbf{x_2}$ are statistically independent so that the covariance matrix $S_{xx} = \begin{bmatrix} S_{x_1x_1} & 0 \\ 0 & S_{x_2x_2} \end{bmatrix}$. Then, it turns out that in this case, $I(\mathbf{X};\mathbf{Y})$ can be achieved using two independent codebooks, one for $\mathbf{x_1}$ and another one for $\mathbf{x_2}$. Further, not only encoding of $\mathbf{x_1}$ and $\mathbf{x_2}$ can be de-coupled, decoding can also be done independently if a generalized decision-feedback equalizer is implemented at the receiver. In effect, equalization and decoding are separated.

The development of GDFE involves three key ideas. The first idea is to recognize that the minimum mean-square error (MMSE) estimation of $\mathbf{x}$ given $\mathbf{y}$ is capacity lossless. Consider the setting shown in Figure 5 where at the output of the vector Gaussian channel $\mathbf{y} = H\mathbf{x} + \mathbf{n}$, an MMSE estimator W is applied to $\mathbf{y}$ to generate $\hat{\mathbf{x}}$. Clearly, the maximum achievable rate after the MMSE estimator is $I(\mathbf{X};\hat{\mathbf{X}})$. The following argument shows that $I(\mathbf{X};\hat{\mathbf{X}}) = I(\mathbf{X};\mathbf{Y})$, i.e. the MMSE estimation process is information-lossless. The MMSE estimator of a Gaussian process is linear, so W represents a matrix multiplication. Further, let the difference between $\mathbf{x}$ and $\hat{\mathbf{x}}$ be $\mathbf{e}$. From linear estimation theory, $\mathbf{e}$ is Gaussian and is independent of $\hat{\mathbf{x}}$. So, if $I(\mathbf{X};\hat{\mathbf{X}})$ is re-written as $I(\hat{\mathbf{X}};\mathbf{X})$, it can be interpreted as the capacity of a Gaussian channel from $\hat{\mathbf{x}}$ to $\mathbf{x}$ with $\mathbf{e}$ as the additive noise:

$$I(\mathbf{X};\hat{\mathbf{X}}) = I(\hat{\mathbf{X}};\mathbf{X}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}, \tag{18}$$

where S_{xx} and S_{ee} are covariance matrices of $\mathbf{x}$ and $\mathbf{e}$ respectively. This mutual information is related to the capacity of the original channel $I(\mathbf{X};\mathbf{Y})$. The key is the following observation made in [19]:

$$I(\mathbf{X};\mathbf{Y}) = H(\mathbf{Y}) - H(\mathbf{Y}|\mathbf{X}) = \frac{1}{2}\log\frac{|S_{yy}|}{|S_{y|x}|} = \frac{1}{2}\log\frac{|S_{yy}|}{|S_{nn}|}, \tag{19}$$

$$I(\mathbf{Y};\mathbf{X}) = H(\mathbf{X}) - H(\mathbf{X}|\mathbf{Y}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{x|y}|} = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}, \tag{20}$$

where $H(\mathbf{Y}|\mathbf{X})$ is the uncertainty in $\mathbf{y}$ given $\mathbf{x}$, so $S_{y|x} = S_{nn}$, and likewise, $H(\mathbf{X}|\mathbf{Y})$ is the uncertainty in $\mathbf{x}$ given $\mathbf{y}$, so $S_{x|y} = S_{ee}$. Since $I(\mathbf{X};\mathbf{Y}) = I(\mathbf{Y};\mathbf{X})$, so $\frac{1}{2}\log(|S_{xx}|/|S_{ee}|) = \frac{1}{2}\log(|S_{yy}|/|S_{nn}|)$. Thus,

$$I(\mathbf{X};\mathbf{Y}) = I(\mathbf{Y};\mathbf{X}) = I(\mathbf{X};\hat{\mathbf{X}}) = I(\hat{\mathbf{X}};\mathbf{X}), \tag{21}$$

which shows that MMSE estimation is capacity-lossless. Figure 5 illustrates the channels associated with each of the four mutual information quantities. In particular, $I(\mathbf{X};\mathbf{Y})$ is the capacity of the channel from $\mathbf{x}$ to $\mathbf{y}$ with $\mathbf{n} = \mathbf{y} - \mathbf{E}[\mathbf{y}|\mathbf{x}]$ as noise. $I(\mathbf{Y};\mathbf{X})$ can be interpreted as the capacity of the channel from $\mathbf{y}$ to $\mathbf{x}$ with $\mathbf{e} = \mathbf{x} - \mathbf{E}[\mathbf{x}|\mathbf{y}]$ as noise. This pair of channels is called the forward and backward channels. The two channels have the same capacity. Another pair of channels relates $\mathbf{x}$ and $\hat{\mathbf{x}}$. They also have the same capacity.

Now write $\hat{\mathbf{x}}^T = [\hat{\mathbf{x}}_1^T \hat{\mathbf{x}}_2^T]$. Suppose that $\mathbf{x_1}$ and $\mathbf{x_2}$ are independently coded with two separate codebooks. Decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$, however, cannot be done on $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ separately.

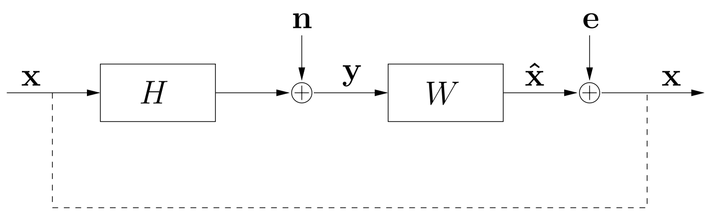

Figure 5: Minimum mean-square error estimation

To see this, write $\mathbf{e_1} = \mathbf{x_1} - \hat{\mathbf{x}}_1$ and $\mathbf{e_2} = \mathbf{x_2} - \hat{\mathbf{x}}_2$. Individual detections on $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ achieve $I(\mathbf{X_1}; \hat{\mathbf{X}}_1)$ and $I(\mathbf{X_2}; \hat{\mathbf{X}}_2)$, respectively. Now, because $\mathbf{e_1}$ and $\mathbf{e_2}$ are independent of $\mathbf{x_1}$ and $\mathbf{x_2}$ respectively, and they are both Gaussian, the argument in the previous paragraph may be repeated to conclude that individual detections on $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ achieve $\frac{1}{2}\log(|S_{x_1x_1}|/|S_{e_1e_1}|)$ and $\frac{1}{2}\log(|S_{x_2x_2}|/|S_{e_2e_2}|)$, respectively. But $\mathbf{e_1}$ and $\mathbf{e_2}$ are not uncorrelated, so by Hadamard's inequality, $|S_{ee}| \le |S_{e_1e_1}| \cdot |S_{e_2e_2}|$, thus

$$\frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1e_1}|} + \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|S_{e_2e_2}|} \le \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}. \tag{22}$$

Thus, although decoding of $\mathbf{x}$ based on $\hat{\mathbf{x}}$ is capacity-lossless, independent decoding of $\mathbf{x_1}$ based on $\hat{\mathbf{x}}_1$ and decoding of $\mathbf{x_2}$ based $\hat{\mathbf{x}}_2$ is capacity-lossy.

The goal of GDFE is to use decision-feedback to facilitate the independent decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$. This is accomplished by a diagonalization of the MMSE error $\mathbf{e}$, while preserving the "information content" in $\hat{\mathbf{x}}$. Toward this end, the MMSE filter W in Figure 5 is broken into two components, creating yet another pair of forward and backward channels. First, let's write down the MMSE filter W explicitly,

$$\begin{aligned} W &= S_{xy}S_{yy}^{-1} && (23)\\ &= S_{xx}H^T(HS_{xx}H^T + I)^{-1} && (24)\\ &= (H^TH + S_{xx}^{-1})^{-1}H^T, && (25) \end{aligned}$$

where (23) follows from standard linear estimation theory, and (25) follows from the matrix inversion lemma [20], which will be used repeatedly in subsequent development,

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}. \tag{26}$$

Now, it is clear that W may be split into a matched filter H^T, and an estimation filter $(H^TH + S_{xx}^{-1})^{-1}$, as shown in Figure 6. This creates the forward channel from $\mathbf{x}$ to $\mathbf{z}$:

$$\mathbf{z} = H^TH\mathbf{x} + H^T\mathbf{n} = R_f\mathbf{x} + \mathbf{n}', \tag{27}$$

and the backward channel from $\mathbf{z}$ to $\mathbf{x}$:

$$\mathbf{x} = (H^TH + S_{xx}^{-1})^{-1}\mathbf{z} + \mathbf{e} = R_b\mathbf{z} + \mathbf{e}, \tag{28}$$

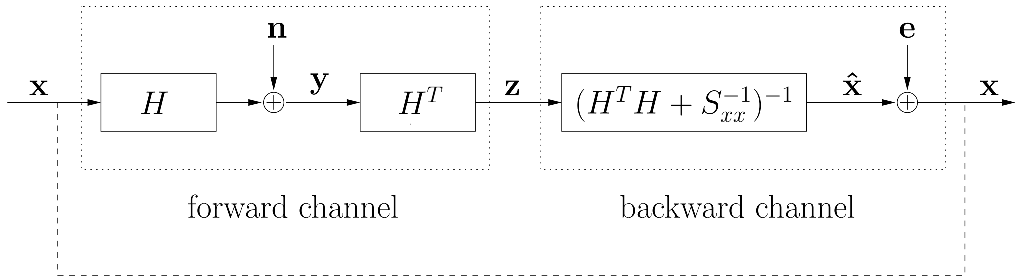

Figure 6: Forward and backward channels

where $R_f = H^T H$ and $R_b = (H^T H + S_{xx}^{-1})^{-1}$. Note that in the forward channel, the covariance matrix of the noise $\mathbf{n}'$ is the same as the channel R_f. The second key idea in GDFE is to recognize that the backward channel also has the same property:

$$\begin{aligned}
\mathbf{E}[\mathbf{e}\mathbf{e}^T] &= \mathbf{E}[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T] \\
&= \mathbf{E}[(\mathbf{x}-S_{xy}S_{yy}^{-1}\mathbf{y})(\mathbf{x}-S_{xy}S_{yy}^{-1}\mathbf{y})^T] \\
&= S_{xx} - S_{xx}H^T(HS_{xx}H^T+I)^{-1}HS_{xx} \\
&= (H^T H + S_{xx}^{-1})^{-1} \\
&= R_b,
\end{aligned} \tag{29}$$

where the matrix inversion lemma (26) is again used.

