Clues to the Paradoxes of Knowability:
Reply to Dummett and Tennant

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In 'Victor's Error' (2001) Dummett considers the semantic anti-realist's conception of truth as knowability. He ponders Fitch's paradox of knowability,[1] which threatens any such conception. Dummett maintains that the anti-realist's error is to offer a blanket characterization of truth, expressed by the following knowability principle: any statement A is true if and only if it is possible to know A. Formally,

$$\mathrm{Tr}(A) \quad \text{iff} \quad \Diamond K(A)$$

To remedy the error, Dummett's proposes the following inductive characterization of truth:

> (i) Tr(A) iff $\Diamond$K(A), if A is a basic statement;
> (ii) Tr(A and B) iff Tr(A) & Tr(B);
> (iii) Tr(A or B) iff Tr(A) v Tr(B);
> (iv) Tr(if A, then B) iff (Tr(A) $\rightarrow$ Tr(B));
> (v) Tr(it is not the case that A) iff ¬Tr(A),
>
> where the logical constant on the right-hand side of each biconditional clause is understood as subject to the laws of intuitionistic logic.[2]

The only other principle in play in Dummett's discussion is

> (+) A iff Tr(A),

which, as he notes, the anti-realist is likely to accept.[3]

With the restriction on the knowability principle, expressed by clause (i), the knowability paradox is blocked. That is because Fitch's paradoxical result requires the substitution of 'B & ¬K(B)' for 'A' in the knowability principle.[4] And the conjunction 'B

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[1] The paradox of knowability derives from more general results found in Fitch (1963).
[2] Excluded here are Dummett's clauses for existential and universal statements. They do not enter into our discussion.
[3] Dummett's discussion involves only the left-to-right formulation of (+). However, we will make use subsequently of the right-to-left formulation.
[4] The paradox may be summarized as follows. Clearly we are non-omniscient, so there is statement B such that B & ¬K(B). From the anti-realist's unrestricted knowability principle, it follows that it is possible to know B & ¬K(B). However, using standard modal resources, it can be shown independently that it is

& ¬K(B)' is not basic. This is obvious if by 'basic' Dummett means 'atomic' or 'truth-functionally simple'.

As it stands, Dummett's treatment of the paradox is unprincipled. The only reason we are given for restricting the knowability principle to basic statements is that it blocks Fitch's result. But that is not our main criticism. Even if the restriction can be well motivated, alternative formulations of the paradox may be developed against Dummett's inductive characterization of truth. Below we consider permutations on Fitch's result that satisfy Dummett's restriction on the knowability principle.

Dummett admits that something more substantial needs to be said about which statements are basic statements. And he says one thing further about how one must proceed toward this end:

> if [t]his inductive characterization of truth is to be comprehensive, the basic statements must include all those that cannot be represented as in any of the forms governed by clauses (ii) to [(v)], or by any supplementary clauses. (Dummett 2001: 2)

Consider epistemic statements of the form 'K(B)'. Are they basic? Dummett's inductive account underdetermines an answer to this question. Such statements are not governed by the provided clauses. So either such epistemic statements are basic, or their truth conditions are to be given by supplementary clauses. Either way, Dummett's account gets into trouble. Here is the dilemma.

First, let us suppose that statements of the form 'K(B)' are basic. Arguably they are, since they are not *truth-functionally* complex. This allows substitutions of 'K(B)' for 'A' in clause (i). Consider the following result:

1. B & ¬K(B)	Assumption
2. Tr(K(B)) → ◇KK(B)	by clause (i)
3. Tr(B) → ◇K(B)	by clause (i)
4. ¬◇KK(B)	from 1 and 2, utilizing (+)
5. ◇K(B)	from 1 and 3, utilizing (+)

logically impossible to know B & ¬K(B). That is because a conjunction is known, only if its conjuncts are known. And to know the left conjunct is to contradict the right conjunct. The anti-realist is forced to admit that we are not non-omniscient. An unwelcome consequence indeed.

