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The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds

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Abstract

The authors use risk-neutral option pricing theory to value the guaranteed minimum death benefit (GMDB) in variable annuities (VAs) and some recently introduced mutual funds. A variety of death benefits, such as return-of-premium, rising floors, and "ratches," are analyzed. Specifically, the authors compute the *fair* insurance risk fee, charged to assets, that funds the embedded option. The authors derive analytic option prices for a simplified exponential mortality model and robust numerical estimates in the case of a properly calibrated Gompertz model. The authors label this contingent claim a Titanic option because its payoff structure is in between European and American style but is triggered by death.

The authors' main objective is to compare theoretical estimates against a cross-section of insurance risk charges, as reported by Morningstar, Inc. The authors' main conclusion is that a simple return-of-premium death benefit is worth between one and ten basis points, depending on gender, purchase age, and asset volatility. In contrast, the median Mortality and Expense risk charge for return-of-premium variable annuities is 115 basis points. Presumably, the remaining markup can be attributed to profits, model imperfections, or, more cynically, to an implicit payment for the tax-deferral privilege.

Introduction

In the United States, a variable annuity (VA) policy is a collection of investment subaccounts that are wrapped with a life insurance contract. The *raison d'être* of the

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VA is the convenient deferral of all income and capital gains taxation until the funds are withdrawn or annuitized at retirement. The tax deferral on investment gains—of increasing long-term benefit—make VAs a popular alternative to fully taxable mutual funds, even though they involve higher management fees. In fact, despite the recent reduction in the long-term capital gains tax rate, sales of variable annuities amounted to $121 billion during 1999 and $100 billion during 1998. Interestingly, an estimated 55 percent of these variable annuity purchases were conducted within qualified, i.e., tax-sheltered, IRAs, 401(k)s, 403(b)s, and 457 retirement savings plans for which the tax deferral is redundant.[1]

However, in addition to the tax deferral—and in contrast to most mutual funds—the variable annuity provides a unique death benefit, possibly justifying its sale within tax-sheltered plans. The variable annuity death benefit stipulates that *at least* the original investment will be returned to the estate or beneficiary of the policy, *regardless* of the performance of the underlying assets in the account. This guaranty will be in-the-money if the VA policyholder (annuitant) dies during a "bear" market and at the same time the investment is showing a paper loss. Recently, some innovative mutual funds have joined with insurance companies to offer a similar death benefit on their investments. Presumably, this innovation has been partially driven by an attempt to compete with the VA market and its implicit investment protection. Of course, the tax deferral can only be provided by VAs.

The coincidence of death and market downturn, as unlikely as it may seem, is the impetus for this study. Specifically, the authors are interested in the economic value and fair cost of this death benefit.

Before embarking on this endeavor, it is important to mention that not all VA contracts and death-protected mutual funds (DPMF) are created equal. Many innovative insurance companies sell VA products with death benefits that are more "generous" than others. By generous the authors mean that the guaranteed minimum death benefit is higher than a simple return of premium plus minimal interest. For example, some products guarantee a more lucrative death benefit that locks in accrued investment gains during the life of the contract. As such, the VA and death-protected mutual fund market is not homogenous, and one must be careful with any generalizations about these products.[2]

Nevertheless, these guarantees do not come free and are paid for by the VA and DPMF holder in the form of an insurance risk charge. In the case of VAs, the fee is included in the so-called Mortality and Expense (M&E) risk charge. The M&E risk charge is a perpetual fee that is deducted from the underlying assets in the VA, above and beyond any fund expenses that would normally be paid for the services of managed money. According to the Morningstar Variable Annuity & Life Performance Report (April 1999), the average M&E risk charge was 113 basis points per annum. For products offering a simple return-of-premium guarantee, the average (median) M&E risk charge was 104 (115) basis points.

[1] Source: *Forbes Magazine*, February 1998, the Variable Annuity Research and Data Service (VARDS), the Life Insurance Market Research Association (LIMRA), and the Tillinghast-Towers Perrin's VALUE Survey.

[2] See Williamson (1999) for a consumer-oriented description of variable annuities and their features.

This study's main research question is therefore: What is a fair cost for a guaranteed minimum death benefit? And how does it compare to what the industry is charging?

From a contingent claims perspective, the guaranteed minimum death benefit (GMDB) on a VA or DPMF is essentially an increasing-strike put option with a stochastic maturity date. The embedded put option guarantees that the beneficiary of the VA or DPMF can "put" (read: sell) the investment contract back to the insurance company at a (possibly) better than market price.

Of course, the "putting" will be advisable only if the value of the account is less than the guaranteed floor, and it can only take place at the time of death—which is obviously unknown in advance. Now, although exercise can take place at any time, this product should be classified as in between European and American, because exercise can only be triggered by (involuntary) death. Therefore, the authors add the "Titanic Option" to the exotic dictionary. Practically speaking, the strike price of this put option will be equal to the original investment, in the event that the guaranteed return is zero. When there is a rising-floor feature, the strike price of the put will be increasing, in time, at the rising-floor rate—which obviously adds to the value of the guaranty. Some contracts allow for periodic resets, which lock in any gains. In this case, the option analogy is equally valid by substituting a floating strike look-back put option instead of the standard European variety. The holder of a look-back put is entitled to "put" the VA a receive a strike price that is equal to the maximum value of the account observed to date. Once again, the maturity is stochastic.

The authors will restrict attention to the single-premium variable annuity (SPVA) and not the flexible premium (FPVA) specification. Indeed, when each periodic premium is individually guaranteed, the total package can always be decomposed into single guarantees. When the sum of premiums is guaranteed, the model provides an upper bound.

A common actuarial approach for valuing these guarantees is to perform Monte Carlo simulations for the future evolution of equity values and interest rates (and possibly mortality), discount the expected death benefit, and then charge a premium that "protects" the insurance company a fixed percent of the time. Critical to such actuarial simulations would be the expected growth rate of the underlying assets in the variable annuity.

In this article—and in stark contrast to a (statistical) actuarial approach—the expected growth rate of the assets in the account is deemed *irrelevant* because of the risk-neutrality assumption in complete markets. In other words, the No Arbitrage value of the GMDB is equal to the price of a suitably parameterized put option, designed with a stochastic time horizon. Of course, the complete markets assumption is always questionable when applied in practice, especially as it pertains to the ability to completely diversify mortality risk. Suffice it to say that the authors' valuation methodology is predicated on the existence of a large enough pool of insured lives. This means that the insurance company can price mortality "risk neutrally" and assume a deterministic pattern for the aggregate death rate. Given the hundreds of billions of dollars invested in variable annuities, and to a lesser extent mutual funds, the authors believe this "largeness" assumption to be justified.

The only feature that is nonstandard in this product is that, in contrast to over-the-counter and exchange-traded derivative securities where payments are made on acquisition, this particular put option is paid by installments. Furthermore, these installments, so to speak, are paid by deducting a fixed *proportional* amount from the account value on a periodic basis. The proportional charges stand in contrast to deducting a fixed *nominal* amount. This perhaps minor-sounding feature leads to unique modeling problems that revolve around estimating the exact present value of the future risk charges. This issue is further complicated by the fact that the policyholder may (find it optimal to) lapse or surrender the contract before maturity. This will reduce the present value of fees, without an equal and opposite reduction in the value of benefits. In practice, of course, surrender charges of 1 percent to 7 percent for the first five to eight years of the policy serve as a strong deterrent to early termination.

Overall, the authors argue that from an economic equilibrium perspective, the proper (or fair) periodic charge for the GMDB should be such that its discounted expected present value is equal to the value of a put option with a stochastic maturity date.

Therefore, the authors are now faced with three tasks. First, it is necessary to compute the present value of the death-benefit risk charges as a fraction of the initial investment. Then the authors must compute the value of the above-mentioned stochastic-maturity put option, assuming an *exogenous* risk charge is deducted from the account. Finally, the authors must solve for a risk charge that equated the costs and benefits. The implied risk charge—which solves the expectation equality—is the *fair* cost for the guaranty. In reality, of course, loading charges, commissions, and profits would increase the price charged for the stochastic-maturity put. Nevertheless, this analysis contributes to the understanding of these products by shedding light on the portion of the additional charge that *should* be attributed to the pure mortality component.

It is important to stress that from an option pricing perspective, a higher insurance risk charge actually increases the probability that the put option will expire in-the-money. One can think of the risk charge as akin to the dividend yield on a stock. It is quite easy to show—and is well known to derivative specialists—that the higher the dividend yield, the more valuable the put option [see, for instance, Hull (1997) for details]. It is therefore factually incorrect to price the embedded put option and then amortize (or charge) the insurance fee over some predetermined time horizon. This will always understate the value of the guaranty.

In sum, the main contribution of this study is twofold. The authors obtain the No Arbitrage and equilibrium valuation of a stochastic-maturity put option where payment is made via a proportional reduction in the value of the underlying asset. With an assumed constant force of mortality, the authors show that the present value of the guaranteed minimum death benefit is the Laplace transform, in time, of the classical Black-Scholes/Merton equation adjusted for dividends, where the dividend is the insurance risk charge. The authors also provide numerical estimates for the *fair* risk charge using a properly calibrated Gompertz specification for the force of mortality.

Second, by assuming a 20 percent volatility (which is slightly above the historical average for VA policies) and a 6 percent interest rate, the authors conclude that a simple return of premium death benefit is worth between one to ten basis points, depending on purchase age. In contrast to this number, the insurance industry is charging a median Mortality and Expense risk charge of 115 basis points, although the numbers do vary widely for different companies and policies.[3] Presumably, the remaining markup can be attributed either to model and market imperfections or, more cynically, to an implicit payment for the tax-deferral privilege. The recently introduced death-protected mutual funds—whose investment gains are not tax sheltered—appear to be more fairly priced. The risk charge is age dependent and much closer to—although still higher than—the above-mentioned one to ten basis point range.

On a historical note, this research resonates with some earlier work by Greene (1973), where the excessive loads on variable annuities were criticized because of their long-term drag on investment returns.