Recall that the goal is to diagonalize the MMSE error $\mathbf{e}$. The third key idea in GDFE is to recognize that the diagonalization may be done causally using a block-Cholesky factorization of R_b, which is both the backward channel matrix and the covariance matrix of $\mathbf{e}$:

$$R_b = G^{-1}\Delta^{-1}G^{-T}, \tag{30}$$

where $G = \begin{bmatrix} I & G_{22} \\ 0 & I \end{bmatrix}$ is a block upper triangular matrix and $\Delta = \begin{bmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{bmatrix}$ is a block-diagonal matrix. The Cholesky factorization diagonalizes $\mathbf{e}$ in the following sense. Define $\mathbf{e}' = G\mathbf{e}$:

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \end{bmatrix} = \begin{bmatrix} I & G_{22} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}. \tag{31}$$

Then, its components $\mathbf{e}_1'$ and $\mathbf{e}_2'$ are uncorrelated because

$$S_{e'e'} = \mathbf{E}[\mathbf{e}'\mathbf{e}'^T] = \mathbf{E}[G\mathbf{e}(G\mathbf{e})^T] = GR_bG^T = \Delta^{-1} \tag{32}$$

is a block-diagonal matrix. Further, the diagonalization preserves the determinant of the covariance matrix:

$$|S_{e'e'}| = |\Delta^{-1}| = |G^{-1}\Delta^{-1}G^{-T}| = |S_{ee}|. \tag{33}$$

This diagonalization can be done directly by modifying the backward canonical channel to form a decision-feedback equalizer. Because the channel and the noise covariance are the same, it is possible to split the channel filter R_b into the following feedback configuration:

$$\begin{aligned}
\mathbf{x} &= R_b\mathbf{z} + \mathbf{e} && (34)\\
\mathbf{x} &= G^{-1}\Delta^{-1}G^{-T}\mathbf{z} + \mathbf{e} && (35)\\
G\mathbf{x} &= \Delta^{-1}G^{-T}\mathbf{z} + G\mathbf{e} && (36)\\
\mathbf{x} &= \Delta^{-1}G^{-T}\mathbf{z} + (I-G)\mathbf{x} + G\mathbf{e}. && (37)
\end{aligned}$$

Writing out the matrix computation explicitly,

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} = \begin{bmatrix} \Delta_{11}^{-1} & 0 \\ 0 & \Delta_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -G_{22}^T & I \end{bmatrix} \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix} + \begin{bmatrix} 0 & -G_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} + \begin{bmatrix} \mathbf{e_1'} \\ \mathbf{e_2'} \end{bmatrix}. \quad (38)$$

It is now clear that the backward canonical channel is split into two individual sub-channels. The sub-channel for $\mathbf{x_2}$ is:

$$\mathbf{x_2} = \Delta_{22}^{-1}(-G_{22}^T\mathbf{z_1} + \mathbf{z_2}) + \mathbf{e_2'} \triangleq \mathbf{x_2'} + \mathbf{e_2'}. \quad (39)$$

And once $\mathbf{x_2}$ is decoded correctly, $G_{22}\mathbf{x_2}$ can be subtracted from the other sub-channel to form:

$$\mathbf{x_1} = \Delta_{11}^{-1}\mathbf{z_1} + \mathbf{e_1'} \triangleq \mathbf{x_1'} + \mathbf{e_1'}, \quad (40)$$

where $\mathbf{x}'$ is defined as $\mathbf{x}' \triangleq \Delta^{-1}G^{-T}\mathbf{z} + (I-G)\mathbf{x}$, and $\mathbf{x}'^T = [\mathbf{x_1'}^T \mathbf{x_2'}^T]$. This suggests the generalized decision-feedback structure as shown in Figure 7. The combination $\Delta^{-1}G^{-T}H^T$ is called the GDFE feedforward filter, and $I-G$ is called the feedback filter.

The GDFE is capacity-lossless. To see this, note that the maximum achievable rate with a GDFE is $I(\mathbf{X};\mathbf{X}')$. But this mutual information, when written as $I(\mathbf{X}';\mathbf{X})$, can also be interpreted as the capacity of the channel $\mathbf{x} = \mathbf{x}' + \mathbf{e}'$. Because $\mathbf{e}' = G\mathbf{e}$ is Gaussian and is independent of $\hat{\mathbf{x}}$, so it is independent of $\mathbf{z}$ and thus independent of $\mathbf{x}'$, then

$$I(\mathbf{X};\mathbf{X}') = I(\mathbf{X}';\mathbf{X}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{e'e'}|}. \quad (41)$$

This is precisely the capacity of the original channel because:

$$I(\mathbf{X};\mathbf{Y}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|} = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{e'e'}|} = I(\mathbf{X};\mathbf{X}'). \quad (42)$$

Further, S_{xx} and $S_{e'e'}$ are both diagonal, i.e. $|S_{xx}| = |S_{x_1x_1}| \cdot |S_{x_2x_2}|$, and $|S_{e'e'}| = |\Delta^{-1}| = |\Delta_{11}^{-1}| \cdot |\Delta_{22}^{-1}| = |S_{e_1'e_1'}| \cdot |S_{e_2'e_2'}|$. So, the GDFE structure has decomposed the vector channel into two sub-channels each of which can be independently encoded and decoded. The capacities of the sub-channels are:

$$R_1 = I(\mathbf{X_1'};\mathbf{X_1}) = \frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1'e_1'}|}, \quad (43)$$

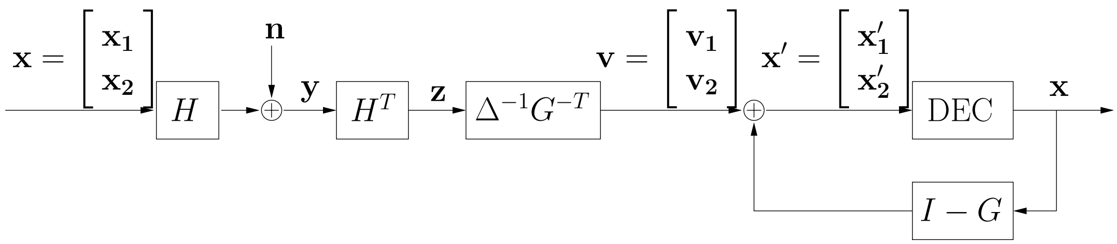

Figure 7: Generalized decision feedback equalizer

and

$$R_2 = I(\mathbf{X_2'}; \mathbf{X_2}) = \frac{1}{2} \log \frac{|S_{x_2x_2}|}{|S_{e_2'e_2'}|}. \tag{44}$$

And the sum capacity is:

$$\begin{aligned} R_1 + R_2 &= I(\mathbf{X_1'}; \mathbf{X_1}) + I(\mathbf{X_2'}; \mathbf{X_2}) && (45) \\ &= \frac{1}{2} \log \frac{|S_{x_1x_1}|}{|S_{e_1'e_1'}|} + \frac{1}{2} \log \frac{|S_{x_2x_2}|}{|S_{e_2'e_2'}|} && (46) \\ &= \frac{1}{2} \log \frac{|S_{xx}|}{|S_{ee}|} && (47) \\ &= I(\mathbf{X}; \mathbf{Y}). && (48) \end{aligned}$$

3.2 Precoding

The decision-feedback equalizer is able to decompose a vector channel into two individual sub-channels that can be independently coded. As long as $\mathbf{x_1}$ and $\mathbf{x_2}$ are statistically independent and the decision-feedback is error-free, the sum capacity of the two sub-channels is the same as the capacity of the original vector channel. Thus, transmitter coordination is not necessary to achieve the mutual information $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y})$. Compared to a vector channel, the capacity loss due to the lack of coordination at the transmitter is just the decrease in mutual information due to the block-diagonal constraint on the input covariance S_{xx}. On the other hand, receiver coordination is required in a decision-feedback equalizer. This is true for two reasons. First, generating $\mathbf{v}$ in Figure 7 requires the entire received vector $\mathbf{y}$. Second, the feedback structure requires the correct codeword from the second sub-channel to be available before the decoding of the first sub-channel can begin. It turns out that the second problem can be averted using ideas from coding for channels with transmitter side information. In this section, a precoding method that essentially moves the feedback operation to the transmitter is described.

