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Computer Graphics from a Geometric Algebra Perspective

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The report presents introductory notions concerning the field of geometric algebra, basic algebraic manipulation techniques, geometric algebra based description of different models of the Euclidean space and the way these notions could be used to specify primitive graphic objects and computer graphics algorithms. The report is intended to be used by students and specialists who are interested to begin the study of the vast domain of geometric algebra and its applications.

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Chapter 1

Introduction

From a formal point of view, an algebra is an associative ring (A, +, ∘), whose additive group (A, +) is in the same time a vector space over a field K (the scalars set). The scalar multiplication of the elements from A can commute with the product of the ring A (that is ∘). The definitions of the above mentioned algebraic structures are specified in the first Appendix of the present report.

An usual case is the algebra of real numbers (where A=R, K=R) whose computation rules were learned at the arithmetic course. According to Eric Weisstein's World of Mathematics server, there are actually known another over 1150 consistent algebras. Some of them constitute models of the Euclidean geometric space and define computation rules that allow a quantitative understanding of the geometry.

In fact the first who tried to use algebraic manipulation in order to symbolically represent geometric properties was René Descartes (1596-1650) the inventor of the *analytic geometry.* He established a mapping between an arbitrary line segment and a number. That last one represented the segment length, without taking into consideration the position/orientation of the segment in the geometric space. Two segments were considered equal and consequently represented by the same real positive valued constant if the second could be obtained from the first through a rotation/translation operation.

The possibility that the numbers specify spatial orientation information as well, was recognized by Herman Günther Grassmann (1809-1877). He considered two line segments as being congruent if one of them could be obtained through a translation transformation from the other. The line segments are different not only by their magnitudes but also by their directions. The

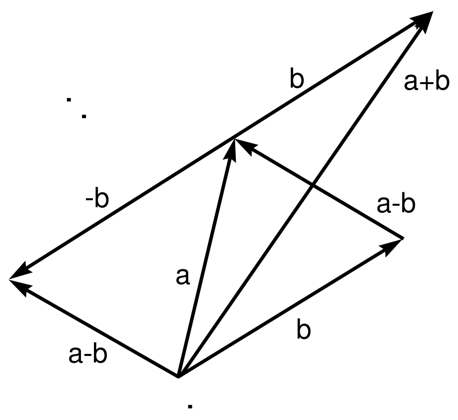

Figure 1.1: The geometric semantic of vector addition

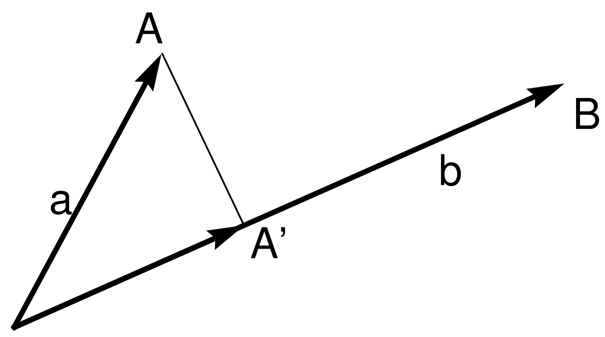

Figure 1.2: The geometric semantic of $\mathbf{a} \bullet \mathbf{b}/\mid \mathbf{b}\mid$

directed line segments are called *vectors*.

One vector designates the class of oriented line segments with the same magnitude (length) and direction but having possibly different positions in the 3D space. Grassmann defined the vector addition and the scalar vector multiplication in a way that today is commonly accepted.

Returning to the previously mentioned algebra definition, and considering $A = R^3$, + the vector addition and $\lambda.\mathbf{v}(\lambda \in K = R)$ the usual scalar-vector multiplication, it is well known that the triple (A,+,.) forms a linear space and (A, +) is an abelian group. The model of the 3D Euclidean space viewed as a linear vector space is largely used in sciences as physics, geometry, computer graphics etc. The vector space V^n has a certain dimension (n) and any vector can uniquely be represented as a combination of n linearly independent vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ that form a basis of the vector space. These only two operations (vector addition and scalar-vector multiplication) made possible the proof (by expressing in a symbolic language) of a broad range of geometric (physics) results.

In order to achieve the enormous computational and modeling framework offered by the manipulation rules characteristic to a full algebra (of vectors) the researchers tried (at the half of the 19-th century) to fit the missing product $\circ$ that could make the triple (A, +, $\circ$) an associative ring. As in the case of vector addition and scalar-vector multiplication that did not represent simply algebraic manipulation rules but had also a related geometric semantic, the $\circ$ product had to have a precise geometrical interpretation.

One possible product would be the *scalar product* (so named because the multiplication of two vectors yields a scalar). Grassmann called it *inner product* and defined $\mathbf{a} \bullet \mathbf{b}$ as being equal to the length of the orthogonal projection of $\mathbf{a}$ onto $\mathbf{b}$ multiplied by the length of $\mathbf{b}$. In terms of Figure 1.2 the length OA' is numerically equal to the value of the inner product between $\mathbf{a}$ and the unit vector collinear with $\mathbf{b}$. Supposing the cos() function already defined:

$$\mathbf{a} \bullet \mathbf{b} = \mid \mathbf{a} \mid\mid \mathbf{b} \mid cos(\theta)$$

The algebraic properties of the inner product are:

- Distributivity over (vector) addition: $\mathbf{a}\bullet(\mathbf{b}+\mathbf{c})=\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\mathbf{c}$
- Commutativity: $\mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}$
- Commutativity with scalar-vector multiplication:

$$\lambda(\mathbf{a} \bullet \mathbf{b}) = \mathbf{a} \bullet (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \bullet \mathbf{b}$$

- Positive definition: $\mathbf{a}\bullet\mathbf{a}=\mid \mathbf{a} \mid^2 \geq 0$

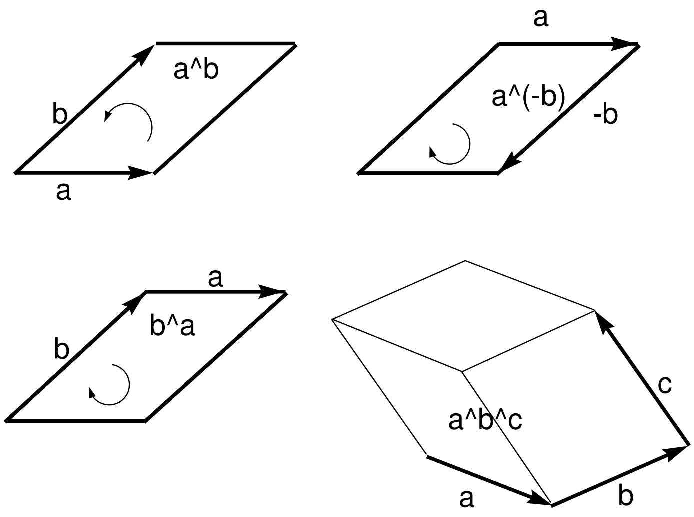

Figure 1.3: The geometric semantic of the outer product

The proofs are elementary and could be entirely based on geometric considerations.

The inner product is closely related to the notions of orthogonality, distance (metric) and could be used to compute lengths (vector modules) or angles between vectors. The whole results of trigonometry could be inferred through computations in the 2D vector algebra where the $\circ$ designates the inner product. Note: (R^3, +, $\bullet$) is not a full algebra because the ring (R^3, +, $\bullet$) is not associative (the inner product is not associative).

Another possible vector product (discovered by Grassmann) is the *outer product*. It emphasizes the dimensionality of the geometric space and allows the formal specification of subspaces (called blades). As we will see, in Geometric Algebra the subspaces (blades) are the elementary computational elements.

Be two vectors **a**, **b**, their outer product denoted $\mathbf{a} \wedge \mathbf{b}$ designates a directed plane segment called *bivector* (or 2-blade or blade of grade 2). The geometric meaning of the outer product is related to the spatial sweeping operation. Usually the bivector $\mathbf{B}$=$\mathbf{a}$∧$\mathbf{b}$ is visualized as the parallelogram spanned by the points of **a** that are supporting a translation of vector **b**. The shape of the area element is not important, some models visualize the bivector as a circular disc. The geometric properties associated to the bivector are: *direction* (two bivectors with the same direction are lying in parallel planes), *orientation* (that corresponds to the sense in which the parallelogram boundary is traversed) and *magnitude* (a scalar numerically equal with the signed area of the parallelogram $\mid \mathbf{a} \wedge \mathbf{b} \mid = \mid \mathbf{a} \mid\mid \mathbf{b} \mid sin(\theta)$).

In the 3D geometric space the vectors can be considered 1-blades (they result through space sweeping by a point), the directed volume elements are 3-blades and the scalars are 0-blades (in fact it is always considered that $\alpha \wedge \beta = \alpha\beta$ and $\alpha \wedge \mathbf{v} = \alpha\mathbf{v}$). In Figure 1.3 the 3-blade designated by $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ has a positive value since **a**, **b**, **c** are forming a right-handed system.

The main algebraic properties of the vector outer product are:

- Associativity: $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$
- Anticommutativity: $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$
- Distributivity over the vector addition: $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$
- Commutativity with scalar-vector multiplication: $\lambda(\mathbf{a} \wedge \mathbf{b}) = (\lambda\mathbf{a}) \wedge \mathbf{b} = \mathbf{a} \wedge (\lambda\mathbf{b})$

Note: The anticommutivity could be completed by: $\mathbf{x} \wedge \mathbf{y} = \mathbf{y} \wedge \mathbf{x}$ iff $\mathbf{x} \wedge \mathbf{y} = 0$ (that is **x** and **y** are coplanar vectors). The outer product is also anti-symmetric.

The validity of the above mentioned properties could easily be proved based on geometric considerations.

The triple (A, +, ∧) forms an algebra over the field R (called *Grassmann algebra* or *exterior algebra*), however the outer product is not inversable. The outer multiplication of vectors will produce a null result if and only if the factors of the product are linearly dependent. Assertions as: "vectors **a** and **b** are collinear" or "vectors **a**, **b**, **c** are coplanar" could be expressed as: $\mathbf{a} \wedge \mathbf{b} = 0$ or respectively $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0$.

The two products (inner and outer product) have complementary significance. They codify different properties of the space: distance versus area, orthogonality versus parallelism. Grassmann research insisted on the (abstract) algebraic properties of those products and partially forgot their (concrete) geometrical meaning.

The discovery of fundamental product that completed the structure of an algebra that could fully express the geometric universe is attributed to William Kingdon Clifford (1845-1879) (though it seems that Grassmann too, tested its properties in his last years). It is called *geometric product* and is defined (in case of vector operands) as:

$$\mathbf{ab} = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

The result of the geometric multiplication is therefore the sum between a scalar and a bivector. These non-homogeneous entities that, in case of the 3D Euclidean space, could be obtained by summing scalars, vectors, bivectors and trivectors are called *multivectors.* The set of multivectors generated by adding and/or multiplying vectors from the Euclidean space forms (together with the multivector sum, the geometric product and the scalar-multivector multiplication) the *Geometric Algebra* of the 3D Euclidean space.

The algebra (M, +, ∘) where M is a set of multivectors, + is the multivector addition (having the usual significance in case that both operands are scalars, vectors, bivectors etc.), ∘ is the geometric product, K is the field of real numbers and the scalar multiplication has the usual semantic, is called a *Clifford algebra.* In contrast with exterior algebra, the Clifford algebra emphasizes a metric (a bilinear form) as it will be detailed in the second chapter of this report. Clifford algebra is sometimes denoted $G_n(V^n)$ in order to explicitly mention the vector space over which the algebra was built. The elementary operands of the Clifford algebra are the subspaces (blades) of the associated n dimensional vector space.

Usually the geometric product is written as a blank. The evaluation of complex unparenthesized expressions is done taking into account that, by convention, the precedence of the outer product is greater than the precedence of the inner product and this last one is at its turn greater than the precedence of the geometric product.

The geometric significance of the vector products was presented here considering the familiar case of the 3D Euclidean space. However the algebraic manipulation rules typical to the inner, outer and geometric products were extended so that they could accept as operands subspaces of arbitrary grade (greater than 3). Through its rules, the Geometric Algebra could be considered a formal language suitable to specify the behavior of various natural systems and capable to provide a unified framework to solve problems covering surprisingly wide areas of applicability (***[Dorst 02a]***, ***[Somm 01]***, ***[Corro 01]***). The scientist that (after almost one hundred years since Clifford passed away) emphasized the huge possibilities offered by the Geometric Algebra as a universal language for geometric calculus is the Cambridge physicist David Hestenes (***[Hest 85]***, ***[Hest 86b]***).

The present paper explores the modalities in which geometric algebra could be used to solve computer graphics problems. Different models of the geometry of the Euclidean space are considered. This becomes by turns a simple linear space, a 4D homogeneous space or a space whose characteristics are described by algebraic operators with a proper geometric semantic.

Chapter 2

Basic Notions

The present chapter focuses on G_3 specific manipulation techniques. Whenever possible, the identities (expressing theorems of the algebraic formalism) were generalized and the formulas valid in G_n were presented and proofed. The understanding of notions as: *projection*, *spinor*, *angle*, *rotation* and *reflection* transformations, *multivector inverse* with respect to the geometric product is necessary for the future developments and understanding of the computer graphics geometry based techniques as they appear from the geometric algebra perspective.

The fundamental product of Clifford algebra, the geometric product, is linear and associative. The distributivity of the geometric product follows as a consequence of the distributivity of its components (the inner and outer product) over the vector addition. The associativity property has not an obvious geometric interpretation.

From: $\mathbf{ab} = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$ and $\mathbf{ba} = \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \bullet \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$ results through addition and respectively subtraction:

$$\mathbf{a} \bullet \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \tag{2.1}$$

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \tag{2.2}$$

or, writing a bit modified:

$$\mathbf{ab} = -\mathbf{ba} + 2(\mathbf{a} \bullet \mathbf{b}) \tag{2.3}$$

$$\mathbf{ba} = \mathbf{ab} - 2(\mathbf{a} \wedge \mathbf{b}) \tag{2.4}$$

The relations 2.1 and 2.2 show that the (vector) inner product is the symmetric part and the outer product is the antisymmetric part of the geometric product. The relations 2.3 and 2.4 are usually applied if reversing the order of two factors in a more complicated product is desired.

Note: The outer product of k vectors is the only antisymmetric function that could be formed with these vectors. The uniqueness is considered modulo a scalar factor. It is defined as:

$$\mathbf{a_1} \wedge \mathbf{a_2} \wedge \cdots \wedge \mathbf{a_k} = \frac{1}{k!} \sum_{\Lambda} (-1)^{\Phi(\Lambda)} \mathbf{a}_{\lambda_1} \mathbf{a}_{\lambda_2} \cdots \mathbf{a}_{\lambda_k}$$

where Λ=$\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k\}$ is an arbitrary permutation of $\{1,\ 2,\ \cdots,\ \text{k}\}$ and $\Phi(\Lambda)$ is a binary valued function determining the parity of the permutation Λ[1]. The choice of the scalar factor value as: $\frac{1}{k!}$ makes practically the module of the outer product $\bigwedge_{i=1}^{n} \mathbf{a_i}$ (where n is the dimension of the geometric space) identical with the value of the determinant of the matrix having as lines the vectors $\mathbf{a_i}$.

[1] A permutation is considered even/odd if its number of inversions is even respectively odd

Generally the geometric product is not commutative however in case of two collinear vectors ($\mathbf{a} \wedge \mathbf{b} = 0$) the geometric product is commutative (**ab**=**ba**=**a•b**); the other extreme case is that of two orthogonal vectors (**a•b**=0) when the geometric product is anticommutative (as the outer product). In cases"in between"(vectors neither orthogonal nor parallel) the operands of the product do not commute.

Once the geometric semantic of the vector products has been emphasized, the relations 2.1 and 2.2 could be interpreted as definitions of the inner and outer products and the geometric product could be stated as the main product and axiomatically defined through its properties. That is the way followed in papers as ***[Hest 99]***.

2.1 The Axiomatic Definition of the Geometric Algebra

A Geometric Algebra (G_n) over a n-dimensional vector space V^n over the field R could be formally defined as the algebra generated by stating the existence of a geometric product with the following properties:

- Associativity
- Multilinearity
- $\forall a \in V^n \Rightarrow \mathbf{a}^2$ is a scalar and $\mathbf{a}^2 = \epsilon_a \mid \mathbf{a} \mid^2$, where $\epsilon_a \in \{-1, 0, 1\}$ is called the signature of $\mathbf{a}$ and $\mid \mathbf{a} \mid \geq 0$ is the module of $\mathbf{a}$. This last rule is called the *contraction rule*.

Notes:

- The multilinearity property of the geometric product is defined by:

$$a_1 a_2 \cdots a_{k-1} (\sum_{j=1}^{r} b_j) a_{k+1} \cdots a_s = \sum_{j=1}^{r} a_1 a_2 \cdots a_{k-1} b_j a_{k+1} \cdots a_s \tag{2.5}$$

  and is equivalent with distributivity

- The spaces in which the squared distance function (a bilinear form denoted sometimes Q(a)(=a^2)) is positive definite and all vectors have consequently a positive signature, are Euclidean (see Appendix 1). The spaces that contain vectors of negative or null signature are denoted by $R^{p,q,r}$ where p, q, r are respectively the number of vectors (from a space basis) of positive, negative or null signature. One example of space that has vectors with negative signature is the Minkowski space time denoted as $R^{3,1}$.

In most chapters of the present report, the case n=3 ($V^n = R^3$) and the function Q(a) corresponding to the square of the Euclidean distance will be considered. All vectors of this Euclidean space will obviously have positive signatures. An element of the Geometric Algebra is called multivector and can be denoted as a sum of elements of different dimensions i.e. blades (scalar+vector+bivector+trivector in the case of G_3 algebra).

In G_3 exist a scalar subspace, a vector subspace (3 dimensional), a bivector subspace (3 dimensional) and a trivector subspace (1-dimensional). If we denote $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ an orthonormal basis of the vector subspace then a possible basis of the bivector space would be $\mathbf{e_1} \wedge \mathbf{e_2}$, $\mathbf{e_2} \wedge \mathbf{e_3}$, $\mathbf{e_3} \wedge \mathbf{e_1}$ and the basis for the trivector subspace would be $\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$. Evidently the basis of the scalar space is 1. In general, for a n-dimensional vector space V^n, the associated Geometric Algebra (named also the Clifford algebra generated by V^n) operates on multivectors having $2^n = \sum_{i=0}^{n} C_n^i$ components, that could be grouped in elements of n linear subspaces, each of them containing the blades of a given grade. The space having as basis k-grade blades is C_n^k

dimensional and its elements are called *k-vectors*. The blades are the elementary computational operands in geometric algebra expressions.

In the following, we will note scalars through Greek lowercase letters α, β, ... vectors through bold lowercase Latin letters **a**, **b**, ... 2-blades through bold capital Latin letters (the first letters of the alphabet) **A**, **B**, ... Sometimes the grade of a blade is explicitly mentioned as in $\mathbf{A_r}$. The unit trivector (unit pseudoscalar of G_3) is denoted by **I** or $\mathbf{I_3}$ and multivectors through **M**, **N** ... . The component of grade k ($k \geq 0$) of the multivector **M** is denoted by $\langle \mathbf{M} \rangle_k$. Usually the scalar component $\langle \mathbf{M} \rangle_0$ is denoted $\langle \mathbf{M} \rangle$.

2.2 Other operators of Geometric Algebra and related terminology

Reversion as a geometric algebra operator means reversing the order of factors in any product that is included in an algebraic expression. The reversion operator is unary and denoted by the character $\sim$. In case of a G_3 multivector the reversion modifies only its bivector and trivector components. Some properties of the reversion operator are:

1. $\lambda^{\sim} = \lambda, \forall \lambda \in R,$
2. $\mathbf{a}^{\sim} = \mathbf{a}, (\mathbf{ab})^{\sim} = \mathbf{b}^{\sim}\mathbf{a}^{\sim} = \mathbf{ba}, \forall \mathbf{a}, \mathbf{b} \in R$
3. $$\mathbf{MN}^{\sim} = \mathbf{N}^{\sim}\mathbf{M}^{\sim}, \forall \mathbf{M}, \mathbf{N} multivectors \tag{2.6}$$
4. $\mathbf{M} = \alpha + \mathbf{a} + \mathbf{A_2} + \beta \mathbf{I_3} \Rightarrow \mathbf{M}^{\sim} = \alpha + \mathbf{a} - \mathbf{A_2} - \beta \mathbf{I_3}$

This last implication follows from the possibility to factorize $\mathbf{I_3}$ as a product of 3 vectors that make up an orthonormal basis.

$\mathbf{I_3} = \mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3} = \mathbf{e_1 e_2 e_3} \Rightarrow \mathbf{I_3^{\sim}} = \mathbf{e_3 e_2 e_1} = -\mathbf{e_3 e_1 e_2} = \mathbf{e_1 e_3 e_2} = -\mathbf{e_1 e_2 e_3} = -\mathbf{I_3}$

due to the fact that the orthonormality implies $\mathbf{e_i}^2 = 1$ and $\mathbf{e_i e_j} = \mathbf{e_i} \wedge \mathbf{e_j}$, $\forall 1 \leq i, j \leq 3$. The same kind of argumentation could be used to proof that, for arbitrary blades:

$$\mathbf{A_r}^{\sim} = (-1)^{r(r-1)/2} \mathbf{A_r}$$

Generally:

$$\mathbf{M}^{\sim} = \sum_k (-1)^{k(k-1)/2} \langle \mathbf{M} \rangle_k \tag{2.7}$$

where the exponent of (-1) equals the numbers of inversions (elementary swap operations) required to transform $\mathbf{e_1}, \ldots \mathbf{e_k}$ into $\mathbf{e_k} \ldots \mathbf{e_1}$

The magnitude of a multivector is:

$$\mid \mathbf{M} \mid = \sqrt{\sum_k \mid \langle \mathbf{M} \rangle_k \mid} \tag{2.8}$$

where:

$$\mid \langle \mathbf{M} \rangle_k \mid = \sqrt{\langle \mathbf{M} \rangle_k \bullet \langle \mathbf{M} \rangle_k} \tag{2.9}$$

or we could equally state:

$$\mid \mathbf{A}^2 \mid = \langle \mathbf{A}^{\sim} \mathbf{A} \rangle (\geq 0) \tag{2.10}$$

The *parity conjugate* of a multivector **M** is denoted $\hat{\mathbf{M}}$ and defined by:

$$\langle \widehat{\mathbf{M}} \rangle_k = (-1)^k \langle \mathbf{M} \rangle_k \tag{2.11}$$

A multivector is called *even/odd* iff[1] it contains only blades of an even/odd grade.

The *dual* of the k-blade $\mathbf{B}$ is denoted as $\mathbf{B}^*$ and defined as $\mathbf{B}^* = \mathbf{B}\mathbf{I}^{-1} = \mathbf{B} \bullet \mathbf{I}^{\sim}$, where $\mathbf{I}$ is the unit pseudoscalar of the geometric algebra. Evidently $\mathbf{I}^{-1} = (-1)^{n(n-1)/2}\mathbf{I} = \mathbf{I}^{\sim}$ and in G_3: $\mathbf{B}^* = -\mathbf{I}\mathbf{B}$. For $\mathbf{B}$ a k-grade blade in G_n: $\mathbf{B}^*{=}\mathbf{B}\mathbf{I}^{-1}{=}(-1)^{n(n-k)}\mathbf{I}^{-1}\mathbf{B}$. The general definition for the dual of a multivector (***[Hest 84]***) is $\mathbf{M}^* = \mathbf{M} \bullet \mathbf{I}^{\sim}$. The dual of a subspace of V^n is interpreted as the orthogonal complement of this subspace relatively to V^n.

Note: It is a well known linear algebra result that, given an orthonormal frame $\{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ of a vectorial space V^n, it exists a dual frame associated with $\{\mathbf{e_1}, \cdots, \mathbf{e_n}\}$ denoted $\{\mathbf{e^1}, \cdots, \mathbf{e^n}\}$ The vectors of the dual frame have the properties: $\mathbf{e^i} \bullet \mathbf{e_j}{=}\delta_{ij}\ \forall 1{\leq}\ i,j \leq n$ and $i \neq j$ (δ_{ij} is the Kronecker symbol). In geometric algebra, the vectors of the dual referential are determined as:

$$\mathbf{e^i} = (-1)^{i+1} \bigwedge_{j=1, j\neq i}^{n} \mathbf{e_j} = \mathbf{e_i} \bullet \mathbf{I_n}^{-1} = \mathbf{e_i}^* \tag{2.12}$$

where $\mathbf{I_n}$ is the pseudoscalar of the geometric algebra corresponding to V^n. The same type of construction could be applied to a frame $\{\mathbf{a_1}, \cdots, \mathbf{a_k}\}$ of a subspace of grade k of V^n. In that case the pseudoscalar $\mathbf{I_n}$ is replaced by the pseudoscalar $\mathbf{I_k}$ of the geometric algebra corresponding to V^k.

The definition of the inner product operating on blades of arbitrary grade (in relation below, $\mathbf{A_r}$ has grade r and $\mathbf{B_s}$ grade s) is:

$$\mathbf{A_r} \bullet \mathbf{B_s} = \langle \mathbf{A_r} \bullet \mathbf{B_s} \rangle_{|r-s|} \tag{2.13}$$

and, for multivectors:

$$\mathbf{M} \bullet \mathbf{N} = \sum_{r\neq 0, s\neq 0} \langle \langle \mathbf{M} \rangle_r \langle \mathbf{N} \rangle_s \rangle_{|r-s|} \tag{2.14}$$

One could see the difference with respect to the *scalar product*:

$$\mathbf{A}.\mathbf{B} = \langle \mathbf{AB} \rangle = \langle \mathbf{AB} \rangle_0 \tag{2.15}$$

the two products act identically if both operands are vectors or, more generally, have equal grade.

Note: The fact that the inner product does not contain as terms products involving scalars, requires special treatment of some cases that could appear during evaluation of complex expressions. By convention the presence of a scalar factor in an inner product means obtaining a null multiplication result. That is why some authors propose the utilization of other products in place of the usual inner product. Thus products have been already successfully defined. For example Lounesto ***[Laun 93]***, Dorst and Mann ***[Dorst 02b]***, ***[Dorst 02c]*** claim that the left contraction defined as:

$$\mathbf{M} \rfloor \mathbf{N} = \sum_{r,s} \langle \langle \mathbf{M} \rangle_r \langle \mathbf{N} \rangle_s \rangle_{s-r} \tag{2.16}$$

is more suitable to be used for geometric algebra based modeling of computer graphics applications. The geometric semantic of the left contraction is "$\mathbf{A} \rfloor \mathbf{B}$ *is a blade representing the orthogonal complement (within the subspace **B**) of the orthogonal projection of A onto B*" In sum 2.16 the negative grades obviously vanish. In case of blade operands, the left contraction $\mathbf{A_r} \rfloor \mathbf{B_s}$ gives a non-null result only for $s \geq r$ and if s=r the left contraction coincides with the scalar product. The contraction is linear in both operands and coincides with the inner product if the operands are vectors.

The inner product produces always a result of grade less than the grade of its operands. It is called a grade-decreasing operator.

[1]iff means if and only if

On the contrary, the outer product of two arbitrary blades is a grade-increasing operator:

$$\mathbf{A_r} \wedge \mathbf{B_s} = \langle \mathbf{AB} \rangle_{r+s} \tag{2.17}$$

and consequently (by multilinearity):

$$\mathbf{M} \wedge \mathbf{N} = \sum_{r,s} \langle \langle \mathbf{M} \rangle_r \langle \mathbf{N} \rangle_s \rangle_{r+s} \tag{2.18}$$

The geometric product of two blades $\mathbf{A_r}$ and $\mathbf{B_s}$ yields a mixed grade result. It contains terms of grade r+s, r+s-2, r+s-4,... $\mid r - s \mid$. Besides the r+s and $\mid r - s \mid$ grade terms (corresponding respectively to the outer and inner product of same operands) the other blades (if they exist) codify the relative position and orientation of $\mathbf{A_r}$ and $\mathbf{B_s}$.

If one of the factors is a vector **a** and the other a k-grade blade $\mathbf{B_k}$, the geometric product $\mathbf{aB_k}$ produces a k-1 and a (k+1)-grade blade. The inner product and the outer product represent also the symmetric and antisymmetric part of the initial product i.e.

$$\mathbf{aB_k} = \mathbf{a} \bullet \mathbf{B_k} + \mathbf{a} \wedge \mathbf{B_k} \tag{2.19}$$

but this time:

$$\mathbf{a} \bullet \mathbf{B_k} = \frac{1}{2}(\mathbf{aB_k} - (-1)^k \mathbf{B_k a}) \tag{2.20}$$

and

$$\mathbf{a} \wedge \mathbf{B_k} = \frac{1}{2}(\mathbf{aB_k} + (-1)^k \mathbf{B_k a}) \tag{2.21}$$

Comparing 2.20 and 2.1 one could observe that the sign of one term alternates depending on grade of **B**; that could be simply justified (in case of grade(**B**)=2) by factorizing the blade $\mathbf{B} = \mathbf{b} \wedge \mathbf{c}$, applying 2.20 and remembering that **aB** - **Ba** is an antisymmetric expression and the vector outer product is anticommutative :

$\mathbf{aB}$-$\mathbf{Ba}$=$\mathbf{a}\bullet(\mathbf{b}\wedge\mathbf{c})$ + $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ - $(\mathbf{b}\wedge\mathbf{c})\bullet\mathbf{a}$ - $(\mathbf{b}\wedge\mathbf{c})\wedge\mathbf{a}$ = $\mathbf{a}\bullet(\mathbf{b}\wedge\mathbf{c})$ - $(\mathbf{b}\wedge\mathbf{c})\bullet\mathbf{a}$ = 2$\mathbf{a}\bullet(\mathbf{b}\wedge\mathbf{c})$

Therefore the grade of **aB**-**Ba** is 1 that corresponds to the grade resulted from the definition 2.13 of the inner product **a•B**.

Depending on the grade of the second argument of the product $\mathbf{aB_k}$ the inner or outer products $\mathbf{a}\bullet\mathbf{B_k}$ and $\mathbf{a} \wedge \mathbf{B_k}$ could be commutative or anticommutative:

$$\mathbf{a} \bullet \mathbf{B_k} = (-1)^{k+1} \mathbf{B_k} \bullet \mathbf{a} \tag{2.22}$$

$$\mathbf{a} \wedge \mathbf{B_k} = (-1)^k \mathbf{B_k} \wedge \mathbf{a} \tag{2.23}$$

2.3 Basic algebraic manipulation techniques

In G_3 there are 10 different possibilities of factors combinations for the geometric product (the grade of the first factor is considered less or equal than the grade of the second factor). The combinations are:

- two scalars
- a scalar and a vector
- a scalar and a bivector
- a scalar and a trivector

- two vectors
- a vector and a bivector
- a vector and a trivector
- two bivectors
- a bivector and a trivector

If one of the factors is scalar, the significance of the product is (more or less) obvious. The multiplication between a scalar and a blade **A** gives as result a blade of same direction as **A** (but the module modified)

The geometric multiplication between two vectors gives as result a multivector with one scalar and one bivector component; the semantic of this type of product was largely explained in the introductory chapter.