Organization of the Article

The remainder of this article is organized as follows. The next section describes the particulars of the VA and DPMF market. The authors discuss the various products and the costs associated with the guarantees. Following that, the authors review some of the previous academic research in the field and then present some notation for the mortality theory. The authors then proceed with the technical core of the study, which develops a risk-neutral option pricing model for the valuation of the GMDB. The key is to locate a "fair" insurance risk charge that equates expected fees and benefits. The authors compute the risk charge under two different distributional assumptions for mortality: exponential and Gompertz. The exponential assumption, although unrealistic in practice, serves as an (easy to obtain) upper bound for the more reasonable Gompertz model. Readers who are simply interested in the output of this model are encouraged to skip directly to the section entitled "Numerical Examples," which provides estimates of the "fair" insurance expense (M&E) charges for a variety of guarantee types. The final section concludes the article with some suggestions for further research.

EMPIRICAL ANALYSIS OF THE MARKET

According to a 1999 survey commissioned by the Committee of Annuity Insurers and conducted by the Gallup organization, the average age at which owners purchased their first (nonqualified) variable annuity is 50. They also note an increase in the number of people under the age of 50 who have recently purchased variable annuities. This age will serve as an anchor in the later part of this analysis, when the authors compute the value of the death benefit assuming a particular purchase age. In the same survey, the gender distribution of annuity owners is roughly 52 percent female versus 48 percent male.

According to the Variable Annuity Research and Data Services (VARDS), sales of variable annuities amounted to $121 billion during 1999 and $100 billion during 1998, compared with $85 billion during 1997 and $72 billion during 1996. As of September 30, 1999, Morningstar Principia Pro (a database maintained by Morningstar, Inc.) reported a total of $690 billion of assets in 389 variable annuity policies, consisting of 6,607 subaccounts. The market for death-protected mutual funds is much smaller—

[3] The one company in the database that charged an M&E risk fee in the one to ten basis point range was TIAA-CREF with an annual charge of seven basis points.

Prudential Investments, SunAmerica, American Skandia, and Putnam Investments are the only four companies currently offering this option on their mutual funds—and very little market data are publicly available on the election rates or the volume of assets under management.

Types of Product

For the purposes of this study, two possible types of guaranteed minimum death benefits exist, as well as combinations thereof that are provided by either a variable annuity or a death-protected mutual fund. These two categories are (1) interest guarantees and (2) market guarantees. More precisely, they can be summarized as follows:

1. The basic product guarantees a return of *at least* the original investment at death. Technically, the payout to the beneficiary is $\mathrm{Max}[S_0, S_T]$, where S_0 denotes the original invested premium at time zero, and S_T is the market value of the account at death. As of September 30, 1999, according to Morningstar, 107 (27.65 percent of) variable annuity policies provide a simple "principal guaranteed."

2. An enhanced version of the product returns *at least* the original investment accrued at a minimally guaranteed interest rate. The typical guaranteed interest rate for this type of variable annuity contract is 5 percent, with very few companies offering more than 6 percent. In any event, the benefit is always capped at some predetermined level. The payout to the beneficiary is $\mathrm{Max}\left[\mathrm{Min}[S_0 e^{rT}, MS_0], S_T\right]$, where r is the (continuously compounded) fixed guaranteed rate, and M is the cap on the guaranteed return. According to Morningstar, 21 (5.43 percent of) variable annuity policies provide a "rising-floor" guaranty.

3. A guaranteed death benefit is based on a suitably defined highest anniversary account value. These anniversary step-up features vary widely from company to company: some policies offer an annual reset, while others require a ten-year wait, but the average is approximately five years. In the language of modern option pricing, the exercise price of the embedded put "floats" and increases to a new (higher) level every few years. The floating is based on the anniversary market value of the policy. The general representation is $\mathrm{Max}[S_{t_i}, S_T]$, where t_i is a suitably defined anniversary. The allowable reset frequency is closely linked to the number of years the surrender charge is in effect. In total, 172 (44.19 percent of) variable annuity policies provide some type of "look-back" guaranty.

4. A total of 36 (9.26 percent of) variable annuity policies give the holder a choice between Item 2 and Item 3. This decision must be made at the time of purchase and the exact features—such as number of years until step-up—vary from policy to policy. Finally, 53 (11.17 percent of) variable annuity policies listed in the Morningstar database report no guaranteed minimum death benefit.[4]

A priori, it is not clear whether one type of guaranty is better than another, because a higher interest rate may be preferable to a longer reset period. In sum, the term "in-

[4] A variable annuity does not require a guaranteed minimum death benefit. The insurance classification is established by the annuitization rates.

terest guaranty" refers to a GMDB in which the original premium is guaranteed to accumulate at a fixed (possibly capped) rate of return. The term "market guaranty" refers to a GMDB in which some degree of market returns are guaranteed in the form of anniversary reset. Most of the death-protected mutual funds provide combinations of both interest-rate and look-back guarantees.

In addition to the preceding guarantees, virtually all variable annuity policies guarantee some sort of living benefit in the form of a guaranteed annuitization rate (GAR). In other words, for those who are interested in annuitizing their accounts, the contract stipulates certain mortality tables and interest rates that will be used in the computation of the annuity payments. Clearly, this guaranty is an additional "option" that is embedded in the variable annuity. The implied interest rates are usually on the order of 3 percent to 4 percent, and the mortality table is the dynamically projected 1983 IAM (Individual Annuity Mortality) Table. The authors therefore proceed under the assumption that this benefit has little value—and is not a significant component of the M&E charge—since it is ignored by pricing actuaries, valuation actuaries, regulators, and the reinsurer.[5] Furthermore, only 2 percent to 3 percent of variable annuity accounts are ever annuitized, according to the Annuity Persistency Study survey conducted by Sondergeld (1997) at LIMRA. See Sharp (1999) as well as Pado and Sham (1997) and Campbell, Davidson, and O'Connor (1997) for discussions about reserving requirements.

However, two additional "living benefit" guarantees exist that *are* of value and have recently been introduced in the variable annuity market. They are known as Guaranteed Minimum Accumulation Benefits (GMAB) and Guaranteed Minimum Income Benefits (GMIB). GMAB guarantees a minimal account value upon termination of the product, not just upon death. GMIB ensures a minimal annuity payment, not just an annuity rate. Both of these benefits are sold as optional riders and are currently limited to a few large insurance companies. GMAB and GMIB are a small, but growing,[6] segment of the variable annuity market and always entail additional charges. As such, they will not be discussed further in this study.

Fees

The authors would like to emphasize that the policyholder does not get an actual "invoice" for management and insurance fees. Rather, the payments are charged to the account and implicitly reduce the amount of assets under management. This can be visualized as the company periodically selling, or surrendering, a (small) portion of the account and then using the proceeds to pay the management and insurance fees.

The total expense ratio (TER) on a variable annuity consists of three parts. The first component is called the Mortality & Expense (M&E) risk charge, the second component is the Administration & Distribution (A&D) risk charge, and the third component is the Fund Expense (FE) charge. Regrettably, it is very difficult to determine—

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[5] Nevertheless, the interested reader should refer to a companion research paper by Milevsky and Promislow (2000) in which the value of "options on mortality" is addressed separately.

[6] See Moody's Special Comments: *Riding the Wild Bull, Variable Annuity Growth Continues (May 1999)* and *Bells and Whistles: Credit Implications of the New Variable Annuities* (October 2000).

and most companies do not disclose—the amount of the M&E risk charge that goes toward funding the M (mortality) as opposed to the E (expense). However, when the breakdown is disclosed in the prospectus, the lion's share of the M&E risk charge is attributed to the mortality guaranty. The purpose of this article, in some sense, is to determine how much of the M&E risk charge *should be* allocated to the guaranteed minimum death benefit. Nevertheless, the sum of the M&E risk charge and the A&D risk charge is sometimes referred to as the insurance risk charge, which goes to the sponsoring company. The insurance expense charge should be contrasted with the fund expense (FE) charge, which goes towards the provision of the underlying asset management services.

According to Morningstar (September 30, 1999), the average M&E risk charge on the 389 variable annuity policies contained in its database was 113 basis points. The average A&D charge was 15 basis points, for a total insurance expense charge of 128 basis points.

From an actuarial perspective, the unisex, uni-age dependent nature of the M&E risk charge for variable annuity is quite puzzling. Clearly, the risk of death is much higher for older males compared to younger females, yet the annual premiums—as a percentage of assets—are the same. If indeed the M&E risk charge is meant to cover mortality risk, why is it not based on mortality risk? Interestingly, the vendors of death-protected mutual funds recognize the nature of the risk and *do* charge age-dependent fees, which range from 10 to 40 basis points.

More intriguing—as we shall see in the formal analysis—the theoretical value of the guaranteed minimum death benefit is an increasing function of the return volatility (standard deviation) of the underlying assets in the subaccounts. In other words, the more volatile the asset returns, the higher the risk charge should be. To this end, Table 1 provides a summary of the standard deviation of historical returns in the 1,547 subaccounts ($110 billion) of the 107 policies offering a return of premium guaranty. Of these 107 policies, the median asset-weighted standard deviation is approximately 18 percent. The 75th percentile is 20.3 percent. (Three-quarters of the subaccounts have experienced investment returns that are less volatile than 20.3 percent). Furthermore, according to Ibbotson Associates (1999), the historical standard deviation for U.S. stocks as an asset class is 20.3 percent. The authors therefore opt to use the 20 percent volatility figure for most of the numerical computations in the subsequent sections.

A simple linear regression of the M&E risk charge on the asset volatility indicates no significant relationship. This is quite puzzling because modern option pricing theory dictates that investors pay for higher volatility, yet this does not seem to be the case in the VA market. In fact, the theoretical value of the embedded GMDB option can be five times more expensive when the assumed volatility is 30 percent (the maximum) compared to 10 percent (the minimum).