First, it is instructive to explicitly compute the achievable rates of the two sub-channels in a generalized decision-feedback equalizer. The GDFE structure assumes independently coded $\mathbf{x_1}$ and $\mathbf{x_2}$, so the individual rates of the sub-channels form an achievable rate-pair in a multiple access channel. This view was take by Varanasi and Guess [4] who derived a similar decision-feedback equalizer for a multiple access channel. Now, let $H = [H_1 H_2]$[1], $\mathbf{n}^T = [\mathbf{n_1}^T \mathbf{n_2}^T]$, and write the vector channel in the form of a multiple access channel:

$$\mathbf{y} = H\mathbf{x} + \mathbf{n} = [H_1 H_2] \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} + \begin{bmatrix} \mathbf{n_1} \\ \mathbf{n_2} \end{bmatrix}. \tag{49}$$

The block Cholesky factorization (30) may be computed explicitly [20]:

$$(S_{xx}^{-1} + H^T H)^{-1} = \begin{bmatrix} S_{x_1x_1}^{-1} + H_1^T H_1 & H_1^T H_2 \\ H_2^T H_1 & S_{x_2x_2}^{-1} + H_2^T H_2 \end{bmatrix}^{-1} = G^{-1}\Delta^{-1}G^{-T}, \tag{50}$$

where

$$G = \begin{bmatrix} I & (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2 \\ 0 & I \end{bmatrix}, \tag{51}$$

and

$$\Delta^{-1} = \begin{bmatrix} (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} & 0 \\ 0 & (S_{x_2x_2}^{-1} + H_2^T H_2 - H_2^T H_1 (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2)^{-1} \end{bmatrix}. \tag{52}$$

Thus,

$$S_{e'_1e'_1} = \Delta_{11}^{-1} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1}, \tag{53}$$

so from (43),

$$I(\mathbf{X_1'}; \mathbf{X_1}) = \frac{1}{2} \log \frac{|S_{x_1x_1}|}{|(S_{x_1x_1}^{-1} + H_1^T H_1)^{-1}|} = \frac{1}{2} \log |H_1 S_{x_1x_1} H_1^T + I|, \tag{54}$$

where the matrix identity $|I + AB| = |I + BA|$ is used. This R_1 is precisely the capacity of the multiple access channel (49) without $\mathbf{x_2}$, i.e.

$$R_1 = I(\mathbf{X_1'}; \mathbf{X_1}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}). \tag{55}$$

Also,

$$\begin{aligned} S_{e'_2e'_2} &= (S_{x_2x_2}^{-1} + H_2^T H_2 - H_2^T H_1 (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2)^{-1}, && (56) \\ &= (S_{x_2x_2}^{-1} + H_2^T (I + H_1 S_{x_1x_1} H_1^T)^{-1} H_2)^{-1}, && (57) \end{aligned}$$

where the matrix inversion lemma is used. So from (44),

$$\begin{aligned} I(\mathbf{X_2'}; \mathbf{X_2}) &= \frac{1}{2} \log \frac{|S_{x_2x_2}|}{|(S_{x_2x_2}^{-1} + H_2^T (I + H_1 S_{x_1x_1} H_1^T)^{-1} H_2)^{-1}|} && (58) \\ &= \frac{1}{2} \log \frac{|H_1 S_{x_1x_1} H_1^T + H_2 S_{x_2x_2} H_2^T + I|}{|H_1 S_{x_1x_1} H_1^T + I|}, && (59) \end{aligned}$$

[1]For the rest of this section only, define $H = [H_1 H_2]$. Elsewhere in the paper, define $H^T = [H_1^T H_2^T]$.

which can be verified by directly multiplying out the respective terms, and by repeated uses of the identity $|I + AB| = |I + BA|$. So,

$$R_2 = I(\mathbf{X_2'}; \mathbf{X_2}) = I(\mathbf{X_2}; \mathbf{Y}). \tag{60}$$

This verifies that the achievable sum rate in the multiple access channel using GDFE is

$$R_1 + R_2 = I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}) = \frac{1}{2} \log |H_1 S_{x_1x_1} H_1^T + H_2 S_{x_2x_2} H_2^T + I| \tag{61}$$

Therefore, the generalized decision feedback equalizer not only achieves the sum capacity in a multiple access channel, it also achieves the individual rates of a corner point in the multiple access capacity region. Interchanging the order of $\mathbf{x_1}$ and $\mathbf{x_2}$ achieves the other corner point. This, together with time-sharing or rate-splitting, allows GDFE to achieve the entire rate region of the multiple access channel.

The decision-feedback structure requires one sub-channel to be decoded correctly before the feedback. In practice, however, error propagation can occur. But, if transmit coordination is also allowed, and both $\mathbf{x_1}$ and $\mathbf{x_2}$ are known at the transmitter, it is possible to use a "writing-on-dirty-paper" approach to pre-subtract the effect of $\mathbf{x_2}$ in $\mathbf{x_1}$. This would completely eliminate the effect of error propagation and partially alleviate the need for receiver coordination. The rest of this section investigates this possibility. The main result is the following.

Theorem 2 *For a Gaussian vector channel $\mathbf{y} = \sum_{i=1}^{K} H_i \mathbf{x_i} + \mathbf{n}$, where $\mathbf{x_i}$'s are statistically independent Gaussian signals, the sum capacity $I(\mathbf{X_1}, \cdots, \mathbf{X_K}; \mathbf{Y})$ with $R_i = I(\mathbf{X_i}; \mathbf{Y}|\mathbf{X_{i+1}}, \cdots, \mathbf{X_K})$, is achievable in two ways: either using a decision-feedback structure where the knowledge of $\mathbf{x_{i+1}}, \cdots, \mathbf{x_K}$ is assumed to be available before the decoding of $\mathbf{x_i}$, or a precoder structure where the knowledge of $\mathbf{x_{i+1}}, \cdots, \mathbf{x_K}$ is assumed to be available before the encoding of $\mathbf{x_i}$.*

Proof: The development leading to the theorem shows that in a generalized decision-feedback equalizer, $I(\mathbf{X_1}; \mathbf{X_1'}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2})$, $I(\mathbf{X_2}; \mathbf{X_2'}) = I(\mathbf{X_2}; \mathbf{Y})$, and $I(\mathbf{X_1}; \mathbf{X_1'}) + I(\mathbf{X_2}; \mathbf{X_2'}) = I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y})$, with the assumption that $\mathbf{x_1}$ and $\mathbf{x_2}$ are Gaussian and no error propagation occurs. Thus, two independent Gaussian random codebooks can be designed on $\mathbf{x_1}$ and $\mathbf{x_2}$ to achieve the desired rate-pair. An induction argument generalizes this result to more than two users. Assume that a GDFE achieves $R_i = I(\mathbf{X_i}; \mathbf{Y}|\mathbf{X_{i+1}}, \cdots, \mathbf{X_K})$ for a K-user multiple access channel. In a $(K+1)$-user channel, users 1 and 2 can first be considered coordinated, and the GDFE result can be applied to the resulting K-user channel, i.e. $R_i = I(\mathbf{X_i}; \mathbf{Y}|\mathbf{X_{i+1}}, \cdots, \mathbf{X_{K+1}})$ for $i = 3, \cdots, K$, and $R_1 + R_2 = I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}|\mathbf{X_3}, \cdots, \mathbf{X_{K+1}})$. Then, a separate two-user GDFE can be applied to user 1 and 2 to get $R_i = I(\mathbf{X_i}; \mathbf{Y}|\mathbf{X_{i+1}}, \cdots, \mathbf{X_{K+1}})$, for $i = 1, 2$.

Now, the same rate-tuple can be shown to be achievable using a precoding structure for channels with side information available at the transmitter. Consider the signal $\mathbf{v}$ as shown in Figure 7. Write $\mathbf{v}^T = [\mathbf{v_1}^T \mathbf{v_2}^T]$. Note that $\mathbf{v_2} = \mathbf{x_2'}$. So, the sub-channel from $\mathbf{x_2}$ to $\mathbf{v_2}$ is the same as before, and

$$R_2 = I(\mathbf{X_2}; \mathbf{V_2}) = I(\mathbf{X_2}; \mathbf{X_2'}) = I(\mathbf{X_2}; \mathbf{Y}). \tag{62}$$

Now, consider the sub-channel from $\mathbf{x_1}$ to $\mathbf{v_1}$ with $\mathbf{x_2}$ available at the transmitter instead of at the receiver. Because $\mathbf{x_2}$ is Gaussian, and it is independent of $\mathbf{x_1}$, Lemma 1 applies. So, the capacity of this sub-channel is just $I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2})$. To compute this conditional mutual information, it is necessary to explicitly write out the interference cancelation step in the forward channel. Since

$$\mathbf{v} = \Delta^{-1} G^{-T} H^T (H\mathbf{x} + \mathbf{n}), \tag{63}$$

using (52) and (51), $\mathbf{v_1}$ can be expressed as:

$$\mathbf{v_1} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + H_2 \mathbf{x_2}) + \mathbf{n_1'}, \tag{64}$$

where $\mathbf{n}' = \Delta^{-1} G^{-T} H^T \mathbf{n}$, and $\mathbf{n}'^T = [\mathbf{n_1'}^T \mathbf{n_2'}^T]$. It can be shown that $\mathbf{n_1'}$ has a covariance matrix:

$$\mathbf{E}[\mathbf{n_1'} \mathbf{n_1'}^T] = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_1 (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1}. \tag{65}$$

So, $\mathbf{v_1}$ is equivalent to

$$\mathbf{v_1} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + H_2 \mathbf{x_2} + \mathbf{n_1}), \tag{66}$$

where $\mathbf{n_1}$ is the component of $\mathbf{n}^T = [\mathbf{n_1}^T \mathbf{n_2}^T]$. On the other hand, $\mathbf{x_1'}$ can be computed explicitly from $\mathbf{x}' = \mathbf{v} + (I - G)\mathbf{x}$.

$$\mathbf{x_1'} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + \mathbf{n_1}). \tag{67}$$

Since $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{n_1}$ are jointly independent, it is now clear that:

$$R_1 = I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2}) = I(\mathbf{X_1}; \mathbf{X_1'}) = I(\mathbf{X_2}; \mathbf{Y}). \tag{68}$$

Therefore, interference cancelation may occur at the transmitter by pre-subtracting $\mathbf{x_2}$ from $\mathbf{x_1}$. Pre-subtraction achieves the exact same capacity as a decision-feedback equalizer. This proof generalizes to the K-user case by an induction argument similar to before. □

Figure 8 and Figure 9 illustrate the two configurations of GDFE. Figure 8 illustrates the decision-feedback configuration. Two independent codes are used separately by the two users. After user 2's codeword is decoded, its effect, namely $(S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2 \mathbf{x_2}$, is subtracted from user 1's signal before user 1's codeword is decoded. This decision-feedback configuration is able to achieve the vector channel capacity using single-user codes in the sense that $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}) + I(\mathbf{X_2}, \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{X_1'}) + I(\mathbf{X_2}; \mathbf{X_2'})$. Figure 9 illustrates the precoder configuration. In this case, user 2 uses a single-user code as usual. User 1's channel is a Gaussian channel with transmitter side information, and it uses a precoder to completely pre-subtract the effect of user 2, namely $H_2 \mathbf{x_2}$. This precoder configuration achieves the vector channel capacity in the sense that $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}) + I(\mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2}) + I(\mathbf{X_2}; \mathbf{V_2})$. In the decision-feedback configuration, user 2's codewords are assumed to be decoded correctly before its interference is subtracted. This requires long codeword length to be used, thus implicitly implies a delay between the decoding of the