Line 1 is the Fitch conjunction, which is true for some statement B since we are non-omniscient. Lines 2 and 3 are substitution instances Dummett's clause (i), on the assumption that 'K(B)' and 'B' are basic. The inference to line 4 is abbreviated. We take the right conjunct of 1, and apply (+). This gives us ¬Tr(K(B)). From line 2, it follows that ¬◇KK(B). Line 5 follows similarly from the left conjunct of 1 conjoined with line 3.

Consider the following closure principle: if a conditional is necessary, then if the antecedent is possible so is the consequent. Since anti-realism is standardly taken to be a necessary thesis, it can be admitted that when the antecedent is possible the consequent is possible as well: ◇A → ◇◇K(A).[5] Substituting 'K(B)' for 'A', this principle entails ◇K(B) → ◇◇KK(B). Then, applying this formula to line 5 gives us ◇◇KK(B).

6. ◇◇KK(B)	from 5, by closure, clause (i) and (+).
7. ◇KK(B)) at w_1	from 6
8. KK(B) at w_2	from 7
9. ◇KK(B) in actuality	by the transitivity of R
10. Contradiction	from 4 and 9

If line 6 is actually true, then there is an accessible world w_1 at which ◇KK(B). And then, at a world w_2, which is accessible from w_1, it is true that KK(B). Now if R is transitive, then w_2 is accessible from the actual world, since w_2 is accessible from w_1 and w_1 is accessible from the actual world. So if R is transitive, in actuality ◇KK(B). This contradicts line 4.

So Dummett's inductive characterization of truth is not sufficient to salvage the analysis of truth as possible knowledge. If 'K(B)' is basic and R is transitive, then Dummett's inductive characterization of truth falls prey to the same problem it was meant to solve. The lesson is that Dummett has not quite put is finger on the source of the problem.

One might object that we simply need to treat R as non-transitive. But this has not yet been argued for. And it would not be very interesting simply to suppose the non-transitivity of R, having no reason other than the threat of the revised Fitch paradox to

[5] Strictly, the necessity of both anti-realism and (+) gives us the necessity of A → ◇K(A), which gives us ◇A → ◇◇K(A).

motivate the supposition. Pending further discussion, the supposition of non-transitivity is ad hoc.

Turning to the second horn of the dilemma, it may be objected that 'K(B)' is to be treated as a non-basic statement, in which case a supplementary clause is owed to the reader. Which constructive condition explains the truth of 'K(B)'? Whatever it is, it is not ruled out a priori that the clause will have as a consequence the KK thesis:

$$\text{(KK)} \quad \Box(K(B) \rightarrow KK(B)).$$

After all, if 'K(B)' is *constructively* true (i.e., if there is a finite and surveyable discourse that verifies 'it is known that B'), it is arguable that this can be turned into a constructive verification of 'KK(B)' (i.e., there is a finite and surveyable discourse that verifies 'it is known that it is known that B'). Further discussion is required to establish the validity of the KK thesis. The suggestion here is merely that it is not, for the constructive anti-realist, an implausible commitment.

It should also be noted that standardly the anti-realist takes '$\Diamond K(A)$' to be factive.[6] Accordingly, the anti-realist embraces the principle

$$\text{(F)} \quad \text{if } \Diamond K(A), \text{ then A, for all A.}$$

With a treatment of 'K(B)' as non-basic, and a commitment to (KK) and (F), a new permutation on Fitch's paradox may be formulated against Dummett's inductive characterization of truth:

1. $B \,\&\, \neg K(B)$	Assumption
2. $\Box(K(B) \rightarrow K(K(B)))$	from (KK)
3. $Tr(B) \rightarrow \Diamond K(B)$	by clause (i)
4. $\Diamond K(B)$	from 1 and 3, utilizing (+)
5. $\Diamond K(K(B))$	from 4 and 2, by closure
6. $K(B)$	from 5, by (F)
7. $K(B) \,\&\, \neg K(B)$	from 1 and 6

Once again we discover a version of the paradox that Dummett's inductive characterization fails to block. Importantly, this version treats 'K(B)' as a non-basic statement.