Quite interestingly, the vendors of Canadian-based segregated mutual funds—which are essentially an enhanced death-protected mutual fund—impose an insurance fee that *does* depend on the volatility of the assets within the account. For example, growth funds, compared to balanced or bond funds, are charged three to four times as much in protection fees. The rationale is that higher volatility increases the (risk-neutral) probability of the option maturing in-the-money. Clearly, the Canadian insurance and actuarial community is alert to the costs of these options and is basing its so-called M&E risk charges accordingly. For more on Canadian segregated mutual funds, see Windcliff, Forsyth, and Vetzel (2000).

Of the 107 policies listed in the Morningstar database on September 30, 1999, that offered a basic return of premium guaranty, the historical volatility of the subaccounts was as follows:

TABLE 1
Historical Volatility

Maximum	75th Percentile	Median	25th Percentile	Minimum
30.65%	20.30%	18.37%	16.31%	10.24%

Finally, according to Morningstar, the average fund expense (FE) charge was 83 basis points,[7] for a total average expense charge (ME+AD+FE=TER) of 211 basis points. The Morningstar averages are not weighted by assets, nor do they distinguish between the various types of death benefit. Also, although the total expense ratios are reported on an annual basis, they are charged and paid monthly (and in some cases daily) based on the average account value during the month.

Of the 107 policies with a basic return of premium guaranty, listed in the Morningstar database as of September 30, 1999, the breakdown of management and insurance fees was as follows:

Therefore, to assist in the analysis of this study, the authors have constructed Table 2, which provides a detailed breakdown of the fee structure charged on variable annuity policies with a premium return guarantee. In contrast to the above-mentioned summary—which is provided by Morningstar—the authors have chosen to weigh the fees by the amount of assets within the policy. This should also provide a better picture of what the average person is paying in practice. The authors also decided to focus on the premium return policies—and not the universe of all policies—because they are homogenous in the structure of the death guaranty.

The average (asset-weighted) M&E risk charge for a simple premium return guaranty is 104 basis points. The 25th percentile is 90 basis points. In other words, 75 percent of the policies charge at least 90 basis points in M&E fees. As the authors will show in the next section—and in contrast to these numbers—this analysis will “value” the GMDB at between five and ten basis points, depending on age. Interestingly, only

[7] This number is lower than the average expense ratio on standard mutual funds, which is approximately 1.43 percent, according to Morningstar, Inc. Therefore, in some sense, the pure cost of managing money is 60 basis points lower for variable annuities than it is for mutual funds. However, when the comparison is done based on asset-weighted expense ratios, the average expense ratio for mutual funds is 101 basis points, while the average fund expense is 68 basis points. (Funds with high expenses tend to have less assets and vice versa.) So, in fact, on an asset-weighted basis, the gap is only 33 basis points. Of course, some of the fees that would normally appear under fund expenses are included in the administration and distribution charge, so a direct comparison may be misleading.

one company in the entire database charges an M&E in this range. The company with the lowest M&E risk charge is TIAA-CREF (an insurance company and pension fund manager based in New York), and it charges seven basis points.

TABLE 2
Management and Insurance Fees

	M&E Charge (in basis points)	A&D Charge (in basis points)	Fund Exp. (in basis points)	Total Exp. (in basis points)
Average	104	13	68	185
Maximum	130	89	169	294
75th Percentile	125	20	79	205
Median	115	10	65	189
25th Percentile	90	0	58	167
Minimum	7	0	7	37

For a slightly different perspective on the data, Table 3 lists the M&E risk charges for the look-back feature as a function of the number of anniversary years between reset dates. Although the average (asset-weighted) M&E risk charge appears to be higher compared to a premium return guaranty in Table 2, there appears to be no significant relationship between the number of years and the average charge. In other words, it is clearly better to reset the guaranty on an annual basis, compared to every five (or ten) years. This type of guaranty should be more expensive—and the current models will quantify this—yet the M&E risk charge does not increase with the frequency of the reset. The authors take this as further evidence that a very small portion of the M&E risk charge is allocated towards providing the enhanced death benefit. More will follow on this point later.

Of the 205 policies listed in the Morningstar database as of September 30, 1999, offering a look-back guaranty, the asset-weighted Mortality and Expense risk fees were as follows:

TABLE 3
Asset-Weighted Mortality and Expense Risk Fees

Step-Up	Number of Policies	Average M&E Charge
1 year	53	119
2 years	0	N.A.
3 years	9	109
4 years	1	125
5 years	48	112
6 years	45	107
7 years	39	118
8 years	7	120
9 years	1	125
10 years	2	139

Upon further examination of the database, some final empirical observations are in order.

- Most people diversify their investments within the variable annuity subaccounts. As such, the GMDB protection is based on the performance of the aggregate policy, as opposed to the performance of the individual subaccounts. This is why the authors insist on computing asset-weighted fees and standard deviation to account for the portfolio nature of this guaranty. In essence, the protection is based on a basket and not on each individual subaccount. Clearly, if the protection were based on each subaccount's performance, the value of the guaranty would be higher. Nevertheless, the ability to dynamically change the structure of the "basket," possibly to the detriment of the insurer, adds another "passport option" dimension to this problem.
- The surrender charges and penalties and the number of years over which they are imposed can be substantial. The authors computed an average surrender charge of 5.6 percent for the typical variable annuity policy, and this charge was imposed for the first 5.8 years of the life of the policy. The surrender charges and penalties are the only thing that stand in the way of "artificially" reestablishing the basis of the guaranty. In theory, the holder of the VA can sell and repurchase the policy each time the account value reaches a new high, thus creating a better guaranty. The interplay between surrender charges and optimal policyholder behavior is quite complex and will not be fully addressed in this study. The interested reader should refer to a subsequent paper by Milevsky and Salisbury (2000) in which the "Real Option" to lapse is dealt with specifically.
- There appears to be a relationship between the number of years over which a surrender charge is imposed, the magnitude of this charge, and the relative "appeal" of the death benefit. More lucrative policies—in the option sense—have higher all-inclusive fees and penalties. Clearly, the insurance companies are protecting themselves by keeping policies in force and increasing the probability of receiving the fees that are meant to pay for the death benefits.
- Overall, the authors detect a very strong "clustering" effect in the M&E risk charge. Although the fund expense (FE) charges vary across subaccounts and policies, the M&E risk charges do not. Despite the summary data provided in Table 2, a full 40 percent of policies impose an M&E risk charge of exactly 125 basis points. This is regardless of age, volatility, or guaranty structure. This appears to be a relic of previous S.E.C. regulations that capped the M&E risk charge at exactly 125 basis points.
- As mentioned earlier, GMAB and GMIB "living" benefits are provided as additional optional riders, and one must pay an additional 10 to 50 basis points (of assets, per annum) for this feature according to a 1999 Tillinghast-Towers Perrin survey. Approximately 15 companies offer this additional feature on their policies.
- Of the 1,706 (out of a universe of 6,607) subaccounts with a reported ten-year compound annual return, there was only *one* subaccount reporting a negative return (–1.16 percent). The average ten-year return was 11.09 percent, with the maximum being 25.1 percent.

- An important difference between variable annuity death benefits and mutual fund death benefits is that only the latter pass to heirs tax-free. VA death benefits are subject to federal and state income tax and thus worth roughly half as much to consumers as a tax-free death benefit. Mutual fund death benefits are tax-free because the product is written as regular term life insurance. Also, mutual fund death benefits are tax-free regardless of whether the mutual funds are inside or outside of a qualified plan, because the term life insurance policy is separate from the mutual fund. In contrast, annuity death benefits are fully taxable whether the annuity is nonqualified or in a qualified plan, because the product is not treated by the tax code as a life insurance contract. Also, variable annuities do not step-up in tax basis at death. In essence, the heirs inherit the annuity *and* the tax liability the annuity has deferred. In contrast, all gains on mutual funds would be passed on tax-free to heirs.

LITERATURE REVIEW

Overall, the research in this study is very similar in spirit to the recent work by Pennacchi (1999), in which he computed the "value" of guarantees in pension funds. The authors use the same approach to price the implicit performance guarantees provided by the vendors of VA and DPMF products.

The landmark theoretical contribution on which most of this research is based is the Brennan and Schwartz (1976) or Boyle and Schwartz (1977) extension of the Black-Scholes and Merton formula (1973) to equity-linked insurance contracts. They assumed a market that is complete in both financial and mortality risk. Therefore, all derivative prices can be expressed as suitable expectations with respect to an appropriate (risk-neutral) probability measure. Within the context of insurance, a complete market assumption implies that vendors can completely diversify their mortality risk by selling enough policies. Recently, researchers have questioned the complete markets assumption for both types of risk. For example, Follmer and Sonderman (1986) have introduced the concept of risk-minimizing strategies as an alternative to (the impossible) risk elimination strategies. This approach has been extended by Moller (1998) to unit-linked insurance contracts as well. Their "option value" is clearly higher than the examples here because of the inability to completely hedge and is dependent on the number of policies issued by the insurance company. Of course, as the number of policies increases, the option value approaches the complete market price. Stated differently, from an individual's perspective, the option has a stochastic maturity date. However, from the insurance company's (risk management) point of view, the aggregate maturity is deterministic, *assuming* a large enough pool of policyholders. The insurance company can then purchase fixed maturity put options—in the right proportions—and completely hedge its exposure to financial market risk. Therefore, given the huge size of the variable annuity market ($680 billion in 1999) and the ability of a large enough reinsurer to arbitrage this market, this article will rely on the complete markets assumption which underlies most of the academic research in this field.

Indeed, there has been an explosion in research activity surrounding the valuation of guarantees in mutual funds and investments.[8] As mentioned in the introduction, a

[8] In September 1999, the Society of Actuaries and the Canadian Institute of Actuaries sponsored a joint conference that was entirely devoted to the topic of investment guarantees in variable annuities and segregated mutual funds. Over a two-day period, 25 research papers were presented to an audience of more than 300 actuaries.

variety of companies in North America have recently introduced very (complex and) lucrative benefits that have reignited the interest of academics and practitioners. Most of the research is nonempirical in nature and attempts to precisely model a particular aspect of these contracts, as opposed to roughly value the products in their entirety.