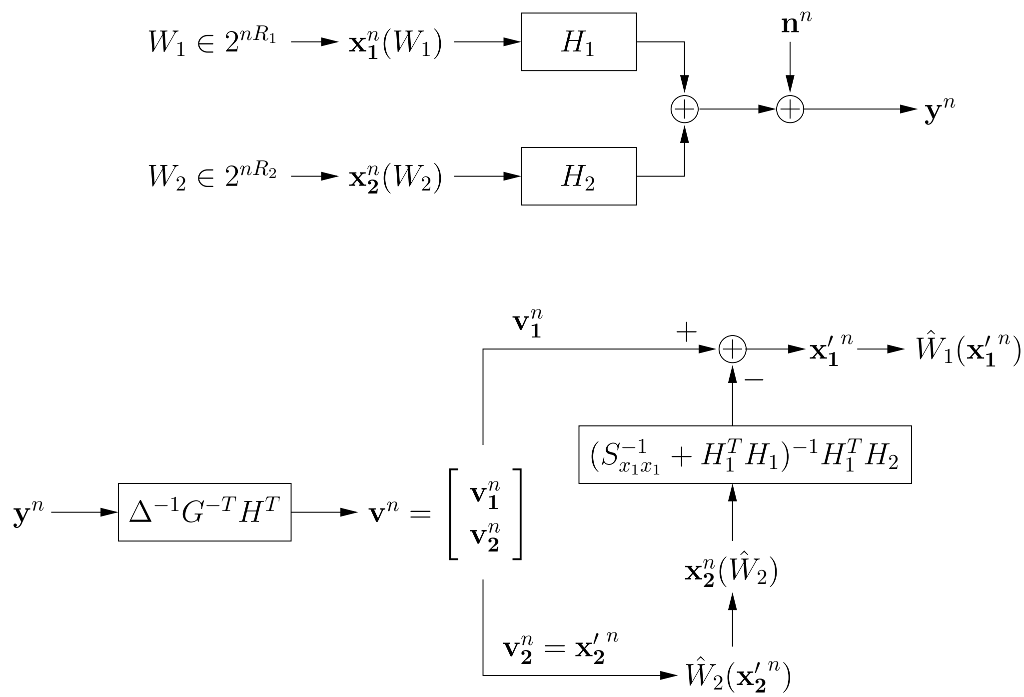

Figure 8: Decision feedback decoding

two users. If a erroneous decision on user 2 were made, the error would propagate. In the precoding configuration, error propagation never occurs. However, because non-causal side information is needed, user 1's message cannot be encoded until user 2's codeword is available, thus implying a delay at the encoder. The two situations are symmetric, and they are both capacity-achieving.

The decision-feedback configuration does not require transmitter coordination. So, it is naturally suited for a multiple access channel. In the precoder configuration, the feedback operation is moved to the transmitter. So, one might hope that it corresponds to a broadcast channel where receiver coordination is not possible. This is, however, not yet true in the present setting. The capacity-achieving precoder requires a feedforward filter which acts on the entire received signal, so receiver coordination is still needed. However, under certain conditions, the feedforward filter degenerates into a diagonal matrix which does not require receiver coordination. The condition under which this happens is the focus of the rest of this paper.

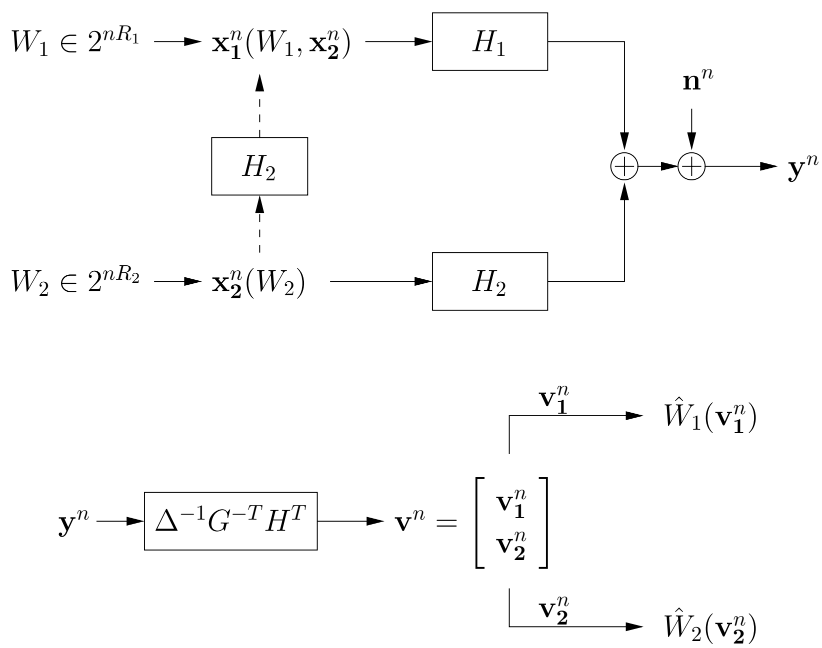

Figure 9: Decision feedback precoding

4 Broadcast Channel Sum Capacity

4.1 Least Favorable Noise

Consider the broadcast channel

$$\begin{bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{n_1} \\ \mathbf{n_2} \end{bmatrix}, \tag{69}$$

where $\mathbf{y_1}$ and $\mathbf{y_2}$ do not cooperate. Fix an input distribution $p(\mathbf{x})$. The sum capacity of the broadcast channel is clearly bounded by the capacity of the vector channel $I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2})$ where $\mathbf{y_1}$ and $\mathbf{y_2}$ cooperate. As recognized by Sato [21], this bound can be further tightened. Because $\mathbf{y_1}$ and $\mathbf{y_2}$ cannot coordinate in a broadcast channel, the broadcast channel capacity does not depend on the joint distribution $p(\mathbf{n_1}, \mathbf{n_2})$ but only on the marginals $p(\mathbf{n_1})$ and $p(\mathbf{n_2})$. This is so because two broadcast channels with the same marginals but different joint noise distribution can interchange their respective codebooks, and retain the same probability of error. Therefore, the sum capacity of a broadcast channel must be bounded by the smallest cooperative capacity of the vector channel:

$$R_1 + R_2 \leq \min I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2}), \tag{70}$$

where the minimization is over all $p(\mathbf{n_1}, \mathbf{n_2})$ which has the same marginal distribution as the actual broadcast channel noise. The minimizing distribution is called the least favorable

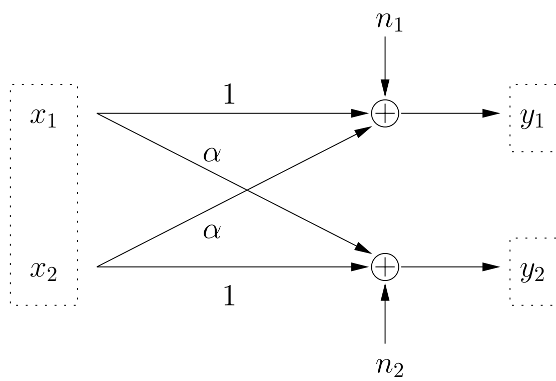

Figure 10: A two-user broadcast channel

noise. Sato's bound is the basis of Caire and Shamai's computation of two-by-two broadcast channel capacity [6].

The following example illustrates Sato's bound. Consider the two-user two-terminal broadcast channel shown in Figure 10, where the channel from x_1 to y_1 and the channel from x_2 to y_2 have unit gain, the cross-over channels have a gain α. Assume that x_1 and x_2 are independent Gaussian signals, and n_1 and n_2 are Gaussian noises, all with unit variance. The broadcast channel capacity is clearly bounded by $I(X_1, X_2; Y_1, Y_2)$, which depends on the cross-over channel gain α, and the correlation coefficient ρ between n_1 and n_2. Consider the case $\alpha = 0$. In this case, the least favorable noise correlation is $\rho = 0$. This is because if n_1 and n_2 were correlated, decoding of y_1 would reveal n_1 from which n_2 can be partially inferred. Such inference is possible, of course, only if y_1 and y_2 can cooperate. In a broadcast channel, where y_1 and y_2 cannot take advantage of such correlation, the capacity with correlated n_1 and n_2 is the same as with uncorrelated n_1 and n_2. Thus, regardless of the actual correlation between n_1 and n_2, the broadcast channel capacity is bounded by the mutual information $I(X_1, X_2; Y_1, Y_2)$ evaluated assuming uncorrelated n_1 and n_2. Consider another case $\alpha = 1$. The least favorable noise is the perfectly correlated noise, i.e. $\rho = 1$. This is because $\rho = 1$ implies $n_1 = n_2$ and $y_1 = y_2$. So, one of y_1 and y_2 is superfluous. If n_1 and n_2 were not perfectly correlated, (y_1, y_2) collectively would reveal more information then y_1 or y_2 alone would. But again, in a broadcast channel, y_1 and y_2 cannot take advantage of joint decoding. So, the sum capacity of the broadcast channel is bounded by the mutual information $I(X_1, X_2; Y_1, Y_2)$ evaluated assuming the least favorable noise correlation $\rho = 1$. This example also illustrates that the correlation of the least favorable noise depends on the correlation structure of the channel. The rest of this section is devoted to a characterization of the least favorable noise correlation.

Consider the Gaussian vector channel $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{n_i}, i = 1, \cdots, K$. Assume for now that $\mathbf{x}$ is a vector Gaussian signal with a fixed covariance matrix S_{xx}, and $\mathbf{n_1}, \cdots, \mathbf{n_K}$ are jointly Gaussian noises each with a marginal distribution $\mathbf{n_i} \sim \mathcal{N}(0, I)$. Both assumptions will be justified later. Then, the task of finding the least favorable noise correlation can be formulated as the following optimization problem. Let $H^T = [H_1^T \cdots H_K^T]$. The minimization problem is:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|} \\ \text{subject to} \quad & S_{nn}^{(i)} = I, i = 1, \cdots, K, \\ & S_{nn} \geq 0, \end{aligned} \tag{71}$$

where S_{nn} is the covariance matrix for $\mathbf{n}$ with $\mathbf{n}^T = [\mathbf{n_1}^T \cdots \mathbf{n_K}^T]$, and $S_{nn}^{(i)}$ refers to the ith block-diagonal term of S_{nn}. In effect, $\mathbf{n_i}$'s are allowed to have arbitrary correlations, and the broadcast channel capacity is bounded by the cooperative capacity associated with the least favorable noise correlation.