The multiplication between a vector and a bivector produces a multivector with one vector and one trivector component. It is possible to write

$$\mathbf{aB} = \mathbf{a} \bullet \mathbf{B} + \mathbf{a} \wedge \mathbf{B} \tag{2.24}$$

where conforming to 2.13 and 2.17 $\mathbf{a}\bullet\mathbf{B}$ represents the vector and $\mathbf{a}\wedge\mathbf{B}$ the trivector part of the result. If we factorize $\mathbf{B}=\mathbf{b}\wedge\mathbf{c}$ we could detail: $\mathbf{a}\bullet(\mathbf{b}\wedge\mathbf{c}) = \frac{1}{2}(\mathbf{a}(\mathbf{b}\wedge\mathbf{c})$ - $(\mathbf{b}\wedge\mathbf{c})\mathbf{a}) = \frac{1}{2}(\mathbf{a}(\mathbf{bc}$-$\mathbf{b}\bullet\mathbf{c})$-$(\mathbf{bc}$-$\mathbf{b}\bullet\mathbf{c})\mathbf{a}) = \frac{1}{2}(\mathbf{abc}$ - $\mathbf{bca}) = \frac{1}{2}(($-$\mathbf{ba}$+$2(\mathbf{a}\bullet\mathbf{b}))\mathbf{c}$ -$\mathbf{b}($-$\mathbf{ac}$+$2\mathbf{a}\bullet\mathbf{c}))$=$(\mathbf{a}\bullet\mathbf{b})\mathbf{c}$-$(\mathbf{a}\bullet\mathbf{c})\mathbf{b}$, therefore:

$$\mathbf{a} \bullet (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \bullet \mathbf{b})\mathbf{c} - (\mathbf{a} \bullet \mathbf{c})\mathbf{b} \tag{2.25}$$

For proof there were applied 2.22 2.3 and 2.4.

Formula 2.20 admits a useful generalization:

$$\mathbf{a} \bullet (\bigwedge_{k=1}^{n} \mathbf{b_k}) = \sum_{k=1}^{n} (-1)^{k+1} (\mathbf{a} \bullet \mathbf{b_k}) (\bigwedge_{i=1, i\neq k}^{n} \mathbf{b}_i) \tag{2.26}$$

The proof could be done by induction. For k=2 the identity is obviously valid (see 2.25). Suppose now that the identity is valid for an arbitrary n and let's try to deduce its validity for n+1. Indeed:

$$\begin{aligned}
\mathbf{a} \bullet (\bigwedge_{k=1}^{n+1} \mathbf{b_k}) &= (-1)^n \mathbf{a} \bullet (\mathbf{b_{n+1}} \wedge (\bigwedge_{k=1}^{n} \mathbf{b_k})) \\
&= (-1)^n (\mathbf{a} \bullet \mathbf{b_{n+1}}) (\bigwedge_{k=1}^{n} \mathbf{b_k}) + (-1)^n (-\mathbf{b_{n+1}} \wedge (\mathbf{a} \bullet \bigwedge_{k=1}^{n} \mathbf{b_k})) \\
&= (-1)^{n+2} (\mathbf{a} \bullet \mathbf{b_{n+1}}) \bigwedge_{k=1}^{n} \mathbf{b_k} + (-1)^n \mathbf{b_{n+1}} \wedge (\sum_{k=1}^{n} \bigwedge_{i=1, i\neq k}^{n} \mathbf{b_i}) \\
&= (-1)^{n+2} (\mathbf{a} \bullet \mathbf{b_{n+1}}) \bigwedge_{k=1}^{n} \mathbf{b_k} - (-1)^{2n-1} \sum_{k=1}^{n} (-1)^{k+1} (\mathbf{a} \bullet \mathbf{b_k}) (\bigwedge_{i=1, i\neq k}^{n-1} \mathbf{b_i}) \\
&= \sum_{k=1}^{n+1} (-1)^{k+1} (\mathbf{a} \bullet \mathbf{b_k}) (\bigwedge_{i=1, i\neq k}^{n} \mathbf{b_i})
\end{aligned}$$

q.e.d.

The proof above used the identity:

$$\mathbf{a} \bullet (\mathbf{b} \wedge \mathbf{C_r}) = (\mathbf{a} \bullet \mathbf{b})\mathbf{C_r} - \mathbf{b} \wedge (\mathbf{a} \bullet \mathbf{C_r}) \tag{2.27}$$

The proof is based on 2.3, 2.20, 2.21, 2.22 2.23 and follows the pattern used for proving 2.25:

$\mathbf{abC_r} = \mathbf{a{\bullet}bC_r} + \mathbf{a} \wedge \mathbf{bC_r} =$ (-$\mathbf{ba}$ + 2($\mathbf{a{\bullet}b}$))$\mathbf{C_r}$= -$\mathbf{b}$($\mathbf{aC_r}$)+2($\mathbf{a{\bullet}b}$)$\mathbf{C_r}$ = -$\mathbf{b}$($(-1)^r\mathbf{C_r a}$ + 2$\mathbf{a{\bullet}C_r}$) + 2($\mathbf{a{\bullet}b}$)$\mathbf{C_r} \Rightarrow \mathbf{abC_r} + (-1)^r\mathbf{bC_r a} = 2(\mathbf{a{\bullet}b})\mathbf{C_r}$ - 2$\mathbf{b}$($\mathbf{a{\bullet}C_r}$) $\Rightarrow \mathbf{a}(\mathbf{b{\bullet}C_r})+(-1)^r(\mathbf{b{\bullet}C_r})\mathbf{a}$ +$\mathbf{a}(\mathbf{b}\wedge\mathbf{C_r})+(-1)^r(\mathbf{b}\wedge\mathbf{C_r})$ $\mathbf{a}$=2$\mathbf{a}$(($\mathbf{b{\bullet}C_r}$)-2$\mathbf{a}\bullet$($\mathbf{b} \wedge\mathbf{C_r}$)=2($\mathbf{a{\bullet}b}$)$\mathbf{C_r}-2\mathbf{b}$($\mathbf{a} \bullet\mathbf{C_r}$) $\Rightarrow \langle \mathbf{a}\bullet(\mathbf{b{\bullet}C_r})+\mathbf{a}\bullet(\mathbf{b}\wedge \mathbf{C_r})\rangle_r = \langle(\mathbf{a{\bullet}b})\mathbf{C_r}$-$\mathbf{b}(\mathbf{a} \bullet\mathbf{C_r})\rangle_r \Rightarrow \mathbf{a}\bullet(\mathbf{b}\wedge\mathbf{C_r}) = (\mathbf{a}\bullet\mathbf{b})\mathbf{C_r} - \mathbf{b}\wedge(\mathbf{a}\bullet\mathbf{C_r})$ q.e.d.

The product between two bivectors has parts of grade 0, 2 and 4 as follows:

- The part of grade 0 $\langle\mathbf{AB}\rangle_0$ is the inner product $\mathbf{A{\bullet}B}$
- The part of grade 4 $\langle\mathbf{AB}\rangle_4$ is the outer product $\mathbf{A}\wedge\mathbf{B}$
- The part of grade 2 is denoted $\mathbf{A}\times\mathbf{B}$ and called the *commutator product* of $\mathbf{A}$ and $\mathbf{B}$

Swapping $\mathbf{A}$ with $\mathbf{B}$, the terms of grade 0 and 4 of the product remain unchanged. This could be proofed factoring each blade i.e. considering $\mathbf{B}=\mathbf{b_1}\wedge\mathbf{b_2}$, $\mathbf{A}=\mathbf{a_1}\wedge\mathbf{a_2}$ where $\mathbf{b_1}$ and $\mathbf{b_2}$ are orthogonal:

$\mathbf{BA}=(\mathbf{b_1}\wedge\mathbf{b_2})(\mathbf{a_1}\wedge\mathbf{a_2}) = \mathbf{b_1b_2}(\mathbf{a_1}\wedge\mathbf{a_2}) = \mathbf{b_1}(\mathbf{b_2}\bullet(\mathbf{a_1}\wedge\mathbf{a_2}) + \mathbf{b_2}\wedge(\mathbf{a_1}\wedge\mathbf{a_2})) = \mathbf{b_1}((\mathbf{b_2}\bullet\mathbf{a_1})\mathbf{a_2} - (\mathbf{b_2}\bullet\mathbf{a_2})\mathbf{a_1} + \mathbf{b_2}\wedge\mathbf{a_1}\wedge\mathbf{a_2}) = (\mathbf{b_2}\bullet\mathbf{a_1})(\mathbf{b_1}\bullet\mathbf{a_2}) + (\mathbf{b_2}\bullet\mathbf{a_1})\mathbf{b_1}\wedge\mathbf{a_2}$ - $(\mathbf{b_2}\bullet\mathbf{a_2})(\mathbf{b_1}\bullet\mathbf{a_1})$ - $(\mathbf{b_2}\bullet\mathbf{a_2})\mathbf{b_1}\wedge\mathbf{a_1} + \mathbf{b_1}\bullet(\mathbf{b_2}\wedge\mathbf{a_1}\wedge\mathbf{a_2}) + \mathbf{b_1}\wedge\mathbf{b_2}\wedge\mathbf{a_1}\wedge\mathbf{a_2}$

$\langle\mathbf{BA}\rangle_0= (\mathbf{b_2}\bullet\mathbf{a_1})(\mathbf{a_2}\bullet\mathbf{b_1})$-$(\mathbf{b_1}\bullet\mathbf{a_1})(\mathbf{b_2}\bullet\mathbf{a_2}) = \langle\mathbf{AB}\rangle_0$

$\langle\mathbf{BA}\rangle_4=\mathbf{b_1}\wedge\mathbf{b_2}\wedge\mathbf{a_1}\wedge\mathbf{a_2} = \mathbf{a_1}\wedge\mathbf{a_2}\wedge\mathbf{b_1}\wedge\mathbf{b_2}=\langle\mathbf{AB}\rangle_4$

$\langle\mathbf{BA}\rangle_2= (\mathbf{b_2}\bullet\mathbf{a_1})\mathbf{b_1}\wedge\mathbf{a_2}$ -$(\mathbf{b_2}\bullet\mathbf{a_2})\mathbf{b_1}\wedge\mathbf{a_1}$ - $(\mathbf{b_1}\bullet\mathbf{a_1})\mathbf{b_2}\wedge\mathbf{a_2} + (\mathbf{b_1}\bullet\mathbf{a_2})\mathbf{b_2}\wedge\mathbf{a_1}$ = -$\langle\mathbf{AB}\rangle_2$

In conclusion the terms of grade 0 and 4 form the symmetric part of the geometric product $\mathbf{A_2B_2}$ but $\mathbf{A_2}\times\mathbf{B_2}$ is the antisymmetric part of $\mathbf{A_2B_2}$

$$\langle\mathbf{AB}\rangle_0 + \langle\mathbf{AB}\rangle_4 = \frac{1}{2}(\mathbf{AB} + \mathbf{BA}) \tag{2.28}$$

$$\mathbf{A}\times\mathbf{B} = \langle\mathbf{AB}\rangle_2 = \frac{1}{2}(\mathbf{AB} - \mathbf{BA}) \tag{2.29}$$

In G_3 the multiplication of two bivectors produces a scalar ($\mathbf{A}\bullet\mathbf{B}$) and a bivector ($\mathbf{A}\times\mathbf{B}$)

Note: The *factorization* of a k-blade $\mathbf{A}$ means writing it as an outer product of at least two blades (with grade less than $\mathbf{A}$). Usually the factors satisfy some orthogonality requirements. The factorization is a useful technique in many proofs. In fact a blade is defined as a multivector that can be written as a geometric product of anticommuting (i.e. orthogonal) vectors. (A blade is consequently a versor.) It is called also a *homogeneous multivector*. Obviously the number of factors of the decomposition product equals the grade of the blade.

The unit right-handed trivector $\mathbf{I_3}$ (in G_3) is called also the unit (dextral) *pseudoscalar* of the algebra. Given $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ one orthonormal basis of the associated vector space $\mathbf{I_3}=\mathbf{e_1}\wedge\mathbf{e_2}\wedge\mathbf{e_3}=\mathbf{e_1e_2e_3}$. The pseudoscalar commutes with every element (blade) of the algebra (fact that justifies its name). For example, if $\mathbf{a}=a_1\mathbf{e_1}+a_2\mathbf{e_2}+a_3\mathbf{e_3}$ is an arbitrary vector:

$\mathbf{I_3a} = \mathbf{e_1e_2e_3}(a_1\mathbf{e_1}+a_2\mathbf{e_2}+a_3\mathbf{e_3}) = a_1\mathbf{e_1e_2e_3e_1}+a_2\mathbf{e_1e_2e_3e_2}+a_3\mathbf{e_1e_2e_3e_3} = a_1\mathbf{e_2e_3}+a_2\mathbf{e_3e_1}+a_3\mathbf{e_1e_2}$ $=\mathbf{aI_3}$

As it could be seen the multiplication result is a bivector.

Given an arbitrary 2-blade $\mathbf{A}$, the result of $\mathbf{I_3A}$ is a vector. Considering $\mathbf{A} = \mathbf{a} \wedge \mathbf{b}$ we obtain:

$$\begin{aligned}
\mathbf{v} &= \mathbf{I_3A} = \mathbf{I_3}(\mathbf{a}\wedge\mathbf{b}) \\
&= \mathbf{e_1e_2e_3}((a_1\mathbf{e_1}+a_2\mathbf{e_2}+a_3\mathbf{e_3})\wedge(b_1\mathbf{e_1}+b_2\mathbf{e_2}+b_3\mathbf{e_3})) \\
&= \mathbf{e_1e_2e_3}(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{e_1}\wedge\mathbf{e_2}+\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{e_2}\wedge\mathbf{e_3}+\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{e_3}\wedge\mathbf{e_1} \\
&= -(a_1b_2-a_2b_1)\mathbf{e_3}-(a_2b_3-a_3b_2)\mathbf{e_1}-(a_3b_1-b_3a_1)\mathbf{e_2} \\
&= -\begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$

which corresponds to the familiar definition of the cross product $\mathbf{a}\times\mathbf{b}$. So that:

$$\mathbf{a}\times\mathbf{b} = -\mathbf{I_3}(\mathbf{a}\wedge\mathbf{b}) = (\mathbf{a}\wedge\mathbf{b})^* \tag{2.30}$$

This vector is indeed perpendicular on the $\mathbf{a}\wedge\mathbf{b}$ plane as is seen from:
$\mathbf{v}\bullet\mathbf{A} = \frac{1}{2}(\mathbf{vA}\text{-}\mathbf{Av}) = 0$. On the other side:

$\mathbf{v}\wedge\mathbf{A} = (v_1\mathbf{e_1}+v_2\mathbf{e_2}+v_3\mathbf{e_3})\wedge(a_1\mathbf{e_1}+a_2\mathbf{e_2}+a_3\mathbf{e_3})\wedge(b_1\mathbf{e_1}+b_2\mathbf{e_2}+b_3\mathbf{e_3}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$
$-(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2+\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2)\mathbf{I_3}$

From the fact that the sign of $\mathbf{I_3}$ coefficient is negative, we could conclude that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$ form a left-handed vector system and consequently $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ form a right-handed system.

Notes:

- In order to differ the notations for cross product and commutator product, the later one was designated by a bold x-shaped cross.
- $\mathbf{a}\times\mathbf{b}$ = -$\mathbf{I_3}(\mathbf{a}\wedge\mathbf{b})(=\mathbf{a}\wedge\mathbf{b}(\text{-}\mathbf{I_3}) = \mathbf{a}\wedge\mathbf{b}\ \mathbf{I_3}^{-1}) \Rightarrow (\mathbf{a}\times\mathbf{b})^2 = \mathbf{I_3}^2(\mathbf{a}\wedge\mathbf{b})^2 = \text{-}(\mathbf{a}\wedge\mathbf{b})^2 = \mid\mathbf{a}\wedge\mathbf{b}\mid^2$, consequently the square of a bivector in G_3 is always a negative scalar.

As is already shown, the multiplication with the pseudoscalar gives the duality transformation of the Geometric Algebra. The dual of the blade $\mathbf{A}$ is denoted by $\mathbf{A}^\star$.

In G_3, $\mathbf{A}^\star$ = -$\mathbf{I_3A} = \mathbf{AI_3}^{-1} = \mathbf{A}\widetilde{\mathbf{I_3}} = \mathbf{A}\bullet\widetilde{\mathbf{I_3}}$. The duality concept is own not only to subspaces but also to the inner and outer products. They are said to be dual to one another (or, in other words, orthogonality and spanning are two dual aspects of the geometric universe):

$$(\mathbf{a}\bullet\mathbf{A})^* = \mathbf{a}\wedge\mathbf{A}^* \tag{2.31}$$

and

$$(\mathbf{a}\wedge\mathbf{A})^* = \mathbf{a}\bullet\mathbf{A}^* \tag{2.32}$$

Indeed, if for example $\mathbf{A}$ has grade 2: $(\mathbf{a}\bullet\mathbf{A})^*$= -$\mathbf{I_3}(\mathbf{a}\bullet\mathbf{A})$=-$\frac{1}{2}\mathbf{I_3}(\mathbf{aA}\text{-}\mathbf{Aa})$= -$\frac{1}{2}(\mathbf{a}(\mathbf{I_3A})\text{-}(\mathbf{I_3A})\mathbf{a})$=$\frac{1}{2}(\mathbf{a}\ \mathbf{A}^*$ - $\mathbf{A}^*\mathbf{a}) = \mathbf{a}\wedge\mathbf{A}^*$

$\mathbf{A}^*$ is a vector so that: $\frac{1}{2}(\mathbf{aA}^*$ - $\mathbf{A}^*\mathbf{a}) = \mathbf{a}\wedge\mathbf{A}^*$

Relation 2.32 could be similarly proved.

2.4 The inverse element

Although not all multivectors are inversable, large subsets among them are. For example the inverse of a vector is:

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mid \mathbf{a} \mid^2} = \frac{\mathbf{a}}{\mathbf{a} \bullet \mathbf{a}} \tag{2.33}$$

A notable class of inversable elements are the *versors*. They are elements that could be written as a product of inversable vectors as for example: $\mathbf{V}=\mathbf{a_1}\mathbf{a_2}\cdots\mathbf{a_n}$. The inverse of a versor is given by:

$$\mathbf{V}^{-1} = \frac{\mathbf{V}^{\sim}}{\mathbf{V}^{\sim}\mathbf{V}} = \frac{\mathbf{V}^{\sim}}{\prod_{i=1}^{n} \mid \mathbf{a}_i \mid^2} \tag{2.34}$$

and that of an arbitrary blade (that is a versor because the blade could be factorized finding an orthonormal basis of the subspace):

$$\mathbf{A}^{-1} = \frac{\mathbf{A}^{\sim}}{\mathbf{A}^{\sim}\mathbf{A}} = \frac{\mathbf{A}^{\sim}}{\mid \mathbf{A} \mid^2} \tag{2.35}$$

The two previous formulas could be deduced by applying:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{2.36}$$

The product is not commutative, consequently the left inverse may be different from the right inverse.

2.5 Projection and Rejection

The fact that, given the product **xa** and the vector **a** then **x** could be unambiguously determined has an elegant geometric interpretation.

If **xa** is given, then both $\mathbf{x}\bullet\mathbf{a}$ and $\mathbf{x}\wedge\mathbf{a}$ have fixed values. The locus of the points (pointed by vectors **x**) that have a fixed value $\kappa = \mathbf{x} \bullet \mathbf{a}$ (where **a** is fixed) is a plane (π) perpendicular to **a** and placed at distance $\mid \mathbf{x} \bullet \mathbf{a} \mid$ from the origin of **a**. The locus of points (pointed by **x**) that have a fixed value $\mathbf{K} = \mathbf{x} \wedge \mathbf{a}$ (where **a** is fixed) is a line (d) parallel to **a**, lying in **K** plane at a distance $\mid \mathbf{x} \wedge \mathbf{a} \mid / \mid \mathbf{a} \mid$ (i.e. $\mid \mathbf{x} \mid sin(\phi)$) and placed on a certain side of **a** in function of $\mid \mathbf{K} \mid$ sign. In Figure 2.1 the plane of **K** has been shaded. The intersection between (π) and (d) gives a unique point P, and consequently a unique vector **x** having the source identical to the source of **a** and pointing P. Writing algebraically $\mathbf{x}=(\mathbf{xa})\mathbf{a}^{-1}=(\mathbf{x}\bullet\mathbf{a})+(\mathbf{x}\wedge\mathbf{a}) =\mathbf{x_{proj}} + \mathbf{x_{perp}}$. The two components of **x** are interpreted as the projection of **x** onto **a** ($\mathbf{x_{proj}}$ or $\mathbf{x}_{\parallel}$) and respectively the rejection of **x** by **a** ($\mathbf{x_{perp}}$ or $\mathbf{x}_{\perp}$).

On the other side, knowing $\mathbf{K}=\mathbf{x}\wedge\mathbf{a}$ and **a** it is not possible to uniquely determine **x** (in Figure 2.1 **x'** equally satisfies $\mathbf{x'}\wedge\mathbf{a}=\mathbf{K}$) and knowing $\kappa = \mathbf{x} \bullet \mathbf{a}$ does not uniquely specify **x** (see for example **x"** that satisfies $\mathbf{x''}\bullet\mathbf{a}=\kappa$).

Note: In Geometric algebra, the projection of a blade **X** on other blade **A** is designated by the projection operator $\mathbf{P_A}(\mathbf{X})$. Using this notation we could write $\mathbf{x}_{\parallel} = \mathbf{P_a}(\mathbf{x})$. Generally:

$$\mathbf{P_A}(\mathbf{X}) = (\mathbf{X} \bullet \mathbf{A}) \bullet \mathbf{A}^{-1} \tag{2.37}$$

This formula holds if grade(**X**)≠0 and grade(**X**)≠grade(**A**). As previously mentioned by convention it is stated that: $\mathbf{A}\bullet\mathbf{B}$=0 if grade(**A**)=0 or grade(**B**)=0. This maintains the validity of 2.37 if grade(**A**)=grade(**B**). A correct formula independent of the grades of the blades is:

$(\mathbf{P_A}(\mathbf{X}) = (\mathbf{X}\rfloor\mathbf{A})\rfloor\mathbf{A}^{-1}$ The same projection and rejection of a vector **x** could be considered relatively to a plane (an important case in G_3) or to a blade of arbitrary grade. It is possible to write: $\mathbf{x}=\mathbf{x}\mathbf{A}\mathbf{A}^{-1} = (\mathbf{x}\bullet\mathbf{A}+\mathbf{x}\wedge\mathbf{A})\mathbf{A}^{-1}=\mathbf{P_A}(\mathbf{x})+\mathbf{P}^{\perp}_{\mathbf{A}}(\mathbf{x})$ where:

$$\mathbf{P_A}(\mathbf{X}) = (\mathbf{X} \bullet \mathbf{A})\mathbf{A}^{-1} \tag{2.38}$$

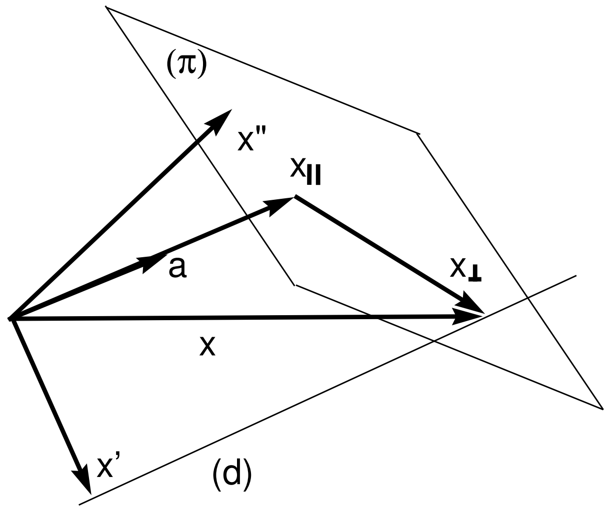

Figure 2.1: Significance of the geometric product invertibility

$$\mathbf{P}^{\perp}_{\mathbf{A}}(\mathbf{X}) = (\mathbf{X} \wedge \mathbf{A})\mathbf{A}^{-1} = (\mathbf{X} \wedge \mathbf{A})\frac{\mathbf{A}^{\sim}}{\mid \mathbf{A} \mid^2} \tag{2.39}$$

represent respectively the component of $\mathbf{x}$ placed in $\mathbf{A}$ (projection of $\mathbf{x}$ onto $\mathbf{A}$) and the component orthogonal to $\mathbf{A}$ and placed in the subspace complementary to $\mathbf{A}$ relatively to V^n (rejection of $\mathbf{x}$ by $\mathbf{A}$).

2.6 Algebraic characterization of subspaces

In a linear vector space V^n, m vectors $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_m}$ are linearly dependent if there are m real constants α_1, α_2, ..., α_m (at least one of them non null) so that $\sum_{i=1}^{m} \alpha_i \mathbf{v_i}$. In the Euclidean space E^3 any four vector system has linearly dependent components. Moreover:

Proposition: If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in E^3$ then they are linearly dependent iff $\mathbf{v_1} \wedge \mathbf{v_2} \wedge \mathbf{v_3}$=0.

Proof: The direct implication is trivial: if $\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{v_3}$=0 and without restricting generality α is considered non null, then:

$$\mathbf{v_1} = -\frac{\beta}{\alpha}\mathbf{v_2} - \frac{\gamma}{\alpha}\mathbf{v_3} = \beta_1\mathbf{v_1} + \gamma_1\mathbf{v_3}$$

in this case $\mathbf{v_1} \wedge \mathbf{v_2} \wedge \mathbf{v_3} = (\beta_1\mathbf{v_2}+\gamma_1\mathbf{v_3})\wedge\mathbf{v_2} \wedge \mathbf{v_3} = \beta_1\mathbf{v_2} \wedge \mathbf{v_2} \wedge \mathbf{v_3} + \gamma_1\mathbf{v_3} \wedge \mathbf{v_2} \wedge \mathbf{v_3}$=0

Supposing that $\mathbf{v_1} \wedge \mathbf{v_2} \neq 0$, the inverse implication follows from:

$(\mathbf{v_1} \wedge \mathbf{v_2})\bullet\ (\mathbf{v_1} \wedge \mathbf{v_2} \wedge \mathbf{v_3})$=0

Indeed:

$(\mathbf{v_1}\wedge\mathbf{v_2})\bullet(\mathbf{v_1}\wedge\mathbf{v_2}\wedge\mathbf{v_3}) = \mathbf{v_1}\bullet(\mathbf{v_2}\bullet(\mathbf{v_1}\wedge\mathbf{v_2}\wedge\mathbf{v_3})) = \mathbf{v_1}\bullet(\mathbf{v_2}\bullet\mathbf{v_1}(\mathbf{v_2}\wedge\mathbf{v_3}))$ - $\mathbf{v_1}\bullet\mathbf{v_2}^2(\mathbf{v_1}\wedge\mathbf{v_3})$ + $\mathbf{v_1}\bullet(\mathbf{v_2}\bullet\mathbf{v_3}(\mathbf{v_1}\bullet\mathbf{v_2})) = (\mathbf{v_2}\bullet\mathbf{v_1})(\mathbf{v_1}\bullet\mathbf{v_2})\mathbf{v_3}$ - $(\mathbf{v_2}\bullet\mathbf{v_1})(\mathbf{v_1}\bullet\mathbf{v_3})\mathbf{v_2}$ - $\mathbf{v_2}^2\mathbf{v_1}^2\mathbf{v_3} + \mathbf{v_2}^2(\mathbf{v_1}\bullet\mathbf{v_3})\mathbf{v_1}$ + $(\mathbf{v_2}\bullet\mathbf{v_3})\mathbf{v_1}^2\mathbf{v_2}$ - $(\mathbf{v_2}\bullet\mathbf{v_3})(\mathbf{v_1}\bullet\mathbf{v_2})\mathbf{v_1} = (\mathbf{v_2}^2(\mathbf{v_1}\bullet\mathbf{v_3})$ - $(\mathbf{v_2}\bullet\mathbf{v_3})(\mathbf{v_1}\bullet\mathbf{v_2}))\mathbf{v_1} + (\mathbf{v_1}^2(\mathbf{v_2}\bullet\mathbf{v_3})$ - $(\mathbf{v_2}\bullet\mathbf{v_1})(\mathbf{v_1}\bullet\mathbf{v_3}))\mathbf{v_2} + ((\mathbf{v_2}\bullet\mathbf{v_1})(\mathbf{v_1}\bullet\mathbf{v_2})$ - $\mathbf{v_1}^2\mathbf{v_2}^2)\mathbf{v_3} = 0$

Thus:

$\alpha = \mathbf{v_2}^2(\mathbf{v_1}\bullet\mathbf{v_3}) - (\mathbf{v_2}\bullet\mathbf{v_3})(\mathbf{v_1}\bullet\mathbf{v_2})$

$\beta = \mathbf{v_1}^2(\mathbf{v_2}\bullet\mathbf{v_3}) - (\mathbf{v_2}\bullet\mathbf{v_1})(\mathbf{v_1}\bullet\mathbf{v_3})$

$\gamma = (\mathbf{v_2}\bullet\mathbf{v_1})(\mathbf{v_1}\bullet\mathbf{v_2}) - \mathbf{v_1}^2\mathbf{v_2}^2$

are the coefficients of the vanishing linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$.

The proof used:

$$(\mathbf{A} \wedge \mathbf{b}) \bullet \mathbf{C} = \mathbf{A} \bullet (\mathbf{b} \bullet \mathbf{C}) \tag{2.40}$$

where grade($\mathbf{A}$)<grade($\mathbf{C}$). This last identity could be derived from:

$$\langle(\mathbf{Ab})\mathbf{C}\rangle_{grade(\mathbf{C})-grade(\mathbf{A})-1} = \langle\mathbf{A}(\mathbf{bC})\rangle_{grade(\mathbf{C})-grade(\mathbf{A})-1}$$

The proposition above admits the following generalization:

Given a m vector system $\mathbf{v_1}$, $\mathbf{v_2}$, ..., $\mathbf{v_m}$ in the Euclidean space E^n, the vectors are linearly dependent iff: $\bigwedge_{i=1}^{m} \mathbf{v_i} = 0$

Based on the previous proposition and on the fact that in E^3 any vector can be represented as a linear combination of only 3 vectors (composing a basis of the Euclidean space) it could be written $\mathbf{x} \wedge \mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$=0 or

$$\mathbf{x} \wedge \mathbf{I_3} = 0, \forall \mathbf{x} \in E^3 \tag{2.41}$$

Relation 2.41 expresses the correspondence between a space and a unit pseudoscalar that gives the orientation of the space (this pseudoscalar is called *tangent* to the space). That is the geometric algebra way to say that $\mathbf{x}$ is a vector of the 3D space. As it is known from the elementary geometry classes, the three coordinates (components or director coefficients) of vector $\mathbf{x}$ are the lengths of its projections on the three orthonormal unit vectors that make up the referential (and the unit pseudoscalar).

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3} = (\mathbf{x} \bullet \mathbf{e_1})\mathbf{e_1} + (\mathbf{x} \bullet \mathbf{e_2})\mathbf{e_2} + (\mathbf{x} \bullet \mathbf{e_3})\mathbf{e_3} \tag{2.42}$$

The identity 2.42 is equivalent to 2.41: $\mathbf{x}\wedge\mathbf{I_3}$=0 $\Leftrightarrow$ $\mathbf{x}\bullet\mathbf{I_3}$=$\mathbf{xI_3}$ $\Leftrightarrow$ $\mathbf{x}$=$(\mathbf{x}\bullet\mathbf{I_3})\mathbf{I_3}^{-1}$ Using identity 2.26 the inner product could be developed. That gives: $\mathbf{x}$= $((\mathbf{x}\bullet\mathbf{e_1})\ \mathbf{e_2} \wedge \mathbf{e_3}$ - $(\mathbf{x}\bullet\mathbf{e_2})\mathbf{e_1} \wedge \mathbf{e_3}$ + $(\mathbf{x}\bullet\mathbf{e_3})\mathbf{e_1} \wedge \mathbf{e_2})\mathbf{I_3}^{\sim}$ =$(\mathbf{x}\bullet\mathbf{e_1})\mathbf{e_1}$ + $(\mathbf{x}\bullet\mathbf{e_2})\mathbf{e_2}$ + $(\mathbf{x}\bullet\mathbf{e_3})\mathbf{e_3}$, i.e. exactly 2.42.

In general, in an arbitrary n-dimensional G_n geometric algebra, every blade determine a vector space (subspace of V^n) whose vectors $\mathbf{x}$ satisfy: $\mathbf{x} \wedge \mathbf{A}$=0 and inverse every vector space determine a unit pseudoscalar that uniquely characterizes the vectors of the space and the space orientation.