For example, the issue of stochastic interest rates, and their effect on the value of guaranty, has been examined by Bacinello and Ortu (1993, 1996); Ekern and Persson (1996); Nielsen and Sandman (1995); Persson and Aase (1997); Aase and Persson (1994); and Miltersen and Persson (1999).

The question of an appropriate stochastic model for the long-term evolution of equity markets has been analyzed by Boyle and Hardy (1997) as well as Bilodeau (1997), and more recently Hardy (1999) and Wirch and Hardy (1999). The actuarial community has looked at the implications of a variety of model assumptions. See, for example, the analyses by Bernard (1993), Mitchell (1994), Brizeli (1998), and Ravindran and Edelist (1996).

More recently, Windcliff, Forsyth, and Vetzal (2000), as well as Boyle, Kolkiewicz, and Tan (1999), have looked at the "reset" feature available in some of the segregated mutual funds using Monte Carlo and numerical PDE approaches. In fact, Windcliff, Forsyth, and Vetzal (2000) conclude that Canadian insurers may be undercharging for this particular guaranty.

In contrast, this study's position in the literature is to (1) obtain simple closed-form expressions in the case of exponential mortality, (2) derive robust numerical estimates of the fair risk charge in the case of realistic mortality, and (3) compare these estimates to actual risk charges in the market.

Mortality: Notation and Terminology

For convenience and simplicity, the authors assume that the future lifetime random variable T can be expressed in continuous time. The authors do not believe this simplification detracts from the main results of the study, since any discrete mortality table can be "fitted" to a continuous function and must be done so for fractional ages in any event. See Carriere (1992) for details. The authors maintain standard actuarial notation wherever possible. Therefore, for an individual aged x, the probability of death before time $t \geq 0$, i.e., before age $x + t$, is denoted by $P(\tilde{T} \leq t) = 1 - ({}_t p_x)$. This is equivalent to $P(\tilde{T} \leq t) = 1 - \exp\{-\int_0^t \lambda(x+s)ds\}$, where $\lambda(x)$ denotes the *force of mortality* at age x. Technically speaking, the authors impose a finite lifetime condition: $\int_0^t \lambda(x+s)ds \to \infty$ as $t \to \infty$. Interchangeably, the authors let $F_x(t) = P(\tilde{T} \leq t)$ denote the *cdf* and $f_x(t)$ denote the *pdf* of the future lifetime random variable. They are related via the well-known relationship: $\lambda(x+t) = f_x(t)/(1 - F_x(t))$. See the classic book by Bowers et al. (1986) for more information. Finally, the net single premium for a life insurance contract that pays $DB(t)$ at death, is

$$\bar{A}_x = \int_0^\infty DB(t)e^{-rt} f_x(t)dt, \quad (1)$$

and the price of a life annuity that pays $1 per year, while still alive, is

$$\overline{a}_x = \int_0^\infty e^{-rt}(1 - F_x(t))dt \,, \tag{2}$$

where r is the risk-free rate (known as the force of interest). Indeed, *if* the variable annuity's death benefit were deterministic, Equation (1) would suffice to obtain its initial value. Dividing by Equation (2) would yield its periodic premium. However, the stochastic nature of the payment at death, $DB(t)$, requires a generalization of Equation (1), while the unique mode of proportional payment forces us to seek an alternative to Equation (2).

In this article, the authors will consider two particular forms of mortality function: an exponential distribution and a Gompertz approximation to the 1994 Group Annuity Mortality (Basic) table. With exponential mortality the authors assume that $\lambda(x) = \lambda$ for all ages. Therefore, $F_x(t) = P(T \le t) = 1 - e^{-\lambda t}$, $f_x(t) = \lambda e^{-\lambda t}$, and $E[T] = 1/\lambda$. Although the exponential distribution is quite convenient to work with—and the authors intend to use it for some results—it obviously has the unrealistic property that the probability of death is constant throughout the human life cycle. A more realistic and popular assumption for the continuous force of mortality—and one that the authors will use for empirical estimates at higher ages—is the Gompertz assumption for which $\lambda(x) = \exp\{(x - m)/b\}/b$. The Gompertz[9] distribution is parametrized by m—which is the modal value—and the dispersion parameter b. Although the Gompertz specification is not very accurate at lower ages, at higher ages—which is the demographic market for variable annuities—the authors find that the implied probabilities of survival are uniformly within 0.75 percent of table values. The exact parameters used depend on the current age of the individual. For example, the 1994 Group Annuity Mortality (Basic) table can be fitted to a Gompertz specification using a nonlinear least squares methodology resulting in parameters m,b. The 1994 GAM table has been selected and is particularly relevant in this case, given its position in reserving requirements for variable annuities.

The Gompertz distribution was calibrated to the 1994 GAM (Basic) table at higher ages. The exact parameters were obtained using a nonlinear least squares minimization procedure.

The parameters listed in Table 4 are used later in the study to obtain the Titanic option values.

PRICING MODEL

As mentioned in the introduction, the authors' objective is to determine a "fair" insurance risk charge that equates fees and benefits. The authors first compute the present value of fees, assuming a fixed-risk charge and then compute the present value of benefits, also assuming a fixed-risk charge. Both expressions are nonlinear in the risk charge. In the final step, the authors equate the two and solves for the proper risk charge using numerical methods. Once that is achieved, the authors focus attention on the issue of lapsation and its effect on both sides of the equation.

[9] See Carriere (1992, 1994) for discussions of the properties of the Gompertz distribution as well as the references therein to support its usage by the actuarial community.

TABLE 4
Gompertz Distribution

	Female			Male		
Age (x)	m	b	$x + Ex\,[T]$	m	b	$x + Ex\,[T]$
30	88.8379	9.213	83.61	84.4409	9.888	78.94
40	88.8599	9.160	83.82	84.4729	9.831	79.31
50	88.8725	9.136	84.21	84.4535	9.922	79.92
60	88.8261	9.211	84.97	84.2693	10.179	81.17
65	88.8403	9.183	85.69	84.1811	10.282	82.25

The Discounted Value of the Insurance Risk Charge

The authors compute the discounted value of the insurance risk charge by treating the stochastic case flows as a contingent claim on the underlying account value. The insurance company can be viewed as having a long position in the continuous-flow "fee" derivative. The derivative remains "alive" as long as the policyholder has not died or lapsed. Consequently, following standard assumptions in the literature, the authors model the general risk-neutral evolution of the asset by

$$dS_t = (r_t - l_t)S_t dt + \sigma(S_t, t)S_t dB_t, \quad S = 1 \tag{3}$$

where B_t is a one dimensional standard Brownian motion, r_t is the instantaneous interest rate (independent of S_t), l_t is the insurance risk charge, and $\sigma(S_t, t)$ represents the (possibly stochastic) volatility. The integral representation is

$$S_t = S_0 + \int_0^t (r_u - l_u)S_u du + \int_0^t \sigma(S_u, u)S_u dB_u, \tag{4}$$

where it should be emphasized that the second (Ito) integral is a martingale. In the special case of geometric Brownian motion, the volatility $\sigma(S_t, t)$ is constant. The authors also define the money market account, denoted by

$$R_t = e^{\int_0^t r_s ds}. \tag{5}$$

Now, let F_t denote the stochastic (discounted to time zero) value of fees collected until time t. By construction, we have

$$dF_t = R_t^{-1} l_t S_t dt.$$

The quantity $l_t S_t dt$ can be viewed as the instantaneous earnings of the insurance company, while the R_t^{-1} factor discounts the quantity to time zero. We are interested in both F_τ and its expectation $E[F_\tau]$, where τ is a *general* stopping time for the process S_t. First, by a simple chain rule, we have

$$\begin{aligned} d(R_t^{-1}S_t) &= -r_t R_t^{-1} S_t dt + R_t^{-1} dS_t \\ &= -r_t R_t^{-1} S_t dt + R_t^{-1}(r_t - l_t) S_t dt + R_t^{-1}\sigma(S_t, t) S_t dB_t \\ &= -R_t^{-1} l_t S_t dt + R_t^{-1}\sigma(S_t, t) S_t dB_t \\ &= -dF_t + R_t^{-1}\sigma(S_t, t) S_t dB_t . \end{aligned} \tag{6}$$

Therefore, by rearranging Equation (6) and recalling that $R_0^{-1}S_0 = 1$, we have

$$\begin{aligned} F_\tau &= \int_0^\tau dF_t = -\int_0^\tau d(R_t^{-1}S_t) + \int_0^\tau R_t^{-1}\sigma(S_t, t) S_t dB_t \\ &= 1 - R_\tau^{-1} S_\tau + \int_0^\tau R_t^{-1}\sigma(S_t, t) S_t dB_t . \end{aligned} \tag{7}$$

The discounted value of the insurance risk charge, up to a stopping time τ subject to standard integrability conditions is $1 - R_\tau^{-1} S_\tau$, plus an Ito (martingale) integral term, whose expectation is clearly zero. This implies

$$E[F_\tau] = 1 - E[R_\tau^{-1} S_\tau]. \tag{8}$$

In specific cases, Equation (7) can be solved to provide the entire distribution of the discounted value of fees. More importantly, Equation (8) can be easily applied to a variety of stochastic maturities. The authors now present some examples.

The Present Value of Fees for a Fixed Time Horizon

When τ is deterministic, $r_t = r, \sigma(S_t, t) = \sigma$, and $l_t = l$, the stochastic differential equation in Equation (3) can be solved to yield

$$S_t = e^{(r - l - \frac{1}{2}\sigma^2)t + \sigma B_t}, \tag{9}$$

and therefore Equation (8) can be simplified to

$$E\left[F_\tau\right] = 1 - E\left[e^{(-l - \frac{1}{2}\sigma^2)\tau + \sigma B_\tau}\right] = 1 - e^{-l\tau}. \tag{10}$$

Notice that the interest rate r and the volatility σ drop out of Equation (10) so that the risk-neutral expected discounted value of fees is invariant to both parameters. For example, with an insurance risk charge of $l = 0.002$ (20 basis points), $E[F_{20}] = 0.039$, which is less than 4 percent of the initial premium.