6 For recent discussions of this anti-realist commitment, see Tennant (forthcoming, 2002) and Wright (2001).

In conclusion, Dummett sketches an anti-realist conception of truth that boasts of having evaded Fitch's paradox of knowability. The important work is done by his restriction on the knowability principle. Dummett's primary error is not that the restriction fails to be motivated in a principled manner, though it does so fail. The more important error consists in Dummett's failure to realize that, even granting the restriction, versions of the paradox threaten his conception of truth. Dummett has not adequately diagnosed the source of the problem.

The results presented herein are problematic for other treatments that restrict the knowability principle in order to evade the paradox of knowability. Neil Tennant (1997: 273-74) offers a restriction less demanding than Dummett's. His proposal may be summarized as follows:

> Every true statement A is knowable, where 'K(A)' is not self-contradictory.

A defence of this clause is all that is needed to block the problematic substitution of 'B & ¬K(B)' for 'A' in the knowability principle. After all, K(B & ¬K(B)) is self-contradictory. Nonetheless, the two versions of the paradox presented herein do not require this, or any other, substitution that violates Tennant's restriction.

Moreover, an additional paradox threatens Tennant's analysis of anti-realism. It hinges on the anti-realist's interpretation of the possibility operator. Let us grant that (KK) is invalid and that $\Diamond$ is non-transitive. This liberates the restriction strategist from the paradoxes developed earlier. One non-transitive candidate is that of epistemic possibility. If a statement A is epistemically possible, then it is consistent with what is known that A.[7]

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[7] The following counter-model illustrates the non-transitivity of epistemic possibility:

> w1: {KA, ¬KKA}
> w2: {¬KA, A}
> w3: {K¬A, ¬A}

Since we are now presupposing the invalidity of KK, w1 does not by itself present a problem. And since we are presupposing Tennant's restriction on the knowability principle, the anti-realist (who takes the knowability principle to be true at w2) is not committed (via Fitch's reasoning) to the impossibility of w2.

Trivially, if it known that ¬A, we may conclude that it is not epistemically possible that A. In symbols,

$$(*) \quad \text{If } K\neg A\text{, then } \neg\Diamond A$$

This principle becomes useful below.

The problem with epistemic possibility is that it renders anti-realism inconsistent with the claim that there are undecid*ed* statements. Worse, it entails that there are no undecided statements, *necessarily*. An undecided statement is one for which neither it nor its negation is known. Formally,

A is *undecided* just in case $\neg KA \ \& \ \neg K\neg A$.

Let us suppose (for our primary reductio) that there is an undecided statement:

$$(1)\ \exists A(\neg KA \ \& \ \neg K\neg A)$$

This is an exceedingly modest assumption. In fact, it is intuitionistically weaker that the non-omniscience thesis, A & ¬KA, appearing in the Fitch paradoxes. If line 1 is true, then some instance of it is true:

$$(2)\ \neg KA \ \& \ \neg K\neg A.$$

Since line 2 does not violate Tennant's restriction (i.e., $K(\neg KA \ \& \ \neg K\neg A)$ is not self-contradictory), we may apply anti-realism to it. It follows from anti-realism that it is possible to know 2:

$$(3)\ \Diamond K(\neg KA \ \& \ \neg K\neg A).$$

Now let the anti-realist suppose for reductio that it is known that A is undecided:

$$(4)\ K(\neg KA \ \& \ \neg K\neg A)$$

Knowing a conjunction entails knowing each of the conjuncts. Therefore,

$$(5)\ K\neg KA \ \ \& \ \ K\neg K\neg A.$$

Applying principle (*) to each of the conjuncts gives us

Now notice that the truths at w2 are consistent with what is known at w1, so w2 is epistemically accessible from w1. Moreover, the truths at w3 are consistent with what is known at w2, so w3 is accessible from w2. Nonetheless, w3 is not accessible from w1, since there is a truth at w3 that contradicts something that is known at w1. So epistemic possibility is not transitive.