2.7 Spinors and Angles

The geometric product of two vectors could be interpreted as a rotation/dilation operator. Indeed, observing that $\mathbf{ab}$=$\mathbf{cd}$ $\Leftrightarrow$ $\mathbf{a} = (\mathbf{cd})\mathbf{b}^{-1}$ we could interpret that $\mathbf{a}$ could be obtained from $\mathbf{b}$ by applying the rotation-dilation operator: $\mathcal{RD} = \frac{1}{|\mathbf{b}|^2}(\mathbf{cd})$. Thus, an even multivector of G_3 could be interpreted as a transformation operator. The geometric product of two vectors (having the above mentioned geometric semantic) is called a *spinor*. Its symmetric (scalar) part gives information about the rotation angle, its antisymmetric (bivector) component gives information about the rotation plane. Until this point of the presentation the even blades were geometrically interpreted only as directed area elements. Rewriting the relation $\mathbf{a} = \frac{\mathbf{cd}}{|\mathbf{b}|^2}\mathbf{b}$ in order to completely separate the dilation factor of the transformation, we must explicitly set off the unit vectors that represent in fact the directions of $\mathbf{c}$ and $\mathbf{d}$.

$$\mathbf{a} = \frac{|\,\mathbf{c}\,||\,\mathbf{d}\,|}{|\,\mathbf{b}\,|^2}(\hat{\mathbf{c}}\hat{\mathbf{d}})\mathbf{b} \tag{2.43}$$

The geometric product of two unit vectors represents a "pure" rotation transformation. It is possible to write: $\mathbf{a}$=$(\hat{\mathbf{c}}\hat{\mathbf{d}})\mathbf{b}$=$(\hat{\mathbf{c}} \bullet \hat{\mathbf{d}})\mathbf{b}$ + $(\hat{\mathbf{c}} \wedge \hat{\mathbf{d}})\mathbf{b}$ = $\mathbf{b}\cos(\theta)$ + $\mathbf{I_{cd}}\sin(\theta)\mathbf{b}$, where $\mathbf{I_c}d$ is the unit bivector of the $\mathbf{c}\wedge\mathbf{d}$ plane. Thus, taking into consideration that the Taylor series developments of cos(θ), sin(θ), e^{θ} are those known from the elementary analysis classes and $(\mathbf{I_{cd}})^2$= -1, we could write the product of two unit vectors as:

$$\hat{\mathbf{c}}\hat{\mathbf{d}} = e^{\mathbf{I_{cd}}\theta} \tag{2.44}$$

and consequently:

$$\mathbf{a} = e^{\mathbf{I_{cd}}\theta}\mathbf{b}$$

The notation $e^{\mathbf{I}\theta}$ that corresponds to a *unitary spinor*, specifies simultaneously the rotation plane (through its unit bivector **I**) and the measure of the rotation angle θ between **a** and **b**. In geometric algebra the angles are represented by bivectors; that is considering 2.44 and denoting $\theta = \mathbf{I_{cd}}\theta$ we could write $\mathbf{a}$=$e^{\theta}\mathbf{b}$ and state that θ is the angle between **a** and **b**. The sense of θ depends on the sense of the associated blade pseudoscalar and the sign of θ determines the sense of rotation.

We could write directly (for **b** and **a** unit vectors) $\mathbf{b}$=$\mathcal{R}\mathbf{a} \Rightarrow \mathcal{R} = \mathbf{b}\mathbf{a}^{-1} \Rightarrow \mathcal{R} = \frac{1}{|\mathbf{a}|^2}\mathbf{b}\mathbf{a} = \mathbf{b}\mathbf{a}$ = cos(θ)-$\mathbf{I_{ab}}sin(\theta)$=$e^{-\mathbf{I_{ba}}\theta}$= $e^{\mathbf{I_{ab}}\theta} \Rightarrow \mathbf{b}$=$e^{\mathbf{I_{ab}}\theta}\mathbf{a}$ where the pseudoscalar of the $\mathbf{a}\wedge\mathbf{b}$ space is oriented from **a** to **b**.

Note: the formal theory defines the sine and respectively cosine between two directions given by their unit vectors, as the magnitudes of the rejection respectively projection of one direction onto the other.

Note: There are three equivalent ways to denote a spinor: S = $\mathbf{uv}$ = α+$\mathbf{I_3}\beta$ = $e^{\mathbf{I_3 u}}$. The last part of the equality emphasizes the axis of rotation **u** and the angle of rotation $|\ \mathbf{u}\ |=|\ \mathbf{A}\ |$ (where $\mathbf{A}$=$\mathbf{I_3 u}$ is the angle of rotation. The equality α+$\mathbf{I_3}\beta$=$e^{I_3 u} \Rightarrow \alpha$=cos($|\ \mathbf{u}\ |$) and β= $\frac{\mathbf{u}}{|\mathbf{u}|}$sin($|\ \mathbf{u}\ |$). α and **u** are forming the *Euler parametrization* of the rotation.

The four dimensional space of spinors is the same with the space of even G_3 multivectors and with the space of quaternions. In fact the quaternion algebra (see Appendix 2) coincides to the even vectors subalgebra of G_3 (this subalgebra is denoted G_3^+).[1]

The same multiple possibilities to ascribe geometric semantic to multivectors occur in G_2 Clifford algebra. Let us refer at the beginning at the geometry of the complex plane. As it is well known, a complex number could be codified by its real and imaginary parts or respectively by its module and argument. These codifications emphasize the two aspects of the complex space: the area (static aspect related to specifying directions using vectors in this space) and the rotation-dilation transform (the operational or transformational aspect). Writing a complex number as $z = \rho e^{\mathbf{I}\theta} = \rho(cos(\theta) + \mathbf{I}sin(\theta))$ allows a relation as: $\mathbf{b}$=$\mathbf{za}$ to be interpreted as: "**b** results from **a** by a counterclockwise rotation of angle θ and a dilation (uniform scaling) of factor ρ".

From the (plane) geometric algebra point of view:

- the vectors in the 2D plane correspond to odd multivectors of the algebra; they emphasize the directional aspects of the 2D geometry

- the spinors are even multivectors (that could be considered as resulted from the geometric multiplication of two vectors); they emphasize the transformational aspects of the 2D geometry

A bijective mapping between the vectors and the spinors plane could be constructively defined. First a unit vector $\mathbf{e_1}$ is arbitrarily chosen. Next every vector **v** in the vector plane will have an associated spinor $\mathbf{e_1 v}$=$|\ \mathbf{v}\ |\ e^{\mathbf{I}\theta}$ in the spinor plane (θ is the angle between $\mathbf{e_1}$ and **v** and **I** is the unit bivector (pseudoscalar) of the plane).

[1]Grassmann realized only in the last years of his life the facts that quaternions are sums of inner and outer products of vectors and that the Hamilton's theory could be elegantly integrated in (Grassmann's) exterior algebra.

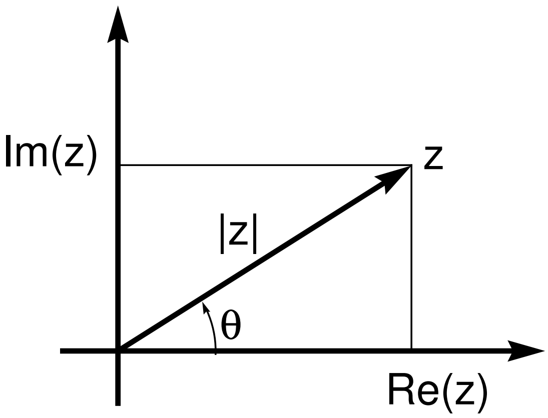

Figure 2.2: The two representations of a complex number

The equation $x^2 = -1$, which in 2D geometric algebra has as solution the unit pseudoscalar (or the renowned $\sqrt{-1}$ conforming to the old terminology related to the complex space) has in 3D geometric algebra two types of solutions. The first one is the unit pseudoscalar and the second is a unit bivector that admits a factorization as: $\mathbf{B}$=$\mathbf{a}\wedge\mathbf{b}$=$\mathbf{ab}$, where $\mathbf{a}$, $\mathbf{b}$ are two orthogonal unit vectors. These solutions have different geometric semantic: the second one (spinor solution) being associated to a specific plane in the space while the first is related to directed volumes. In G_3 a spinor is related to a specific plane i.e. the rotation is specified by indicating a rotation plane not a rotation axis.

Given two arbitrary blades $\mathbf{A}$ and $\mathbf{B}$ it is possible to define the angle between them i.e. the *dihedral angle* between their planes. This angle is characteristic to a spinor that rotates $\mathbf{A}$ onto $\mathbf{B}$ around their intersection line. Geometrically, the dihedral angle of two planes $\mathbf{A}$ and $\mathbf{B}$ is measured in a plane perpendicular to their intersection line. In Figure 2.3, $\mathbf{u}$ is the unit vector specifying the direction of the intersection line[1]. $\mathbf{a}$ and $\mathbf{b}$ are two vectors normal to $\mathbf{u}$ and therefore placed in a plane normal to $\mathbf{u}$. $\mathbf{a}$ and $\mathbf{b}$ are respectively included in $\mathbf{A}$ and $\mathbf{B}$ and consequently the dihedral angle coincides with the angle between $\mathbf{a}$ and $\mathbf{b}$. The reader may observe that rotating $\mathbf{A}$ around the axis $\mathbf{u}$ with the angle θ causes the superposition of $\mathbf{A}$ onto $\mathbf{B}$ and the bivector $\mathbf{A}$ has an opposite orientation related to $\mathbf{B}$. The spinor defined by the exponential from 2.44 and specifying the rotation of one vector upon another find its correspondent in the case of bivector rotation. The geometric product of two bivectors could equally be interpreted as a spinor:

$$\mathbf{AB} = e^{\mathbf{Iu}} = e^{\mathbf{u}^*} \tag{2.45}$$

where $\mathbf{u}^* = \mathbf{Iu}$ is the bivector of the plane normal to $\mathbf{u}$ (the dual of $\mathbf{u}$)

In G_3 the geometric product of two bivectors $\mathbf{A}$, $\mathbf{B}$ has one scalar component $\mathbf{A} \bullet \mathbf{B}$ and one bivector component equal to the commutator product $\mathbf{A} \times \mathbf{B}$. Thus 2.45 could be written:

$$\mathbf{AB} = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \times \mathbf{B} = cos(\mid \mathbf{u} \mid) + \mathbf{Iu}sin(\mid \mathbf{u} \mid) \tag{2.46}$$

Considering $\mathbf{A}$ and $\mathbf{B}$ unit bivectors and noting $\mathbf{u_1}$ the unit vector specifying the intersection direction, it is possible to factorize $\mathbf{A}$, $\mathbf{B}$ as: $\mathbf{A}$= -$\mathbf{au_1}$ and $\mathbf{B}$= -$\mathbf{u_1 b}$ where $\mathbf{a}$ and $\mathbf{b}$ are unit vectors. Consequently the identity 2.46 becomes:

$$\mathbf{ab} = cos(\mid \mathbf{u} \mid) + \mathbf{Iu}sin(\mid \mathbf{u} \mid) = e^{\mathbf{Iu}} = e^{\mathbf{I}|\mathbf{u}|\mathbf{u_1}}$$

[1]The intersection of two planes specified by their bivectors $\mathbf{A}$, $\mathbf{B}$ has the direction given by the cross product: $\mathbf{A}^* \times \mathbf{B}^* = (\mathbf{Ia}) \times (\mathbf{Ib})$

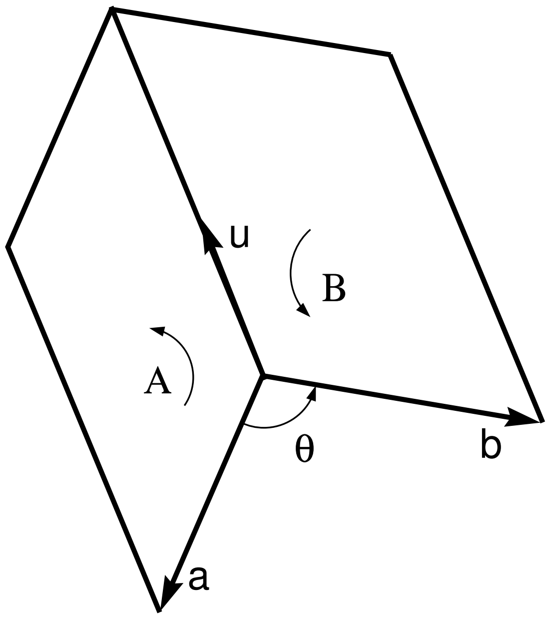

Figure 2.3: The dihedral angle of two planes

that is the spinor $e^{\mathbf{Iu}}$ performs the rotation of **a** onto **b** in their plane which is normal to **u** and corresponds precisely to the rotation of **A** onto **B** by an angular amount equal to their dihedral angle.

The magnitude of **u** gives the radian measure of the rotation angle. $\theta = \mid \mathbf{u} \mid$ =arccos($\mathbf{A} \bullet \mathbf{B}$) the measure of θ could equally be established as follows:

In the case of grade(**A**)=grade(**B**) the inner product coincides with the scalar product. $\mathbf{A} \bullet \mathbf{B} = \mathbf{A}.\mathbf{B}$ Both products are linear; supplementary the scalar product (of multivectors) is also commutative. In conformity with 2.10:

$$\begin{aligned} \mid \mathbf{A}+\mathbf{B} \mid^2 &= (\mathbf{A}+\mathbf{B})^{\sim}(\mathbf{B}+\mathbf{A}) = \mathbf{A}^{\sim}.\mathbf{A} + \mathbf{A}^{\sim}.\mathbf{B} + \mathbf{B}^{\sim}.\mathbf{A} + \mathbf{B}^{\sim}.\mathbf{B} \\ &= \mid \mathbf{A} \mid^2 + \mid \mathbf{B} \mid^2 + 2\mathbf{A}^{\sim}.\mathbf{B} \end{aligned} \tag{2.47}$$

due to the fact that: $\mathbf{A}^{\sim}.\mathbf{B} = \langle \mathbf{A}^{\sim}\mathbf{B} \rangle_0 = \langle (\mathbf{A}^{\sim}\mathbf{B})^{\sim} \rangle_0 = \langle \mathbf{B}^{\sim}\mathbf{A} \rangle_0 = \mathbf{B}^{\sim}.\mathbf{A}$

The quantity: $cos(\theta_{AB}) = \frac{\mathbf{A}^{\sim}.\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \in [-1, 1]$ determines the measure of the angle between **A** and **B**. If **A** and **B** are bivectors, θ_{AB} is the measure of their dihedral angle. If **A** and **B** are considered vectors relation 2.47 could be written as: $\mid \mathbf{a}+\mathbf{b} \mid^2 = \mid \mathbf{a} \mid^2 + \mid \mathbf{b} \mid^2 + 2 \mid \mathbf{a} \mid\mid \mathbf{b} \mid \cos(\theta)$ which is known as the cosine theorem from the elementary trigonometry classes. The fact that: -1 $\leq \frac{\mathbf{A}^{\sim}.\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \leq 1 \Leftrightarrow (\mathbf{A}^{\sim}.\mathbf{B})^2 \leq \mid \mathbf{A} \mid^2 \mid \mathbf{B} \mid^2$ is the Cauchy-Buniakowsky-Schwarz inequality and could be easily deduced from the obvious relation $\mid \mathbf{A} + \lambda\mathbf{B} \mid^2 \geq 0, \forall\lambda \in R$. Substituting $\lambda = \frac{\mathbf{A}^{\sim}.\mathbf{B}}{|\mathbf{B}|^2}$ in $(\mathbf{A}+\lambda\mathbf{B})^{\sim}.(\mathbf{A}+\lambda\mathbf{B}) \geq 0$ results:

$$\begin{aligned} \mathbf{A}^{\sim}.\mathbf{A} + 2\lambda\mathbf{A}^{\sim}.\mathbf{B} + \lambda^2\mathbf{B}^{\sim}.\mathbf{B} &= \mid \mathbf{A} \mid^2 + \mid \mathbf{B} \mid^2 \frac{(\mathbf{A}^{\sim}.\mathbf{B})^2}{\mid \mathbf{B} \mid^4} - 2\frac{\mathbf{A}^{\sim}.\mathbf{B}}{\mid \mathbf{B} \mid^2}(\mathbf{A}^{\sim}.\mathbf{B}) \\ &= \mid \mathbf{A} \mid^2 + \frac{(\mathbf{A}^{\sim}.\mathbf{B})^2}{\mid \mathbf{B} \mid^2} - 2\frac{(\mathbf{A}^{\sim}.\mathbf{B})^2}{\mid \mathbf{B} \mid^2} \geq 0 \\ \Rightarrow \mid \mathbf{A} \mid^2 \geq \frac{(\mathbf{A}^{\sim}.\mathbf{B})^2}{\mid \mathbf{B} \mid^2} &\Rightarrow (\mathbf{A}^{\sim}.\mathbf{B})^2 \leq \mid \mathbf{A} \mid^2 \mid \mathbf{B} \mid^2 \end{aligned}$$

q.e.d.

Another multivector angle heavily used in computer graphics problems is the angle between a line and a plane. In geometric algebra the directions of those two entities are specified respectively by a unit vector **a** and a unit bivector **B**. The rotation transformation necessary to make **a** lay onto the plane of **B** is done around the axis that is the orthogonal complement within **B** of the component of **a** lying in **B**. $\mathbf{a}_{\|} = P_{\mathbf{B}}(\mathbf{a}) = (\mathbf{a} \bullet \mathbf{B})\mathbf{B}^{-1}$ =**a**cos(θ). In other words the rotation plane is the dual of **u** and the spinor expressing this transformation is $e^{\mathbf{Iu}}$. In order to find the relation that links the geometric product of the two entities (between which we want to specify the angle as a bivector) with the spinor that makes one of them superposed onto the other we write:

$$\mathbf{aB} = \mathbf{u_1} e^{\mathbf{Iu}} \tag{2.48}$$

where $\mathbf{u_1}$ is the unit vector of the rotation axis direction.

$\mathbf{aB} = \mathbf{u_1}(\cos(|\,\mathbf{u}\,|) + \mathbf{I u}\sin(|\,\mathbf{u}\,|)) = \mathbf{u_1}\cos(|\,\mathbf{u}\,|) + \mathbf{I}\,|\,\mathbf{u}\,|\sin(|\,\mathbf{u}\,|) \Rightarrow \mathbf{a \bullet B} = \mathbf{u_1}\cos(|\,\mathbf{u}\,|)$ and $\mathbf{a \wedge B} = \mathbf{I}|\,\mathbf{u}\,|\sin(|\,\mathbf{u}\,|)$.

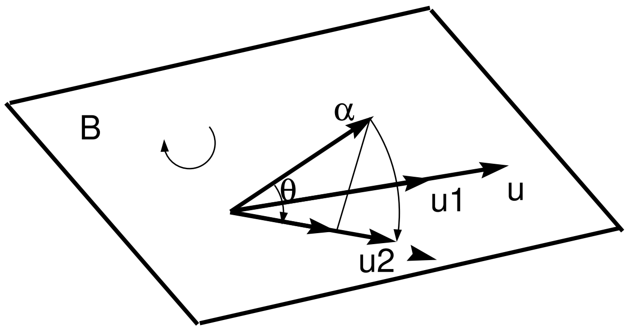

Figure 2.4: The angle between a line and a plane

In a plane, the right multiplication between a vector and the unit pseudoscalar (of the plane) is equivalent to a $\pi/2$ rotation in the sense of the above mentioned pseudoscalar. Thus, considering **a** and **B** unitary, $(\mathbf{a} \bullet \mathbf{B})\mathbf{B} = \mathbf{u_1}\cos(\theta)\mathbf{B} = \cos(\theta)\mathbf{u_1 B} = \cos(\theta)\mathbf{u_2}$. Multiplying 2.48 by **a** results: $\mathbf{B} = \mathbf{a u_1} e^{\mathbf{Iu}} = \mathbf{a} e^{-\mathbf{Iu}}\mathbf{u_1} = \mathbf{-u_2 u_1}$ Therefore $\mathbf{u_2}$ is a unit vector orthogonal to $\mathbf{u_1}$ and lying in the **B** plane i.e. **B** could be factored as $\mathbf{B}=\mathbf{u_1 u_2}$. $\mathbf{u_2}$ could be interpreted as the vector obtained by rotating **a** around **u** with the angular amount θ.

2.8 Orthonormal transformations

The *orthonormal transformations* of the 3D space are satisfying the following requirements:

- They are $V^3 \rightarrow V^3$ transformations
- They are linear transformations: $\mathrm{O}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha O(\mathbf{u}) + \beta O(\mathbf{v}), \forall \alpha, \beta \in R, \mathbf{u}, \mathbf{v} \in V^3$
- They preserve the lengths[1] (or equivalently they preserve the value of the inner product of vectors): $\mathrm{O}(\mathbf{u}) \bullet \mathrm{O}(\mathbf{v}) = \mathbf{u} \bullet \mathbf{v}, \forall \mathbf{u}, \mathbf{v} \in V^3$

This class of transformations includes the *rotations* and the *reflections* (called also *mirroring transformations*[2])

If these transformations are expressed matriceally as $\mathbf{v} = M_{3*3}\mathbf{u}$ than the matrix M_{3*3}:

[1]A transformation preserving length is called an *isometry*. As an example, shearing, a widely used computer graphics transformation is not an isometry.

[2]Translation is not an orthonormal transformation. It does not satisfy the linearity property

- is a unitary matrix: $|\det(M_{3*3})|$=1
- its columns are orthonormal vectors
- its inverse equals its transpose: $M_{3*3}^{-1} = M_{3*3}^{T}$

If the determinant value is negative the transformation is a reflection otherwise it is a rotation.

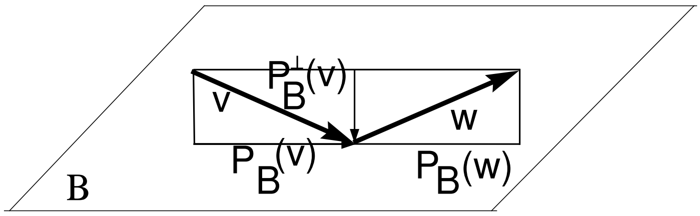

Figure 2.5: Reflection of a vector by a plane

The reflection of vector **v** relative to a plane given by its unit bivector **B** produces the vector **w** having the same projection on **B** as **v** and the opposite rejection comparing to $\mathbf{v}_\perp$. Though $P_\mathbf{B}(\mathbf{w})$=$P_\mathbf{B}(\mathbf{v})$=$(\mathbf{v} \bullet \mathbf{B})\mathbf{B}^{-1}$ and $P^{\perp}_\mathbf{B}(\mathbf{w})$ = -$P^{\perp}_\mathbf{B}(\mathbf{v})$ = -$(\mathbf{v} \wedge \mathbf{B})\mathbf{B}^{-1}$.

$\mathbf{w}$ = $P_\mathbf{B}(\mathbf{v})$-$P^{\perp}_\mathbf{B}(\mathbf{v})$ = $(\mathbf{v} \bullet \mathbf{B} - \mathbf{v} \wedge \mathbf{B})\mathbf{B}^{-1}$ = $(-\mathbf{B} \bullet \mathbf{v} - \mathbf{B} \wedge \mathbf{v})\mathbf{B}^{-1}$ = -$\mathbf{B}\mathbf{v}\mathbf{B}^{-1}$. Identities 2.22 and 2.23 were applied when commuting the operands of the products.

The same type of proof is valid if the plane is specified by its normal unit vector **n**:

$$\mathbf{w} = -P_\mathbf{n}(\mathbf{v}) + P^{\perp}_\mathbf{n}(\mathbf{v}) = -(\mathbf{v} \bullet \mathbf{n})\mathbf{n}^{-1} + (\mathbf{v} \wedge \mathbf{n})\mathbf{n}^{-1} = (-\mathbf{n} \bullet \mathbf{v} - \mathbf{n} \wedge \mathbf{v})\mathbf{n}^{-1} = -\mathbf{n}\mathbf{v}\mathbf{n}.$$

The fact that **n** is unitary or not does not matter: **w**= -$\mathbf{n}\mathbf{v}\mathbf{n}^{-1}$= -**nvn** = -$\mathbf{n_1}\mathbf{v}\mathbf{n_1}$, where $\mathbf{n}_1$ is the normalized **n**.

In fact the identity:

$$\mathcal{M}_\mathbf{A}(\mathbf{v}) = \mathbf{Reflection}_\mathbf{A}(\mathbf{v}) = -\mathbf{A}\mathbf{V}\mathbf{A}^{-1} \qquad (2.49)$$

that computes the mirrored **v** relative to an arbitrary subspace specified by the blade **A**, holds (modulo a sign change) for any grade of **A**.

Reflection of a bivector **B**=$\mathbf{a} \wedge \mathbf{b}$ by another bivector (denoted **C**) is determined mirroring separately each of the two factors of **B**:

$$\begin{aligned}
\mathcal{M}_\mathbf{C}(\mathbf{B}) &= \mathcal{M}_\mathbf{C}(\mathbf{a} \wedge \mathbf{b}) = \mathcal{M}_\mathbf{C}(\mathbf{a}) \wedge \mathcal{M}_\mathbf{C}(\mathbf{b}) \\
&= (-\mathbf{C}\mathbf{a}\mathbf{C}^{-1}) \wedge (-\mathbf{C}\mathbf{b}\mathbf{C}^{-1}) = \frac{1}{2}(\mathbf{C}\mathbf{a}\mathbf{C}^{-1}\mathbf{C}\mathbf{b}\mathbf{C}^{-1} - \mathbf{C}\mathbf{b}\mathbf{C}^{-1}\mathbf{C}\mathbf{a}\mathbf{C}^{-1}) \\
&= \frac{1}{2}(\mathbf{C}\mathbf{a}\mathbf{b}\mathbf{C}^{-1} - \mathbf{C}\mathbf{b}\mathbf{a}\mathbf{C}^{-1}) = \mathbf{C}\frac{\mathbf{a}\mathbf{b} - \mathbf{b}\mathbf{a}}{2}\mathbf{C}^{-1} \\
&= \mathbf{C}\mathbf{a} \wedge \mathbf{b}\mathbf{C}^{-1} = \mathbf{C}\mathbf{B}\mathbf{C}^{-1} \qquad (2.50)
\end{aligned}$$

Therefore a grade change of the reflected entity implies a change of sign in the mirroring formula. The correct formula results from developing the outer product (depending on the grade of its factors) as a symmetric or antisymmetric part of a geometric product. The general formula giving the reflection of an arbitrary blade **B** in a 2-grade blade **C** is :

$$\mathcal{M}_\mathbf{C}(\mathbf{B}) = (-1)^{grade(\mathbf{B})}\mathbf{C}\mathbf{B}\mathbf{C}^{-1} \qquad (2.51)$$

An analogous switching sign appears when the grade of **B** is fixed but the grade of **C** varies. That is, for arbitrary grade of **V** and **B**:

$$\mathcal{M}_{\mathbf{C}}(\mathbf{V}) = (-1)^{grade(\mathbf{C})+grade(\mathbf{V})}\mathbf{C}\mathbf{V}\mathbf{C}^{-1} \tag{2.52}$$

As previously mentioned, in geometric algebra the rotations are specified by their planes not by their axes, as we were accustomed from the computer graphics classes. The canonical form of the other orthonormal transformation, the rotation, is deduced starting from the following

Theorem: The rotation of a vector **v** relatively to a plane (π) with an angle θ can be accomplished finding a factorization for the bivector of (π): **B=mn**, where **m**, **n** are unit vectors forming an angle equal with $\frac{\theta}{2}$. The rotation is then equivalent with two successive reflections relative to the planes normal to **m** and respectively **n**.

Proof: The transformation resulting through the composition of two reflections is orthonormal. The orthonormality conditions could be successively verified.

$$\begin{aligned}\mathcal{R}(\mathbf{v}) &= \mathcal{M}_{\mathbf{m}}(\mathcal{M}_{\mathbf{n}}(\mathbf{v})) = -\mathbf{m}(-\mathbf{nvn})\mathbf{m} \\ &= \mathbf{mnvnm} = (\mathbf{mn})\mathbf{v}(\mathbf{nm}) = (\mathbf{mn})\mathbf{v}(\mathbf{mn})^{\sim}\end{aligned} \tag{2.53}$$

Noting **v' = -nvn** and **v" = -mv'm** we easily observe that $\mathcal{R}(\mathbf{v})$ is a vector.

$\mathcal{R}(\mathbf{v})$ is a linear transformation: $\mathcal{R}(\alpha\mathbf{v_1}+\beta\mathbf{v_2}) = \mathbf{mn}(\alpha\mathbf{v_1}+\beta\mathbf{v_2})\mathbf{nm} = \alpha\mathbf{mnv_1nm}+\beta\mathbf{mnv_2nm}$ $= \alpha\mathcal{R}(\mathbf{v_1})+\beta\mathcal{R}(\mathbf{v_2})$

The transformation preserves the inner product (of vectors):

$\mathcal{R}(\mathbf{v_1})\bullet\mathcal{R}(\mathbf{v_2}) = (\mathbf{mnv_1nm}) \bullet (\mathbf{mnv_2nm}) = \frac{1}{2}$ $(\mathbf{mnv_1nmmnv_2nm}+ \mathbf{mnv_2nmmnv_1nm})$ $= \frac{1}{2}$ $(\mathbf{mnv_1v_2nm} + \mathbf{mnv_2v_1nm}) = \mathbf{mn}(\mathbf{v_1} \bullet \mathbf{v_2})\mathbf{nm} = (\mathbf{v_1} \bullet \mathbf{v_2})\mathbf{mnnm}=\mathbf{v_1} \bullet \mathbf{v_2}$

We continue the proof showing that det($\mathcal{R}$)=1 that is the transformation resulted through the composition of two reflections is a rotation. The determinant value[1] (in G_3) is given by:

$$det(\mathcal{R}) = \mathcal{R}(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z})/(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) \tag{2.54}$$

Starting from $\mathcal{R}(\mathbf{xyz}) = \mathbf{mnxyznm} = \mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{y})\mathcal{R}(\mathbf{z})$ and selecting the trivector part of the last identity, results: $\langle\mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{y})\mathcal{R}(\mathbf{z})\rangle_3$ ($= \mathcal{R}(\mathbf{x})\wedge\mathcal{R}(\mathbf{y})\wedge\mathcal{R}(\mathbf{z}) = \mathcal{R}(\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z})$) $= \langle\mathbf{mnxyznm}\rangle_3$ $= \mathbf{mnx} \wedge \mathbf{y} \wedge \mathbf{znm} = \mathbf{mnI_3nm} = \mathbf{mnnmI_3} = \mathbf{I_3} = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} \Rightarrow \mathcal{R}(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} \Rightarrow$ det($\mathcal{R}$)=1

The fact (used above) that the transformation $\mathcal{R}$ is an outermorphism results from $\mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{y})$ $= \mathbf{mnxnmmnynm} = \mathbf{mnxynm}$ after selecting the terms of grade two: $\mathcal{R}(\mathbf{x}) \wedge \mathcal{R}(\mathbf{y}) = \mathbf{mnx} \wedge \mathbf{ynm} = \mathcal{R}(\mathbf{x} \wedge \mathbf{y})$

In conclusion the transformation $\mathcal{R}(\mathbf{x})$ is indeed a rotation. It finally remains to compute the rotation angle. Decomposing **x** in its components parallel and respectively perpendicular to the rotation plane (**m**$\wedge$**n**) i.e. $\mathbf{x}=\mathbf{x}_{\|} + \mathbf{x}_{\perp}$ and remembering that $\mathbf{x}_{\|} \wedge \mathbf{I_{mn}}=0 \Rightarrow \mathbf{xI_{mn}} = \mathbf{x}\bullet\mathbf{I_{mn}}= -\mathbf{I_{mn}} \bullet \mathbf{x}= -\mathbf{I_{mn}x}$ and $\mathbf{x}_{\perp} \bullet \mathbf{I_{mn}}=0 \Rightarrow \mathbf{xI_{mn}}=\mathbf{x} \wedge \mathbf{I_{mn}}=\mathbf{I_{mn}} \wedge \mathbf{x} =\mathbf{I_{mn}x}$ we could easily deduce that:

$$\begin{aligned}\mathbf{v}" &= (cos(\theta/2) - \mathbf{I_{mn}}sin(\theta/2)\mathbf{v})(\mathbf{x}_{\|} + \mathbf{x}_{\perp})(cos(\theta/2) + \mathbf{I_{mn}}sin(\theta/2)) \\ &= \mathbf{x_{perp}} + \mathbf{x}_{\|}(cos(\theta) + \mathbf{I_{mn}}sin(\theta))\end{aligned}$$

This proofs that the rotation is done in **m**$\wedge$**n** plane (the orthogonal component is not changed) and the rotation angle is double of the angle between **m** and **n**. The product **mn** is a unitary spinor and could be written:

$$\mathcal{S}_{\mathbf{mn}} = \mathbf{mn} = e^{\mathbf{I_{mn}}\theta/2}$$

[1] Any linear transformation f:$E^3 \rightarrow E^3$ induces a linear transformation $\underline{\mathrm{f}}$:$G_3 \rightarrow G_3$, in the geometric algebra G_3 associated to E^3. The defining property of the natural induced transformation is: $\underline{\mathrm{f}}(\mathbf{x}\wedge\mathbf{y})= \underline{\mathrm{f}}(\mathbf{x})\wedge\underline{\mathrm{f}}(\mathbf{y})$. Using the linearity property of $\underline{\mathrm{f}}$, one could easily proof that: $\underline{\mathrm{f}}$ preserves the grade of its arguments and the outer product of multivectors. One such transformation is called *outermorphism.* The determinant of the linear transformation is defined through: $\underline{\mathrm{f}}(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) = \underline{\mathrm{f}}(\mathbf{x})\wedge\underline{\mathrm{f}}(\mathbf{y})\wedge\underline{\mathrm{f}}(\mathbf{z})= \alpha\mathbf{I_3}=\det(\underline{\mathrm{f}})\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$.