In writing down the optimization problem (71), it had been tacitly assumed that the minimizing S_{nn} is strictly positive definite, so that $|S_{nn}| > 0$. This requires justification. In fact, this is not true in general. For example, for the two-user broadcast channel considered earlier, the least favorable noise with $\alpha = 1$ has a covariance matrix equal to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which is singular. In fact, whenever the minimizing S_{nn} is singular, it must also be that $|HS_{xx}H^T + S_{nn}| = 0$, as otherwise the mutual information goes to infinity. But $|HS_{xx}H^T + S_{nn}|$ cannot be zero unless $|HS_{xx}H^T|$ is zero. Thus, a sufficient condition for avoiding a singularity in S_{nn} is that $|HS_{xx}H^T| > 0$. The assumption that minimizing S_{nn} is non-singular is made throughout the rest of this paper.

The following lemma characterizes an optimality condition for the least favorable noise. For now, the input signal is restricted to be Gaussian with a fixed covariance matrix. As it will be shown later, this restriction is without loss of generality.

Lemma 2 *Consider a Gaussian vector broadcast channel* $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{n_i}, i = 1, \cdots, K$, *where* $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, *and* $\mathbf{n_i} \sim \mathcal{N}(0, I)$. *Then, the least favorable noise distribution that minimizes* $I(\mathbf{X}; \mathbf{Y_1}, \cdots, \mathbf{Y_K})$ *is such that* $\mathbf{n_1}, \cdots, \mathbf{n_K}$ *are jointly Gaussian. Further, if the minimizing* S_{nn} *is non-singular, then, the least favorable noise has a covariance matrix* S_{nn} *such that* $S_{nn}^{-1} - (HS_{xx}H^T + S_{nn})^{-1}$ *is a block-diagonal matrix, where* $H^T = [H_1^T \cdots H_K^T]$. *Conversely, any Gaussian noise with a covariance matrix* S_{nn} *satisfying the diagonalization condition and* $S_{nn}^{(i)} = I$ *is a least favorable noise.*

Proof: Fix a Gaussian input distribution $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, and fix a noise covariance matrix S_{nn}. Let $\mathbf{n} \sim \mathcal{N}(0, S_{nn})$ be a Gaussian random vector, and let $\mathbf{n}'$ be any other random vector with the same covariance matrix, but with possibly a different distribution. Then, $I(\mathbf{X}; H\mathbf{X} + \mathbf{N}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{N}')$. This fact was proved in [22] and [23]. So, for a Gaussian input, Gaussian noise is the least favorable distribution among all distributions. Thus, to minimize $I(\mathbf{X}; \mathbf{Y_1}, \cdots, \mathbf{Y_K})$, it is without loss of generality to restrict attention to $\mathbf{n_1}, \cdots, \mathbf{n_K}$ that are jointly Gaussian. In this case, the cooperative capacity is just $\frac{1}{2} \log |HS_{xx}H^T + S_{nn}|/|S_{nn}|$. So, the least favorable noise is the solution to the optimization problem (71).

The objective function in the optimization problem is convex in the set of semi-definite matrices S_{nn}. The constraints are convex in S_{nn}, and they satisfy the constrained quantification condition. Thus, the Karush-Kuhn-Tucker (KKT) condition is a necessary and sufficient condition for optimality. To derive the KKT condition, form the Lagrangian:

$$L(S_{nn}, \Psi_1, \cdots, \Psi_K, \Phi) = \log |HS_{xx}H^T + S_{nn}| - \log |S_{nn}| + \sum_{i=1}^{K} \text{tr}(\Psi_i(S_{nn}^{(i)} - I)) - \text{tr}(\Phi S_{nn}), \quad (72)$$

where $(\Psi_1, \cdots, \Psi_K)$ are the dual variables associated with the block-diagonal constraints, and Φ is the dual variable associated with the semi-definite constraint. ($\Psi_1, \cdots, \Psi_K, \Phi$ are positive semi-definite matrices.) The coefficient $\frac{1}{2}$ is omitted for simplicity. Setting $\partial L / \partial S_{nn}$ to zero:

$$0 = \frac{\partial L}{\partial S_{nn}} = (HS_{xx}H^T + S_{nn})^{-1} - S_{nn}^{-1} + \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix} - \Phi. \quad (73)$$

The minimizing S_{nn} is assumed to be positive definite, so by the complementary slackness condition, $\Phi = 0$. Thus, at the optimum, the following block-diagonal condition must be satisfied:

$$S_{nn}^{-1} - (HS_{xx}H^T + S_{nn})^{-1} = \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix}. \quad (74)$$

Conversely, this block-diagonal condition, together with the constraints in the original problem form the KKT condition, which is sufficient for optimality to hold. □

Note that the diagonalization condition may be written in a different form. If assuming, in addition, that $HS_{xx}H^T$ is non-singular, and $\Psi_1, \cdots, \Psi_K$ are invertible, (74) may be rewritten using the matrix inversion lemma as follows:

$$S_{nn} + S_{nn}(HS_{xx}H^T)^{-1}S_{nn} = \begin{bmatrix} \Psi_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \Psi_K^{-1} \end{bmatrix}. \quad (75)$$

Curiously, this equation resembles a Ricatti equation. Although neither condition (74) nor condition (75) appears to have a closed-form solution, the diagonalization condition allows the GDFE feedforward filter to be evaluated in terms of the dual variables, which would reveal the structure of the GDFE corresponding to the least favorable noise.

4.2 GDFE with Least Favorable Noise

The main result of this paper is to show that the cooperative capacity of the vector Gaussian channel with a least favorable noise is achievable for the Gaussian broadcast channel. Toward this end, it will be shown that a generalized decision feedback precoder designed for the least

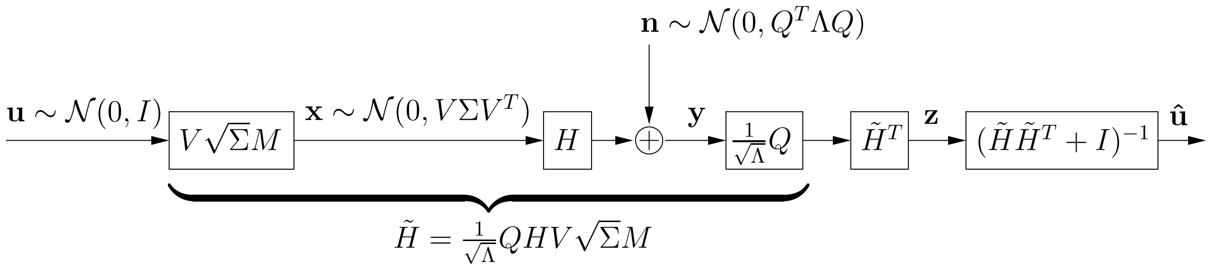

Figure 11: GDFE with transmit filter

favorable noise does not require receiver coordination in the sense that not only the feedback can be moved to the transmitter by precoding, but the feedforward filter has a block-diagonal structure that eliminates the need for coordination.

Consider a Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{n}$. For now, assume that $\mathbf{x}$ is Gaussian, and in addition, assume that H is a square matrix. If the noise covariance matrix S_{nn} is not block-diagonal, an implementation of the generalized decision feedback equalizer requires noise whitening as a first step. Suppose that the noise covariance matrix has an eigenvalue decomposition:

$$S_{nn} = Q^T \Lambda Q, \tag{76}$$

where Q is an orthogonal matrix, and Λ is a diagonal matrix, then $\frac{1}{\sqrt{\Lambda}}Q$ is the appropriate noise whitening filter. If in addition, the transmitter covariance matrix S_{xx} is also not block-diagonal, then a Gaussian source $\mathbf{u}$ and a transmit filter B can be created such that $S_{uu} = I$ and $\mathbf{x} = B\mathbf{u}$. Let

$$S_{xx} = V\Sigma V^T \tag{77}$$

be an eigenvalue decomposition of the transmit covariance matrix S_{xx}. The appropriate transmit filter has the form:

$$B = V\sqrt{\Sigma}M \tag{78}$$

where M is an arbitrary orthogonal matrix. A different generalized decision-feedback equalizer can be designed for each different choice of M. The objective is to show that under the least favorable noise correlation, there exists an orthogonal matrix M such that the feedforward section of the GDFE is block-diagonal.

Lemma 3 *Consider the Gaussian vector channel* $\mathbf{y} = H\mathbf{x} + \mathbf{n}$*, where* H *is a square matrix and* $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$*. Fix a Gaussian source* $\mathbf{u} \sim \mathcal{N}(0, I)$*. There exists a transmit filter* B *such that* $\mathbf{x} = B\mathbf{u}$ *has a covariance matrix* S_{xx} *and the induced generalized decision-feedback equalizer has a block-diagonal feedforward filter if and only if the noise covariance matrix* S_{nn} *is such that* $S_{nn}^{-1} - (S_{nn} + HS_{xx}H^T)^{-1}$ *is block-diagonal.*

Proof: Fix $S_{uu} = I$. Let $S_{xx} = V\Sigma V^T$ be an eigen-value decomposition where Σ is a diagonal matrix and V is an orthogonal matrix. To induce the appropriate transmit covariance with $\mathbf{x} = B\mathbf{u}$, the transmit filter B must be of the form $B = V\sqrt{\Sigma}M$, where M is an orthogonal matrix, so that $S_{xx} = BS_{uu}B^T = V\Sigma V^T$. The rest of the proof shows that it is possible to find an appropriate M to make the GDFE feedforward filter block-diagonal if and only if the noise covariance matrix satisfies the diagonalization condition.