The Present Value of Fees for a Stochastic Lifetime Horizon

When τ is stochastic but independent of S_t, Equation (8) leads to expectations with respect to both random variables. In this study, $\tau = T$, where T is the future lifetime random variable, with probability density function $f_x(t)$. Once again, with $r_t = r, \sigma(S_t, t) = \sigma$ and $l_t = l$, we condition on age x to obtain:

$$E_x\left[F_T\right] = 1 - E_x\left[e^{(-l - \frac{1}{2}\sigma^2)T + \sigma B_T}\right]$$

$$= 1 - E_x\left[E\left[e^{(-l-\frac{1}{2}\sigma^2)T+\sigma B_T}\,|T = t\right]\right]$$

$$= 1 - E_x[e^{-lT}] = 1 - \int_0^\infty e^{-lt} f_x(t)dt\,, \tag{11}$$

which is identified as (one minus) the Laplace transform of the future lifetime random variable evaluated at l.

Exponential Lifetime

For example, when $f_x(t) = \lambda e^{-\lambda t}$ (exponential future lifetime), Equation (11) leads to

$$E_\lambda[F_T] = 1 - \lambda\int_0^\infty e^{-(l+\lambda)t}dt = \frac{l}{\lambda + l}. \tag{12}$$

The λ subscript replaces the current age (x) as the "conditioning" variable. A 65-year-old with an expected future lifetime of $E_\lambda[T] = \lambda^{-1} = 20$ years, and with $l = 0.02$ leads to $E_\lambda[F_T]$ =0.2857. When $l = 0.002$, $E_\lambda[F_T]$ = 0.038. Notice that both values are strictly lower than (a naïve application of) $1 - e^{l/\lambda}$. This is a consequence of Jensen's inequality, according to which $1 - E_\lambda[e^{-lT}] < 1 - e^{lE_\lambda[T]}$.

Gompertz Lifetime

Although the exponential lifetime assumption results in a very simple expression for the discounted value of fees, the empirical section of the study will employ the (more realistic) Gompertz specification of the function $f_x(t)$. According to Equation (11), we have

$$E_x[F_T] = 1 - \int_0^\infty e^{-lt} f_x(t)dt$$

$$= 1 - \int_0^\infty e^{-lt}\lambda(x + u, m, b)e^{-\int_0^u \lambda(x+s,m,b)ds}du$$

$$= lb\Gamma(-bl, b\lambda(x, m, b))e^{(m-x)l+b\lambda(x,m,b)}\,, \tag{13}$$

where $\lambda(x, m, b) = \exp\{(x - m)/b\}/b$, is the Gompertz force of mortality at age x, and $\Gamma(a, b)$ is defined as the incomplete Gamma function. The quantity $1 - E_x[F_T]$ can be identified as the net single premium for a life insurance policy under a force of interest l and future lifetime density $f_x(t)$. This allows us to use some of the formulas derived by Carriere (1994) and others for Gompertz mortality.

General Random Variable

When τ = Min[T,K], where K is some termination date, Equation (11) leads to

$$E_x[F_T] = 1 - E_x[e^{-l\,\mathrm{Min}[T,K]}] = 1 - \int_0^\infty e^{-lt} f_x(t)dt$$

$$= 1 - \int_0^K e^{-lt} f_x(t)dt - (1 - F_x(k))e^{-lK} \ . \tag{14}$$

The integral represents the discounted value of the insurance fees paid, conditional upon death occurring before the policy termination time K. The second component represents the probability of survival to time K, multiplied by the discounted value of fees.

The Present Value of Fees up to the First Passage Time

If we postulate that policies are (rationally) surrendered or lapsed when the account increases by a pre-specified level, we must calculate the discounted value of fees collected until the *first passage time* of the underlying account value S_t to a linear barrier denoted by $h > S_0$.

According to Equation (8), under a geometric Brownian motion assumption with $r_t = r, \sigma(S_t, t) = \sigma$, and $l_t = l$, the discounted value of fees is

$$E\left[F_{\mathrm{T}}\right] = 1 - E\left[R_{\mathrm{T}}^{-1} h\right] = 1 - hE\left[e^{-r\mathrm{T}}\right], \tag{15}$$

where T is the first passage time. Once again, we are faced with a Laplace tranform, but this time it is evaluated at r. More specifically, the stochastic time horizon in question is

$$\mathrm{T} = \underset{0 \le t < \infty}{\mathrm{Min}}\left\{t; S_t \ge h\right\} = \underset{0 \le t < \infty}{\mathrm{Min}}\left\{t; \left(r - l - \frac{1}{2}\sigma^2\right)t + \sigma B_t \ge \ln[h]\right\}, \tag{16}$$

which is the first passage time of a Brownian motion with (positive) drift to a level $l_n[h]$. The random variable T is Inverse Gaussian distributed, provided that $r - l - \sigma^2/2 > 0$ and $l_n[h] > 0$, both of which are assumed. As indicated by Equation (15), we only require the Laplace transform of this random variable, which is given by

$$E[e^{-r\mathrm{T}}] = \exp\left\{\frac{\ln[h]}{\sigma^2}\left(r - l - \frac{1}{2}\sigma^2 - \sqrt{(r - l - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}\right)\right\}. \tag{17}$$

See Yaksick (1995) for a brief and elegant derivation of the Laplace transform of the first passage time random variable, using the optional sampling theorem. Finally, we have

$$E[F_{\mathrm{T}}] = 1 - E[e^{-r\mathrm{T}}] = 1 - h^{(1+\frac{1}{\sigma^2}\left(r-l-\frac{1}{2}\sigma^2-\sqrt{(r-l-\frac{1}{2}\sigma^2)^2+2\sigma^2 r}\right)}. \tag{18}$$

In this case, both the interest rate and the volatility are inputs to the discounted value. Intuitively, this is because the expected hitting time will strongly depend on the drift and volatility of the diffusion. For example, when $\sigma = 0.20$, $r = 0.06$, $l = 0.005$, and $h = 1.25$, we obtain that $E[F_T] = 0.0145$. In other words, if the policyholder lapses as soon as the value of the account increases by 25 percent, the expected present value of fees is 1.45 percent of the initial investment. If, however, $l = 0.003$ (30 basis points),

$E[F_T] = 0.0085$. The same procedure—resulting in a more complex Laplace transform—can be used to obtain an expression for the present value of fees, when τ $-$Min$[K, T, T]$.

The Present Value of Death Benefits

The authors start by examining the rising-floor feature products, since the "rate" can always be set to zero, thus covering the return-of-premium case as well. The authors initiate the discussion by assuming there is no cap on the rising-floor, which only forces minor modifications on the final formula. The analysis then extends to the look-back product.

Valuation of Rising-Floor GMDB

As mentioned in the introduction, the authors intend to keep the cost and benefit side of the analysis separate, only to be equated at the final stage. The authors now focus on the benefits side. From a modeling point of view, it is important to note that *if* the VA (DPMF) policyowner survives until time $t = K$, he or she receives a payment equal to S_K, even if it is less than the guaranteed minimum death benefit. In other words, the only mechanism for using the guaranty is death. Those products offering GMAB and GMIB benefits would have an "option like" payout at $t = K$ as well. Therefore, if the random future lifetime $T \le K$, the beneficiary of the policy is entitled to

$$\text{Death Payment} = \text{Max}[S_T, e^{gT}] = e^{gT} + \text{Max}[S_T - e^{gT}, 0] = S_T + \text{Max}[e^{gT} - S_T, 0]$$
$$= \text{Account Value} + \text{GMDB Option Payoff}, \qquad (19)$$

where, once again, the initial investment is normalized to a value of $S_0 = 1$ and $0 \le g \le r$ is the guaranteed growth (rising-floor) rate. In this model without a cap, the guaranteed rising-floor rate, g, cannot *exceed* the risk-free rate because of internal arbitrage considerations. Clearly, one cannot secure a guaranteed return greater than the risk-free rate *plus* upside equity participation. Therefore, Equation (19) can be analyzed in one of two symmetric ways. Motivated by the concept of put-call parity, the VA (or DPMF) can either be viewed as a zero-coupon bond with a face value of e^{gT} together with a call option struck at e^{gT} or it can be treated as a position in the underlying security together with a put option struck at e^{gT}. Regardless of the perspective, the economic position is the same. The authors proceed to value the GMDB as a put option.

General Valuation

As introduced, the fund dynamics are

$$dS_t = (\mu - l)S_t dt + \sigma S_t dB_t \Leftrightarrow S_t = e^{\left(\mu - l - \frac{1}{2}\sigma^2\right)t + \sigma B_t}, \qquad (20)$$

which is essentially a dividend yielding security, where the dividend yield is the insurance charge l. The growth rate μ is net of any fund expenses (FE) payable to the company managing the assets in the VA (or PDMF). All that is included in l is the insurance risk charge that pays for the death benefit. The stochastic maturity makes this a nontrivial, yet tractable, application of the Black-Scholes/Merton formula. Specifically, the authors claim that the Rising Floor GMDB option value is equal to

$$E_x[V^g] = E\left[e^{-rT}\int_0^\infty \text{Max}[e^{gT} - S_T, 0]d\Psi_T\right], \tag{21}$$

where ψ_T is the log-normal (risk neutral) state-price density function. To be absolutely clear, Equation (21) contains expectations with respect to two stochastic variables. One is the price path, and the other is the maturity of the option. The justification for taking both expectations lies in the doubly complete markets assumption underlying this model. Complete markets for the underlying security are standard in the Black-Scholes/Merton model. However, for the time to maturity, this is a more strict assumption. Insurance completeness implies the ability of the insurance company to completely diversify away all mortality risk and therefore assume that death payments will be made in a deterministic fashion. This assumption was first used by Boyle and Schwartz (1977), and the authors rely on it for this analysis as well. As mentioned in the literature review, other researchers have recently looked at alternative incomplete-market formulations. Of course, as the number of policies increases, the option value approaches the complete market price. However, given the size of the variable annuity market ($690 billion in 1999) and the ability of a large enough reinsurer to arbitrage this market, the complete markets assumption is reasonable and consistent with most of the research in the field. Therefore, using conditional expectations, we obtain an explicit GMDB option value:

$$E_x[V^g] = E\left[E\left[e^{-rT}\int_0^\infty \text{Max}[e^{gT} - S_T, 0]d\Psi_T \,|T = t\right]\right]. \tag{22}$$