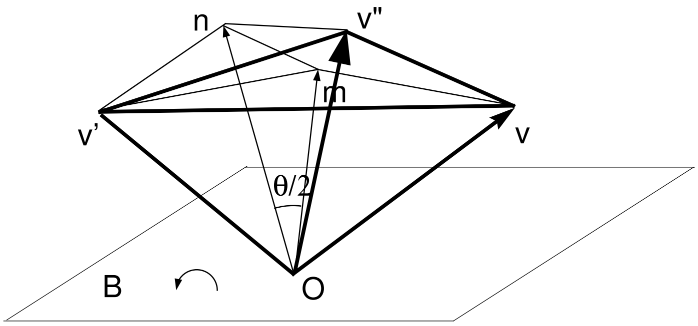

Figure 2.6: Rotation as composition of two reflections

The unit bivector associated to the rotation plane ($\mathbf{m} \wedge \mathbf{n}$) is $\mathbf{I_{mn}} = \frac{\mathbf{m}\wedge\mathbf{n}}{sin(\theta/2)}$ because $\mid \mathbf{m} \wedge \mathbf{n} \mid^2$ = $(\mathbf{m} \wedge \mathbf{n})^{\sim}.(\mathbf{m} \wedge \mathbf{n})$=$(\mathbf{nm}$- $\mathbf{n}\bullet\mathbf{m}).(\mathbf{m}\wedge\mathbf{n})$ = $\langle \mathbf{nm}(\mathbf{m}\wedge\mathbf{n})\rangle_0$ = $\langle \mathbf{n}(\mathbf{m}\bullet(\mathbf{m}\wedge\mathbf{n})+\mathbf{m}\wedge\ (\mathbf{m}\wedge\mathbf{n}))\rangle_0$ = $\langle \mathbf{n}\bullet(\mathbf{m}\bullet(\mathbf{m}\wedge\mathbf{n}))\rangle_0$ = $\langle \mathbf{n}\bullet((\mathbf{m}\bullet\mathbf{m})\mathbf{n}$- $(\mathbf{m}\bullet\mathbf{n})\mathbf{m})\rangle_0$ = $(\mathbf{m}\bullet\mathbf{m})(\mathbf{n}\bullet\mathbf{n})$-$(\mathbf{m}\bullet\mathbf{n})^2$ = 1-$cos^2(\theta/2)$ = $sin^2(\theta/2)$

The theorem above gives the canonical form of the rotation transformation:

$$\mathcal{R}(\mathbf{V}) = e^{\mathbf{I}\theta/2}\mathbf{V}e^{-\mathbf{I}\theta/2} \tag{2.55}$$

This formula remains valid for any multivector $\mathbf{V}$ (because rotation is linear and outermorphism). If $\mathcal{S}$ is a unitary spinor then $\mathcal{S}^{\sim} = \mathcal{S}^{-1}$ and the canonical form of a rotation transformation could equally be specified as: $\mathcal{R}(\mathbf{x}) = \mathcal{S}^{\sim}\mathbf{x}\mathcal{S} = \mathcal{S}_1\mathbf{x}\mathcal{S}_1^{\sim}$, where $\mathcal{S}_1 = \mathcal{S}^{-1}$.

Note: The fact that, in an Euclidean space E^n, any orthonormal transformation could be expressed as a composition of at most n reflections (relative to different hyperplanes of the space) is known as the Cartan-Dieudonné theorem.

Note: Due to the fact that spinors are sums of even blades the product between a spinor and a vector is always commutative.

Chapter 3

Other models of the Euclidean space

The present chapter introduces the main characteristics of other two geometric algebra based models of the Euclidean space:

- the *homogeneous model* (denoted HM or 4DGA) that corresponds to the coordinate free geometric algebra treatment of the homogeneous space (the space well known through its homogeneous coordinates from the computer graphics course)
- the *conformal model* (denoted CM or 5DGA) where every conformal transformation (translation, rotation, inversion) finds a spinor based representation, fact that simplifies considerably the algebraic treatment of the operational aspects of the geometry.

The description of every model will detail (as in the previous chapter) the representational aspect as well as the transformational aspect. The method used to derive these new models starting from the classical geometric algebra is the projective split, it is described in ***[Hest 91b]***, ***[Hest 99]***, ***[Li99]***. In the present report, further detailing of the above-mentioned models requires supplementary explanations of some related mathematical notions.

3.1 Preliminary notions

Definition: The *direct sum* of two vector spaces $\mathbf{A}$ and $\mathbf{B}$ over the field R is the vector space $\mathbf{A}\oplus\mathbf{B}$ with elements $\{\mathbf{a}\oplus\mathbf{b} \mid \mathbf{a}\in\mathbf{A}, \mathbf{b}\in\mathbf{B}\}$. Every operation specific to a vector space (i.e. addition and scalar multiplication) between two elements of $\mathbf{A}\oplus\mathbf{B}$ (let them be respectively $\mathbf{a_1} \oplus \mathbf{b_1}$ and $\mathbf{a_2} \oplus \mathbf{b_2}$) is defined separately for each term of the sum:

$(\mathbf{a_1} \oplus \mathbf{b_1})+(\mathbf{a_2} \oplus \mathbf{b_2})=(\mathbf{a_1} + \mathbf{a_2}) \oplus(\mathbf{b_1} + \mathbf{b_2})$, $\forall \mathbf{a_1}, \mathbf{a_2} \in \mathbf{A}, \mathbf{b_1}, \mathbf{b_2} \in \mathbf{B}$

$\lambda(\mathbf{a} \oplus \mathbf{b}) = \lambda\mathbf{a} \oplus \lambda\mathbf{b}$, $\forall\lambda \in R$, $\mathbf{a} \in \mathbf{A}$, $\mathbf{b} \in \mathbf{B}$.

If $\{\mathbf{ea_1}, \mathbf{ea_2}, ..., \mathbf{ea_n}\}$ and $\{\mathbf{eb_1}, \mathbf{eb_2}, ..., \mathbf{eb_m}\}$ are bases of $\mathbf{A}$ respectively $\mathbf{B}$ then, following the above definition, results trivially that the space $\mathbf{A} \oplus \mathbf{B}$ admits as basis $\{\mathbf{ea_1}, ..., \mathbf{ea_n}, \mathbf{eb_1}, ..., \mathbf{eb_m}\}$. From the geometric algebra point of view, the direct sum of subspaces corresponds to a blade factorization. Writing $\mathbf{A}=\mathbf{B}\wedge\mathbf{C}$ is equivalent to $\mathbf{A}=\mathbf{B}\oplus\mathbf{C}$. In section 2.1, treating about the axiomatic definition of geometric algebra, it was briefly mentioned the possibility that the vector space V^n, lying under the algebra, have basis vectors with negative or null signature. A *null vector* (i.e. vector with null signature) is an element of V^n whose square is 0. The orthogonal transformations over a vector space form a group. If the vector space is a *Minkowski space* ($R^{n,1}$) i.e. its basis contains one vector of negative signature, the corresponding group of orthogonal transformations is called the *Lorentz group* of transformations, associated to the Minkowski space. A geometric algebra whose subjacent vector space has a basis with null and non-null vectors is called a *degenerated* geometric algebra. In case of a non-degenerate geometric algebra, the value of the corresponding squared pseudoscalar has a sign depending on the number of vectors with negative signature from the basis of the associated vector space.

Proposition: The Minkowski space $R^{1,1}$ admits two types of bases the first formed from two non-null vectors $\mathbf{e}_+$, $\mathbf{e}_-$ and the second formed from null vectors which will be denoted as $\mathbf{e_0}$, $\mathbf{e}_\infty$. The properties of the above-mentioned basis vectors are:

$\mathbf{e}_+{}^2$ = 1, $\mathbf{e}_-{}^2$ = -1 (the Minkowski characteristic of the vector space)

$\mathbf{e}_+ \bullet \mathbf{e}_-$=0 (the non-null basis is orthonormal)

$\mathbf{e_0}{}^2$=$\mathbf{e}_\infty{}^2$=0,

$\mathbf{e_0} \bullet \mathbf{e}_\infty$= -1 (the normalization condition for the null basis)

Proof: The properties of the non-null basis are directly derived from the definition of the Minkowski space. Concerning the null-basis we could chose:

$$\mathbf{e_0} = \frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+) \tag{3.1}$$

$$\mathbf{e}_\infty = \mathbf{e}_- + \mathbf{e}_+ \tag{3.2}$$

The conditions specified in the proposition above are then immediately verified:

$$\begin{aligned}
\mathbf{e_0}{}^2 &= \frac{1}{4}(\mathbf{e}_+{}^2 + \mathbf{e}_-{}^2 - \mathbf{e}_+ \bullet \mathbf{e}_- - \mathbf{e}_- \bullet \mathbf{e}_+) = 0 \\
\mathbf{e}_\infty{}^2 &= \mathbf{e}_-{}^2 + \mathbf{e}_+{}^2 + 2\mathbf{e}_- \bullet \mathbf{e}_+ = 0 \\
\mathbf{e_0} \bullet \mathbf{e}_\infty &= \frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+) \bullet (\mathbf{e}_- + \mathbf{e}_+) = \frac{1}{2}(\mathbf{e}_-{}^2 - \mathbf{e}_+{}^2) = -1
\end{aligned}$$

Any vector $\mathbf{v}\in R^{1,1}$, $\mathbf{v}=\alpha\mathbf{e}_+ + \beta\mathbf{e}_-$ could be written as a linear combination of the null-basis elements, $\mathbf{v}=\lambda\mathbf{e_0}+\mu\mathbf{e}_\infty$, where $\lambda = \beta - \alpha$, $\mu = (\alpha + \beta)/2$. The verification is straightforward.

Therefore, in $R^{1,1}$ Minkowski space any vector could be expressed as linear combination of the null basis elements.

The classical geometric algebra based model of the Euclidean space, model that is mainly based on the geometric semantic of algebraic operators specified in the first two chapters of the present report has a main shortcoming, the separate treatment required by the origin comparatively to the others points of the space. The vector labeling the origin is not invertible. This deficiency was resolved by the introduction of the homogeneous coordinates. The procedure is detailed also in Appendix 3 and consists in immersing the original Euclidean space E^n as a hyperplane of a n+1 dimensional space called the homogeneous space. The visual representation (see Figure A3.1) is most clear for n=2. Practically the points from E^n are projections of points from the homogeneous space.

Be P an arbitrary point from E^n. Denoting by $\mathbf{e_0}$ the homogeneous (i.e. R^{n+1}) vector pointing to the origin of E^n, $\mathbf{p}$ the homogeneous vector pointing to P and $\vec{p}$ the vector specifying the position of P relatively to the origin of the Euclidean space E^n, we could state that we represent directly P through the vector $\mathbf{p}$ where

$$\mathbf{p} = \mathbf{e_0} + \vec{p} \tag{3.3}$$

This helps us to recover the position of P in E^n as:

$$\vec{p} = \mathbf{p} - \mathbf{e_0} = P_{E^n}(\mathbf{p}) = (\mathbf{p} \bullet \mathbf{I_n}) \bullet \mathbf{I_n}{}^{-1} \tag{3.4}$$

as the projection of $\mathbf{p}$ on E^n.

The same type of reasoning stays behind the projective split method.

Definition: A *projective space* of R^{n+1} is the space of all one dimensional vector subspaces from R^{n+1} (i.e. of all lines passing through the origin of R^{n+1}). It is usually denoted by P^n.

The *projective split* is a method (invented by David Hestenes ***[Hest 91b]***) that consists of establishing a linear mapping between the vectors of a higher dimensionality space V^m and a part of the multivectors of a lower dimensionality geometric algebra G_n $(m > n)$. In the most simple case m=n+1; that corresponds to a classical projective split relative to a non-null vector and this generates the homogeneous model of E^n (i.e. V^n). In other important case m=n+2 the original space is a Minkowski space ($R^{n+1,1}$) and the split is done relatively to a bivector (the one formed through the outer-multiplication of the null basis vectors) that is called the *conformal split*. The original Minkowski space $R^{n+1,1}$ is considered as the direct sum $R^n \oplus R^{1,1}$.

3.2 The homogeneous model

Let's start with the presentation of the classical projective split. Suppose that the Euclidean space to be represented is E^n (usually n=3). This type of projective split establishes a mapping between V^{n+1} (R^{n+1}) and the vectors and scalars from G_n as follows:

- An arbitrary non-null vector (denoted $\mathbf{e_0}$) from $V^{n+1} = G^1_{n+1}$ is chosen; without restraining generality we will further consider that $\mathbf{e_0}$ is a unit vector ($\mid \mathbf{e_0} \mid$=1).
- All the points (labeled by vectors $\mathbf{x} \in R^{n+1}$) from a ray passing through the homogeneous origin (1-blade of G_{n+1}) could be considered forming an equivalence class whose representative element is labeled by the homogeneous (and normalized) vector: $\frac{\mathbf{x}}{\mathbf{x} \bullet \mathbf{e_0}}$.
- With every vector $\mathbf{x} \in V^{n+1}$ we associate the rejection with respect to $\mathbf{e_0}$ of the normalized $\mathbf{x}$. The mapping $V^{n+1} \to E^n$ is defined by:

$$\vec{x} = \frac{\mathbf{x} \wedge \mathbf{e_0}}{\mathbf{x} \bullet \mathbf{e_0}} \mathbf{e_0}^{-1}$$

The points from the above-mentioned ray are all equivalent homogeneous representations of the Euclidean point labeled by $\vec{x}$, where:

$$\vec{x} = \frac{\mathbf{x} \wedge \mathbf{e_0}}{\mathbf{x} \bullet \mathbf{e_0}} \mathbf{e_0}^{-1} = \frac{\mathbf{x} \wedge \mathbf{e_0}}{\mathbf{x} \bullet \mathbf{e_0}} \mathbf{e_0} \tag{3.5}$$

Practically, the rejections of all vectors $\mathbf{x} \in V^{n+1}$ with respect to $\mathbf{e_0}$ are forming themselves a space dual to $\mathbf{e_0}$. Therefore the homogeneous representation of an Euclidian space E^n is obtained by embedding it in an n+1 homogeneous space, fixing the origin of E^n with the unit vector $\mathbf{e_0}$ and considering E^n as a hyperplane normal to $\mathbf{e_0}$. This hyperplane is the translation of vector $\mathbf{e_0}$ of the blade dual to $\mathbf{e_0}$.

From the representational point of view, the homogeneous model associates to a point $\vec{x}$ from the Euclidean space the point $\mathbf{x}=\vec{x} + \mathbf{e_0}$ from the homogeneous space and inversely, every point $\mathbf{x}$ from the homogeneous space has the corresponding point labeled $\vec{x} = \frac{\mathbf{x} \wedge \mathbf{e_0}}{\mathbf{x} \bullet \mathbf{e_0}}$ in the Euclidean space.

The homogeneous space construction method presented above considers E^n isomorphic with the projective space P^n of R^{n+1} because, as it was already explained, every ray passing through the origin of the homogeneous space has associated one and only one point in E^n.

Note: Take for example the space E^2. The 2D lines, have the implicit description ax+by+c=0. The coefficients a, b, c could be considered the homogeneous coordinates of a 2D line. The homogeneous space is in this case the 3D space. If the 2D lines are passing through a fixed point (x_0, y_0) their implicit specification would be: a(x-x_0)+b(y-y_0)=0. Consequently we could establish an isomorphism between a two-dimensional vector space and the pencil of lines passing through (x_0, y_0) that themselves form a space with coordinates a, b. Each line of the pencil is identified

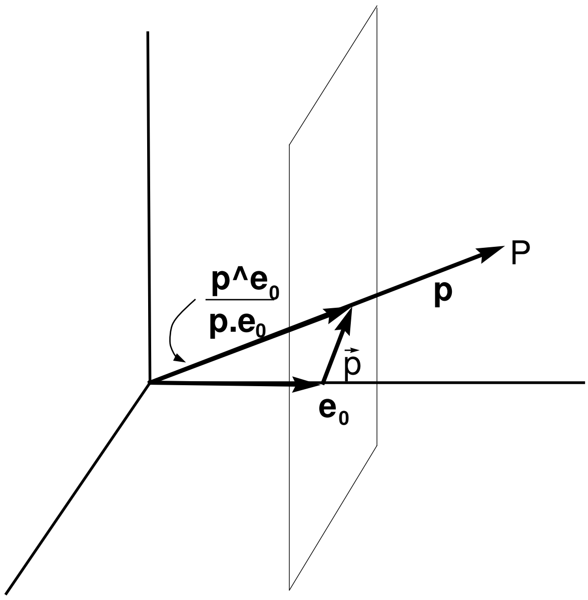

Figure 3.1: The spatial mapping characteristic to the homogeneous model

by its coefficients a, b. The propositions above specify, in a coordinate based language, the fact that the projective space is in fact a pencil of lines passing through a fixed point. These lines are projections of rays passing through the homogeneous origin.

The study of the projective split consists not only in establishing the linear mapping $V^{n+1} \rightarrow V^n$ that state in fact the representation of Euclidean entities in the homogeneous space but also in specifying its effects on the geometric transformations. From the transformational point of view, the main results mentioned in ***[Hest 91b]*** are:

Theorem: Every orthogonal transformation in vector space $V^{p,q}$ can always be written in a canonical form:

$$\mathbf{Sx} = \mathbf{Sx}\hat{\mathbf{S}}^{-1}$$

where $\mathbf{S}$ is an even versor (spinor) of the associated geometric algebra $G_{p,q}$.

Section 2.8 suggests a way to prove this theorem.

The projective split (characteristic to the homogeneous model) relative to the unit vector $\mathbf{e_0}$ relates the group of spinors S(p,q) (called also the spin group of $V^{p,q}$) to S(p+1,q) if $\mathbf{e_0}^2$=1 or to S(p,q+1) if $\mathbf{e_0}^2$= -1.

Theorem: Considering the projective split relative to the unit vector $\mathbf{e_0}$, every spinor $\mathbf{S}_{n+1} \in G^+_{n+1}$ admits a factorization:

$$\mathbf{S}_{n+1} = \mathbf{S}_0\mathbf{S}_n \tag{3.6}$$

where $\mathbf{S}_n$ is a spinor from G^+_n and $\mathbf{S}_0$=$\mathbf{ue}_0$ with $\mathbf{u} \in V^{n+1}$.

Proof:

The relation 3.6 expresses the effect of the projective split over the spinors of G^+_{n+1}. The transformation described by $\mathbf{S}_n$ keeps $\mathbf{e_0}$ unchanged. Therefore if the global effect of $\mathbf{S_{n+1}}$ over $\mathbf{e_0}$ is known, it must be the same as the effect of $\mathbf{S}_0$ over $\mathbf{e_0}$. Let $\mathbf{v}$ be the vector resulted from applying the orthogonal transformation to $\mathbf{e_0}$ i.e. $\mathbf{v}$=$\mathbf{S}_{n+1}\mathbf{e_0}\mathbf{S}^{-1}_{n+1}$. We could immediately state that: $\mathbf{v}$=$\mathbf{S}_0\mathbf{S}_n\mathbf{e_0}\mathbf{S}_n^{-1}\mathbf{S}_0^{-1}$ =$\mathbf{S}_0\mathbf{e_0}\mathbf{S}_0^{-1}$ =$\mathbf{ue_0e_0e_0^{-1}u^{-1}}$ =$\mathbf{ue_0^{-1}u^{-1}}$ =$\mathbf{ue_0u}^{-1}$ (remember $\mathbf{e_0}$ is a unitary vector) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, the simplest versor that transforms $\mathbf{u}$ onto $\mathbf{v}$ is: $\mathbf{R}$=$(\mathbf{u}$+$\mathbf{v})\mathbf{u}$. The same effect is obtained by the unitary versor $\mathbf{R}$=$(\mathbf{u}$+$\mathbf{v})\mathbf{u}/|\ \mathbf{u}+\mathbf{v}\ |$. The affirmation above follows from: $\mathbf{RuR}^{-1} = (\mathbf{u}+\mathbf{v})\mathbf{uu}((\mathbf{u}+\mathbf{v})\mathbf{u})^{-1} = (\mathbf{u}+\mathbf{v})\mathbf{uu}(\mathbf{v}(\mathbf{u}+\mathbf{v}))^{-1}$

$= (\mathbf{u}+\mathbf{v})\mathbf{u}\mathbf{u}(\mathbf{u}+\mathbf{v})^{-1}\mathbf{v}^{-1} = \mathbf{v}^{-1} = \mathbf{v}$. Consequently, using for the spinor $\mathbf{S}_0$ the value $(\mathbf{e_0}+\mathbf{v})\mathbf{e_0}$ or $\mathbf{v}(\mathbf{e_0}+\mathbf{v})$ implies that $\mathbf{S}_n$ preserves invariant the vector $\mathbf{e_0}$. Indeed: $\mathbf{S}_n\mathbf{e_0}\mathbf{S}_n^{-1} = (\mathbf{S}_0^{-1}\mathbf{S}_{n+1})\mathbf{e_0}(\mathbf{S}_0^{-1}\mathbf{S}_{n+1})^{-1} = \mathbf{S}_0^{-1}\mathbf{S}_{n+1}\mathbf{e_0}\mathbf{S}_{n+1}^{-1}\mathbf{S}_0 = \mathbf{S}_0^{-1}\mathbf{v}\mathbf{S}_0 = \mathbf{e_0}^{-1}(\mathbf{e_0}+\mathbf{v})^{-1}\mathbf{v}\mathbf{v}(\mathbf{e_0}+\mathbf{v}) = \mathbf{e_0}^{-1} = \mathbf{e_0}$. In conclusion $\mathbf{S}_0 = \mathbf{u}\mathbf{e_0}$ where $\mathbf{u}=\mathbf{e_0}+\mathbf{v}$ or remembering that the transformational effect of a spinor is independent of the scale of its elementary factors, we could equally state that:

$$\mathbf{S}_0 = \mathbf{u}\mathbf{e_0} = \frac{(\mathbf{e_0}+\mathbf{v})\mathbf{e_0}}{\mid \mathbf{e_0}+\mathbf{v} \mid} \tag{3.7}$$

i.e.

$$\mathbf{u} = \frac{\mathbf{e_0}+\mathbf{v}}{\mid \mathbf{e_0}+\mathbf{v} \mid} \tag{3.8}$$

Theorem: An Euclidean translation of vector $\mathbf{t}$ is equivalent to a *shear transformation*[1] in the homogeneous space, given by:

$$\mathbf{T}_t(\mathbf{x}) = \mathbf{x} + \mathbf{e_0} \bullet (\mathbf{x} \wedge \mathbf{t}) \tag{3.9}$$

the homogeneous vector $\mathbf{t}$ is supposed to satisfy the condition $\mathbf{t} \bullet \mathbf{e_0}=0$ that is it specifies a direction from the Euclidean space.

Proof: The condition satisfied by $\mathbf{t}$ implies: $\mathbf{e_0} \bullet (\mathbf{x} \wedge \mathbf{t}) = (\mathbf{e_0} \bullet \mathbf{x})\mathbf{t}\text{-}(\mathbf{e_0} \bullet \mathbf{t})\mathbf{x} = (\mathbf{x} \bullet \mathbf{e_0})\mathbf{t}$. Noting $\mathbf{x'}=\mathbf{T}_t(\mathbf{x})$, the Euclidean point associated to $\mathbf{x'}$ is:

$$\begin{aligned}
\vec{x}' &= \frac{\mathbf{x}' \wedge \mathbf{e_0}}{\mathbf{x}' \bullet \mathbf{e_0}}\mathbf{e_0} = \frac{(\mathbf{x} + \mathbf{e_0} \bullet (\mathbf{x} \wedge \mathbf{t})) \wedge \mathbf{e_0}}{\mathbf{x} \bullet \mathbf{e_0}}\mathbf{e_0} = \\
&= \frac{\mathbf{x} \wedge \mathbf{e_0}}{\mathbf{x} \bullet \mathbf{e_0}}\mathbf{e_0} + \frac{(\mathbf{e_0} \bullet \mathbf{x})(\mathbf{t} \wedge \mathbf{e_0})}{\mathbf{x} \bullet \mathbf{e_0}}\mathbf{e_0} - \frac{(\mathbf{e_0} \bullet \mathbf{t})(\mathbf{x} \wedge \mathbf{e_0})}{\mathbf{x} \bullet \mathbf{e_0}}\mathbf{e_0} \\
&= \vec{x} - \mathbf{e_0} \bullet (\mathbf{t} \wedge \mathbf{e_0}) + \frac{\mathbf{e_0} \bullet \mathbf{t}}{\mathbf{x} \bullet \mathbf{e_0}}\mathbf{e_0} \bullet (\mathbf{x} \wedge \mathbf{e_0}) \\
&= \vec{x} - (\mathbf{e_0} \bullet \mathbf{t})\mathbf{e_0} + \mathbf{t} = \vec{x} + \mathbf{t} = \vec{x} + \vec{t}
\end{aligned}$$

3.3 The conformal model

The projective split applied to $R^{n+1,1}$ that is seen as a direct sum $R^n \oplus R^{1,1}$ constitutes the foundation of the conformal model. In this model a vector from the Euclidean space E^n is regarded as the rejection, relative to the null plane (of bivector $\mathbf{E}=\mathbf{e_0} \wedge \mathbf{e}_\infty$) of the corresponding vector from the Minkowski space [2].

The main properties of the null bivector are:

$$\mathbf{E}^2 = (\mathbf{e_0} \wedge \mathbf{e}_\infty)(\mathbf{e_0} \wedge \mathbf{e}_\infty) = (\frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+)(\mathbf{e}_- + \mathbf{e}_+))^2 = (-1)^2 = 1 \tag{3.10}$$

$$\mathbf{E}^\sim = \mathbf{e}_\infty \wedge \mathbf{e_0} = -\mathbf{e_0} \wedge \mathbf{e}_\infty = -\mathbf{E} \tag{3.11}$$

$$\mathbf{E}^{-1} = \frac{\mathbf{E}}{\mid \mathbf{E} \mid^2} = \mathbf{E} \tag{3.12}$$

[1] A shear transformation specifies the translation of each hyperplane normal to a specified shearing axis, with an amount proportional with the distance of the plane to the origin (i.e. the plane directance). In the case of the theorem above, the shearing axis has direction $\mathbf{e_0}$. When $\mathbf{x}$ is normalized i.e. $\mathbf{x} \bullet \mathbf{e_0}=1$ the shearing becomes a translation. In a coordinate based language the shear relative to Oz axis is described by: x'=x+az; y'=y+bz; z'=z; where (a,b,0) is the shear vector.

[2] The description of the conformal model from this section is based on a presentation done by Leo Dorst at the University of Amsterdam, the 4-th July 2002 and on ***[Hest 91b]***

$$\mathbf{e}_\infty \mathbf{E} = \mathbf{e}_\infty \bullet \mathbf{E} = \mathbf{e}_\infty \bullet (\mathbf{e_0} \wedge \mathbf{e}_\infty) = -\mathbf{e}_\infty \tag{3.13}$$

$$\mathbf{e_0 E} = \mathbf{e_0} \bullet \mathbf{E} = \mathbf{e_0} \bullet (\mathbf{e_0} \wedge \mathbf{e}_\infty) = \mathbf{e_0} \tag{3.14}$$

$$\mathbf{e_0} \bullet \mathbf{e}_\infty = -1 \tag{3.15}$$

Straight from the definition of the direct sum of subspaces one could state that any vector $\mathbf{x}$ of the conformal space $R^{n+1,1}$ could be written as:

$$\mathbf{x} = \vec{x} + \alpha \mathbf{e_0} + \beta \mathbf{e}_\infty \tag{3.16}$$

The Euclidean vector $\vec{x}$ is the rejection of $\mathbf{x}$ by $\mathbf{E}$:

$$\vec{x} = (\mathbf{x} \wedge \mathbf{E})\mathbf{E}^{-1} = (\mathbf{x} \wedge \mathbf{E}) \bullet \mathbf{E} = -(\mathbf{x} \wedge \mathbf{E}) \bullet \mathbf{E}^{\sim} = -(\mathbf{x} \wedge \mathbf{E})\mathbf{E}^{\sim} \tag{3.17}$$

Substituting $\mathbf{x}$ with $\mathbf{e_0}$ in 3.17 results: $\vec{x} = (\mathbf{e_0} \wedge (\mathbf{e_0} \wedge \mathbf{e}_\infty)) \bullet \mathbf{E}$=0 i.e. $\mathbf{e_0}$ represents the Euclidean origin. The fact that

$$\lim_{-(\mathbf{x} \bullet \mathbf{e_0}) \to \infty} \frac{\mathbf{x}}{-(\mathbf{x} \bullet \mathbf{e_0})} = \mathbf{e}_\infty$$

means that $\mathbf{e}_\infty$ labels the point at infinity. The computation of the previous limit is based on 3.26.

As it is clearly revealed by Hestenes (***[Hest 91b]***) the null basis decomposition will be used especially to specify representational aspects of the conformal split while the non-null basis will be mainly used to specify orthogonal transformations in the conformal model. Since $\mathbf{E}$ is a bivector from 3.17 results $\vec{x} \in R^n$.

The coefficients α and β that make up the conformal representation of $\vec{x}$ are determined by enforcing the conditions:

$$\mathbf{x}^2 = 0 \tag{3.18}$$

$$\mathbf{x} \bullet \mathbf{e}_\infty = -1 \tag{3.19}$$

The condition 3.18 implies that the Euclidean points are represented by null conformal vectors. The set of all null vectors of the conformal space is named the *null cone* of the space. The condition 3.19 could be written $\mathbf{x} \bullet \mathbf{e}_\infty = \mathbf{e_0} \bullet \mathbf{e}_\infty \Leftrightarrow \mathbf{x} \bullet (\mathbf{e_0} - \mathbf{e}_\infty) = 0$. The Euclidean space will consequently be placed at the intersection of the null cone with the plane $\mathbf{x} \bullet (\mathbf{e_0} - \mathbf{e}_\infty) = 0$; this surface is called a *horosphere*.