The GDFE configuration is as shown in Figure 11. Let $S_{nn} = Q^T \Lambda Q$, so the noise whitening filter is $\frac{1}{\sqrt{\Lambda}}Q$. The transmit filter and the noise whitening filter create the following effective channel:

$$\widetilde{H} = \frac{1}{\sqrt{\Lambda}} QHV\sqrt{\Sigma}M. \tag{79}$$

The GDFE depends on the following Cholesky factorization:

$$\begin{aligned} G^{-1}\Delta^{-1}G^{-T} &= (\widetilde{H}^T\widetilde{H} + I)^{-1} && (80)\\ &= \left(M^T\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma}M + I\right)^{-1} && (81)\\ &= M^T\left(\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma} + I\right)^{-1}M. && (82) \end{aligned}$$

Now, choose a square matrix R, such that

$$R^TR = \left(\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma} + I\right)^{-1}. \tag{83}$$

(For example, R can be chosen to be a triangular matrix using Cholesky factorization.) Because the right-hand side of the above is positive definite, if a square matrix C satisfies $C^TC = \left(\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma} + I\right)^{-1}$, it must be of the form $C = UR$ where U is an orthogonal matrix [24]. In particular, from (82), $\Delta^{-\frac{1}{2}}G^{-T}M^T = UR$. Then, the Cholesky factorization can be written as:

$$G^{-1}\Delta^{-1}G^{-T} = M^TR^TU^TURM, \tag{84}$$

where URM is block-lower-triangular. For a fixed M, it is possible to choose a U to make URM block-triangular. Such a U can be found via a block QR-factorization of RM. Similarly, if U is fixed and M is allowed to vary, for each particular U, there exists a M that makes URM block-triangular. Such a M is found by a block QR-factorization of $(UR)^T$.

The feedforward filter of a GDFE, now denoted as F, can be computed as follows:

$$\begin{aligned} F &= \Delta^{-1}G^{-T}\widetilde{H}^T\frac{1}{\sqrt{\Lambda}}Q && (85)\\ &= \Delta^{-\frac{1}{2}}URMM^T\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}Q && (86)\\ &= \Delta^{-\frac{1}{2}}UR\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}Q. && (87) \end{aligned}$$

It shall be shown next that the condition under which there exists a suitable UR to make the feedforward filter F block-diagonal is the same as the diagonalization condition on the noise covariance matrix.

Now, assume that $S_{nn}^{-1} - (S_{nn} + HS_{xx}H^T)^{-1}$ is block-diagonal. Then,

$$\begin{aligned}
\begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix} &= S_{nn}^{-1} - (S_{nn} + HS_{xx}H^T)^{-1} \qquad (88) \\
&= Q^T\Lambda^{-1}Q - (Q^T\Lambda Q + HV\Sigma V^T H^T)^{-1} \\
&= Q^T\Lambda^{-\frac{1}{2}}\left(I - \left(I + \Lambda^{-\frac{1}{2}}QHV\Sigma V^T H^T Q^T \Lambda^{-\frac{1}{2}}\right)^{-1}\right)\Lambda^{-\frac{1}{2}}Q \\
&= Q^T\Lambda^{-1}QHV\sqrt{\Sigma}\left(I + \sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1}QHV\sqrt{\Sigma}\right)^{-1}\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1}Q
\end{aligned}$$

where the matrix inversion lemma is used in the last step. Now, substituting (83) into the above gives:

$$Q^T\Lambda^{-1}QHV\sqrt{\Sigma}R^T R\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1}Q = \begin{bmatrix} \Psi_1 & & 0 \\ & \ddots & \\ 0 & & \Psi_K \end{bmatrix}. \qquad (89)$$

Because H is assumed to be a square matrix, $R\sqrt{\Sigma}V^T H^T Q^T$ is also square. So, it must be of the form $U'D$, where U' is an orthogonal matrix and D is any particular square root of $\text{diag}\{\Psi_1, \cdots, \Psi_K\}$. Because $\Psi_1, \cdots, \Psi_K$ are positive semi-definite, D can be chosen to be $\text{diag}\{\sqrt{\Psi}_1, \cdots, \sqrt{\Psi}_K\}$. Thus, there exists an orthogonal matrix U' such that

$$R\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1}Q = U' \begin{bmatrix} \sqrt{\Psi}_1 & & 0 \\ & \ddots & \\ 0 & & \sqrt{\Psi}_K \end{bmatrix}. \qquad (90)$$

But, this is exactly the diagonalization condition for F. By choosing $U = U'^T$ in (87), F becomes:

$$\begin{aligned}
F &= \Delta^{-\frac{1}{2}}U'^T R\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1}Q \qquad (91) \\
&= \Delta^{-\frac{1}{2}} \begin{bmatrix} \sqrt{\Psi}_1 & & 0 \\ & \ddots & \\ 0 & & \sqrt{\Psi}_K \end{bmatrix}. \qquad (92)
\end{aligned}$$

which is block-diagonal. Finally, an appropriate transmit filter B can be found by finding an M that makes URM block lower-triangular. This is possible by the following QR-factorization: $R^T U^T = MK$, where K is upper-triangular, and M is orthogonal. Then, $URM = K^T$ is lower-triangular. In particular, it is block lower-triangular.

Conversely, if there exists a transmit filter that makes F block-diagonal, then a suitable U can be found in (87). Further, by setting $U' = U^T$ in (90), the appropriate $\sqrt{\Psi}_1, \cdots, \sqrt{\Psi}_K$ can be found to satisfy the noise covariance diagonalization condition. Therefore, the feed-forward filter is block diagonal if and only if the noise covariance diagonalization condition is satisfied. $\square$

Combining Lemma 2 and Lemma 3, it is clear that with the least favorable noise, there exists a GDFE structure with a block-diagonal feedforward filter. This, together with a precoder, eliminates the need for coordination at the receiver. Thus, a rate equal to the cooperative capacity with the least favorable noise correlation, i.e. $\min_{S_{nn}} \frac{1}{2}\log|HS_{xx}H^T + S_{nn}|/|S_{nn}|$ is achievable in the broadcast channel. This rate is achieved under a fixed input covariance S_{xx}. So, one would expect that the capacity of the broadcast channel to be the above rate maximized over all S_{xx} subject to a power constraint. This is proved next.

4.3 Sum Capacity

The development so far contains the simplifying assumption that the input distribution is Gaussian. To see that the restriction is without loss of generality, the following fact is useful. Consider for a moment the mutual information expression $I(\mathbf{X}; H\mathbf{X}+\mathbf{N})$. If some $(p(\mathbf{x}), p(\mathbf{n}))$ is such that

$$I(\mathbf{X}'; H\mathbf{X}' + \mathbf{N}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{N}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{N}') \tag{93}$$

for all $p(\mathbf{x}') \in \mathcal{K}_x$ and $p(\mathbf{n}') \in \mathcal{K}_n$, where $\mathcal{K}_x$ and $\mathcal{K}_n$ are some constraint sets for the input and noise distributions, then $(p(\mathbf{x}), p(\mathbf{n}))$ is called a saddle-point. The main result concerning the saddle-point is the following fact due to Diggavi [23].

Lemma 4 ([23]) *Consider the mutual information expression $I(\mathbf{X}; H\mathbf{X} + \mathbf{N})$ where $p(\mathbf{x}) \in \mathcal{K}_x$ and $p(\mathbf{n}) \in \mathcal{K}_n$ are convex constraints. Then, there exists a saddle-point whose distributions are Gaussian.*

The proof of this result can be found in [23]. The proof goes as follows. First, it is shown that the search for the saddle-point can be restricted to Gaussian distributions without loss of generality. Then, the mutual information can be written as $\frac{1}{2}\log|HS_{xx}H^T + S_{nn}|/|S_{nn}|$. Because $\log|\cdot|$ is a concave function over the set of positive definite matrices, $\frac{1}{2}\log|HS_{xx}H^T + S_{nn}|/|S_{nn}|$ is convex in S_{nn} and concave in S_{xx}. The constraints are convex. So, from a minimax theorem in game theory [25], there exists a saddle-point (S_{xx}, S_{nn}) such that

$$\frac{1}{2}\log\frac{|HS'_{xx}H^T + S_{nn}|}{|S_{nn}|} \leq \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|} \leq \frac{1}{2}\log\frac{|HS_{xx}H^T + S'_{nn}|}{|S'_{nn}|}, \tag{94}$$

for all (S'_{xx}, S'_{nn}) in the constraint sets.

A saddle-point (when it exists) is the solution to the following max-min problem:

$$\max_{p(\mathbf{x})} \min_{p(\mathbf{n})} I(\mathbf{X}; H\mathbf{X} + \mathbf{N}). \tag{95}$$

This can be easily seen as follows. Suppose $(\mathbf{X}, \mathbf{N})$ is a saddle-point. Then, $\min_{p(\mathbf{n}'')} I(\mathbf{X}'; H\mathbf{X}' + \mathbf{N}'') \leq I(\mathbf{X}'; H\mathbf{X}'+\mathbf{N}) \leq I(\mathbf{X}; H\mathbf{X}+\mathbf{N})$. So $\max_{p(\mathbf{x}')} \min_{p(\mathbf{n}')} I(\mathbf{X}'; H\mathbf{X}'+\mathbf{N}') \leq I(\mathbf{X}; H\mathbf{X}+\mathbf{N})$. On the other hand, fixing $p(\mathbf{x})$ gives $\min_{p(\mathbf{n})} I(\mathbf{X}; H\mathbf{X} + \mathbf{N}') = I(\mathbf{X}; H\mathbf{X} + \mathbf{N})$. So,

$\max_{p(\mathbf{x'})} \min_{p(\mathbf{n'})} I(\mathbf{X'}; H\mathbf{X'}+\mathbf{N'}) = I(\mathbf{X}; H\mathbf{X}+\mathbf{N})$. By the same argument, the saddle-point is also the solution to the min-max problem:

$$\min_{p(\mathbf{n})} \max_{p(\mathbf{x})} I(\mathbf{X}; H\mathbf{X} + \mathbf{N}). \tag{96}$$

For any arbitrary function $f(x,y)$, it is always true that $\min_x \max_y f(x,y) \geq \max_y \min_x f(x,y)$. However, the existence of a saddle-point implies that max-min equals min-max, i.e.

$$\max_{S_{xx}} \min_{S_{nn}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|} = \min_{S_{nn}} \max_{S_{xx}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|}. \tag{97}$$

It turns out that max-min corresponds to achievability, min-max corresponds to the converse, and the saddle-point corresponds to the capacity of the Gaussian vector broadcast channel.