The inner expectation in Equation (22) is the (strike modified) Black-Scholes/Merton price of a put option. The second expectation is obtained by integrating over the future lifetime random variable. Specifically,

$$\begin{aligned} &E\left[e^{-rT}\int_0^\infty \text{Max}[e^{gT} - S_T, 0]d\Psi_T \,|T = t\right] = \\ &BSM_p\left(t, g|\sigma, r, l\right)\Big| = e^{(g-r)t}\text{N}\left(-\xi_2\sqrt{t}\right) - e^{-lt}\text{N}\left(-\xi_1\sqrt{t}\right), \end{aligned} \tag{23}$$

where

$$\xi_1 = \frac{r - g - l + \sigma^2/2}{\sigma}, \qquad \xi_2 = \frac{r - g - l - \sigma^2/2}{\sigma},$$

and $N(x)$ is the cdf of the standard normal distribution. The specific notation $BSM_P(t, g|\sigma, r, l)$ should draw attention to the option strike price increasing at a rate of g. This is in contrast to the classical Black-Scholes/Merton formula, where the strike price is fixed. Finally, the value of the stochastic-maturity put option is

$$E_x[V^g] = \int_0^K f_x(t)BSM_P(t, g|\sigma, r, l)dt, \tag{24}$$

where K is the maximum term of the contract, and $f_x(t)$ is the probability density function (pdf) of the future lifetime random variable. Intuitively, the price of the rising-floor GMDB-put option is the mortality weighted average price of a deterministic put—with dividend yield *l*—over the interval 0 to *K*. The expression in Equation (24), aside from the rising-floor rate *g*, was originally derived by Boyle and Schwartz (1977) within the context of equity-linked life insurance. In fact, more recently, the exponential and Erlang-maturity American-style put has been examined by Carr (1998) in the context of approximating the early exercise boundary of a deterministic-maturity American put. On a related note, Jennergen and Naslund (1996) examined the effect of stochastic maturity on call prices but did not obtain any closed-form expressions.

The current objective is to obtain simplified analytical expressions and numerical estimates for the option while accounting for the unique asset-based payment features.

A few stylized facts should emerge by visually examining Equation (24). Both $f_x(t)$ and $BSM_P(t,g|\sigma,r,l)$ go to zero as $t \to \infty$. However, neither of the functions is monotonically decreasing. This implies that although the value of the rising-floor GMDB option is very much age dependent, it does not necessarily increase in age. A priori, one would expect that higher purchase ages—which are synonymous with lower life expectancy—would translate into higher values for the guaranty. However, the *K* truncation on the upper bound of integration and the unimodality of $BSM_P(t,g|\sigma,r,l)$ can reduce the value of the guaranty at higher ages. In fact, as we shall see in the empirical analysis, when $K = 75$ (most common feature), $E_x[V^g]$ is highest at ages 50 to 60. It seems this age group experiences a large enough probability of death, within the time range at which the put value is at its highest. *Ceteris paribus*, a higher value of *K* increases the (risk-neutral) probability that the policyholder will die (and use the put) before the termination of the product. However, a higher K increases the present value of risk fees collected, which serves to reduce the required annual risk charge.

In the most general mortality case, the integral Equation (24) can be evaluated only using numerical methods. With exponential mortality, however, an explicit evaluation of the integral is possible.

Rising-Floor: Exponential Future Lifetime

In the case of exponential future lifetime, Equation (24) becomes

$$E_\lambda[V^g] = \int_0^K \lambda e^{-\lambda t} BSM_P(t,g|\sigma,r,l)dt\,, \tag{25}$$

where the λ subscript denotes dependence on the (inverse) life expectancy parameter. Interestingly, when $K = \infty$, Equation (25) is the Laplace (Carson) transform of the Black-Scholes/Merton equation—with respect to time to maturity—evaluated at the force of mortality. Substituting Equation (23) into Equation (25), we obtain

$$E_\lambda[V^g] = \int_0^K \lambda e^{(g-r-\lambda)t} \mathrm{N}(-\xi_2\sqrt{t})dt - \int_0^K \lambda e^{-(l+\lambda)t} \mathrm{N}(-\xi_1\sqrt{t})dt\,. \tag{26}$$

Despite the fact that Equation (26) does not contain an explicit mention of the age at purchase, the force of morality λ—and its inverse, the life expectancy—will implicitly determine the initial age.

Equation (26) can be simplified to produce a tractable expression for the option value. In the Appendix, the authors show that

$$\int_0^K e^{at}\int_{-\infty}^{b\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dxdt = -\frac{1}{a}\left(\frac{1}{2} - e^{aK}\mathrm{N}(b\sqrt{K})\right) + \frac{\mathrm{N}\left(-b\sqrt{K}\sqrt{1-2a/b^2}\right) - \frac{1}{2}}{a\sqrt{1-2a/b^2}}. \quad (27)$$

This allows the authors to use the appropriate values of $a = (g - r - \lambda)$ or $a = (l + \lambda)$ and $b = (l + g - r + \sigma^2/2)/\sigma$ or $b = (l + g - r - \sigma^2/2)/\sigma$.

Substituting Equation (27) into Equation (26) and using the appropriate a,b, leads to a closed-form expression for $E_\lambda[V^g]$. Note that $a < 0$, which is required for convergence, as long as $g < r + \lambda$. In particular, when $K = \infty$, we get

$$E_\lambda[V^g] = \frac{\lambda}{2(r-g+\lambda)}\left(1 - \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r-g+\lambda)}}\right) - \frac{\lambda}{2(l+\lambda)}\left(1 - \frac{\xi_1}{\sqrt{\xi_1^2 + 2(l+\lambda)}}\right), \quad (28)$$

where

$$\xi_1 = \frac{r-g-l+\sigma^2/2}{\sigma}, \text{ and } \xi_2 = \frac{r-g-l-\sigma^2/2}{\sigma}.$$

Figure 1 provides a graphic illustration of the relationship between the Titanic (exponential) put option as a percent of the initial investment value, using Equation (28). It is compared to the price of a generic put, presented in Equation (23). For comparison purposes, the Titanic option is chosen to have a life expectancy of $\lambda^{-1} = t$, both with parameters $\sigma = 0.20, l = 0.01, r = 0.06,$ and $g = 0.0$ (and there is no terminal date, so $K = \infty$). As one can see, the price of the generic put option is more expensive for shorter (expected) maturities, compared to the Titanic option. However, at approximately $\lambda^{-1} = t \approx 14$ years, the relationship is inverted and the Titanic option becomes more valuable. In fact, the generic put decays towards zero quite rapidly, whereby the Titanic put is worth much more at longer (expected) maturities.

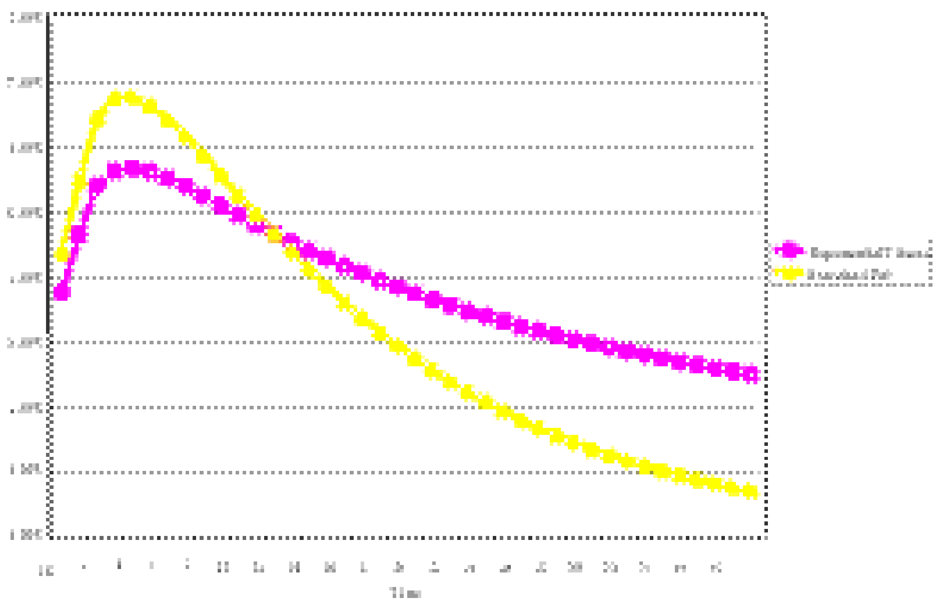
Finally, when $r = g$ (the rising-floor rate is exactly equal to the risk-free rate) and $l = 0$, Equation (28) collapses to

$$E_\lambda[V^g] = \frac{1}{\sqrt{1 + 8\lambda/\sigma^2}}. \quad (29)$$

Equation (29) indicates that (eight times) the force of mortality has an inverse relationship with the volatility squared. Intuitively, an increase in asset volatility—or decrease in the force of mortality (younger ages)—will increase the present value of the death benefit.

To summarize this discussion, when future lifetime is exponential, the single initial premium for a rising-floor guaranteed minimum death benefit option (Type 1 & 2) is given by Equation (28). It is a function of the life expectancy ($1/\lambda$), insurance charge (l), guaranteed rising-floor rate (g), risk-free rate (r), and volatility (σ). The authors further reiterate that the simplified formula in Equation (28) applies only when future lifetime is exponentially distributed and $K = \infty$ — in other words, when there is no maximum term for the contract. Of course, for $K = \infty$, a similar closed-form expression is obtainable from the expression in the Appendix. In practice, though, the approximation $K = \infty$ is quite accurate for middle-aged investors because both the put value and the probability of survival contribute negligibly to $E_\lambda[V^g]$ in later years.