In order to determine the values for α and β the two conditions above are worked: $\mathbf{x}^2 = 0 \Leftrightarrow (\vec{x}+\alpha\mathbf{e_0}+\beta\mathbf{e}_\infty)(\vec{x}+\alpha\mathbf{e_0}+\beta\mathbf{e}_\infty) = 0 \Leftrightarrow \vec{x}^2 + \alpha(\vec{x}\mathbf{e_0}+\mathbf{e_0}\vec{x}) + \beta(\vec{x}\mathbf{e}_\infty+\mathbf{e}_\infty\vec{x}) + \alpha\beta(\mathbf{e_0}\mathbf{e}_\infty+\mathbf{e}_\infty\mathbf{e_0})=0$ $\Leftrightarrow$

$$\vec{x}^2 + 2\alpha\vec{x} \bullet \mathbf{e_0} + 2\beta\vec{x} \bullet \mathbf{e}_\infty - 2\alpha\beta = 0 \tag{3.20}$$

$\mathbf{x} \bullet \mathbf{e}_\infty$= -1 $\Leftrightarrow \vec{x} \bullet \mathbf{e}_\infty + \alpha\mathbf{e_0} \bullet \mathbf{e}_\infty$= -1 $\Leftrightarrow$

$$\vec{x} \bullet \mathbf{e}_\infty = \alpha - 1 \tag{3.21}$$

$\mathbf{x} \wedge \mathbf{E}$=$\mathbf{xE}$-$\mathbf{x} \bullet \mathbf{E} \Leftrightarrow (\mathbf{x} \wedge \mathbf{E})\ \mathbf{E} = \mathbf{xE}^2$-$(\mathbf{x} \bullet \mathbf{E})\mathbf{E} \Leftrightarrow \vec{x}$=$\mathbf{x}$-$(\mathbf{x} \bullet (\mathbf{e_0} \wedge \mathbf{e}_\infty))\mathbf{E}$ = $\mathbf{x}$ - $((\mathbf{x} \bullet \mathbf{e_0})\mathbf{e}_\infty$ - $(\mathbf{x} \bullet \mathbf{e}_\infty)\mathbf{e_0})\mathbf{E} \Leftrightarrow \vec{x}$=$\mathbf{x}$+$(\mathbf{x} \bullet \mathbf{e_0})\mathbf{e}_\infty$+ $(\mathbf{x} \bullet \mathbf{e}_\infty)\mathbf{e_0} \Leftrightarrow$

$$\mathbf{x} = \vec{x} - (\mathbf{x} \bullet \mathbf{e}_\infty)\mathbf{e_0} - (\mathbf{x} \bullet \mathbf{e_0})\mathbf{e}_\infty \tag{3.22}$$

Consequently:

$$\begin{aligned} \alpha &= -(\mathbf{x} \bullet \mathbf{e}_\infty) = 1 \\ \beta &= -(\mathbf{x} \bullet \mathbf{e_0}) \end{aligned} \tag{3.23}$$

But -$\beta = \mathbf{x} \bullet \mathbf{e_0}$=($\vec{x}$ + $\alpha\mathbf{e_0}$ + $\beta\mathbf{e}_\infty$)$\bullet\mathbf{e_0}$ = $\vec{x} \bullet \mathbf{e_0}$ - $\beta \Rightarrow$

$$\vec{x} \bullet \mathbf{e_0} = 0 \tag{3.24}$$

The substitution of $\vec{x} \bullet \mathbf{e_0}$ and $\vec{x} \bullet \mathbf{e}_\infty$ in 3.20 gives finally: $\vec{x}^2$ + $2\beta(\alpha - 1)$-$2\alpha\beta$=0 i.e.

$$\beta = \frac{\vec{x}^2}{2} \tag{3.25}$$

This completes the validity proof of 3.16 and gives the conformal representation of the Euclidean point as:

$$\mathbf{x} = \vec{x} + \mathbf{e_0} + \frac{\vec{x}^2}{2}\mathbf{e}_\infty \tag{3.26}$$

Conditions 3.21 and 3.24 directly imply:

$$\vec{x} \bullet \mathbf{e}_+ = \vec{x} \bullet \mathbf{e}_- = 0 \tag{3.27}$$

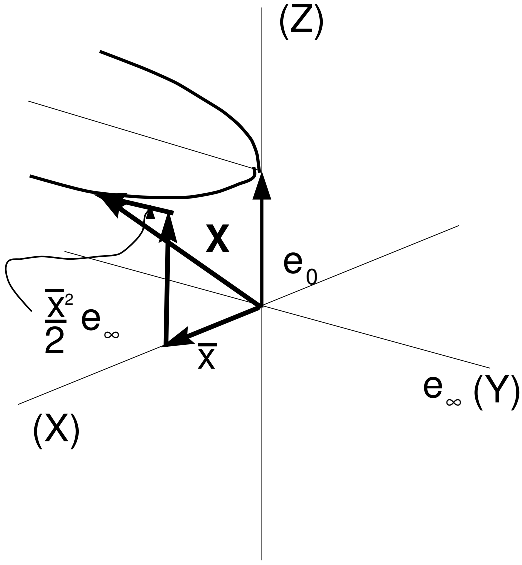

Figure 3.2: Visualization of the conformal representation of an 1D space

Note: The Figure 3.2 visualizes the representation of an Euclidean one-dimensional space in a 3D conformal space. The horosphere is in this case a parabola placed in the plane $\vec{x} \bullet \mathbf{e_0}$ i.e. a plane normal to $\mathbf{e_0}$. The Euclidean space is "the x axis" $\mathbf{e_0}$ is placed along "the z axis" and the infinity is "measured" along the y axis. Condition 3.19 shows that the vectors $\mathbf{x}$ and $\mathbf{e}_\infty$ could be considered as having "opposite senses"

Now, using 3.16, we could find a formula for the inner product in the geometric algebra associated to the conformal space.

$$\begin{aligned}\mathbf{x}\bullet\mathbf{y} &= (\vec{x}+\mathbf{e_0}+\frac{\vec{x}^2}{2}\mathbf{e}_\infty)\bullet(\vec{y}+\mathbf{e_0}+\frac{\vec{y}^2}{2}\mathbf{e}_\infty)\\ &= \vec{x}\bullet\vec{y}+(\vec{x}+\vec{y})\bullet\mathbf{e_0}+(\frac{\vec{x}^2}{2}+\frac{\vec{y}^2}{2})\mathbf{e_0}\bullet\mathbf{e}_\infty+\frac{\vec{x}^2}{2}(\vec{y}\bullet\mathbf{e}_\infty)+\frac{\vec{y}^2}{2}(\vec{x}\bullet\mathbf{e}_\infty)\\ &= -\frac{\vec{x}^2}{2}-\frac{\vec{y}^2}{2}+\vec{x}\bullet\vec{y}\end{aligned}$$

i.e.

$$\mathbf{x}\bullet\mathbf{y} = -\frac{1}{2}(\vec{x}-\vec{y})^2 \tag{3.28}$$

This last formula shows that the inner product in $G_{n+1,1}$ is related to the Euclidean distance between the points represented by $\mathbf{x}$ and $\mathbf{y}$. It can be verified that the relation 3.28 is independent of the chosen normalization condition (3.19 or $\mathbf{x}\bullet\mathbf{e}_\infty$=1). Finally, due to $\mathbf{x}^2$=$\mathbf{y}^2$=0 results: $\mathbf{x}\bullet\mathbf{y} = -\frac{\mathbf{x}^2}{2}-\frac{\mathbf{y}^2}{2}+\mathbf{x}\bullet\mathbf{y} = -\frac{1}{2}(\mathbf{x}-\mathbf{y})^2$ i.e. the conformal transformation is isometric.

Note: Different publications referring to the conformal model are characterizing the conformal mapping by different normalization conditions. For example condition 3.19 is substituted by: $\mathbf{x}\bullet\mathbf{e}_\infty$=1. This implies the modification of 3.26 as: $\mathbf{x} = \vec{x}+\mathbf{e_0}-\frac{\vec{x}^2}{2}\mathbf{e}_\infty$. Other papers use a modified definition of the null vector basis. Relations 3.1 and 3.2 are substituted by $\mathbf{e_0}$=$\mathbf{e}_+$+$\mathbf{e}_-$ and respectively $\mathbf{e}_\infty = \mathbf{e}_+ - \mathbf{e}_-$. These modifications lead to: $\mathbf{x}$=2$\vec{x}$ - $\mathbf{e_0}$ + $\vec{x}^2\mathbf{e}_\infty$. These kinds of changes do not affect the qualitative aspects of the conformal mapping theory.

We continue with the transformational aspects of the conformal split. As it is well known from the Chasles theorem (see Appendix 3) every Euclidean transformation can be considered resulted from the composition of a rotation around an axis (of arbitrary orientation) and a translation. The orthonormal transformations (expressed by the spinor canonical representation) preserve the null vectors; that is we could apply the transformations characterized by the conformal spinor $\mathbf{S}$ and see what effects they have over the corresponding Euclidean vector.

Indeed, let's note

$$\mathbf{y} = \mathbf{S}\mathbf{X}\hat{\mathbf{S}}^{-1} \tag{3.29}$$

the vector resulted by applying an orthonormal transformation to $\mathbf{x}$, then $\mathbf{y}^2 = \mathbf{S}\mathbf{X}\hat{\mathbf{S}}^{-1}\mathbf{S}\mathbf{x}\hat{\mathbf{S}}^{-1}$ $= \alpha\mathbf{S}\mathbf{x}\mathbf{x}\hat{\mathbf{S}}^{-1} = 0$.

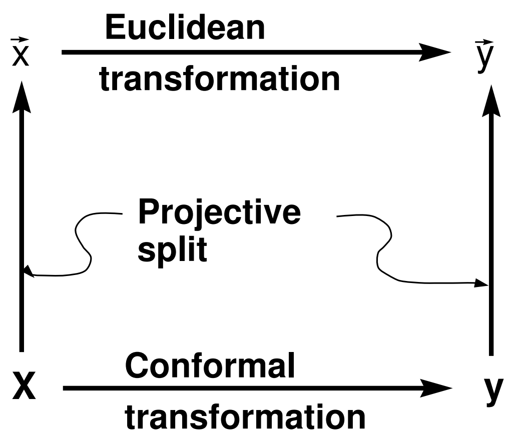

Figure 3.3: The effect of conformal transformations

The conformal transformation 3.29 has a certain effect over the Euclidean space i.e. $\vec{y} = f(\vec{x})$. This effect could be deduced by computing $\mathbf{S}(\vec{x}+\mathbf{e_0}+\frac{\vec{x}^2}{2}\mathbf{e}_\infty)\hat{\mathbf{S}}^{-1}$ and expressing the result as $\lambda(\vec{y} + \mathbf{e_0} + \frac{\vec{y}^2}{2}\mathbf{e}_\infty)$ where λ is a scalar factor that could be directly computed with: $\lambda = -\mathbf{e}_\infty \bullet \mathbf{y}$. This type of reasoning will be further applied to study some important geometric transformations:

Proposition: The Euclidean translation of vector $\vec{a}$ is expressed in the conformal space by the orthogonal transformation of spinor: $\mathbf{T}$=1-$\frac{1}{2}\vec{a}\mathbf{e}_\infty$

Proof: $\mathbf{TxT}^{-1} = (1\text{-}\frac{1}{2}\vec{a}\mathbf{e}_\infty)\mathbf{x}(1\text{+}\frac{1}{2}\vec{a}\mathbf{e}_\infty) = (1\text{-}\frac{1}{2}\vec{a}\mathbf{e}_\infty)\ (\vec{x}\text{+}\mathbf{e_0}\text{+ }\frac{\vec{x}^2}{2}\mathbf{e}_\infty)\ (1\text{+ }\frac{1}{2}\vec{a}\mathbf{e}_\infty) = \vec{x}\text{+}\mathbf{e_0}\text{+}\frac{\vec{x}^2}{2}\mathbf{e}_\infty$ - $\frac{1}{2}\vec{a}\mathbf{e}_\infty\vec{x}$ - $\frac{1}{2}\vec{a}\mathbf{e}_\infty\mathbf{e_0} + \frac{1}{2}\vec{x}\vec{a}\mathbf{e}_\infty + \frac{1}{2}\mathbf{e_0}\vec{a}\mathbf{e}_\infty + \frac{1}{2}\vec{x}^2\mathbf{e}_\infty\vec{a}\mathbf{e}_\infty$ - $\frac{1}{4}\vec{a}\mathbf{e}_\infty\vec{x}\vec{a}\mathbf{e}_\infty$ - $\frac{1}{4}\vec{a}\mathbf{e}_\infty\mathbf{e_0}\vec{a}\mathbf{e}_\infty$ - $\frac{1}{8}\vec{x}^2\vec{a}\mathbf{e}_\infty\mathbf{e}_\infty\vec{a}\mathbf{e}_\infty$

But, taking into consideration 3.24, 3.21 and 2.3 we could deduce that: $\mathbf{e}_\infty\vec{a}\mathbf{e}_\infty$= -$\vec{a}{\mathbf{e}_\infty}^2$+2($\mathbf{e}_\infty\bullet$ $\vec{a}$)$\mathbf{e}_\infty$=0, , $\mathbf{e}_\infty\mathbf{e_0}\mathbf{e}_\infty$=-2$\mathbf{e}_\infty$, $\vec{a}\mathbf{e}_\infty$=-$\mathbf{e}_\infty\vec{a}$, $\vec{a}\mathbf{e_0}$=-$\mathbf{e_0}\vec{a}$ That gives us the possibility to continue the previous development:

$$\begin{aligned}
\mathbf{TXT}^{-1} &= \vec{x} + \mathbf{e_0} + \frac{\vec{x}^2}{2}\mathbf{e}_\infty + (\vec{x}\bullet\vec{a})\mathbf{e}_\infty + \vec{a} - \frac{1}{2}\vec{a}\mathbf{e}_\infty\mathbf{e_0}\vec{a}\mathbf{e}_\infty \\
&= \vec{x} + \vec{a} + \mathbf{e_0} + \frac{1}{2}\vec{x}^2\mathbf{e}_\infty + \frac{1}{2}\vec{a}^2\mathbf{e}_\infty + (\vec{x}\bullet\vec{a})\mathbf{e}_\infty \\
&= \vec{x} + \vec{a} + \mathbf{e_0} + \frac{1}{2}(\vec{x}+\vec{a})^2\mathbf{e}_\infty \qquad (3.30)
\end{aligned}$$

The result from 3.30 corresponds to the conformal point $\mathbf{y}$ representing the Euclidean point $\vec{x}+\vec{a}$ that results from $\vec{x}$ through a translation of vector $\vec{a}$ q.e.d.

Note: The spinor that characterizes the conformal translation could be written as $e^{-\frac{1}{2}\vec{a}\mathbf{e}_\infty}$. This results through a Taylor series development of the exponential and from ${\mathbf{e}_\infty}^2$=0.

Proposition: The conformal reflection of vector $\vec{v}$ corresponds to a reflection in the Euclidean space relative to the same vector. The conformal reflection relative to the hyperplane $\vec{n} + \delta\mathbf{e}_\infty$ (a hyperplane with normal $\vec{n}$ and directance δ) has the associated versor $\mathbf{v}$=$\vec{n} + \delta\mathbf{e}_\infty$ and corresponds to an Euclidean transformation $T_{\delta\vec{n}}M_{\vec{n}}T_{-\delta\vec{n}}$ where $T_{\vec{a}}$ is a translation of vector $\vec{a}$ and $M_{\vec{n}}$ a reflection (mirroring) relative to the vector $\vec{n}$ that passes through the origin.

Proof: The first statement results from: $-\vec{v}\mathbf{x}\vec{v}$ = $\mathbf{x}$-2($\vec{v}\bullet\mathbf{x}$)$\vec{v}$ = $\mathbf{x}$ - 2($\vec{v}\bullet$($\vec{x}$+$\mathbf{e_0}$+$\frac{\vec{x}^2}{2}\mathbf{e}_\infty$)$\vec{v}$=$\vec{x}$-2($\vec{v}\bullet\vec{x}$)$\vec{v}$+ $\mathbf{e_0}$+$\frac{\vec{x}^2}{2}\mathbf{e}_\infty$ =$\vec{x}$-2($\vec{v}\bullet\vec{x}$)$\vec{v}$+ $\mathbf{e_0}$+ $\frac{(\vec{x}-2(\vec{v}\bullet\vec{x})\vec{v})^2}{2}\mathbf{e}_\infty$, where $\vec{v}$ was considered a unit vector. Let's pass now to the general case. We consider $\vec{n}$ a unit vector then $\hat{n} = -\vec{n}$, $\hat{\mathbf{v}} = -\vec{n} - \delta\mathbf{e}_\infty$ and $\hat{\mathbf{v}}^{-1} = \hat{\mathbf{v}}$. The conformal transformation is: $\mathbf{vx}\hat{\mathbf{v}}^{-1}$=-$\mathbf{xv}\hat{\mathbf{v}}^{-1}$+2($\mathbf{v}\bullet\mathbf{x}$)$\mathbf{v}$= -$\mathbf{x}$+2($\vec{n}$+$\delta\mathbf{e}_\infty$)$\bullet$($\vec{x}$ +$\mathbf{e_0}$+$\frac{\vec{x}^2}{2}\mathbf{e}_\infty$)$\mathbf{v}$= -$\mathbf{x}$+2($\vec{n}\bullet\vec{x}$)$\mathbf{v}-2\delta\mathbf{v}$

The hyperplane represented by the vector $\mathbf{v}$ has the equation $\mathbf{x}\wedge\mathbf{v}^*$=0 $\Leftrightarrow$ $\mathbf{x}\bullet\mathbf{v}$=0 $\Leftrightarrow$ ($\vec{x}$+$\mathbf{e_0}$+ $\frac{\vec{x}^2}{2}\mathbf{e}_\infty$) $\bullet$ ($\vec{n} + \delta\mathbf{e}_\infty$)=0 $\Leftrightarrow$ $\vec{x}\bullet\vec{n} = \delta$

$$\begin{aligned}
\mathbf{vx}\hat{\mathbf{v}} &= (\vec{n}+\delta\mathbf{e}_\infty)(\vec{x}+\mathbf{e_0}+\frac{1}{2}\vec{x}^2\mathbf{e}_\infty)(-\vec{n}-\delta\mathbf{e}_\infty) \\
&= -\vec{n}\mathbf{x}\vec{n} + 2\delta\vec{n} - 2\delta(\vec{n}\bullet\vec{x}-\delta)\mathbf{e}_\infty \\
&= -\vec{n}(\mathbf{x}-\delta\vec{n})\vec{n} + \delta\vec{n} - 2\delta(\vec{n}\bullet\vec{x}-\delta)\mathbf{e}_\infty \\
&= \vec{n}(\vec{x}-\delta\vec{n})\vec{n} + \delta\vec{n} - 2\delta(\vec{n}\bullet\vec{x}-\delta)\mathbf{e}_\infty
\end{aligned}$$

That corresponds exactly to the transformation sequence described in the text of the proposition, where $\vec{a} = -\delta\vec{n}$.

Proposition: The Euclidean rotation (around an axis that passes through the origin) specified by a spinor $\mathbf{S}$ has the corresponding conformal transformation specified by the same spinor $\mathbf{S}$. If the rotation axis passes through an arbitrary Euclidean point $\vec{c}$, the corresponding versor in the conformal space is $\mathbf{S}$+($\mathbf{c}\times\mathbf{S}$)$\mathbf{e}_\infty$ where $\times$ denotes the commutator product.

Proof: $\mathbf{SxS}^{-1}$=$\mathbf{S}$($\vec{x}$+$\mathbf{e_0}$+$\frac{\vec{x}^2}{2}\mathbf{e}_\infty$)$\mathbf{S}^{-1}$=$\mathbf{S}\vec{x}\mathbf{S}^{-1}$+$\mathbf{S}$ $\mathbf{e_0}\mathbf{S}^{-1}$+$\frac{\vec{x}^2}{2}\mathbf{S}\mathbf{e}_\infty\mathbf{S}^{-1}$=
$\mathbf{SxS}^{-1}$+$\mathbf{e_0}$+$\frac{\vec{x}^2}{2}\mathbf{e}_\infty$ q.e.d.

The second part of the proposition could be derived expressing the rotation as the composition of two reflections with respect to two planes that are passing through $\mathbf{c}$. The spinor is then

$\mathbf{S} = (\mathbf{n_1} + (\mathbf{n_1} \bullet \mathbf{c})\mathbf{n_1})(\mathbf{n_2} + (\mathbf{n_2} \bullet \mathbf{c})\mathbf{n_2})$, where $\mathbf{n_1}$, $\mathbf{n_2}$ are the plane normals and $\mathbf{n_1} \bullet \mathbf{c}$, $\mathbf{n_2} \bullet \mathbf{c}$ the corresponding directances.

Definition: If P is a point exterior to the circle (C) (centered in C), its inverse P' relative to (C) is the foot of the altitude in the right triangle CPT. This triangle has CP as hypotenuse and the vertex T the foot of the tangent from P to (C).

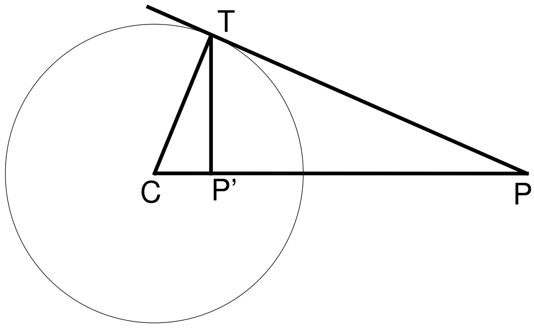

Figure 3.4: The inversion transformation in a 2D space

If we consider the circle centered in the origin and of radius ρ then from the similarity of the triangles ΔTCP' and ΔCPT we obtain: CP'=$\frac{\rho^2}{CP}$, analogously for an arbitrary centered circle (the center labeled by the vector $\vec{c}$) and P placed in the point labeled by the vector $\vec{p}$, P' will be obtained by composing a translation of vector -$\vec{c}$, an inversion relative to the origin and a translation of vector $\vec{c}$ i.e.

$$\vec{p'} = \frac{\rho^2}{\vec{p} - \vec{c}} + \vec{c} \tag{3.31}$$

Proposition: The Euclidean inversion relative to the origin is represented by a conformal reflection relative to the vector $\mathbf{e}_+$ (the vector with positive signature from the non-null orthonormal basis of $R^{1,1}$). The Euclidean inversion relative to a (conformal) sphere centered in $\mathbf{c}$ and of radius ρ is represented in the conformal space by a reflection relative to the spherical vector $\mathbf{s}$=$\mathbf{c}$-$\frac{1}{2}\rho^2\mathbf{e}_\infty$ (see 4.29)

Proof: The first part of the proposition is proofed as follows:

From 3.1 and 3.2 we could express $\mathbf{e}_+$ and $\mathbf{e}_-$ as combinations of the null basis vectors:

$$\mathbf{e}_+ = \frac{\mathbf{e}_\infty - 2\mathbf{e_0}}{2} \tag{3.32}$$

$$\mathbf{e}_- = \frac{\mathbf{e}_\infty + 2\mathbf{e_0}}{2} \tag{3.33}$$

We could also state that: -$\mathbf{e}_+\mathbf{x}\mathbf{e}_+$=$\mathbf{x}$-2($\mathbf{e}_+ \bullet \mathbf{x}$)$\mathbf{e}_+$ =$\mathbf{x}$-($\mathbf{e}_\infty - 2\mathbf{e_0}\bullet(\vec{x} + \mathbf{e_0} + \frac{\vec{x}^2}{2}\mathbf{e}_\infty$)$\mathbf{e}_+$=$\mathbf{x}$-(-1+$\vec{x}^2$) $\frac{\mathbf{e}_\infty-2\mathbf{e_0}}{2}$=$\vec{x}$+$\vec{x}^2\mathbf{e_0}$ +$\frac{\mathbf{e}_\infty}{2}$=$\vec{x}^2$($\frac{\vec{x}}{\vec{x}^2}$+$\mathbf{e_0}$ +$\frac{1}{2\vec{x}^2}\mathbf{e}_\infty$)

Taking into account that $\vec{x}^{-1} = \frac{\vec{x}}{\vec{x}^2}$, the last part of the previous relation contains the conformal representation of $\mathbf{x}^{-1}$, modulo a scalar scaling factor ($\vec{x}^2$).

For the general case of the proposition, the resulting vector is given by: $\mathbf{s}\mathbf{x}\mathbf{s}^{-1}$=$\mathbf{x}$-2($\mathbf{s} \bullet \mathbf{x}$)$\mathbf{s}^{-1}$ =$\frac{1}{\rho^2}$($\vec{c}$+$\mathbf{e_0}$+$\frac{\vec{c}^2-\rho^2}{2}\mathbf{e}_\infty$) ($\vec{x}$+$\mathbf{e_0}$+$\frac{1}{2}\rho^2\mathbf{e}_\infty$) ($\vec{c}$+$\mathbf{e_0}$+$\frac{\vec{c}^2-\rho^2}{2}\mathbf{e}_\infty$) =$\mathbf{x}$+($\vec{x}-\vec{c}$)^2-$\rho^2$=$\frac{(\vec{x}-\vec{c})^2}{\rho^2}$ (($\frac{\rho^2}{\vec{x}-\vec{c}}$+$\vec{c}$)+$\mathbf{e_0}$+$\frac{1}{2}$ ($\frac{\rho^2}{\vec{x}-\vec{c}} + \vec{c})^2\mathbf{e}_\infty$)

That corresponds to an Euclidean inversion with respect to the circle centered in $\vec{c}$ and of radius ρ.

Proposition: The Euclidean scaling transformation of factor ρ is represented in the conformal space by the spinor $\mathbf{V}$=$\frac{\rho}{2}$((1+$\mathbf{E}$)$\frac{1}{\rho}$+(1-$\mathbf{E}$)ρ)

Proof: A scaling could be regarded as resulting from the composition of two inversions. In order to simplify let's consider that the center of inversions is the origin and the two circles have respectively the radius 1 and ρ, this will correspond to a uniform scaling of factor ρ relative to the origin. The versor of the composed transformation is computed in conformity with one of the propositions above: $\mathbf{V}$=($\mathbf{e_0}$-$\frac{1}{2}\mathbf{e}_\infty$)($\mathbf{e_0}$-$\frac{1}{2}\rho^2\mathbf{e}_\infty$)=-$\frac{\rho}{2}$(-1+$\mathbf{E}$)-$\frac{1}{2}$(-1-$\mathbf{E}$) =$\frac{1}{2}$((1+$\mathbf{E}$)+(1-$\mathbf{E}$)$\rho^2$) If we consider $\mathbf{V}$=$\frac{1}{2}\rho$((1+$\mathbf{E}$)$\frac{1}{\rho}$+(1-$\mathbf{E}$)ρ) and consequently $\mathbf{V}^{-1}$=$\frac{1}{2\rho}$ ((1+$\mathbf{E}$)ρ+ (1-$\mathbf{E}$)$\frac{1}{\rho}$)

Indeed, applying the same technique as above (and 2.20) results:

$$\mathbf{V}\mathbf{x}\mathbf{V}^{-1} = \mathbf{x} - 2(\mathbf{x} \bullet \mathbf{V})\mathbf{V}^{-1} \qquad (3.34)$$

We note equally that: $\mathbf{V} \bullet \mathbf{e_0} = \mathbf{e_0}$, $\mathbf{V} \bullet \mathbf{e}_\infty = \rho^2\mathbf{e}_\infty$, $\mathbf{e_0}\mathbf{V}^{-1} = 1$, $\mathbf{e}_\infty\mathbf{V}^{-1} = \frac{1}{\rho^2}$, $(1+\mathbf{E})^2$=2(1+$\mathbf{E}$) and $(1-\mathbf{E})^2$=2(1-$\mathbf{E}$)

$$\begin{aligned}\mathbf{x} \bullet \mathbf{V} &= \frac{\rho}{2}(\vec{x} + \mathbf{e_0} + \frac{\vec{x}^2}{2}\mathbf{e}_\infty)(\frac{1+\mathbf{E}}{\rho} + (1-\mathbf{E})\rho) \\ &= (\frac{1+\mathbf{E}}{2} + \frac{(1-\mathbf{E})\rho^2}{2})\vec{x} + \mathbf{e_0} + \frac{\vec{x}^2\rho^2}{2}\mathbf{e}_\infty\end{aligned}$$

By substitution in 3.34 results:

$$\begin{aligned}\mathbf{V}\mathbf{x}\mathbf{V}^{-1} &= \mathbf{x} - \frac{1}{\rho}((\frac{1+\mathbf{E}}{2} + \frac{(1-\mathbf{E})\rho^2}{2})\vec{x} + \mathbf{e_0} + \frac{\vec{x}^2\rho^2}{2}\mathbf{e}_\infty) \\ &\quad ((1+\mathbf{E})\rho + (1-\mathbf{E})\frac{1}{\rho}) = -\mathbf{x}\end{aligned}$$

we could interpret the result above in this way: the multiplication of $\mathbf{x}$ with a scalar factor λ implies the multiplication with $\frac{1}{\lambda}$ (i.e. a scaling) of its Euclidean image.

Note: The versor $\mathbf{V}$ can be written also as: $e^{-\frac{\lambda}{2}\mathbf{E}}$ = ch($\frac{\lambda}{2}$)+$\mathbf{E}$sh($\frac{\lambda}{2}$)

The propositions above allow us to formulate the following:

Theorem: The orthogonal transformations (that can be expressed by spinors) from the conformal space correspond to conformal transformations in the Euclidean space.

The content of this last theorem gives us the motivation that stayed behind the choice of the name of the *conformal* model.

Note: The inversion is an example of transformation that is not orthogonal but preserves angles (i.e. is conformal).

Chapter 4

Objects and Methods

Computer Graphics (called also *graphic synthesis*) emphasizes the possibilities to model different graphical objects (constitutive elements of a scene) as well as the techniques required to appropriately visualize these models. It is an area of study placed at the confluence of various fields of science:

- *Geometry* that allows the description of object boundaries through points, curves and surfaces (geometric modeling) as well as the specification of different measurable (distance based) objects characteristics and of operations (reunion, intersection) between these models. The methods specific to different types of geometry (*projective geometry*, *differential geometry*, *combinatorial geometry*) are heavily used for solving various computer graphics problems as: implementing the transformations of the visualization pipeline, deforming the objects models (*morphing* and *warping* of graphical objects) or efficiently generating images of geometric models in the discrete image space (*rasterization*).

- *Physics* that allows description of material properties of the object surfaces and of the laws permitting an accurate simulation of the visual characteristics of those surfaces, taking into account the interaction with the incident radiation and the properties of the viewing device (camera, eye).

- *Numerical methods* help finding efficient and robust algorithms that could manage the sometimes huge demand of computational resources necessary to accurately render a given scene.

- *Statistics* based approaches or *fractal* based methods are used to produce effects that increase the realism of visual representations (examples are various types of texture generators or the rendering techniques based on particle systems)

- The *signal analysis* methods find sometimes their application in image synthesis see for example the Fourier based texture synthesis, the wavelet based approaches for image modeling or various antialiasing techniques

- *Data structures and algorithms* have a broad range of applications in computer graphics field concerning the object space methods (for improved efficiency in retrieving data from huge collections of geometric information or for building hierarchical solid models) and image space methods as well (examples are found in hidden surface removal algorithms or different techniques used to accelerate the rendering).

The Euclidean geometry studies the properties of arbitrary sets of points from the Euclidean space (The definition of an Euclidean space is given in Appendix 1.) The geometric algebra associated to the 3D Euclidean space is G_3; the set of vectors of this algebra, denoted G_3^1, is isomorphic to the set of points of the Euclidean space. Every such vector labels a point of the Euclidean space (excepting the origin). That is the classical geometric algebra based model of the Euclidean space and it will be used preponderantly in the present report. Other geometric algebra based models of the Euclidean space as the *homogeneous model* and the *conformal model* have been described in the previous chapter. These models allow even more elegant and comprehensive descriptions of the geometric relations specific to the Euclidean space. The geometric algebra models of the Euclidean space specify mainly the intrinsic relations between the points that make up the objects of the scene without explicitly emphasizing coordinates. (The distances to the coordinate planes are specifying extrinsically the relationships between objects.)