Theorem 3 *The sum capacity of a Gaussian vector broadcast channel $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{n_i}$, $i = 1, \cdots, K$, under a power constraint P is a saddle-point of the mutual information expression $\frac{1}{2}\log|HS_{xx}H^T + S_{nn}|/|S_{nn}|$, where $H^T = [H_1^T \cdots H_K^T]$, whenever the noise covariance matrix S_{nn} at the saddle-point is non-singular. Here, the saddle-point is computed with S_{nn} constrained to the set of positive definite matrices whose block-diagonal entries are covariance matrices of $\mathbf{n_1}, \cdots, \mathbf{n_K}$, and S_{xx} constrained to the set of positive semi-definite matrices such that* $\text{trace}(S_{xx}) \leq P$.

Proof: First, the converse: Sato's outer bound states that the broadcast channel sum capacity is bounded by the capacity of a discrete memoryless channel with receiver cooperation whose noise marginal distribution conforms to $p(\mathbf{n_i})$. The capacity of the discrete memoryless channel is $\max_{p(\mathbf{x})} I(\mathbf{X}; \mathbf{Y_1}, \cdots, \mathbf{Y_K})$, where the maximization is over the power constraint. The sum capacity is then bounded by the capacity of the channel with the least favorable noise correlation, i.e.

$$C \leq \min_{p(\mathbf{n})} \max_{p(\mathbf{x})} I(\mathbf{X}; H\mathbf{X} + \mathbf{N}), \tag{98}$$

where the minimization is over all noise distributions whose marginals are the same as the actual noise. The constraint on the input distribution is $\mathcal{K}_x = \{p(\mathbf{x}) : \mathbf{E}[\mathbf{x}^T\mathbf{x}] \leq P\}$. The constraint on the noise distribution is $\mathcal{K}_n = \{p(\mathbf{n}) : \text{marginal distributions equal to } p(\mathbf{n_i})\}$. Both constraints are convex. So, by Lemma 4, a saddle-point of the mutual information $I(\mathbf{X}; H\mathbf{X} + \mathbf{N})$ exists. The saddle-point is the solution to the min-max problem. Further, the saddle-point distributions are Gaussian. So, the outer bound can be written as

$$C \leq \min_{S_{nn}} \max_{S_{xx}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|}, \tag{99}$$

where S_{xx} belongs to the set of positive semi-definite matrices satisfying the power constraint $\text{trace}(S_{xx}) \leq P$, and S_{nn} belongs to the set of noise covariance matrices with $S_{nn}^{(i)} = \mathbf{E}[\mathbf{n_i}\mathbf{n_i}^T]$, $i = 1, \cdots, K$, on the block-diagonal entries.

Next, the achievability: the existence of a saddle-point implies that min-max equals max-min. So, it is only necessary to show that

$$C \geq \max_{S_{xx}} \min_{S_{nn}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|}. \tag{100}$$

The solution to this max-min problem is again the saddle-point. The input and noise distributions corresponding to the saddle-point are Gaussian. Therefore, the development leading to the theorem, which restricts consideration to Gaussian inputs, is without loss of generality. Further, Lemma 3 requires the channel matrix to be square. If there are more receive antennas than transmit antennas, zeros can be padded to H without affecting capacity, so H can be made square. If there are more transmit antennas than receive antennas, because S_{xx} is a water-filling covariance matrix with respect to H, the rank of S_{xx} is bounded by the number of receive antennas. Then, the null space of S_{xx} may be deleted, and H can be made equivalent to a square matrix. In either case, the condition in Lemma 3 that H is square can be satisfied.

Now, at the saddle-point, S_{nn} is a least favorable noise for S_{xx}. So, by Lemma 2 and the assumption $S_{nn} > 0$, it must satisfy the condition that $S_{nn}^{-1} - (S_{nn} + HS_{xx}H^T)^{-1}$ is block-diagonal. By Lemma 3, this implies that there is an appropriate transmit filter B such that a GDFE designed for this B and S_{nn} has a block-diagonal feedforward filter. Consider now the precoding configuration of the GDFE. The feedforward section is block-diagonal. The feedback section is moved to the transmitter. So, the decoding of $\mathbf{y_1}, \cdots, \mathbf{y_K}$ are completely independent of each other. Further, because the feedback filter is block-diagonal, the GDFE receiver is oblivious of the correlation between $\mathbf{n_i}$'s. Thus, although the actual noise distribution may not have the same joint distribution as the least favorable noise, because the marginal distributions are the same, a GDFE precoder designed for the least favorable noise performs as well with the actual noise. By Theorem 2, this GDFE precoder achieves $I(\mathbf{X}; H\mathbf{X} + \mathbf{N})$. Therefore, the outer bound is achievable. □

The condition that the saddle-point $S_{nn} > 0$ limits the applicability of the theorem somewhat. As mentioned before, a sufficient condition for $S_{nn} > 0$ is that $HS_{xx}H^T > 0$. However, this sufficient condition applies only to broadcast channel with more transmit antennas than receive antennas.

Note that the GDFE transmit filter B designed for the least favorable noise also identifies the sum capacity-achieving $\{S_i\}$ as in Theorem 1. Let $B = [B_1 \cdots B_K]$. Set $S_1 = B_1 B_1^T$, $\cdots$, $S_K = B_K B_K^T$. Then, it is easy to verify that the sum capacity is achieved with $R_i = \frac{1}{2} \log |\sum_{k=i}^{K} H_i S_k H_i^T + I| / |\sum_{k=i+1}^{K} H_i S_k H_i^T + I|$.

Theorem 3 suggests the following game-theory interpretation of the vector broadcast channel. A signal player chooses a S_{xx} to maximize $I(\mathbf{X}; H\mathbf{X} + \mathbf{N})$ subject to a power constraint. A fictitious noise player chooses a S_{nn} to minimize $I(\mathbf{X}; H\mathbf{X} + \mathbf{N})$. Because different receivers do not coordinate and are ignorant of the noise correlation, the fictitious noise player is able to choose a least favorable noise correlation. The Nash equilibrium of this mutual information game is precisely the sum capacity of the Gaussian vector broadcast channel.

The saddle-point property of the Gaussian broadcast channel sum capacity implies that the capacity achieving (S_{xx}, S_{nn}) is such that S_{xx} is the water-filling covariance matrix for S_{nn}, and S_{nn} is the least favorable noise covariance matrix for S_{xx}. In fact, the converse is also true. If a set of (S_{xx}, S_{nn}) can be found such that S_{xx} is the water-filling covariance for S_{nn} and S_{nn} is the least favorable noise for S_{xx}, then (S_{xx}, S_{nn}) constitutes a saddle-point. This is due to the fact that the mutual information is a concave-convex function, and the two KKT conditions, corresponding to the two optimization problems are, collectively, sufficient and necessary at the saddle-point [26] [27]. Thus, the computation of the saddle-point is equivalent to solving the water-filling and the least favorable noise problems simultaneously.

One might suspect that the following algorithm may be able to find a saddle-point numerically. The idea is to iteratively compute the best input covariance matrix S_{xx} for the given noise covariance, and compute the least favorable noise covariance matrix S_{nn} for the given input covariance. When the process converges, both KKT conditions are satisfied, so the limit must be a saddle-point of $\frac{1}{2}\log|HS_{xx}H^T + S_{nn}|/|S_{nn}|$. However, such an iterative min-max procedure is not guaranteed to converge for a general game even when the pay-off function is concave-convex. But, the iterative procedure appears to work well in practice for this particular problem. The convex-concave nature of the problem also suggests that general-purpose numerical convex programming algorithms can be used to solve the least favorable noise problem, or to solve for a saddle-point directly with polynomial complexity [28] [29].

4.4 Example

The following numerical example illustrates the computation of the saddle-point (S_{xx}, S_{nn}) and the construction of a precoder. Consider the following broadcast channel:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.0 & -0.3 & 0.2 \\ -0.4 & 2.0 & 0.5 \\ -0.1 & 0.2 & 3.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_2 \end{bmatrix}, \tag{101}$$

where y_1, y_2, and y_3 are uncoordinated receivers, and n_1, n_2, and n_3 are i.i.d. noise signals drawn from a Gaussian random variable $\mathcal{N}(0,1)$. The total power constraint is set to 5. The iterative algorithm described at the end of the previous section is used to solve for the saddle point (S_{xx}, S_{nn}). The water-filling step is standard. The least favorable noise problem is solved using an interior-point method. The algorithm converged in 3-4 steps. The numerical solution is:

$$S_{xx} = \begin{bmatrix} 1.0762 & -0.2327 & -0.0074 \\ -0.2327 & 1.8635 & 0.0387 \\ -0.0074 & 0.0387 & 2.0603 \end{bmatrix}, \quad S_{nn} = \begin{bmatrix} 1.0000 & -0.1286 & 0.0493 \\ -0.1286 & 1.0000 & 0.0311 \\ 0.0493 & 0.0311 & 1.0000 \end{bmatrix}. \tag{102}$$

To verify that the above solution satisfies the KKT conditions:

$$S_{nn}^{-1} - \left(S_{nn} + HS_{xx}H^T\right)^{-1} = \Psi = \begin{bmatrix} 0.4859 & 0 & 0 \\ 0 & 0.8701 & 0 \\ 0 & 0 & 0.9422 \end{bmatrix} \tag{103}$$

and

$$H^T \left(HS_{xx}H^T + S_{nn}\right)^{-1} H = 0.4597I. \tag{104}$$

The vector channel capacity with the least favorable noise correlation is:

$$C = \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|} = 2.8952. \tag{105}$$

The objective is to design a generalized decision-feedback precoder that achieves the vector channel capacity without receiver coordination. This is accomplished by finding an appropriate transmit filter $B = V\Sigma^{\frac{1}{2}}M$ which would induce a diagonal feedforward filter in a GDFE. Following the proof of Lemma 3, compute the eigen-decomposition $S_{xx} = V\Sigma V^T$ and $S_{nn} = Q^T\Lambda Q$. Then, compute R as a square root of the following as in (83):

$$R^T R = \left(\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} + I\right)^{-1}. \tag{106}$$