FIGURE 1
The Standard Put Option With Maturity *T*



Rising-Floor: Gompertz Future Lifetime

Exponential future lifetime—despite its simplicity—is clearly not realistic in practice. Later, it will be shown that it provides an upper bound for the value of the GMDB. However, for empirical comparisons, the authors use a Gompertz approximation, which at higher ages is quite an accurate description of mortality experience. The GMDB value is

$$E_x[V^g] = \int_0^K \lambda(x+t) e^{-\int_0^t \lambda(x+s)ds} BSM_P(t, g|\sigma, r, l)dt, \tag{30}$$

where $\lambda(x) = \exp\{(x-m)/b\}/b$ for suitable values of x, m, and b. The authors must resort to numerical methods since the integral in Equation (30) cannot be solved in closed form.

Capping the Rising-Floor

When the rising-floor is capped by a fixed maximum percentage of the initial premium, denoted by M, the GMDB option payoff becomes $\mathrm{Max}\left[\mathrm{Min}\left[M, e^{gT}\right] - S_T\right]$. It is more convenient to locate the time t = λ, at which point the rising-floor ceases. In other words, since $e^{gT} = M$ forces $\tau = \ln[M]/g$, we have that the GMDB Option Payoff at death is

$$\begin{cases} \mathrm{Max}\left[e^{gT} - S_T, 0\right] \text{ if } T \le \ln[M]/g \\ \mathrm{Max}[M - S_T, 0] \text{ if } T > \ln[M]/g . \end{cases} \quad (31)$$

The valuation expression must now be modified to

$$E_x[V^{\hat{g}}] = \int_0^{\mathrm{Min}[K, \ln[M]/g]} f_x(t) BSM_P(t, g \,|\, \sigma, r, l)dt + \int_{\mathrm{Min}[k, \ln[M]/g]}^{K} f_x(t) BSM_P(t, M \,|\, \sigma, r, l)dt, \quad (32)$$

where $f_x(t)BSM_P(t, g|\sigma, r, l)$, in the second integral, denotes the Black-Scholes/Merton put option pricing formula with an exercise price of M. The $\hat{g}$ superscript on $E_x[V^{\hat{g}}]$ indicates that the guaranteed minimum payoff will cease to increase after a certain pre-specified amount of time. Also, the upper and lower bound in the integral accounts for the possibility that $K < \ln[M]/g$, in which case the contribution form of that portion is zero.

Valuation of Look-Back GMDB Option

With a look-back feature, the strike price of the embedded put option increases *stochastically* during the life of the contract. The frequency with which the strike price is updated depends on the particular policy in question. As shown earlier, it can range anywhere from one to ten years. Furthermore, as in the rising-floor case, the strike price ceases to increase once a pre-specified age has been reached. Therefore, although it is quite cumbersome to present a specific formula for each product, the authors now present a general upper-bound for the value of a look-back guaranteed minimum death benefit. This is achieved by assuming that the strike price is continuously updated, as opposed to being updated at discrete contract-anniversary intervals of varying frequency. One can think of the continuous look-back as corresponding to a VA (or DPMF) whose basis is artificially reestablished each time a new high is achieved by selling and then repurchasing the contract at the new level. Clearly, the transaction costs and deferred sales charges might be prohibitive, but once again, this model provides an upper bound on the value of the guaranteed minimum death benefit. Under this specification, the option payoff at death is

$$\text{GMDB Option Payoff at Death} = \mathop{\mathrm{Max}}_{0 \le u \le T} S_u - S_T . \quad (33)$$

This "exotic" is known as the floating-strike look-back put option and was originally valued in closed-form by Goldman, Sosin, and Gatto (1979). Or see Hull (1997, Chapter 18) for another possible source. The authors therefore denote this option, with a deterministic horizon, by $GSG_P(t|\sigma,r,l)$, where t is the maturity date and l, σ, r are the usual insurance charge (dividend yield), volatility, and interest-rate parameters. The price of this option is given by

$$GSG_P(t|\sigma,r,l) = e^{-rt}\mathrm{N}(-\xi_2\sqrt{t}) - e^{-lt}\mathrm{N}(-\xi_1\sqrt{t}) + \eta\left[e^{-lt}\mathrm{N}(\xi_1\sqrt{t}) - e^{-rt}\mathrm{N}(\xi_3\sqrt{t})\right], \quad (34)$$

where

$$\eta = \frac{\sigma^2}{2(r-l)},\ \xi_1 = \frac{(r-l+\sigma^2/2)}{\sigma},\ \xi_2 = \xi_1 - \sigma,\ \xi_3 = \xi_1 - \frac{2(r-l)}{\sigma}. \quad (35)$$

For example, when $\sigma = 0.20, l = 0.01, r = 0.06, \text{and } t = 1$, we have $GSG_p = 0.1414$, which is 14 percent of the initial investment. This can be compared to the standard Black-Scholes/Merton price, with the same parameters, of $BSM_p = 0.0551806$. Clearly, the look-back feature is more expensive given the ability to "sell" at the highest achieved price to date.

Finally, applying the general valuation expression from Equation (24), we have

$$\text{Look-back GMDB Option Value} = E_x[V^L] = \int_0^K f_x(t) GSG_P(t|\sigma,r,l)dt, \quad (36)$$

which is the No Arbitrage value of the Step-Up Guaranteed Minimum Death Benefit when the future lifetime pdf is $f_x(t)$, and K is the maximum term of the contract.

Look-Back: Exponential Future Lifetime

As before, when future lifetime is exponentially distributed, and the maximum term of the contract $K = \infty$, Equation (36) becomes

$$E_\lambda[V^L] = \int_0^\infty \lambda e^{-\lambda t} GSG_P(t|\sigma,r,l)dt. \quad (37)$$

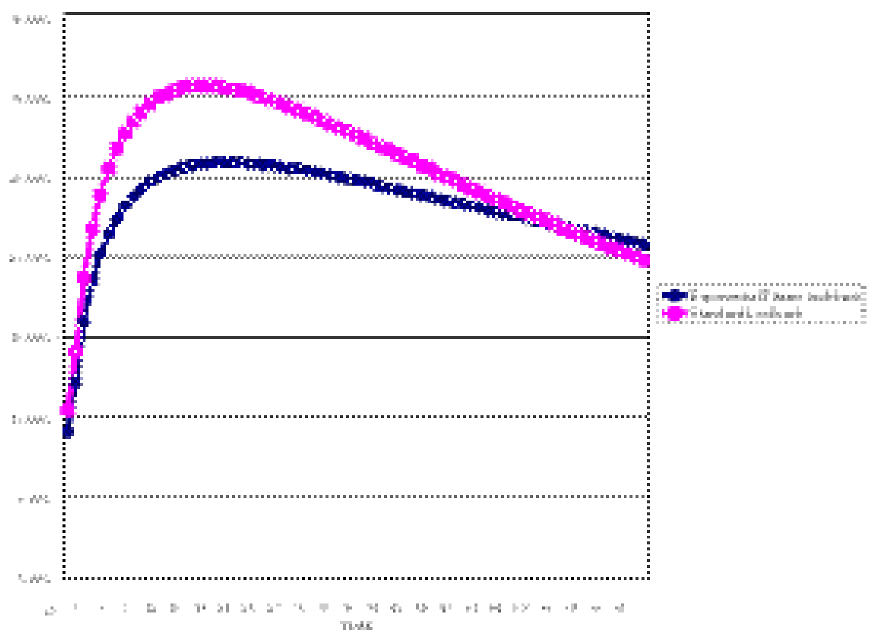
Again, the authors use the λ subscript to denote dependence on the (inverse) life expectancy parameter, as opposed to age. Using the same manipulation performed for the standard put (see Appendix for proof), we obtain a closed-form expression for the Titanic (exponential) Look-back option:

$$E_\lambda[V^L] = \frac{\lambda(1-\eta)}{2(r+\lambda)}\left(1 - \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r+\lambda)}}\right) - \frac{\lambda}{2(l+\lambda)}\left(1 - \eta - \frac{\xi_1(1+a)}{\sqrt{\xi_1^2 + 2(l+\lambda)}}\right), \quad (38)$$

where η, ξ_1, and ξ_2 are the same as in Equation (35). Notice, also, the similarity with Equation (28). For example, with $\sigma = 0.20, l = 0.01, \text{and } r = 0.06$ when (life expectancy) $\lambda^{-1} = 5$, the GMDB value is $V^L_{1/5} = 0.214852$; when $\lambda^{-1} = 15$, the GMDB value is $V^L_{1/15} = 0.257573$. However, the value of the GMDB is not necessarily monotonic in λ; for example, when $\lambda^{-1} = 40$, we obtain (a slightly lower) $V^L_{1/40} = 0.242645$.

Figure 2 is a plot of the Titanic (exponential) look-back option as a percent of the initial investment value, using Equation (38). It is compared to the price of a vanilla look-back, presented in Equation (34). For comparison purposes, the Titanic option is chosen to have a life expectancy of $\lambda^{-1} = t$, both with parameters $\sigma = 0.20$, $l = 0.01$, and $r = 0.06$. (There is no terminal date, so $K = \infty$.) As one can see, the price of the vanilla option is more expensive for shorter (expected) maturities compared to the Titanic option. However, at approximately $\lambda^{-1} = t = 60$ years, the relationship is inverted and the Titanic option becomes more valuable. Compared to the standard put displayed in the previous section, the Titanic look-back crosses the vanilla look-back much later.

FIGURE 2
The Standard Look-Back Option With Maturity



Finally, when $r = l = 0$, we obtain an expression for the Titanic (exponential) look-back option

$$E_\lambda[V^L] = \frac{\sigma^2}{4\lambda} + \sqrt{\left(\frac{\sigma^2}{4\lambda} + 1\right)^2 - 1}, \tag{39}$$

which is similar in structure to the price of the Titanic Put with $r = g$ and $l = 0$, originally derived in Equation (29). In fact, rearranging Equation (39), one can see that $E_\lambda[V^L] = 2E_\lambda[V^0]/(1 - E_\lambda[V^0])$. The Titanic look-back is, at the very least, twice as expensive as the Titanic Put, when $r = g = l = 0$.