The present chapter describes the geometric algebra based methods to specify the primitive 2D or 3D objects that compose a scene, as well as fundamental methods to accomplish usual operations between these primitive objects (e.g. distance determination or boolean operations as intersection). Two new geometric algebra operators *meet* and *join* are introduced. The classical geometric techniques are sometimes mentioned and compared with those based on Geometric Algebra.

4.1 Primitive geometric objects

From a classical analytical geometry point of view, a surface in 3D space could be represented:

- Implicitly:

$$f(x, y, z) = 0 \tag{4.1}$$

- Explicitly:

$$z = f(x, y) \tag{4.2}$$

- Parametrically:

$$\begin{aligned} x &= f(u, v) \\ y &= g(u, v) \\ z &= h(u, v) \end{aligned} \tag{4.3}$$

  where $u_{min} \leq u \leq u_{max}$, $v_{min} \leq v \leq v_{max}$

In the last case (x, y, z) is named the *generic point* of the surface and u, v are called the *parametric coordinates* of the generic point. If the domain of the two parameters is finite ($[u_{min}, u_{max}] \times [v_{min}, v_{max}]$), the generic point covers a surface patch.

The same (coordinate based) methods of representation apply in the case of 3D curves with the mention that the generic point coordinates depend on only one parameter. Implicitly a curve could be represented as the intersection of two surfaces i.e. as a system of two equations describing, each of them, one surface.

4.1.1 The line

A line passing through two points A and B is given by:

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A} \tag{4.4}$$

The position of the generic point (P) of the line is:

$$P = Au + B(1-u), where\ u \in R \tag{4.5}$$

If the parameter u∈[0,1], the generic point covers the segment BA.

Using the classical geometric algebra model, if a line passes through the origin and has the direction specified through a vector **u** it will be implicitly described by **x**∧**u**=0, where **x** is the vector that labels the generic point of the line (see section 2.6).

An arbitrary line could be completely specified by one point labeled **a** (through which it passes) and a non-null vector **u** that gives its direction. The implicit form of the line equation is then:

$$(\mathbf{x}-\mathbf{a}) \wedge \mathbf{u} = 0 \tag{4.6}$$

Indeed, the vectors **x-a** and **u** have the same direction and consequently span together a null area.

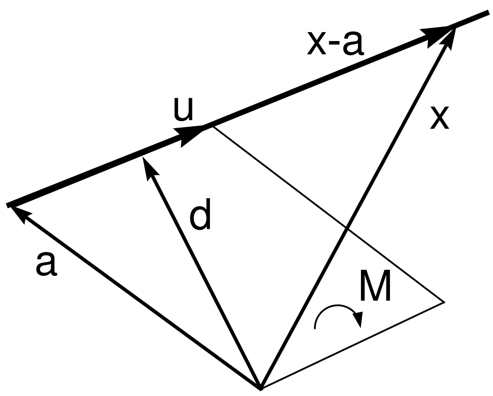

Figure 4.1: The line geometry specified through geometric algebra means

Equation 4.6 could be written:

$$\mathbf{x} \wedge \mathbf{u} = \mathbf{M} \tag{4.7}$$

where **M**=**a**∧**u** is the moment of the line. We can solve this last equation for **x** as follows: **xu**=**M**+**x**•**u** ⇒

$$\mathbf{x} = \mathbf{M}\mathbf{u}^{-1} + (\mathbf{x} \bullet \mathbf{u})\mathbf{u}^{-1} = \mathbf{M} \bullet \mathbf{u}^{-1} + \frac{\mathbf{x} \bullet \mathbf{u}}{\mid \mathbf{u} \mid^2}\mathbf{u} \tag{4.8}$$

because $\mathbf{M} \wedge \mathbf{u}^{-1} = \mathbf{a} \wedge \mathbf{u} \wedge \frac{\mathbf{u}}{|\mathbf{u}|^2}$.

The vector $\mathbf{M} \bullet \mathbf{u}^{-1}$ is orthogonal to **u**:

$(\mathbf{M} \bullet \mathbf{u}^{-1})\bullet\mathbf{u} = ((\mathbf{a}\wedge\mathbf{u})\bullet\mathbf{u}^{-1})\bullet\mathbf{u} = -(\mathbf{u}^{-1}\bullet(\mathbf{a}\wedge\mathbf{u}))\bullet\mathbf{u} = \frac{1}{|\mathbf{u}|^2}(-(\mathbf{u}\bullet\mathbf{a})\mathbf{u}+ \mathbf{u}^2\mathbf{a})\bullet\mathbf{u} = \frac{1}{|\mathbf{u}|^2}((\mathbf{u}\bullet\mathbf{a})\mathbf{u}^2 - \mathbf{u}^2(\mathbf{a}\bullet\mathbf{u})) = 0$

and its module is numerically equal with the distance from the origin to the closest point of the line (and to the module of the rejection of **a** by **u**: $P_{\mathbf{u}}^{\perp}(\mathbf{a})=(\mathbf{a}\wedge\mathbf{u})\mathbf{u}^{-1}$)

The vector:

$$\mathbf{d} = \mathbf{M} \bullet \mathbf{u}^{-1} = \frac{1}{\mid \mathbf{u} \mid^2}(\mathbf{a} - (\mathbf{u} \bullet \mathbf{a})\mathbf{u}) \tag{4.9}$$

is called the *directance* of the line and the equation 4.8 rewritten as:

$$\mathbf{x} = \mathbf{d} + \lambda\mathbf{u} \tag{4.10}$$

is the parametric form of the line equation in the geometric algebra variant. The parametric coordinate is given by: $\lambda = \frac{\mathbf{x}\bullet\mathbf{u}}{|\mathbf{u}|^2}$. An analogous parametric equation would be: $\mathbf{x}=\mathbf{a}+\lambda\mathbf{u}^{-1}$ fact that is trivially deduced from 4.6 writing this time the parametric coordinate $\lambda=(\mathbf{x}\text{-}\mathbf{a})\bullet\mathbf{u}$.

The line in the homogeneous model

Let **p** and **q** be two arbitrary (distinct) points in the homogeneous space. The outer product **p**∧**q** will project in the Euclidean space as a line. In order to determine the exact geometric characteristics of the line with the homogeneous representation **p**∧**q** i.e. the direction of the line in the Euclidean space and one point (**a**) through which the line passes, we could deduce an expression for **p**∧**q** separating the bivector part of the geometric product **pq**.

$\mathbf{pq}{=}\mathbf{pe_0}^2 q{=}(\mathbf{e_0}{+}\vec{p})\mathbf{e_0}\mathbf{e_0}\ (\mathbf{e_0}{+}\vec{q}){=}(1{+}\vec{p}\mathbf{e_0})(1{+}\mathbf{e_0}\vec{q}){=}\ 1{+}\vec{p}\mathbf{e_0}{+}\mathbf{e_0}\vec{q}{+}\vec{p}\vec{q}{=}\ 1{+}\vec{p}\bullet\mathbf{e_0}{+}\vec{p}\wedge\mathbf{e_0}{+}\mathbf{e_0}\bullet \vec{q}{+}\mathbf{e_0}\wedge\vec{q}{+}\vec{p}\bullet\vec{q}{+}\vec{p}\wedge\vec{q}\ {=}1{+}\vec{p}\bullet\vec{q}{+}\mathbf{e_0}\wedge(\vec{q}-\vec{p}){+}\vec{p}\wedge\vec{q}\ {=}1{+}\vec{p}\bullet\vec{q}{+}\mathbf{e_0}\wedge(\vec{q}-\vec{p}){+}\frac{1}{2}(\vec{p}-\vec{q})\ (\vec{p}+\vec{q}) \Rightarrow$

$$\mathbf{p}\wedge\mathbf{q} = \langle\mathbf{pq}\rangle_2 = \mathbf{e_0}\wedge(\vec{q}-\vec{p}) + \frac{1}{2}(\vec{p}-\vec{q})\wedge(\vec{p}+\vec{q}) \tag{4.11}$$

The development of **p**∧**q** in terms of Euclidean vectors contains the line direction $\vec{q}-\vec{p}$ and the moment of the line $\frac{1}{2}(\vec{p}-\vec{q})(\vec{p}+\vec{q})$ that corresponds to a line passing through the Euclidean point $\frac{\vec{p}+\vec{q}}{2}$.

More general, starting from the homogeneous representation **p**∧**q** where **p** and **q** are labeling arbitrary homogeneous points (not only points from E^n) one could deduce the line direction using 3.5 as follows:

$$\begin{aligned} \vec{q}-\vec{p} &= \frac{\mathbf{q}\wedge\mathbf{e_0}}{\mathbf{q}\bullet\mathbf{e_0}}\mathbf{e_0} - \frac{\mathbf{p}\wedge\mathbf{e_0}}{\mathbf{p}\bullet\mathbf{e_0}}\mathbf{e_0} \\ &= \frac{(\mathbf{p}\bullet\mathbf{e_0})(\mathbf{q}\wedge\mathbf{e_0})\bullet\mathbf{e_0} - (\mathbf{q}\bullet\mathbf{e_0})(\mathbf{p}\wedge\mathbf{e_0})\bullet\mathbf{e_0}}{(\mathbf{p}\bullet\mathbf{e_0})(\mathbf{q}\bullet\mathbf{e_0})} \\ &= \frac{\mathbf{e_0}\bullet(\mathbf{p}\wedge\mathbf{q})}{(\mathbf{p}\bullet\mathbf{e_0})(\mathbf{q}\bullet\mathbf{e_0})} \end{aligned} \tag{4.12}$$

An analogous development gives the position of the point the line passes through, as function of the homogeneous points from the **p**∧**q** representation:

$$\frac{\vec{p}+\vec{q}}{2} = \frac{1}{2}(\frac{\mathbf{p}\wedge\mathbf{e_0}}{\mathbf{p}\bullet\mathbf{e_0}} + \frac{\mathbf{q}\wedge\mathbf{e_0}}{\mathbf{q}\bullet\mathbf{e_0}})\mathbf{e_0} = -\mathbf{e_0} + \frac{1}{2}(\frac{\mathbf{p}}{\mathbf{p}\bullet\mathbf{e_0}} + \frac{\mathbf{q}}{\mathbf{q}\bullet\mathbf{e_0}}) \tag{4.13}$$

The line directance is obtained from 4.9 substituting **a** by (**p**+**q**)/2 and **u** by **q**-**p**:

$$\mathbf{d} = \frac{1}{\mid\mathbf{q}-\mathbf{p}\mid^2}(\frac{1+\mathbf{q}^2-\mathbf{p}^2}{2}\mathbf{p} + \frac{1+\mathbf{p}^2-\mathbf{q}^2}{2}\mathbf{q}) \tag{4.14}$$

The scalar part of the development of **pq** gives: **p**•**q**- $\vec{p}\bullet\vec{q}$ =1

Synthetically a line has in homogeneous model the equation **x**∧**p**∧**q**=0. This line passes through the Euclidean point given by 4.13 and has direction given by 4.12.

The line in the conformal model

In the Euclidean space, the equation of the line that passes through the points $\vec{a}$ and $\vec{b}$ is: $(\vec{x}\wedge\vec{a})\wedge(\vec{b}\wedge\vec{a}) = 0$. It has the correspondent in the conformal space:

$$\mathbf{x}\wedge\mathbf{e}_\infty\wedge\mathbf{a}\wedge\mathbf{b} = 0 \tag{4.15}$$

Considering the representation 3.26 of the generic point of the line, one could notice that the corresponding vector (denoted **x**) contains only three linear independent components ($\mathbf{e_0}$, $\mathbf{e}_\infty$ and the vector giving the line direction; this last one could be considered the first vector of the non-null basis of the conformal space). That implies that any outer product **x**∧**a**∧**b**∧**c**, where **a**, **b**, **c** label three arbitrary points of the line, will vanish. On the other side, if **a**, **b**, **c** are non-collinear, the equation $\mathbf{x}\wedge\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}=0$ represent (as we will soon see) a circle that passes through the (conformal) origin. As $\mathbf{e}_\infty$ could be considered collinear with any two points $\mathbf{a}$, $\mathbf{b}$ we practically deduce that for completely specifying a line in the conformal space, only two points are necessary and the line implicit equation is 4.15. On the other side one line transforms through inversion in a circle and $\mathbf{e}_\infty$ transforms through inversion in $\mathbf{e_0}$. As we saw in the previous chapter an Euclidean inversion (relative to the origin) is conformaly represented by a reflection relative to $\mathbf{e}_+$. Consequently the line 4.15 is the inverse of the circle $\mathbf{x}\wedge\mathbf{e_0}\wedge\frac{\mathbf{a}}{|\mathbf{a}|^2}\wedge\frac{\mathbf{b}}{|\mathbf{b}|^2}=0$. This last proposition could be derived from: $\mathbf{e}_+(\mathbf{a}\wedge\mathbf{b})\mathbf{e}_+ = \frac{1}{2}$ ($\mathbf{e}_+\mathbf{a}\mathbf{e}_+\mathbf{e}_+\mathbf{b}\mathbf{e}_+$-$\mathbf{e}_+\mathbf{b}\mathbf{e}_+\mathbf{e}_+\mathbf{a}\mathbf{e}_+$) $= \frac{1}{2}$ $\mathbf{e}_+(\mathbf{ab}-\mathbf{ba})\mathbf{e}_+ = \mathbf{e}_+\mathbf{a}\mathbf{e}_+ \wedge \mathbf{e}_+\mathbf{b}\mathbf{e}_+$.

4.1.2 The plane

The plane is analytically described by its implicit equation:

$$Ax + By + Cz + D = 0 \tag{4.16}$$

The coefficients A, B, C are in fact the components of the plane normal. Indeed, in the traditional approach, given a Cartesian (i.e. orthonormal) frame, a plane is completely specified by its normal vector $\mathbf{n}=(n_1, n_2, n_3)$ and one point in plane labeled by the vector $\mathbf{a}$. The implicit equation could then be derived from: $(\mathbf{x}\text{-}\mathbf{a})\bullet\mathbf{n}=0 \Leftrightarrow n_1x_1+n_2x_2+n_3x_3\text{-}a_1n_1\text{-}a_2n_2\text{-}a_3n_3=0$ and identifying the coefficients of this last equation with those of 4.16 one could easily observe the signification of A, B and C.

A plane passing through three points P_1, P_2 and P_3 has its generic point given by the following parametric equation:

$$P = uP_1 + vP_2 + (1-u-v)P_3 \tag{4.17}$$

If $0\leq$u,v≤ 1 then P positions cover the interior of the triangle $\Delta P_1P_2P_3$. One could immediately notice that the three scalar coefficients are in fact the *barycentric coordinates*[1] of P in the plane determined by P_1, P_2, P_3. The geometric algebra implicit description of a plane specified by a point (labeled by the vector $\mathbf{a}$) and by its direction (given by the *tangent bivector* $\mathbf{U}$) is:

$$(\mathbf{x}-\mathbf{a})\wedge\mathbf{U} = 0 \tag{4.18}$$

or

$$\mathbf{x}\wedge\mathbf{U} = \mathbf{T} \tag{4.19}$$

where $\mathbf{T}=\mathbf{a}\wedge\mathbf{U}$ is a trivector constituting the moment of the plane. Similarly to the line case, the *plane directance* is the rejection of $\mathbf{a}$ (or any other point of the plane) by $\mathbf{U}$

$$\mathbf{d} = \mathbf{P}^{\perp}_{\mathbf{U}}(\mathbf{a}) = \mathbf{T}\mathbf{U}^{-1} = (\mathbf{a}\wedge\mathbf{U})\mathbf{U}^{-1} \tag{4.20}$$

One could observe that applying 2.32 (the duality transformation of an outer product) the plane equation 4.18 becomes the familiar $(\mathbf{x}\text{-}\mathbf{a})\bullet\mathbf{U}^*=(\mathbf{x}\text{-}\mathbf{a})\bullet\mathbf{n}=0$ mentioned some paragraphs above.

Note: In an n dimensional Euclidean space E^n a plane described by $(\mathbf{x}\text{-}\mathbf{a})\wedge\mathbf{A}_r=0$ is called hyperplane if r=n-1. If the hyperplane passes through the origin (i.e. is a n-1-blade) it is called a *hyperspace*.

[1] In the 3D Euclidean space every 4 non-coplanar points A, B, C, D are said to form an *affine frame*. The position of an arbitrary point P could be unambiguously specified relatively to a fixed affine frame by: P=λ_AA+λ_BB+λ_CC+λ_DD, where the scalars λ_A, λ_B, λ_C, λ_D are called the *barycentric coordinates* of the point P and satisfy the condition:$\lambda_A + \lambda_B + \lambda_C + \lambda_D$=1. If all λ coefficients are positive (and consequently less than 1) then P is interior to the tetrahedron ABCD. An analogous definition is valid in the 2D space.

If the plane is given by three non-collinear points labeled respectively by the vectors **a**, **b** and **c**, its equation is deduced by the request that the vectors **b-a** and **c-a** (which are precisely lying in the plane) be coplanar with **x-a**, where **x** is the position vector of an arbitrary point in the plane. The condition is most easily expressed using the outer product:

$$(\mathbf{x}-\mathbf{a}) \wedge (\mathbf{b}-\mathbf{a}) \wedge (\mathbf{c}-\mathbf{a}) = 0 \tag{4.21}$$

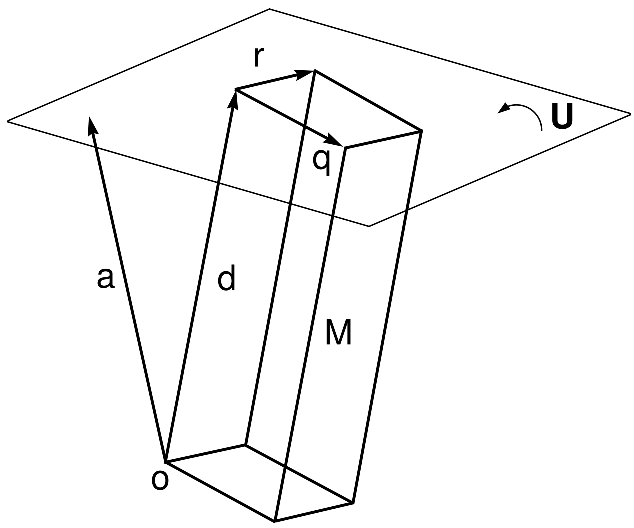

Figure 4.2: The plane U=q∧r, its directance and moment

Otherwise speaking the tangent of the plane specified by 4.21 is (**b-a**)∧(**c-a**) and its moment is **a**∧(**b-a**)∧(**c-a**)= **a**∧**b**∧**c**.

Note: If **x**, **a**, **b**, **c** label arbitrary points of the Euclidean 3D space then the volume of the parallelepiped of sides (**b-a**), (**c-a**) and (**x-a**) (i.e. |(**x-a**)∧(**b-a**)∧(**c-a**)|) equals 6 times the volume of the tetrahedron having as vertices the points labeled **x**, **a**, **b** and **c**.

The plane in the homogeneous model

A plane tangent direction in the classical 3DGA model is described by a bivector. The moment of the plane is a trivector. In the homogeneous space the plane representation must have grade 3 and be specified by the outer product of three homogeneous points **p**∧**q**∧**r**. If **p**, **q**, **r** are normalized we could deduce the Euclidean characteristics of the plane **p**∧**q**∧**r** following a technique similar to that used for the line. We separate the trivector part of the geometric product **pqr** as follows:

$$\begin{aligned}
\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r} &= \langle\mathbf{pqr}\rangle_3 = \langle\mathbf{e_0}\wedge(\vec{q}-\vec{p})\mathbf{r} + (\vec{p}\wedge\vec{q})\mathbf{r}\rangle_3\\
&= \mathbf{e_0}\wedge(\vec{q}-\vec{p})\wedge(\mathbf{e_0}+\vec{r}) + \vec{p}\wedge\vec{q}\wedge(\mathbf{e_0}+\vec{r})\\
&= \mathbf{e_0}\wedge(\vec{p}\wedge\vec{q}+\vec{q}\wedge\vec{r}+\vec{r}\wedge\vec{p}) + \vec{p}\wedge\vec{q}\wedge\vec{r}
\end{aligned} \tag{4.22}$$

Lets return briefly to the 3DGA plane model. Working the outerproducts from 4.21 we obtain: **x**∧(**p**∧**q** +**q**∧**r**+**r**∧**p**)+**p**∧**q**∧**r**=0. Further we denote **U**=**p**∧**q**+**q**∧**r**+**r**∧**p** and observe that **p**∧**q**∧**r**=(**p-r**)∧ (**q-p**)∧(**r-p**)=$\frac{1}{2}$ (**p-r**)∧**U**. Coming back again to the homogeneous model, the identification of the development 4.22 with 4.18 allows us to declare that the homogeneous

representation **p**$\wedge$**q**$\wedge$**r** corresponds to the Euclidean plane passing through the point: $\frac{1}{2}(\vec{p}-\vec{r})$=$\frac{1}{2}$(**p-r**) and having the tangent blade
$(\vec{p}\wedge\vec{q}+\vec{q}\wedge\vec{r}+\vec{r}\wedge\vec{p})$ = (**p-e₀**)$\wedge$(**q-e₀**)$\wedge$ (**r-e₀**) = **p**$\wedge$**q**+**q**$\wedge$**r**+**r**$\wedge$**p**.

The plane directance is found applying 4.20:

$\mathbf{d}=\frac{1}{2}((\mathbf{p}\text{-}\mathbf{r})\wedge(\mathbf{p}\wedge\mathbf{q}+\mathbf{q}\wedge\mathbf{r}+\mathbf{r}\wedge\mathbf{p}))\bullet(\mathbf{p}\wedge\mathbf{q}+\mathbf{q}\wedge\mathbf{r}+\mathbf{r}\wedge\mathbf{p})^{-1}=(\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r})\bullet(\mathbf{p}\wedge\mathbf{q}+\mathbf{q}\wedge\mathbf{r}+\mathbf{r}\wedge\mathbf{p})^{-1}$

The plane in the conformal model

Following the reasoning presented for the line, one could deduce that the plane is essential a sphere that passes through the point at infinity; so the treatment of the plane will be postponed to the subsection referring to the sphere. Briefly the plane as well as the sphere are represented in Minkowski space $R^{n+1,1}$ by one vector **s**. If $\mathbf{s}\bullet\mathbf{e}_\infty$=0 then **s** represents a plane; if $\mathbf{s}\bullet\mathbf{e}_\infty<0$ then **s** represents a sphere. In the former case **s** could be considered as:

$$\mathbf{s}=\vec{n}+\delta\mathbf{e}_\infty \tag{4.23}$$

Until this assumption, the equation $\mathbf{x}\bullet\mathbf{s}$=0 becomes equivalent with: $\vec{x}\bullet\vec{n}-\delta$=0 that is the equation of an Euclidean plane with normal $\vec{n}$ and directance δ with respect to the Euclidean origin.

4.1.3 The circle and the sphere

A circle centered in (C) and of radius ρ placed in a plane of bivector **A** has the parametric equation:

$$\mathbf{x}=\mathbf{c}+\rho e^{\mathbf{A}\phi} \tag{4.24}$$

where $\phi\in[0,2\pi]$

The corresponding implicit equation is:

$$(\mathbf{x}-\mathbf{c})^2=\rho^2 \tag{4.25}$$

or $\mid\mathbf{x}-\mathbf{c}\mid=\rho$

Another form of parametric equation that express the position of a point on a circle arc (having as extremities two a-priori given points labeled **a** and **b**) is deduced by noticing that the arc is the locus of points that see the chord **b-a** under a constant angle φ (see Figure 4.3). If **x** is the vector labeling the generic point of the arc, then: **b-x**=(**a-x**)$\lambda e^{\mathbf{I}\phi}$ where the scaling factor $\lambda=\pm\frac{|\mathbf{b}-\mathbf{x}|}{|\mathbf{a}-\mathbf{x}|}$ could have positive or negative values as well (there are two arcs of the same circle, having same extremities) and **I** specifies the circle plane. When $\lambda\in(-\infty,\infty)$ the point **x** sweeps the circle (centered on the perpendicular bisector of the chord **b-a**) that contains the points from where the chord **b-a** is seen under the angle ϕ or $2\pi-\varphi$. The circle radius is deduced from ΔDCA as: $\rho=\mid\vec{AC}\mid=\frac{|\vec{AD}|}{sin(\varphi)}=\frac{|\mathbf{b}-\mathbf{a}|}{2sin(\varphi)}$ and the center is given by:

$$\mathbf{c}=\mathbf{a}+\vec{AC}=\mathbf{a}+(\vec{AD}e^{\mathbf{I}(\frac{\pi}{2}-\varphi)})\frac{1}{sin(\varphi)}$$

Sometimes a circle could be considered resulting from the intersection between a plane whose direction is specified by its bivector **B** and a sphere, whose points satisfy the implicit equation: $(\mathbf{x}\text{-}\mathbf{c})^2=\rho^2$. $(\mathbf{x}\wedge\mathbf{I_3}=0)$

If the plane with direction **B** passes through **c**, the circle is a great circle of the sphere and its points satisfy also the condition: (**x-c**)$\wedge$**B**=0.

If the plane **B** passes through the point labeled **b** then its directance **d** with respect to **c** can be computed as the rejection of **b-c** by **B**: $\mathbf{d}=((\mathbf{b}\text{-}\mathbf{c})\wedge\mathbf{B})\mathbf{B}^{-1}$. Consequently the center of the

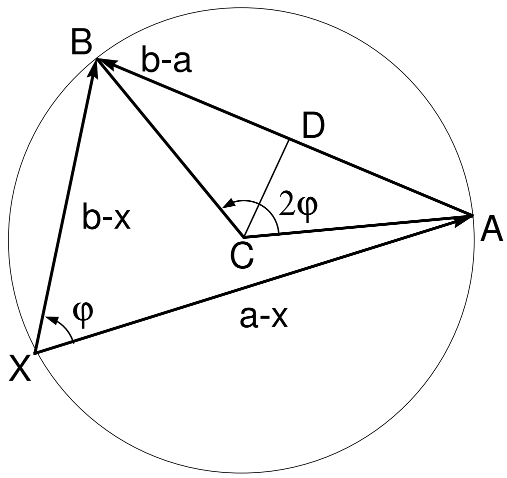

Figure 4.3: The circle as locus of points that subtend a constant angle

intersection circle is **q**=**b**+$P_{\mathbf{B}}$(**c**-**b**)=**c**+**d** and another expression for directance is: **d**=**b**-**c**+((**c**-**b**)•**B**)$\mathbf{B}^{-1}$

If d≤r then the intersection circle exists and its radius magnitude is given by: $r_c = \sqrt{r^2 - d^2}$

The circle equation has the form 4.24 where the center is positioned in **q**=**c**+**d** and the radius $\mathbf{r_c}$ is:

$$\mathbf{r_c} = r_c \frac{P_{\mathbf{B}}(\mathbf{b} - \mathbf{c})}{\mid P_{\mathbf{B}}(\mathbf{b} - \mathbf{c}) \mid} \tag{4.26}$$

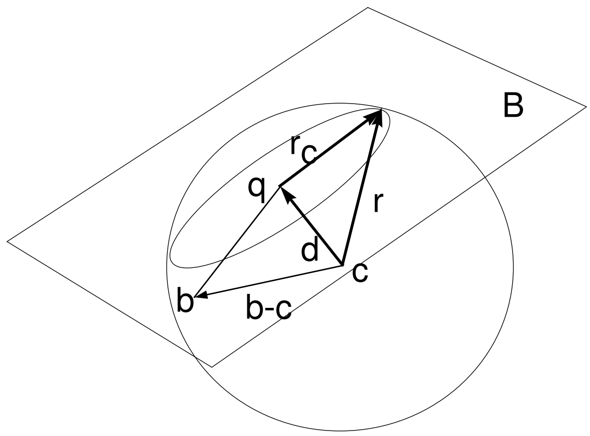

Figure 4.4: The circle at the intersection of a sphere and a plane

The sphere in the conformal model

Due to the significance of the scalar product specific to this model (see 3.28), the equation $(\vec{x} - \vec{c})^2 = \rho^2$ could be rewritten $-\frac{1}{2}(\vec{x} - \vec{c})^2 = -\frac{1}{2}\rho^2$ i.e.

$$\mathbf{x} \bullet \mathbf{c} = -\frac{1}{2}\rho^2 \tag{4.27}$$

That is the implicit equation of the sphere in the conformal model. It could be written equivalently as $\mathbf{x} \bullet \mathbf{c}$ - $\frac{1}{2}\rho^2\mathbf{x} \bullet \mathbf{e}_\infty$=0 or:

$$\mathbf{x} \bullet \mathbf{s} = 0 \tag{4.28}$$

where the conformal vector

$$\mathbf{s} = \mathbf{c} - \frac{1}{2}\rho^2\mathbf{e}_\infty \tag{4.29}$$

identifies uniquely the sphere i.e. its center and radius.

Indeed, the radius could be recovered with:

$$\begin{aligned} \mathbf{s}^2 &= (\mathbf{c} - \frac{1}{2}\rho^2\mathbf{e}_\infty)(\mathbf{c} - \frac{1}{2}\rho^2\mathbf{e}_\infty) = (\vec{c} + \mathbf{e_0} + \frac{\vec{c}^2 - \rho^2}{2}\mathbf{e}_\infty)(\vec{c} + \mathbf{e_0} - \frac{\vec{c}^2 - \rho^2}{2}\mathbf{e}_\infty) \\ &= \vec{c}^2 + \frac{\vec{c}^2 - \rho^2}{2}\mathbf{e_0} \bullet \mathbf{e}_\infty + \frac{\vec{c}^2 - \rho^2}{2}\vec{c} \bullet \mathbf{e}_\infty = \vec{c}^2 - (\vec{c}^2 - \rho^2) = \rho^2 \end{aligned} \tag{4.30}$$

and the center of the sphere with:

$$\begin{aligned} -\frac{1}{2}\mathbf{s}\mathbf{e}_\infty\mathbf{s} &= -\frac{1}{2}(\mathbf{c} - \frac{1}{2}\rho^2\mathbf{e}_\infty)\mathbf{e}_\infty(\mathbf{c} - \frac{1}{2}\rho^2\mathbf{e}_\infty) \\ &= -\frac{1}{2}\mathbf{c}\mathbf{e}_\infty\mathbf{c} = -\frac{1}{2}(-\mathbf{e}_\infty\mathbf{c} + 2\mathbf{c} \bullet \mathbf{e}_\infty)\mathbf{c} = \mathbf{c} \end{aligned} \tag{4.31}$$

The implicit equation 4.28 could be written dually as:

$$\mathbf{x} \wedge \mathbf{s}^* = 0 \tag{4.32}$$

where $\mathbf{s}^*$ is a n+1 blade (the dual of a vector in the n+2 dimensional Minkowski space) that means that $\mathbf{s}^*$ could be written as the outer product: $\mathbf{s}^* = \mathbf{p_0} \wedge \mathbf{p_1} \wedge \ldots \wedge \mathbf{p_n}$. This remark leads to the following consequence:

Given n+1 points (labeled $\mathbf{p_0}$, $\mathbf{p_1}$, …, $\mathbf{p_n}$) in the conformal space $R^{n+1,1}$ their outerproduct represents a plane or a sphere in the Euclidean space. If the n+1 vectors $\mathbf{p_i}$ are linearly dependent, only n of them determine completely a plane, the n+1-th point will in this case be considered $\mathbf{e}_\infty$ i.e. the plane passing through n points is represented by $\mathbf{e}_\infty \wedge \mathbf{p_1} \wedge \ldots \wedge \mathbf{p_n}$.