$$(6)\ \neg\Diamond KA\ \&\ \neg\Diamond K\neg A$$

Given the assumption of anti-realism, we derive the contradiction $\neg A\ \&\ \neg\neg A$. So the anti-realist must reject our assumption at line 4.

$$(7)\ \neg K(\neg KA\ \&\ \neg K\neg A)$$

Resting only on the assumption of anti-realism, which the anti-realist takes to be known, line 7 is now known:

$$(8)\ K\neg K(\neg KA\ \&\ \neg K\neg A).$$

But then, by (*), it is epistemically impossible to know that A is undecided:

$$(9)\ \neg\Diamond K(\neg KA\ \&\ \neg K\neg A).$$

But this contradicts line 3, which rests merely upon anti-realism and line 2. Line 2 is the instance of the undecidedness claim at line 1. A contradiction then rests on anti-realism conjoined with undecidedness. The anti-realist must reject the claim of undecidedness:[8]

$$(10)\ \neg\exists A(\neg KA\ \&\ \neg K\neg A).$$

Since anti-realism is taken to be a necessary thesis, it must be admitted by the anti-realist that, necessarily, there are no undecided statements:

$$(11)\ \Box\neg\exists A(\neg KA\ \&\ \neg K\neg A).$$

Line 11 says, necessarily, no statement is such that it and its negation are not known. Denying in this way that there is an undecided statement boasts of a kind of epistemic completeness that we are in no position to endorse a priori. After all, it is more likely that we will leave some stones unturned. Denying undecidedness in favor of epistemic completeness is bad enough, but things are worse. We see that anti-realism (a necessary thesis) entails the necessity of that completeness. But whether a statement is known is often a contingent matter. And so, whether such statements are undecided is a contingent matter as well.

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[8] Percival (1990) provides an analogous criticism against a defence proposed in Williamson (1988). Williamson suggests that the intuitionistic consequences of conjoining the knowability principle with the Fitch conjunction, $B\ \&\ \neg K(B)$, are harmless once constructively interpreted. In particular he suggests that

In defence of Tennant's strategy, one might object that 'K(¬KA & ¬K¬A)' is (or ought to be) self-contradictory. It entails line 5 of the above result:

(5) K¬KA & K¬K¬A.

Consider the left conjunct (i.e., "it is known that A is unknown"). Upon the development of a constructive interpretation of the knowledge operator, it may turn out that 'knowing that A is unknown' entails 'it is known that ¬A'. Formally,

(**) If K¬KA, then K¬A.

In that case, the left conjunct of 5 would contradict the right conjunct.

But there are clear counterexamples to (**). Let 'A' state that there is a particular fossil buried in a particular remote location, and suppose that we do not know whether A. And suppose that astronomers have learned that the sun just went supernova and that we have seven minutes before the destruction of the Earth and all its inhabitants. In that situation we may know that A is not known (by anyone ever) without knowing ¬A. It would be crazy to think that A is false (i.e., the fossil is not there), just because the explosion of the sun will soon destroy all of the evidence. And yet, that is what we should think, if (**) is valid.

Whether the undecidedness paradox threatens Tennant's characterization of truth is underdetermined. Not enough has been said about the logic of ◊ and K. Nevertheless, we are left with some clues about how the restriction strategist is to proceed. (KK) had better be invalid, if ◊K(A) is factive. Anti-realist possibility should not be transitive. And (*) should be invalid. That is, knowing the negation of A had better not imply the impossibility of A, as it does with the epistemic treatment of possibility.

Our conclusion is that the restriction strategies proposed thus far are insufficient to treat the real problem. The paradoxes presented herein turn on basic logic — and the ways in

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B → ¬¬K(B) holds under a constructivist interpretation. Percival shows, much to the dismay of the constructive intuitionist, that B → ¬¬K(B) is intuitionisically inconsistent with undecidedness.

which □ operates on epistemic statements. If a restriction strategy can be vindicated, this will be known only after we have formally analysed the anti-realist's notion of possibility.[9]

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