In particular, R can be found by a Cholesky factorization. In this example, because S_{xx} is the water-filling covariance, the matrix V diagonalizes the channel, so that $R^T R$ is already diagonal. So, finding an R is trivial. Numerically,

$$R = \begin{bmatrix} 0.2191 & 0 & 0 \\ 0 & 0.3451 & 0 \\ 0 & 0 & 0.7312 \end{bmatrix}. \tag{107}$$

The next step is to find an orthogonal matrix U, such that $UR\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} Q$ is diagonal. The proof of Lemma 3 shows that U can be found as follows:

$$U = \Psi^{-\frac{1}{2}} Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} R^T = \begin{bmatrix} 0.0115 & -0.2220 & 0.9750 \\ 0.3147 & 0.9263 & 0.2072 \\ 0.9491 & -0.3045 & -0.0805 \end{bmatrix}. \tag{108}$$

The final step to find an orthogonal matrix M such that URM is lower-triangular. This is done by computing the QR-factorization of $R^T U^T = MK$, where K is upper-triangular, and M is orthogonal. Then, $URM = K^T$ is lower-triangular. In this example,

$$R^T U^T = MK = \begin{bmatrix} -0.0035 & -0.2010 & -0.9796 \\ 0.1069 & -0.9740 & 0.1995 \\ -0.9943 & -0.1040 & 0.0249 \end{bmatrix} \begin{bmatrix} -0.7170 & -0.1167 & 0.0466 \\ 0 & -0.3410 & 0.0666 \\ 0 & 0 & -0.2262 \end{bmatrix}. \tag{109}$$

This gives the appropriate M for the desired transmit filter $B = V\Sigma^{\frac{1}{2}}M$.

Now, design a generalized decision-feedback equalizer for the effective channel

$$\widetilde{H} = \frac{1}{\sqrt{\Lambda}} QHV\sqrt{\Sigma}M = \begin{bmatrix} -0.7439 & 2.2489 & 0.0128 \\ 0.1698 & 0.8505 & 4.3105 \\ 0.6027 & 1.4311 & -0.8596 \end{bmatrix}, \tag{110}$$

an input covariance $S_{uu} = I$, and a noise covariance $S_{nn} = I$. Compute $G^{-1}\Delta^{-1}G^{-T} = (\widetilde{H}^T\widetilde{H} + I)^{-1}$:

$$G = \begin{bmatrix} 1 & -0.3423 & 0.1051 \\ 0 & 1 & 0.2947 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1.9454 & 0 & 0 \\ 0 & 8.6009 & 0 \\ 0 & 0 & 19.5512 \end{bmatrix}. \tag{111}$$

As expected, the choice of transmit filter makes the feedforward filter a diagonal matrix:

$$F = \Delta^{-1}G^{-T}\widetilde{H}^T \frac{1}{\sqrt{\Lambda}} Q = \begin{bmatrix} -0.4998 & 0 & 0 \\ 0 & -0.3181 & 0 \\ 0 & 0 & -0.2195 \end{bmatrix}. \tag{112}$$

First, let's compute the capacities of individual sub-channels in the GDFE feedback configuration. The effective channel is $\mathbf{u}' = FHB\mathbf{u} + (I - G)\mathbf{u} + F\mathbf{n}$:

$$\begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 0.4860 & 0 & 0 \\ -0.0398 & 0.8837 & 0 \\ -0.0105 & 0.0151 & 0.9489 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -0.4998n_3 \\ -0.3181n_2 \\ -0.2195n_1 \end{bmatrix}. \tag{113}$$

Thus, the capacities of the three sub-channels are:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{0.4860^2}{0.4998^2}\right) = 0.3327 \tag{114}$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{0.8837^2}{0.3181^2 + 0.0398^2}\right) = 1.0759 \tag{115}$$

$$R_3 = \frac{1}{2}\log\left(1 + \frac{0.9489^2}{0.0105^2 + 0.0151^2 + 0.133^2}\right) = 1.4865. \tag{116}$$

The sum capacity is $R_1 + R_2 + R_3 = 2.8952$, which agrees with the vector channel capacity.

Now, compute the capacity of individual sub-channels in the precoding configuration. The effective channel is $\mathbf{y} = HB\mathbf{u} + \mathbf{n}$:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -0.9723 & 0.6847 & -0.2101 \\ 0.1251 & -2.7785 & -0.9265 \\ -0.0480 & -0.0687 & -4.3222 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} n_3 \\ n_2 \\ n_1 \end{bmatrix}. \tag{117}$$

Decoding u_3 from y_3, the capacity is:

$$R_3 = \frac{1}{2}\log\left(1 + \frac{4.3222^2}{1 + 0.0480^2 + 0.0687^2}\right) = 1.4865. \tag{118}$$

The signal from u_3 may be pre-subtracted from u_2, leading to:

$$R_2 = \frac{1}{2}\log\left(1 + \frac{2.7785^2}{1 + 0.1251^2}\right) = 1.0759. \tag{119}$$

The signals from u_2 and u_3 may be pre-subtracted from u_1, leading to:

$$R_1 = \frac{1}{2}\log(1 + 0.9723^2) = 0.3327. \tag{120}$$

Therefore, without receiver coordination, a sum capacity of $R_1 + R_2 + R_3 = 2.8952$ is also achievable. In fact, it is now possible to identify the appropriate transmit covariance matrices for each user as in Theorem 1. Let B_1, B_2 and B_3 be the column vectors of the transmit filter $B = [B_1 B_2 B_3]$. Then information bits u_1, u_2 and u_3 are modulated with covariance matrices $S_1 = B_1 B_1^T$, $S_2 = B_2 B_2^T$ and $S_3 = B_3 B_3^T$. Let H_1, H_2 and H_3 be the row vectors of the channel $H^T = [H_1^T H_2^T H_3^T]$. Then, by Theorem 1, the following rates are achievable:

$$R_1 = \frac{1}{2}\log\left(H_1 S_1 H_1^T + 1\right) = 0.3327 \tag{121}$$

$$R_2 = \frac{1}{2}\log\left(\frac{H_2 S_2 H_2^T + H_2 S_1 H_2^T + 1}{H_2 S_1 H_2^T + 1}\right) = 1.0759 \tag{122}$$

$$R_3 = \frac{1}{2}\log\left(\frac{H_3 S_3 H_3^T + H_3 S_2 H_3^T + H_3 S_1 H_3^T + 1}{H_3 S_2 H_3^T + H_3 S_1 H_3^T + 1}\right) = 1.4865. \tag{123}$$

This again verifies that $R_1 + R_2 + R_3 = 2.8952$ is achievable with no coordination at the receiver side.

5 Conclusions

A principle aim of this paper to illustrate the value of cooperation in a Gaussian vector channel. Consider the channel $\mathbf{y} = H\mathbf{x} + \mathbf{n}$, where the vector signals $\mathbf{x}$ and $\mathbf{y}$ represent multiple transmitter and receiver terminals. Let S_{nn} be the noise covariance matrix. When coordination is available both at the transmitter and at the receiver, the channel capacity under a power constraint is the solution to the following optimization problem.

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|} \\ \text{subject to} & \text{trace}(S_{xx}) \leq P, \\ & S_{xx} \geq 0. \end{array} \tag{124}$$

When coordination is available at the receiver, but not at the transmitter, the sum capacity is still a maximization of an $I(\mathbf{X};\mathbf{Y})$, but with an additional constraint:

$$\begin{array}{ll} \text{maximize} & \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{nn}|}{|S_{nn}|} \\ \text{subject to} & \text{trace}(S_{xx}) \leq P, \\ & S_{xx}(i,j) = 0, \forall (i,j) \text{ uncoordinated} \\ & S_{xx} \geq 0. \end{array} \tag{125}$$

Here $S_{xx}(i,j)$ denotes the (i,j) entry of S_{xx}, and by convention, each terminal always coordinates with itself. Thus, in terms of capacity, the value of cooperation at the transmitter lies in the ability for the transmitter to send out correlated signals.

In a broadcast channel with coordination at the transmitter and no coordination at the receiver, the capacity is now the solution to a minimax problem, (assuming that the solution is such that $S_{nn} > 0$):

$$\begin{aligned} \max_{S_{xx}} \min_{S'_{nn}} \quad & \frac{1}{2} \log \frac{|HS_{xx}H^T + S'_{nn}|}{|S'_{nn}|} \\ \text{subject to} \quad & \text{trace}(S_{xx}) \leq P, \\ & S'_{nn}(i,j) = S_{nn}(i,j), \forall (i,j) \text{ coordinated} \\ & S_{xx}, S_{nn} \geq 0. \end{aligned} \tag{126}$$

Because of the lack of coordination, the receivers cannot distinguish between different noise correlations. So the capacity is as if the noise has a least favorable correlation. Thus, the value of cooperation at the receiver lies in its ability to recognize and to take advantage of the correlation among the noise signals from different receivers.

When full coordination is possible at both the transmitter and at the receiver, a Gaussian vector channel can be decomposed into non-interfering scalar sub-channels that can be independently encoded and decoded. With coordination at one-side only, the vector channel can only be decomposed into a series of scalar sub-channels each interfering into subsequent sub-channels. Thus, from a coding point of view, the value of coordination lies in its ability to eliminate the need to either pre-subtract or post-subtract interference. When coordination is not possible, the generalized decision-feedback equalizer has emerged as a unifying structure that is able to achieve both the multiple access channel capacity and the broadcast channel sum-capacity.

To summarize, this paper deals with a class of non-degraded Gaussian vector broadcast channels. The sum capacity is characterized as a saddle-point of a Gaussian mutual information game where the signal player chooses a signal covariance matrix to maximize the mutual information, and a noise player chooses a fictitious noise correlation to minimize the mutual information. This capacity is achieved using the precoding configuration of a generalized decision-feedback equalizer. These results hold under the condition that the noise covariance matrix at the saddle-point is full rank.

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