Suppose $\bigwedge_0^n \neq 0$ the equation of a sphere/plane is then:

$$\mathbf{x} \wedge \bigwedge_0^n \mathbf{p_i} = 0 \Leftrightarrow \mathbf{x} \bullet (\bigwedge_0^n \mathbf{p_i})^* = 0 \Leftrightarrow \mathbf{x} \bullet (\mathbf{p_0} \bullet (\mathbf{p_1} \bullet \ldots \bullet \mathbf{p_n})^*) \ldots) = 0 \tag{4.33}$$

4.1.4 The ellipse and the ellipsoid

The ellipse is sometimes defined as the locus of points having the sum of distances to two a-priori fixed points (called focus points) constant. In the conventional way of thinking geometry, the ellipse placed in the xOy plane centered in (x_c, y_c), with semiaxes a, b parallel to the coordinate axes has the implicit equation:

$$\frac{(x - x_C)^2}{a^2} + \frac{(y - y_C)^2}{b^2} = 1 \tag{4.34}$$

and the parametric equations:

$$\begin{aligned} x &= x_C + acos(\theta) \\ y &= y_C + bsin(\theta) \end{aligned} \tag{4.35}$$

where $\theta \in [0, 2\pi]$

An ellipsoid whose axes are parallel to the coordinate axes has the implicit equation:

$$\frac{(x-x_C)^2}{a^2}+\frac{(y-y_C)^2}{b^2}+\frac{(z-z_C)^2}{c^2}=1 \tag{4.36}$$

and the parametric equations:

$$\begin{aligned} x &= x_C + acos(\theta)cos(\varphi) \\ y &= y_C + bcos(\theta)sin(\varphi) \\ z &= z_C + csin(\theta) \end{aligned} \tag{4.37}$$

where $\theta \in [0, 2\pi]$, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Using the geometric algebra language, the generic point of an ellipse having the semiaxes specified by two orthogonal vectors **a**, **b** (**a•b**=0) and centered in the point labeled **c** has its position given by:

$$\mathbf{x} = \mathbf{a}cos(\varphi) + \mathbf{b}sin(\varphi) + \mathbf{c} \tag{4.38}$$

the plane of the ellipse (**a**∧**b**) is arbitrary oriented.

Analogously the parametric equation of an ellipsoid centered in **c** and having the semiaxes specified by **u**, **v**, **w** is:

$$\begin{aligned} \mathbf{x} &= c + \mathbf{u}cos(\theta)cos(\varphi) \\ \mathbf{y} &= \mathbf{c} + \mathbf{v}cos(\theta)sin(\varphi) \\ \mathbf{z} &= \mathbf{c} + \mathbf{w}sin(\theta) \end{aligned}$$

where, $\theta \in [0, 2\pi]$, $\varphi \in [-\pi/2, \pi/2]$ and the vectors **u**, **v**, **w** form an orthogonal frame i.e. **u•v**=**v•w**=**w•u**=0 and **u**∧**v**∧**w**≠0.

The ellipse is a particular case of a *conic section* that is a curve resulted by sectioning a conic surface by a plane

If the sectioning plane is:

- perpendicular to the cone axis the section is a *circle*
- parallel to one generator line of the conic surface the section is a *parabola*
- not normal to the cone axis neither parallel to one generator line the section could be an ellipse or a *hyperbola* (this last one resulting if the both nappes of the conic surface are intersected by the plane).

A conic section could be generally defined as the locus of points (P) for which the ratio of the distance from P to a fixed point (called *focus*) and a fixed line (called *directrix*) is constant. The value of the constant ratio is called the *eccentricity* of the conic section. If we denote by **f** the focus of the conic section, by **d** the directance of the directrix (l) with respect to the focus, by $\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|}$ the unit vector specifying the directance direction and by ε the eccentricity, we could write the implicit equation of a general conic section as:

$$\frac{\mid \mathbf{x}-\mathbf{f} \mid}{\mid \mathbf{d} \mid -(\mathbf{x}-\mathbf{f}) \bullet \hat{\mathbf{d}}} = \varepsilon \tag{4.39}$$

The case $0 < \varepsilon < 1$ corresponds to an ellipse, the case ε=1 corresponds to a parabola and for $\varepsilon > 1$ results a hyperbola. The conic sections admit parametric descriptions as second order polynomials.

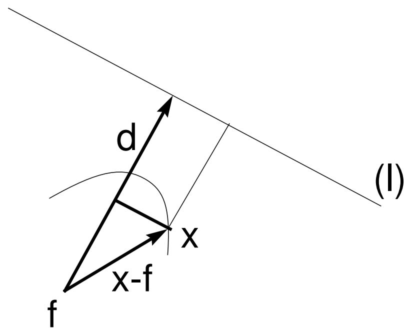

Figure 4.5: The defining property of a conic section

4.1.5 The triangle and the tetrahedron. Simplexes.

A k-dimensional *simplex* in a vector space V^n is the convex hull of a set of k+1 points $\mathbf{a_0}$, $\mathbf{a_1}$, ..., $\mathbf{a_k} \in V^n$ chosen so that at least k of them are linearly independent. In the following, the points $\mathbf{a_1}$, ..., $\mathbf{a_k}$ will be always considered linearly independent. The special point $\mathbf{a_0}$, called also the *base point* of the simplex could or not be placed in the *tangent blade* of the simplex . If the base point lies in the tangent blade of the simplex, $\bigwedge_{i=1}^{k}(\mathbf{a_i} - \mathbf{a_0})$=$\bigwedge_{i=1}^{k} \overline{\mathbf{a_i}}$ that least one is called *degenerate*. When k=2 the non-degenerate simplex $\mathbf{a_0}$, ..., $\mathbf{a_k}$ is a triangle; for k=3 it is a is a tetrahedron.

If the tangent plane (subspace) of a simplex is denoted $\mathbf{A_k}$, its equation could be written conforming to 4.19 as: $\mathbf{x}\wedge\mathbf{A_k}$=constant. The constant is exactly $\mathbf{a_0} \wedge \mathbf{A_k}$=$\mathbf{M_k}$=$\bigwedge_{i=0}^{k} \mathbf{a_i}$; and $\mathbf{M_k}$ is the moment of the simplex tangent subspace.

A simplex could equally be defined with the help of the barycentric coordinates. Indeed a vector labeling an arbitrary point interior to the simplex could be written as:$\mathbf{p}$=$\sum_{i=0}^{k} \lambda_i \mathbf{a_i}$, where $0 \leq \lambda_i \leq 1$ $\forall$i=1, ..., k, and $\sum_{i=0}^{k} \lambda_i = 1$

The simplex (directed) volume is given by $\frac{1}{k!} \bigwedge_{i=1}^{k}(\mathbf{a_i} - \mathbf{a_0})$ where $\overline{\mathbf{a_i}} = \mathbf{a_i} - \mathbf{a_0}$, $(1 \leq i \leq n)$ are the sides (edges) of the simplex relatively to $\mathbf{a_0}$.

Simplexes could be used to approximate arbitrary surfaces of V^n, following the model (heavily used in computer graphics) of the polygonal (triangular) meshes that approximate surfaces (generally smooth) in 3D space. As the reader familiar with computer graphics methods already knows, the surface modeling with polygonal meshes requires informations about polygons normals (faces normals, vertices normals) in order to accurate shade every polygon of the mesh. The faces and the normals associated to a simplex could be elegantly expressed in geometric algebra terms. In the subspace of a non-degenerate k-simplex, subspace characterized by the basis $\mathbf{a_0}$, $\mathbf{a_1}$, ..., $\mathbf{a_k}$ and having the pseudoscalar: $\mathbf{I_{k+1}} = \mathbf{a_0} \wedge \ldots \wedge \mathbf{a_k}$, one could construct the dual frame conforming to the definition 2.12. In this case, the product: $\mathbf{a_i}^* \bullet \mathbf{I_{k+1}} = (-1)^{i+1} \bigwedge_{j=0, j\neq i}^{k} \mathbf{a_j}$ corresponds to a k-blade (in fact a k-1 simplex) that is the face opposite to $\mathbf{a_i}$, of the original simplex. We could denote this face as: $F_i(\mathbf{I_{k+1}})$, $0 \leq i \leq k$. The vector $\mathbf{a_i}^*$ satisfies the condition:

$$\mathbf{a_i}^* F_i(\mathbf{I_k}) = \mathbf{a_i}^* \bullet (\mathbf{a_i}^* \bullet \mathbf{I_k}) = (\mathbf{a_i}^* \wedge \mathbf{a_i}^*) \bullet \mathbf{I_k} = 0$$

and consequently is normal to the face $F_i(\mathbf{I_{k+1}})$. In the same time $\mathbf{a_i}^*$ has similar orientation with $F_i(\mathbf{I_{k+1}})$ because $\mathbf{a_i}^* \wedge F_i(\mathbf{I_{k+1}})$=$\mathbf{I_{k+1}}$ so, the vectors of the dual frames are corresponding to the well known outward pointing normals at the faces of a polyhedron from the 3D computer graphics problems.

The analogies between the above-mentioned notions, the field of algebraic topology and the extension of the theory to the continuous field (towards a simplicial integral calculus) are developed in ***[Sobc 92]***.

4.2 Meet and Join operations with subspaces

Two important operators, that have an interesting geometric semantic will be further defined. They are used especially with blade operands.

Definition: Be **A** and **B** two blades in G_n. The *meet* of **A** and **B**, denoted **A**∩**B**, is the blade **M** of maximal grade that is simultaneously factor of **A** and **B**.

We could therefore write: **A**=**K**∧**M**, **B**=**M**∧**L**. The meet is defined modulo a scalar factor. Similarly:

Definition: The *join* operator of two blades **A**, **B**, denoted **A**∪**B**, produces the blade of lowest grade that includes both **A** and **B**.

This last definition means: **A**∪**B**=**K**∧**M**∧**L**.

The operations meet and join have a semantic similar to the set operators: intersection and reunion, that is why the notation above was chosen. They could also be compared respectively to the greatest common divisor and the lowest common multiple of two integers.

As it was noticed in section 2.7, when the intersection of two planes was determined, the line direction of the intersection is **n**×**m**, where **n** and **m** are the normals of the two planes. Let's note the plane bivectors by **A** and **B**. In G_3 the plane normal is in the same time the dual of the plane tangent bivector, therefore the vector expressing the direction of the intersection is: $\mathbf{n}\times\mathbf{m}=\mathbf{A}^* \times \mathbf{B}^*=(\mathbf{A}\wedge \mathbf{B})\mathbf{I}^{-1}=(\mathbf{A}^* \wedge \mathbf{B}^*)^*$.

According to the previously mentioned definitions we can state:

$$\mathbf{A} \cap \mathbf{B} = (\mathbf{A}^* \wedge \mathbf{B}^*)^* \tag{4.40}$$

The example above emphasizes the fact that the meet operator gives only the direction of plane intersection (as a 1-blade) but not the exact position of the intersection line. That is because its operands specify only the tangents of arbitrary planes without fixing their exact locations.

According to 2.32:

$$\mathbf{A} \cap \mathbf{B} = \mathbf{A}^* \bullet \mathbf{B} \tag{4.41}$$

This relation is valid if grade(**A**)+grade(**B**)≥n (n is the grade of the space relative to which the dual is computed).

Note: If we would like to determine the intersection of two lines considered 1-blades (i.e. knowing only their directions) the meet is not helpful. Observe that in this case grade(**A**)+grade(**B**)< n $(2 < 3)$. When grade(**A**)+grade(**B**)=n (for example the intersection between a 1-blade and a 2-blade in G_3) the result is 0, that is the two blades intersect in the origin (their common subspace is the origin).

The meet is associative: $(\mathbf{A}\cap\mathbf{B})\cap\mathbf{C} = (\mathbf{A}^*\wedge\mathbf{B}^*)^*\cap\mathbf{C} = ((\mathbf{A}^*\wedge\mathbf{B}^*)\wedge\mathbf{C}^*)^* = (\mathbf{A}^*\wedge(\mathbf{B}^*\wedge\mathbf{C}^*))^* = (\mathbf{A}^* \wedge (\mathbf{B} \wedge \mathbf{C})^*)^* = \mathbf{A}\cap(\mathbf{B}\cap\mathbf{C})$

4.3 Graphic methods

4.3.1 2D Geometry

There are multiple modeling methodologies of the Euclidean geometry, based on geometric algebra. The models associate a different geometric semantic to some of the products (the case of the inner product is especially meaningful). The computational rules (identities) remain the same and lead sometimes to different formulations (in each model of geometry) of the same basic geometric notions. The present section describes the modalities to specify some basic operations, extracted from the basic geometric graphics gems (***[Glass 90]***) in different models of Euclidean geometry, based on geometric algebra. The models referred below are the *classical geometric algebra model* the *homogeneous model* and the *conformal model* that were described in the previous chapter and specified in papers like ***[Hest 86b]***, ***[Hest 91b]*** or ***[Li99]***. These models will be named further 3DGA (or 2DGA if the modeled geometry concerns the two-dimensional space), HM (or 4DGA) and CM (or 5DGA).

Normalization

Glassner denotes this operation as:

$$\mathbf{A} \leftarrow \frac{\mathbf{A}}{V2.\mathit{Length}(\mathbf{A})}$$

V2 signifies the two-dimensional space where the process take place. The bold capital letters denote vectors in Glassner notation. The corresponding notation in Geometric Algebra is: $\mathbf{a} \leftarrow \frac{\mathbf{a}}{|\mathbf{a}|}$

Implicit to explicit conversion of a line representation

Glassner defines the line implicit representation as **N**·P+c=0, where P is the generic point of the line, **N** the normal vector and c a scalar. The dot is considered as dot, inner or scalar product a globalized semantic which was salutary detailed in the geometric algebra specification.

Glassner defines the line explicit description as: P=U+**V**t, where P is the generic point of the line, U is an arbitrary fixed point on the line (specifying the line position) V is a vector specifying the line direction and t a scalar (real) parameter. Usually such a description is called a parametrical description of the line but, during this section we will preserve Glassner's original terminology.

Starting from the implicit representation

$$\mathbf{x} \bullet \mathbf{n} + c = 0 \tag{4.42}$$

(which finds an identical correspondent in the geometric algebra) and applying the duality transformation to both parts of the identity, we obtain: $\mathbf{x}\wedge\mathbf{n}^{*}+c^{*}=0$ i.e. an equation of the same form as 4.7. That allows the identification of $\mathbf{n}^*$ as the line direction vector and c^* as the line moment. (The dual transformation is applied in a two dimensional space (whose pseudoscalar is the bivector of the plane in which the line is placed). The Glassner implicit representation coincides with the implicit representation (from 4.1) of a line in 2DGA.

In order to obtain the explicit representation, we compute first the line directance with respect to the origin, which will give the position of **U**. The line directance is a vector having the same direction as **n** and passing through the origin, therefore $\alpha\mathbf{n}$. Substituting in 4.42 results: $\alpha = -\frac{c}{|\mathbf{n}|^2}$ that gives further:

$$\mathbf{d} = -\frac{c}{\mid \mathbf{n} \mid^2}\mathbf{n} \tag{4.43}$$

The line direction is obtained as the orthogonal complement (dual) of **n** with respect to the plane bivector. If this last one is denoted **A** then the line direction is: **b**=**n***= **nA**$^{-1}$. This finally leads to the line explicit (parametric) equation: **x**=**d**+λ**n*** or:

$$\mathbf{x} = -\frac{c}{\mid \mathbf{n} \mid^2} + \lambda \mathbf{n}^*, \lambda \in R \tag{4.44}$$

Explicit to implicit conversion of a line representation

In this case the directance (**d**) and the line direction vector (**b**) are known. Both parts of the line explicit equation: **x**=**d**+α**b** could be multiplied (by inner product) with **b*** (the vector specifying the line normal). That gives $\mathbf{x} \bullet \mathbf{b}^*=\mathbf{d}\bullet\mathbf{b}^*+\alpha\mathbf{b} \bullet \mathbf{b}^* \Leftrightarrow \mathbf{x}\bullet\mathbf{b}^*=\mathbf{d} \bullet \mathbf{b}^*$, and by identification with 4.42 we obtain: **n**=**b*** and c= **-d** $\bullet$ **b***.

Line tangent to a circle at a given point

The point P, the center Q and the circle radius ρ are initially specified. We obtain first the explicit description of the line. If $\mathbf{n} = \frac{\mathbf{p-q}}{|\mathbf{p-q}|} = \frac{\mathbf{p-q}}{\rho}$ is the unit vector of the normal direction then the directance of the tangent is: $\mathbf{d} = \mathbf{p} - \mathbf{q} + (\mathbf{q} \bullet \mathbf{n})\mathbf{n}$ and the line direction is **b**=**n***.

The complete determination of the implicit form of the tangent line requires finding the value of the scalar c. Corresponding to the previous subsection it is:

$$\begin{aligned} c &= -\mathbf{d} \bullet \mathbf{b}^* = -\mathbf{d} \bullet \mathbf{n} = - \mid \mathbf{d} \mid\mid \mathbf{n} \mid cos(0) = - \mid \mathbf{d} \mid \\ &= -(\mid \mathbf{p} - \mathbf{q} \mid + \mathbf{q} \bullet \mathbf{n}) = -((\mathbf{p} - \mathbf{q}) \bullet \mathbf{n} + \mathbf{q} \bullet \mathbf{n}) = -\mathbf{p} \bullet \mathbf{n} \end{aligned}$$

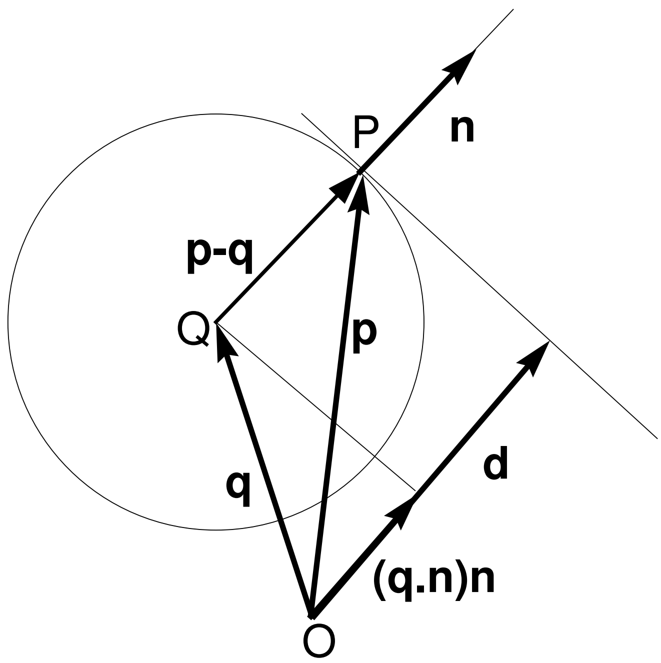

Figure 4.6: Line tangent to a circle

The intersection of a circle and a line

The circle is given by its center Q and radius ρ and the line by its explicit description $\mathbf{x}$=$\mathbf{d}$+$\alpha\mathbf{b}$, where $\alpha \in$R. We suppose, without affecting the generality of the reasoning, that $\mathbf{b}$ is a unit vector. The line directance with respect to Q is $\mathbf{d_Q}$=$\mathbf{d}$-($\mathbf{q}\bullet\mathbf{b}^*$)$\mathbf{b}^*$. The problem has real solutions if $\mid \mathbf{d_Q} \mid \leq \rho$. In this case the point of the line placed nearest Q is H. Its vector label is: $\mathbf{h}$=$\mathbf{q}$+$\mathbf{d_Q}$=$\mathbf{q}$+$\mathbf{d}$-($\mathbf{q} \bullet \mathbf{b}^*$) $\mathbf{b}^*$. From ΔQHA we easily deduce the length HA and consequently the complete specification of the vector:

$$\mathbf{e} = \vec{HA} = \mathbf{b}\sqrt{\rho^2 - \mathbf{d_Q}^2}$$

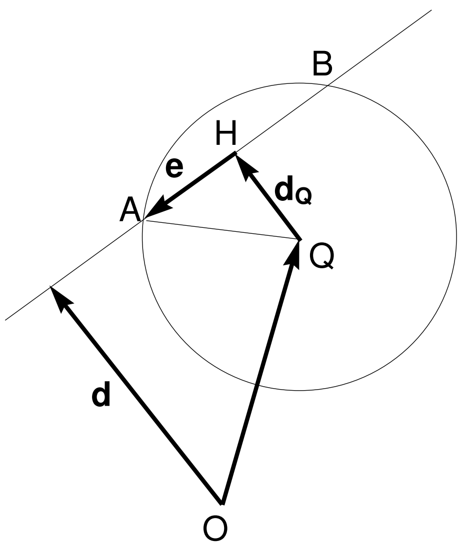

Figure 4.7: Intersection between a line and a circle

The intersection points follow immediately:

$$\mathbf{x}_{\pm} = \mathbf{h} \pm \mathbf{e} = \mathbf{h} \pm \rho\sqrt{1-\delta^2}\mathbf{b}$$

where $\delta = \frac{|\mathbf{d_Q}|}{\rho}$

Note: An alternative way to deduce the value of $\mathbf{e}$ is the following: $\mathbf{e}$ is the projection of $\vec{QA}$ on $\mathbf{b}$ and results from the rotation of $\mathbf{d_Q}$ with the angle $\varphi = \angle$HQA. That is: $\mathbf{e}$=($\mathbf{r}\bullet\mathbf{b}$)$\mathbf{b}^{-1}$, where $\mathbf{r} = \mathbf{d_Q}e^{\mathbf{I}\varphi}$ and φ=arccos(δ).

Line-circle/sphere intersection in the conformal model

As it was mentioned in the section referring to graphic objects, the line is represented by $\mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b}$ and a sphere centered in $\mathbf{c}$ and of radius ρ is represented by the vector $\mathbf{c}$-$\frac{1}{2}\rho^2\mathbf{e}_\infty$. The implicit (dual and direct) equations are respectively:

$$\mathbf{x} \wedge \mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b} = 0 \tag{4.45}$$

and

$$\mathbf{x} \bullet (\mathbf{c} - \frac{1}{2}\rho^2\mathbf{e}_\infty) = 0 \tag{4.46}$$

The characteristics of the line represented by the conformal tri-vector $\mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b}$ could be deduced by separation of the part of grade 3 from the geometric product $\mathbf{e}_\infty\mathbf{ab}$:

$$\begin{aligned}\mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b} &= \langle \mathbf{e}_\infty \mathbf{ab}\rangle_3 = \langle \mathbf{e}_\infty(\vec{a} + \mathbf{e_0} + \frac{1}{2}\vec{a}^2\mathbf{e}_\infty)(\vec{b} + \mathbf{e_0} + \frac{1}{2}\vec{b}^2\mathbf{e}_\infty)\rangle_3 \\ &= \mathbf{e}_\infty \wedge \vec{a} \wedge \vec{b} + \vec{a} \wedge \mathbf{E} - \mathbf{E} \wedge \vec{b} \\ &= \mathbf{e}_\infty \wedge \vec{a} \wedge \vec{b} + \mathbf{E} \wedge (\vec{a} - \vec{b}) \end{aligned} \quad (4.47)$$

From the last equation we could separate the line direction by a scalar multiplication with $\mathbf{e_0}$ and after that with $\mathbf{e}_\infty$. The result is:

$$\begin{aligned}\mathbf{e}_\infty \bullet (\mathbf{e_0} \bullet (\mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b})) &= (\mathbf{e}_\infty \wedge \mathbf{e_0}) \bullet (\mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b}) = -\mathbf{E} \bullet (\mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b}) \\ &= \mathbf{e}_\infty \bullet (\mathbf{e_0} \bullet (\mathbf{e}_\infty \wedge \vec{a} \wedge \vec{b} + \mathbf{E} \wedge (\vec{a} - \vec{b}))) \\ &= \mathbf{e}_\infty \bullet (-\vec{a} \wedge \vec{b} + \mathbf{e_0} \wedge (\vec{a} - \vec{b})) = \vec{b} - \vec{a} \end{aligned} \quad (4.48)$$

The term $\mathbf{e}_\infty\wedge\vec{a}\wedge\vec{b}$ from 4.47, represents the codification of the line moment ($\vec{a}\wedge\vec{b}=\vec{a}\wedge(\vec{b}-\vec{a})$); it corresponds to an Euclidean line that passes through the Euclidean point $\vec{a}$ (and consequently through $\vec{b}$ because the line direction is $\vec{b}-\vec{a}$). That is, in conformity with 3.17, the point having the conformal representation $(\mathbf{a} \wedge \mathbf{E}) \bullet \mathbf{E}$.

The intersection between the two subspaces is computed using the meet operator, whose value is given by 4.41 i.e. $(\mathbf{c}\text{-}\frac{1}{2}\rho^2\mathbf{e}_\infty)^*\bullet(\vec{n} + \delta\mathbf{e}_\infty) = (\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r}\wedge\mathbf{s})\bullet(\vec{n}+\delta\mathbf{e}_\infty) = -(\vec{n} + \delta\mathbf{e}_\infty)\bullet(\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r}\wedge\mathbf{s}) = -(\vec{n}\bullet\vec{p}+\delta)(\mathbf{q}\wedge\mathbf{r}\wedge\mathbf{s}) + (\vec{n}\bullet\vec{q}+\delta)(\mathbf{p}\wedge\mathbf{r}\wedge\mathbf{s}) - (\vec{n}\bullet\vec{r}+\delta)(\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{s}) + (\vec{n}\bullet\vec{s}+\delta)(\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r})$. The dual of the sphere vector was expressed as $\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r}\wedge\mathbf{s}$ i.e. the outer product of 4 points on the sphere.

Line tangent to two non-intersecting circles

Without restraining the generality, we assume the first circle centered in the origin and the second having the center placed in the point labeled **v**. The direction of **v** specifies the center line. There are two main cases (the interior and the exterior tangents) that are presented in Figure 4.8.

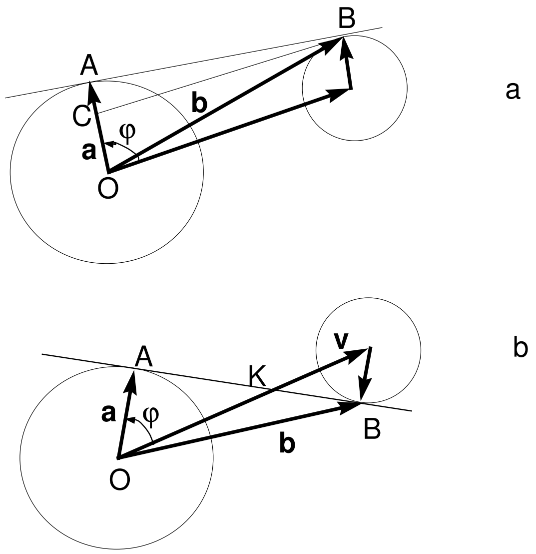

Figure 4.8: Line tangent to non-intersecting circles

Let R and r be the lengths of the two radius. From ΔBAC we deduce:

$$\varphi = arccos(\frac{R-r}{\mid \mathbf{v} \mid}) \quad (4.49)$$

The radius **a** passing through one tangency point could be obtained from **v** through a rotate/scale transform:

$$\mathbf{a} = R\frac{\mathbf{v}}{\mid \mathbf{v} \mid}e^{\mathbf{I}\varphi} \tag{4.50}$$

where **I** denotes the plane bivector. A similar expression could be deduced for **b**, the vector labeling the second tangency point:

$$\mathbf{b} = \mathbf{v} + r\frac{\mathbf{v}}{\mid \mathbf{v} \mid}e^{\mathbf{I}\varphi} \tag{4.51}$$

The exterior tangent passing through **a** and **b** has the direction given by **b-a** and the line directance given by **a**. There is also a symmetric solution obtained by substituting φ with -φ in the two relations above.

The second case depicted in Figure 4.8b is the so-called interior tangent. From the similarity of the triangles ΔOKA and ΔVKB results: $\frac{OK}{VK}=\frac{OK}{|\mathbf{v}|-OK}=\frac{R}{r}$. That gives further:

$$\varphi = arccos(\frac{R}{OK}) = arccos(\frac{R+r}{\mid \mathbf{v} \mid})$$

Following the reasoning used for the exterior tangent we obtain:

$$\mathbf{a} = R\frac{\mathbf{v}}{\mid \mathbf{v} \mid}e^{\mathbf{I}\varphi} \tag{4.52}$$

and

$$\mathbf{b} = \mathbf{v} - r\frac{\mathbf{v}}{\mid \mathbf{v} \mid}e^{-\mathbf{I}\varphi} \tag{4.53}$$

That determines completely the tangency line. Of course there is a symmetric solution obtained by replacing the angle φ with its opposite.

Point on line nearest point

The line (l) is given by its direction vector **u** and a point on line, labeled **a**. Writing the line equation in its dual form (i.e. emphasizing the line normal) results: $\mathbf{x}\bullet\mathbf{u}^*$=c. The point on (l) placed nearest P is in fact the intersection between (l) and the normal to (l) passing through P. The generic point on this normal is given by: $\mathbf{x}=\mathbf{p}+\alpha\mathbf{u}^*$. The value of α for the intersection point results from a simple substitution: $\mathbf{p}\bullet\mathbf{u}^*+\alpha(\mathbf{u}^*)^2=\mathrm{c} \Rightarrow \alpha = \frac{\mathbf{c}-\mathbf{p}\bullet\mathbf{u}^*}{(\mathbf{u}^*)^2}$ We can easily make a new substitution and determine the point Q on (l) placed most closely to P as:

$$\mathbf{q} = \mathbf{p} + \frac{c - \mathbf{p} \bullet \mathbf{u}^*}{\mathbf{u}^* \bullet \mathbf{u}^*}\mathbf{u}^* \tag{4.54}$$

The same type of reasoning could be applied to find the intersection between two lines; the expression giving the generic point of the second line is substituted in the implicit equation of the first line. The result from 4.54 could be used to determine the distance from a point P to a given line.

4.3.2 3D Geometry

Line-plane intersection

The line (l) is given by a point P and its direction vector **u**. The generic point on the line is: $\mathbf{x}=\mathbf{p}+\lambda\mathbf{u}$. The plane is specified dually by its normal and a point A in plane. Its implicit

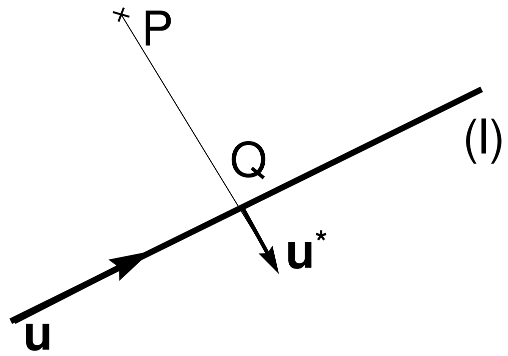

Figure 4.9: Point on a line nearest to a given point

equation is (**x-a**)•$\mathbf{U}^*$=0, where **U** is the bivector tangent to the plane and consequently $\mathbf{U}^*$ is the plane normal. The substitution of **x** leads to a equation in λ that is easily solved:

$$\lambda = \frac{(\mathbf{a} - \mathbf{p}) \bullet \mathbf{U}^*}{\mathbf{u} \bullet \mathbf{U}^*}$$

That gives the position of the intersection point:

$$\mathbf{q} = \mathbf{p} + \frac{(\mathbf{a} - \mathbf{p}) \bullet \mathbf{U}^*}{\mathbf{u} \bullet \mathbf{U}^*}\mathbf{u} \tag{4.55}$$

Point on plane nearest point

The plane is specified implicitly (see 4.18) through its bivector **U** and one fixed point **a**. We will use the dual variant of the equation. The line that passes through P and is normal to the plane is $\mathbf{x}=\mathbf{p}+\lambda\mathbf{U}^*$. The substitution of **x** gives: $\lambda = \frac{(\mathbf{a}-\mathbf{p})\bullet\mathbf{U}^*}{(\mathbf{U}^*)^2}$ and the formula for the point we are looking for is:

$$\mathbf{q} = \mathbf{p} + \frac{(\mathbf{a} - \mathbf{p}) \bullet \mathbf{U}^*}{(\mathbf{U}^*)^2}\mathbf{U}^*$$

that is a result similar to that obtained in 4.54.

Point on sphere nearest point

Suppose that the sphere (S) is centered in **c** it has the length of the radius ρ and P is the given point. Then the point Q we are looking for is placed on the line that passes through **c** and has the direction **p-c**; moreover the distance between C and Q must be equal to ρ. In conclusion:

$$\mathbf{q} = \mathbf{c} + \frac{\mathbf{p} - \mathbf{c}}{\mid \mathbf{p} - \mathbf{c} \mid}\rho$$

General treatment of the 3D intersection problems

This type of problems can be elegantly resolved using the meet operator. The relation usually used is 4.41 but it must be carefully applied taking into account the grade conditions. In the following we refer at the $R^{4,1}$ conformal representation of R^3.

In the conformal model, every blade with Minkowski signature (i.e. that has as factor one vector with negative signature) dually represents an Euclidean sphere or a plane. These Euclidean entities are not necessary subspaces i.e. they have completely arbitrary positions in the Euclidean space. If the grade of the conformal blade is r then the represented Euclidean entity has grade r-2. As we already know, one Euclidean sphere or one plane can be directly specified by a conformal vector **s** (using the implicit equation **x**•**s**=0) or dually represented by the 4-blade $\mathbf{s}^*$ (using the equation $\mathbf{x} \wedge \mathbf{s}^*=0$). In case of a sphere, **s** takes the form: $\mathbf{c} - \frac{1}{2}\rho^2\mathbf{e}_\infty$ and in case of a plane $\mathbf{s}=\vec{n} - \delta\mathbf{e}_\infty$. If $\mathbf{s}^*$ (a r-grade blade) contains as factor $\mathbf{e}_\infty$ then **s** represents a plane (see 4.33). In the opposite case ($\mathbf{s}^* \wedge \mathbf{e}_\infty \neq 0$) **s** represents a sphere. It is possible that $\mathbf{s}^*$ constitute a blade with grade less than 4 (and consequently its dual **s** has grade greater than 1) then it corresponds to a circle (grade($\mathbf{s}^*$)=3), or a pair of points (grade($\mathbf{s}^*$)=2) called also 1D circle if its factors are not containing $\mathbf{e}_\infty$ or to a line (when grade($\mathbf{s}^*$)=3) if the factorization contains $\mathbf{e}_\infty$.

That is a huge advantage of the conformal model to represent arbitrary spheres and planes as subspaces; this makes possible the use of the meet operator (that normally works only over subspaces) to compute intersections between arbitrary Euclidean spheres (or circles) and planes (or lines).

For example, in the conformal model, the line-sphere intersection is computed in the following way: Let the sphere be represented by $\mathbf{s}=\mathbf{c}-\frac{1}{2}\rho^2\mathbf{e}_\infty$ and the line by $\mathbf{L}^*=\mathbf{e}_\infty \wedge \mathbf{a} \wedge \mathbf{b}$; their intersection will be the blade $\mathbf{s} \cap \mathbf{L}=\mathbf{s}^* \bullet \mathbf{L}$. The subspace $\mathbf{s}^*$ is grade 4 (dual of a vector in the 5D Minkowski space) and **L** is obviously of grade 2; the formula above will consequently produce a 2-grade conformal blade

Bibliography

[Corro 01] E.B. Corrochano, G. Sobczyk (editors), *Geometric Algebra with Applications in Science and Engineering*, Birkh*ä*user, Boston, 2001.

[Dorst 02a] L. Dorst, C. Doran, J. Lasenby (editors), *Aplications of Geometric Algebra in Computer Science and Engineering*, Birkhäuser, Boston, 2002.

[Dorst 02b] L. Dorst, S. Mann, *Geometric Algebra: A Computational Framework for Geometrical Aplications*, IEEE Computer Graphics and Applications, pp. 24-31, May/June 2002.

[Dorst 02c] L. Dorst, *The Inner Products of Geometric Algebra*, appeared in [Dorst 02a], pp. 34-46

[Eber 01] D. Eberly, *3D Game Engine Design - A Practical Approach to Real-Time Computer Graphics*, Academic Press, 2001.

[Glass 90] *Graphics Gems*, vol. I (editor A.S. Glassner) Academic Press Professional, Harcourt Brace, 1990.

[Hart 00] R. Hartley, A. Zisserman, *Multiple View Geometry in Computer Vision*, Cambridge University Press, 2000.

[Hest 84] D. Hestenes, *Clifford Algebra to Geometric Calculus - A Unified Language for Mathematics and Physics*, D. Reidel Publishing Company, Dordrecht, 1984.

[Hest 85] D. Hestenes, *Clifford Algebras and their Applications in Mathematical Physics*, D. Reidel publishing company, J. S. R. Chisholm and A. K. Common editors, Dordrecht, NATO ASI series C vol 183, pp. 1-23, 1985.

[Hest 86a] D. Hestenes, *New Foundation for Classical Mechanics*, D. Reidel publishing company, Dordrecht, 1986.

[Hest 86b] D. Hestenes, A Unified Language for Mathematics and Physics, appeared in: *Clifford Algebras and their Applications in Mathematical Physics*, (editors J.S.R. Chisholm, A.K. Common), pp. 1-26, Kluwer Academic Publishers, Dordrecht/Boston, 1986.

[Hest 91a] D. Hestenes, R. Ziegler, Projective Geometry with Cliford Algebra, *Acta Applicandae Mathematicae*, vol 23, pp. 25-36, Kluwer Academic Publishers, 1991

[Hest 91b] D. Hestenes. The Design of Linear Algebra and Geometry, *Acta Applicandae Mathematicae*, vol. 23., pp. 65-93, Kluwer Academic Publishers, 1991

[Hest 99] D. Hestenes, Hongbo Li, Alyn Rockwood, *New Algebraic Tools for Classical Geometry*, available at http://modelingnts.la.asu.edu/pdf/CompGeom-ch1.pdf appeared in [Somm 01]

[Li99] Hongbo Li, D. Hestenes, A. Rockwood, *Generalized Homogeneous Coordinates for Computational Geometry*, available at http://modelingnts.la.asu.edu/pdf/CompGeom-ch1.pdf appeared in [Somm 01]

[Laun 93] P. Lounesto, *Marcel Riesz's work on Clifford algebras*, appeared in: Clifford Numbers and Spinors (editors E. Bolinder, P. Lounesto), Kluwer Academic Publishers, pp. 215-241, 1993.

[Maill 90] P. Maillot, *Using quaternions for coding 3D transformations*, appeared in Graphics Gems vol. 1, (editor A. Glassner), Academic Press, Cambridge MA, 1990.

[Sobc 92] G. Sobczyk, Simplicial Calculus with Geometric Algebra, appeared in: *Clifford Algebras and their Applications in Mathematical Physics*, Kluwer Academic Publishers, Dordrecht/Boston, 1992

[Somm 01] G. Sommer (editor), *Geometric Computing with Clifford Algebras*, Springer Verlag, Berlin Heidelberg, 2001

[Shoe 85] K. Shoemake, *Animating rotation with quaternion calculus*, ACM SIGGRAPH Proceedings of the 12-th Annual Conference in Computer Graphics and Interactive Techniques, vol. 19 (3), July 1985.

.1 Appendix 1 - Useful Definitions

Definition: Given a set A and one internal composition law + defined over A, the couple (A, +) forms a *group* if:

1. the addition is associative
2. $\exists 0 \in A$ so that $a + 0 = 0 + a = a$, $\forall a \in A$ (0 is called the identity element of the group)
3. $\forall a \in A \Rightarrow \exists - a \in A$ so that $a + (-a) = (-a) + a = 0$ (-a is called the inverse (opposite) of a)

If the internal composition law is also commutative, the group is called *commutative* or *abelian*. If only properties (1) and (2) are valid, the resulted algebraic structure is called *semi-group* or *monoid*.

Definition: Given a set A and two internal composition laws +, $\circ$ defined over A, the triple (A,+, $\circ$) forms a *ring* if:

1. (A, +) is an abelian group (i.e. the properties of closure, associativity, commutativity, identity and inverse are satisfied by the addition)
2. the product is distributive over the addition i.e. $a \circ (b+c) = a \circ b + a \circ c$, $(a+b) \circ c = a \circ c + b \circ c$

If the product is associative then the ring is associative. If the product is commutative then the ring is commutative.

Definition: A *vector space* V is a set whose elements could be added or multiplied by scalars and the results of such operations are still elements of V. The following rules are characteristic to a vector space:

1. Addition associativity: **a**+(**b**+**c**)=(**a**+**b**)+**c**= **a**+**b**+**c**, $\forall$ **a**, **b**, **c** $\in$ V
2. Addition commutativity: **a**+**b**=**b**+**a**, $\forall$ **a**, **b**$\in$ V
3. There is a null element: **a**+0=0+**a**=**a**, $\forall$ **a**$\in$V
4. There is an opposite element: $\forall$**a**$\in$V $\Rightarrow \exists$ -**a**$\in$V so that **a**+(-**a**)=(-**a**) +**a**=0
5. Scalar multiplication is associative: $\lambda(\mu\mathbf{a})=(\lambda\mu)\mathbf{a}$, $\forall \lambda, \mu \in$ R, **a**$\in$ V
6. There is a unit scalar: 1**a**=**a**1=**a**, $\forall$ **a**$\in$V
7. Two distributivity laws: $\lambda(\mathbf{a}+\mathbf{b})=\lambda\ \mathbf{a}+\lambda\ \mathbf{b}$; $(\lambda + \mu)\mathbf{a}=\lambda\ \mathbf{a}+\mu\ \mathbf{a}$, $\forall \lambda, \mu \in R$, **a**, **b**$\in$ V

Definition: A *metric space* is a set X together with a function

$$d : X^2 \to R^+$$

(called distance function). A *distance function* has the following properties:

1. Symmetry: d(x, y)=d(y, x)
2. $\forall$x, y $\in$R $\Rightarrow$ d(x,y)=0 iff x=y
3. Triangle inequality: d(x,y)$\leq$d(x,z)+d(z,y), $\forall$x,y$\in$R

Definition: An *Euclidean space* E^n is a set whose elements (called points) can be one to one mapped to the vectors of a n dimensional vector space and is in the same time a metric space where the distance between two points a, b (called Euclidean distance) is given by: $\sqrt{(a-b)^2}$

Definition: A function $f(x_1, x_2, \ldots, x_n)$ of n independent variables is called *symmetric* if its value is not affected by any permutation of its arguments.

.2 Appendix 2 - Quaternions

The quaternions were discovered by sir Wiliam Rowan Hamilton (1805-1865) who looked for a system of numbers that was able to represent rotations in the 3D space. They were remembered to the computer graphics community by Ken Shoemake in a famous paper ***[Shoe 85]***.

A quaternion is represented by four real numbers (w, x, y, z) and is designated by the following notation:

q= w + x$\mathbf{i}$ + y$\mathbf{j}$ + z$\mathbf{k}$ = w + $\mathbf{u}$

Sometimes w is called the *real part* of the quaternion and u=x$\mathbf{i}$+y$\mathbf{j}$+z$\mathbf{k}$ is called a *pure quaternion*. The space of pure quaternions is isomorphic with the 3D space and practically the rotation/translation transformations can be applied to pure quaternions. $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are primitive elements which satisfy the following properties (useful for defining the quaternion multiplication): $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$; $\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}$; the product of the primitive elements is anticommutative and $\mathbf{ijk} = 1$ (the original Hamilton definition stated $\mathbf{ijk} = -1$).

In fact, as we saw in the second chapter, the space of pure quaternions is practically identical to the space of 2-blades, and quaternions are forming a 4 dimensional space as that of the even multivectors of the G_3 Geometric algebra. $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ correspond respectively to $\mathbf{e_1} \wedge \mathbf{e_2}$, $\mathbf{e_2} \wedge \mathbf{e_3}$ and $\mathbf{e_3} \wedge \mathbf{e_1}$. In the original Hamilton definition the system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ corresponded to a left-handed bivectors system.

With these properties satisfied, the result of quaternion multiplication is:

$$\begin{aligned} q_1 q_2 &= (w_1 + x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})(w_2 + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\ &= (w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2) + (w_1x_2 + w_2x_1 + y_1z_2 - z_1y_2)\mathbf{i} + \\ &\quad (w_1y_2 + w_2y_1 - x_1z_2 + z_1x_2)\mathbf{j} + (w_1z_2 + w_2z_1 + x_1y_2 - y_1x_2)\mathbf{k} \end{aligned} \tag{56}$$

Other elements associated with a quaternion are:

- its conjugate: q^*=w-x$\mathbf{i}$-y$\mathbf{j}$-z$\mathbf{k}$
- its module: $\mid q \mid^2 = w^2 + x^2 + y^2 + z^2$
- its (multiplicative) inverse: $q^{-1} = q^* / \mid q \mid^2$

Quaternion addition has an obvious definition, equally obvious is the existence of the opposite element. It can be easily verified that the quaternion space forms with the addition and the multiplication a non-commutative ring.

Rewriting the relation 56 and considering the pure quaternions as usual vectors (here however is an inconsistency because $\mathbf{i}^2$=1 if $\mathbf{i}$ would be the unit vector of the x axis) we obtain:

$$\begin{aligned} q_1 q_2 &= (w_1 + \mathbf{u_1})(w_2 + \mathbf{u_2}) \\ &= w_1w_2 - \mathbf{u_1} \bullet \mathbf{u_2} + w_1(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\ &\quad + w_2(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \\ &\quad + (y_1z_2 - y_2z_1)\mathbf{i} - (x_1z_2 - x_2z_1)\mathbf{j} + (x_1y_2 - y_1x_2)\mathbf{k} \\ &= w_1w_2 - \mathbf{u_1} \bullet \mathbf{u_2} + w_1\mathbf{u_2} + w_2\mathbf{u_1} + \mathbf{u_1} \times \mathbf{u_2} \end{aligned} \tag{57}$$

where:

$$\mathbf{u_1} \times \mathbf{u_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

It is easy to verify that: $(q_1q_2)^* = (q_2)^*(q_1)^*$.

In the quaternion space the unit quaternions could be written as:

$q = cos(\phi) + \mathbf{u}sin(\phi) = e^{u\phi}$, where **u** is a pure quaternion of module 1. In this case, indeed: $\mid \mathbf{u} \mid^2 = cos^2\phi + \mathbf{u}^2 sin^2\phi = 1$. The unit quaternions are representing rotations in the 3D space as could be seen from the following

Theorem: The unit quaternion $q = cos\phi + \mathbf{u}sin\phi$ represents a rotation of angle 2ϕ around the axis whose direction is given by the unit vector **u**. The result of the rotation of point **p** is $R(\mathbf{p}) = q\mathbf{p}q^*$, where R(**p**) is a pure quaternion corresponding to the rotated position of **p**.

Proof:

Let's note for the beginning that: $q = cos\phi + sin\phi(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})$

Repeatedly applying 57 results:

qpq^*=(cos ϕ + **u** sin ϕ) **p** (cos ϕ - **u** sin ϕ) = (cos ϕ + **u** sin ϕ) ((**p•u**)sin ϕ + cos ϕ **p** - sin(ϕ)(**p** × **u**)) = (**p•u**)sin ϕ cos ϕ - sin ϕ **u•**(cos ϕ **p** - sin ϕ (**p** × **u**)) + $cos^2\phi$ **p** - cos(ϕ)sin(ϕ)(**p** × **u**) + $sin^2\phi$ (**p•u**) **u** + sin ϕ cos ϕ **u** × **p** - $sin^2\phi$ **u** × (**p** × **u**) =$cos^2\phi$ **p** + $sin^2\phi$ (**p•u**)**u** +2 sin ϕ cos ϕ **u** × **p** +$sin^2\phi$(**u•p**)**u**-(**u•u**)**p**)= ($cos^2\phi$ **p** - $sin^2\phi$ (**u•u**))**p** + 2$sin^2\phi$ (**u•p**)**u** + sin(2ϕ)**u** × **p**= cos(2ϕ)**p** + (1-cos(2ϕ))(**u•p**)**u** + sin(2ϕ)**u** × **p** i.e.

$$qpq^* = cos(2\phi)\mathbf{p} + (1 - cos(2\phi))(\mathbf{u} \bullet \mathbf{p})\mathbf{u} + sin(2\phi)\mathbf{u} \times \mathbf{p} \quad (58)$$

In the above relations it was used the identity:

$$\mathbf{u} \times (\mathbf{p} \times \mathbf{u}) = (\mathbf{u} \bullet \mathbf{u})\mathbf{p} - (\mathbf{u} \bullet \mathbf{p})\mathbf{u} = \mathbf{p} - (\mathbf{u} \bullet \mathbf{p})\mathbf{u} \quad (59)$$

valid for a unit vector **u**.

The proof of 59 is trivial **u** × (**p** × **u**) is orthogonal to **u** and **p** × **u** and has the module $\mid \mathbf{u} \mid\mid \mathbf{p} \mid\mid \mathbf{u} \mid sin(\phi)$ (where ϕ is the angle between **u** and **p**). On the other side **p**-(**u•p**)**u** is placed in the plane $\mathbf{p} \wedge \mathbf{u}$ (therefore orthogonal to **p** × **u**) and orthogonal to **u**. OA (see Figure 10) has exactly the length $\mathbf{p} - (\mathbf{p} \bullet \mathbf{u})\mathbf{u}$. In conclusion the two vectors have the same orientation and the same module $\mid AP \mid = OPsin(\phi)$ i.e. they are equal.

The same fact could be proved through geometric algebra means:

a × (**b**×**c**) = -**a•**(**I**(**b** × **c**))=-**a•**(**I**(-**I**(**b**∧**c**)))= **a•I**^{2}(**b**∧**c**)= -**a•**(**b**∧**c**)= -(**a•b**)**c**+(**a•c**)**b**

where **a**×**b**=-**a•**(**Ib**) results from:

I(**ab**)=(**Ia**)**b**= (**aI**)**b**=**a**(**Ib**) ⇔ **I**(**a•b**+**a**∧**b**) = **a•**(**Ib**)+**a**∧(**Ib**) ⇒

$<\mathbf{I}(\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\wedge\mathbf{b})>_1 = <\mathbf{a} \bullet(\mathbf{Ib})+\mathbf{a}\wedge(\mathbf{Ib})>_1$ ⇒ **I**(**a**∧**b**)=**a•**(**Ib**)⇔

-**a**×**b**=**a•**(**Ib**) ⇔ **a**×**b** = -**a•**(**Ib**) q.e.d.

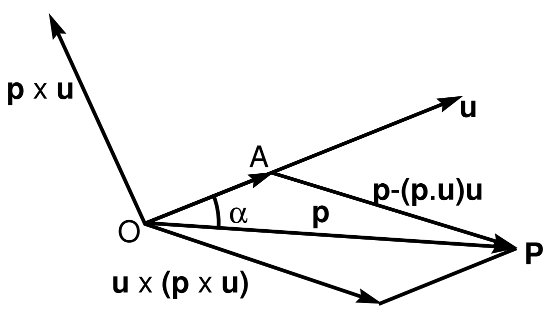

Figure 10: **u**×(**p**×**u**)=**p**-(**p•u**)**u**

The formula that gives a rotation around u with angle ϕ is deduced from Figure 11.

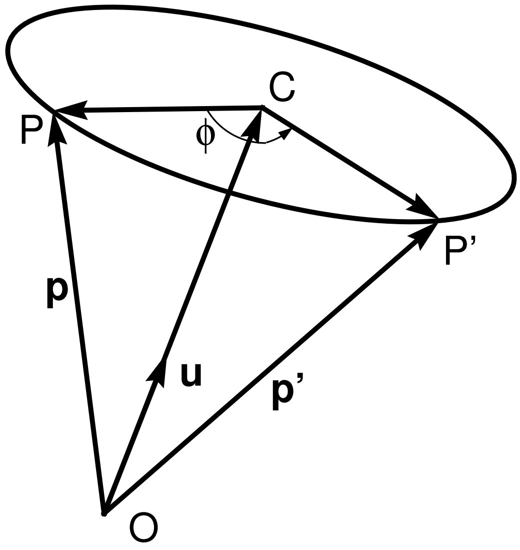

Figure 11: Rotation of a point

$OP' = \mathbf{r_{P'}} = OC + CP' = (\mathbf{r_p} \bullet \mathbf{u})\mathbf{u} + cos(\phi)CP + sin(\phi)\mathbf{v}$
$CP = \mathbf{r_P} - (\mathbf{r_P} \bullet \mathbf{u})\mathbf{u}$
The vector **v**=**u**×CP = **u**×($\mathbf{r_P}$-($\mathbf{r_P}$•**u**)**u**) = **u**×$\mathbf{r_P}$ has the same length with CP and is normal to CP.

$$\begin{aligned}\mathbf{p}' &= (\mathbf{p} \bullet \mathbf{u})\mathbf{u} + (\mathbf{p} - (\mathbf{p} \bullet \mathbf{u})\mathbf{u})cos\phi + sin\phi \mathbf{u} \times (\mathbf{p} - \mathbf{u}(\mathbf{p} \bullet \mathbf{u})) \\ &= \mathbf{p}cos\phi + (\mathbf{p} \bullet \mathbf{u})(1 - cos\phi)\mathbf{u} + sin\phi(\mathbf{u} \times (u))\end{aligned} \tag{60}$$

Finally from 58 and 60 could be concluded the truth of the theorem.

A rotation/translation transformation of a point P could be represented through quaternions as: **p' = u + r p r**

where:

- **u** is a pure quaternion that corresponds to the translation vector
- **p**, **p'** are pure quaternions corresponding to the initial and respectively transformed point
- **r** is a unit quaternion corresponding to the rotation

A rotation/translation transformation is therefore represented (using quaternions) by 7 numbers and not by 16 as in the case of a matrix transformation in the homogeneous space.

If the angle of rotation ϕ and the rotation axis (u_1, u_2, u_3) are known, the four coefficients of the corresponding quaternion (that codifies the above mentioned rotation) are: $w_1 = cos(\phi/2)$, $x_1 = u_1 sin(\phi/2)$, $x_2 = u_2 sin(\phi/2)$, $x_3 = u_3 sin(\phi/2)$. Inverse, given the unit quaternion q=w+x**i**+y**j**+z**k** we can find the angle of rotation as ϕ=2arccos(w) and the axis of rotation has the coefficients (x/sin(ϕ/2), y/sin(ϕ/2), z/sin(ϕ/2)), where $sin(\phi/2) = \sqrt{(1 - w^2)}$.

If the rotation/translation transformation is denoted by RT(**u**, **r**) the quaternion corresponding to the composition of two such transformations RT($\mathbf{u_1}, \mathbf{r_1}$) and RT($\mathbf{u_2}, \mathbf{r_2}$) is computed as:

$$\begin{aligned}\mathbf{u_1} + \mathbf{r_1}(\mathbf{u_2} + \mathbf{r_2}\mathbf{p}\mathbf{r_2}^*)\mathbf{r_1}^* &= RT(\mathbf{t_1}, \mathbf{r_1}) \circ RT(\mathbf{t_2}, \mathbf{r_2}) \\ &= RT(\mathbf{t_1} + \mathbf{r_1}\mathbf{t_2}\mathbf{r_1}^*, \mathbf{r_1}\mathbf{r_2})\end{aligned} \tag{61}$$

An object movement on a given trajectory could be expressed through a sequence of couples (**u**, **r**) specifying the position/orientation of the instance referential associated to the object, relatively to the global referential. If some samples are chosen on the trajectory and the translation t_i respectively the rotation (axis u_i and angle ϕ_i) that make possible the transformation between positions i and i+1 are known, let's note them:

$(t_i, r_i) = (t_i, cos(\phi_i/2) + u_i sin(\phi_i/2))$

The movement can be simulated by repeatedly applying relation 61 and (after determining each new pair (t, r) corresponding to one step on the global trajectory) accumulating the partial transformations to the global transformation supported by the object. The instance transformation matrix is computed from the resulted quaternion and eventually multiplied with the other matrices of the visualization pipeline.

Proposition: The vector expression giving a rotation of angle ϕ around the axis **u**=(u_1, u_2, u_3) specified in 59 could be written matriceally as:

$$P' = RP \tag{62}$$

where:

$$R = I + sin(\phi)S + (1 - cos(\phi))S^2, and S = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix}$$

Proof: trivial by direct verification.

The above proposition allows to deduce the rotation matrix associated to a given quaternion q=w + **u**. First ϕ=2arccos(w) is computed then the rotation axis coefficients: x/$\sqrt{(1-w^2)}$, y/$\sqrt{(1-w^2)}$, z/$\sqrt{(1-w^2)}$ and finally the transformation matrix is computed using 62.

.3 Appendix 3 - Classes of linear transformations

As it is well known, a linear transformation of the geometric Euclidean space can be expressed by a matrix multiplication in the homogeneous space.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{14} \\ & \dots & & \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{63}$$

If the basic Euclidean space is n-dimensional (E^n) then the corresponding homogeneous space (H^{n+1}) is n+1 dimensional and any line passing through the origin of the n+1-dimensional homogeneous space projects in a point of the original n-dimensional Euclidean space[1]. See Figure A3-1 for example, there the plane (π) (i.e. z=1) is the considered Euclidean space E^n (n=2) and the 3D space is the homogeneous space. Every point (P) in the Euclidean space is labeled by a vector (**p**) of the homogeneous space (including the former origin which is labeled by $\mathbf{e_0}$). In the same manner a line with offset in the Euclidean space is obtained by intersecting a plane passing through the origin of the homogeneous space (i.e. a 2-blade of the homogeneous space) with plane (π) (i.e. the initial Euclidean data space).

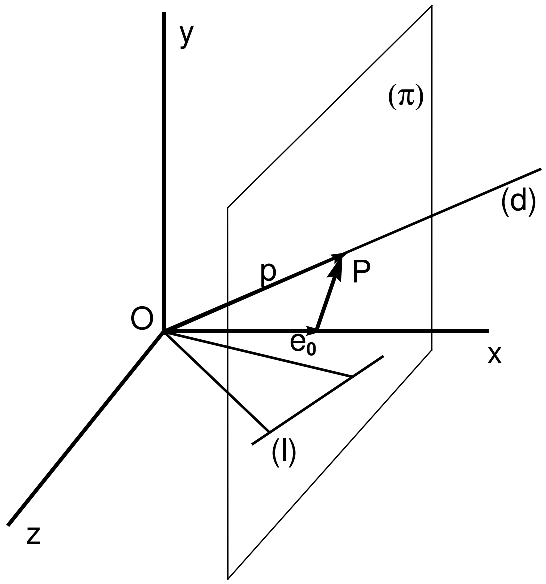

Figure 12: The correspondence between the Euclidean space E^n and the homogeneous space in case n=2

There are some important classes of linear transformations of the homogeneous space, they actually form a hierarchy of transformations and preserve invariant different quantities characteristics to the initial and transformed sets of points. In the following the case n=3 will be considered.

If the homogeneous matrix transformation is non-singular the corresponding Euclidean transformation is called a *projectivity.* It is the most general linear transformation in the homogeneous space. Since every point of the original space has a corresponding line (passing through the homogeneous origin) in the homogeneous space, practically the "encoded" geometrical information

[1] The usage of homogeneous coordinates has as purpose to eliminate the distinction between the origin and the other points of an Euclidean space. The method consists in removing the origin from E^n and placing it in a higher dimensional space.

from one homogeneous 4-tuple (4-dimensional point) is given by the ratio of the first three components with respect to the fourth. That is any homogeneous point $[x,y,z,w]^T$ (where w has a non null value) encodes the point of the Euclidean space having the coordinates [x/w, y/w, z/w]. This implies practically that two projectivities having the associated matrices M and αM ($\alpha \in R^*$) specify the same effect when applied to the Euclidean space. Otherwise speaking a projective transformation has 15 degrees of freedom and the element m_{44} could be almost always (i.e. when $m_{44} \neq 0$) considered 1. The division by m_{44} of all the matrix elements corresponds to the so-called *global scaling transformation*. The values that are completely determining the transformation effect are practically the ratios $\frac{m_{ij}}{m_{44}}$. A projectivity preserve invariant the cross ratio of four collinear points A, B, C, D i.e. the ratio $\frac{AB.CD}{AC.BD}$

The *isometry* is a transformation $R^3 \rightarrow R^3$ that preserves the Euclidean distances between corresponding points. An important category of isometries are the Euclidean transformations (called sometimes *rigid object displacements*). They are orientation preserving transformations and the general form of a matrix expressing an Euclidean transformation in the homogeneous model is: $\begin{bmatrix} \mathbf{R}_{3\times3} & \mathbf{T}_{3\times1} \\ \mathbf{0}^T_{1\times3} & 1 \end{bmatrix}$ where $R_{3\times3}$ is an orthonormal transformation ($R^T = R^{-1}$ and det(R)=1). The sign of det(R) gives the characteristic of *sense preserving* or *sense reversing* transformation.

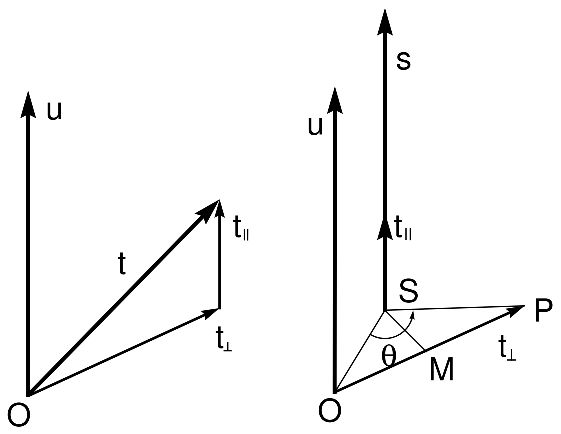

Figure 13: The screw axis for an Euclidean transformation

A general Euclidean transformation has 6 degrees of freedom. Such a transformation is a composition between a rotation around an arbitrary axis **u** (passing through O) with an angle θ and a translation of vector **t**. This is equivalent to a rotation around an axis **s** (passing through a point S) having the same direction as **u** but different spatial positioning and a translation of vector $\mathbf{t}_{\|}=P_{\mathbf{u}}(\mathbf{t})$. This result known as the *screw movement theorem* (or the *Chasles theorem*) implies that an Euclidean transformation has 6 degrees of freedom. The degrees of freedom are (see Figure A3.2): 2 for the translation required to find the proper position of S (in the plane passing through O and normal to **u**), 3 to accomplish the rotation around the screw axis and finally one degree of freedom due to the translation along the screw axis with length $\mathbf{t}_{\|}$. The point S is placed in the plane passing through O and normal to **u**, on the perpendicular bisector of OP and observes OP under an angle θ.

The invariants preserved by this category of transformations are: distances between corresponding point pairs, angles between corresponding lines, areas and volumes.

Note: A transformation that preserves angles is called *conformal mapping*.

Another class of transformations are *similarities*. They include besides the above mentioned elementary transformations that characterize an isometry, a supplementary uniform scaling.

Similarities preserve the shapes of geometric objects that is the ratios of lengths between corresponding pairs of points. In this case the matrix $RS_{3\times3} = \mathrm{s}R_{3\times3}$ where $R_{3\times3}$ is the matrix of an orthonormal transformation; this gives one more degree of freedom than an Euclidean transformation i.e. totally 8 degrees of freedom.

An important class of projectivities are the *affine transformations*. One thus transformation corresponds to nonsingular linear transformation of the geometric Euclidean space, followed by a single translation. The form of an affinity transformation matrix is $\begin{bmatrix} A_{3\times3} & T_{3\times1} \\ 0^T_{1\times3} & 1 \end{bmatrix}$, where $A_{3\times3}$ is an arbitrary nonsingular matrix. The affinities could be considered as resulting from the composition of an arbitrary rotation with an non-uniform scaling (applied relatively to an arbitrary referential). Such transformations preserve collinearity, parallelism, ratio of lengths of parallel segments, centroids and volume ratios. An affinity could equally be orientation preserving or reversing according to the sign of det(A) and has 12 degrees of freedom (the corresponding homogeneous matrix has always the last line [0, 0, 0, 1] in case of an affine transformation).

Finally, returning to the most general class of linear homogeneous space transformations, the projectivities, it ought to be mentioned that the general form of the corresponding homogeneous matrix is: $\begin{bmatrix} A_{3\times3} & T_{3\times1} \\ V^T_{1\times3} & 1 \end{bmatrix}$. The vector $V^T_{1\times3}$ is non-null only for the general projectivities. It is responsible for some deformations; for example *ideal points*[2] are mapped to finite points (*vanishing points*).

[2] An ideal point, called also point at infinity has the homogeneous representation $[x, y, z, 0]^T$ and specify in fact an Euclidean direction.

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