Look-Back: Gompertz Future Lifetime

As in the case of the standard put, the numerical comparisons will be performed with the Gompertz distribution. The look-back GMDB option value is

$$E_x[V^L] = \int_0^K \lambda(x+t)e^{-\int_0^t \lambda(x+s)ds} GSG_p(t|\sigma,r,l)dt\ , \tag{40}$$

where $\lambda(x) = \exp\{(x-m)b\}/b$ for suitable values of x, m, and b. Once again, the integral in Equation (40) can be obtained only by using numerical techniques.

Equating Fees and Benefits

This brings us to the crucial question of equilibrium. What insurance fee will find the GMDB using the option pricing perspective? The answer involves "locating" a value of l that will solve the identity:

$$\text{Economic Profit} = E_x[F_T] - E_x[V] = 0\ , \tag{41}$$

where $E_x[F_T]$ is obtained from Equation (8) and $E_x[V]$ is obtained from Equation (24). A potentially important point is that Equation (41) does not necessarily assure us of the existence of a zero profit insurance charge. In other words, depending on the type of guaranty $E_x[V]$ in Equation (41), Q may exceed the $E_x[F_T]$ regardless of how high the value of l is. Stated differently, increasing the insurance charge will obviously increase the present value of the insurance fees, but it will also increase the present value of the GMDB. Recall that a higher (dividend yield) value of l increases the price of the put in a Black-Scholes/Merton framework. An equilibrium insurance charge may not exist that funds the death benefit.

Of course, in the standard case, where the only thing guaranteed is the initial premium, plus a possible rising-floor, the authors find that a very small insurance fee is enough to fund the benefit.

NUMERICAL EXAMPLES AND DATA

In this section the authors provide some numerical estimates of the value of the guaranteed minimum death benefit as (1) a percentage of the initial investment and (2) a proportional fee. The authors derive the values using an exponential mortality specification as well as a Gompertz-fitted annuity mortality table.

Exponential Mortality

Fix the volatility of the investments in the tax-sheltered annuity at $\sigma = 0.20$ and assume an $r = 0.06$ in the economy. When the force of mortality is $\lambda = 1/35$, which is a remaining expected lifetime of 35 years (or roughly a 50-year-old female), we obtain the following results: The economic profit of the insurance company is 27.58 percent of the initial investment when the *insurance risk charge* is 125 basis points (l = 0.0125) and there is no rising-floor feature (g = 0). This is because the present value of the insurance fee is $E_\lambda[F_T]$ = 30.43 percent of the initial premium, while the cost of the put option is only $E_x[V]$ = 2.85 percent of the initial premium. The economic profit is exactly zero when the insurance risk charge is 6.3 basis points (l = 0.00063).

Both the present value of the insurance fee and the value of the put option is $E_\lambda[F_T] = E_\lambda[V] = 2.158$ percent. Another way of stating this fact is that the *equilibrium* risk charge is 6.3 basis points.

In contrast, when $\lambda = 1/30$, which is a remaining expected lifetime of 30 years (or roughly a 50-year-old male), we obtain the following. The economic profit is 24.13 percent of the initial premium when the *insurance risk charge* is 125 basis points (l = 0.0125) and there is no rising-floor. This is because the present value of the insurance fee is $E_\lambda[F_T] = 27.27$ percent of the initial premium, while the cost of the put option is only $E_\lambda[V] = 3.146$ percent of the initial premium. This means that roughly one-quarter of the initial investment premium goes toward the economic profits of the insurance company! Finally, the economic profit is zero when the insurance risk charge is 8.2 basis points (l = 0.00082).

Gompertz-Parameterized Mortality

In this section, the authors look at Gompertz death rates, fitted to the 1994 GAM table. The following tables display the equilibrium cost of insuring a variable annuity as a function of (1) the age of the investor at the time of purchase and (2) the guaranteed rising-floor rate. Both the initial cost—as a percentage—and the proportional risk charge—in basis points—are presented. The implicit assumptions are that the VA is terminated (annuitized or liquidated) at age 75, the volatility of the assets is λ = 20 percent, while the interest rate is r = 6 percent. The volatility and interest-rate assumptions are consistent with current market conditions.

Table 5 shows the percent of premium versus annual cost of the guaranteed minimum death benefit, assuming a 200 percent cap on the 5 percent rising floor, a volatility of 20 percent, an interest rate of 6 percent, and a termination age of 75.

TABLE 5
Female

Purchase Age	g = 0%		g = 5%		Look-Back	
30 yrs	0.14%	0.30 bp	0.76%	1.77 bp	6.32%	15.1 bp
40 yrs	0.27%	0.80 bp	1.47%	4.45 bp	6.11%	18.9 bp
50 yrs	0.48%	2.00 bp	2.52%	10.84 bp	5.63%	24.6 bp
60 yrs	0.71%	5.00 bp	2.98%	21.6 bp	4.50%	32.8 bp
65 yrs	0.71%	7.60 bp	2.10%	22.5 bp	3.35%	36.1 bp

Clearly, the value of the guaranty is a function of age but does not necessarily increase in age, as explained earlier. The table displays both the single premium value of the guaranty as well as the basis point charge, which is the empirically measurable quantity. For example, a 50-year-old male purchasing a variable annuity should pay 3.5 basis points until death (or age 75) to fund the return-of-premium GMDB. A 50-year-old female would pay 2 basis points for the same benefit. In the event of a 4 percent rising-floor, the numbers increase to 20 basis points for males and 12 basis points for females. The look-back feature should cost 41.8 basis points for a male and 25 basis points for a female, both at age 50. The large difference in equilibrium risk charge, between the male and female in this case, can be attributed to the fact that females are more likely to live to $t = K$, and therefore the probability of the look-back ever being exercised is greatly reduced. This fact also illustrates the sensitivity of the look-back price—much more so than the rising-floor price—to the exact maturity date of the product. Of course, it is crucial to emphasize that no existing product in the market offers a "full" continuous time look-back feature for the GMDB. Indeed, the closest feature is a one-year reset that is offered by less than a quarter of the policies. The norm is five to seven years, which greatly reduces the price of the option.

Table 6 shows the percent of premium versus annual cost of the guaranteed minimum death benefit, assuming a 200 percent cap on the 5 percent rising floor, a volatility of 20 percent, an interest rate of 6 percent, and a termination age of 75.

TABLE 6
Male

Purchase Age	g = 0%		g = 5%		Look-Back	
30 yrs	0.25%	0.40 bp	1.34%	3.24 bp	9.9%	25.0 bp
40 yrs	0.47%	1.30 bp	2.51%	7.96 bp	9.5%	31.6 bp
50 yrs	0.82%	3.50 bp	4.22%	19.2 bp	8.95%	41.8 bp
60 yrs	1.18%	8.70 bp	4.89%	37.5 bp	7.25%	56.4 bp
65yrs	1.18%	13.0 bp	3.47%	39.3 bp	5.47%	62.5 bp

Although Table 5 and Table 6 are based on a volatility of 20 percent with an interest rate of 6 percent, the model was used on a variety of capital market scenarios. It seems—as one would expect—that the value of the guaranty is most sensitive to the volatility assumption. Indeed, when σ = 30 percent (instead of 20 percent), a 50-year-old male should pay 10.4 basis points and a 50-year-old female should pay 6.0 basis points. However, as shown earlier, the empirical evidence seems to indicate that a volatility of 20 percent is an upper bound for three-quarters of the policies. This is mainly because of the basket-like structure of the variable annuity, which acts to reduce the aggregate volatility of the assets. Clearly, optimal behavior on the part of the policyholder would be to create separate policies for each of the subaccounts and to invest in the most volatile assets. To that end, the authors include Table 7 and Table 8, which provide some indication of the sensitivity of the results to volatility and interest rates. Applying this model with higher volatilities than required serves as a possible check for model imperfections. Indeed, even at ridiculously high level of volatilities, the mortality risk charge is less than 30 basis points for the typical age of 50.

Table 7 shows the percent of premium versus annual cost of the basic guaranteed minimum death benefit, assuming an interest rate of 6 percent and a termination age of 75.

TABLE 7
Volatility Sensitivity

Volatility	Female 50 Yrs. Old		Male 50 Yrs. Old	
σ = 10%	0.02%	0.50 bp	0.05%	0.03 bp
σ = 15%	0.17%	0.70 bp	0.30%	1.20 bp
σ = 20%	0.48%	2.00 bp	0.82%	3.50 bp
σ = 30%	1.41%	6.00 bp	2.34%	10.40 bp
σ = 50%	3.41%	14.00 bp	5.60%	25.60 bp

In terms of sensitivity to K, for example, for a female age 65 (σ = 20%, r = 6%, g = 0%), when $K + x = 75$, we get 7.5 basis points; when $K + x = 85$, we get 9.5 basis points; and when $K + x = 100$, we get 10.9 basis points. In general, the equilibrium risk charge increases mildly because of the higher probability of it being exercised—this despite the higher present value of fees.

Table 8 shows the up-front versus annual cost of the basic guaranteed minimum death benefit, assuming a volatility of 20 percent and a termination age of 75.

TABLE 8
Interest-Rate Sensitivity

Interest Rate	Female 50 Yrs. Old		Male 50 Yrs. Old	
r = 4%	1.10%	4.90 bp	1.90%	8.50 bp
r = 5%	0.75%	3.10 bp	1.20%	5.42 bp
r = 6%	0.48%	2.00 bp	0.82%	3.50 bp
r = 7%	0.30%	1.24 bp	0.50%	2.16 bp
r = 8%	0.20%	0.78 bp	0.35%	1.36 bp

Table 9 compares the percent-of-premium cost of a generic put option, compared to the cost of an up-and-out put option. The volatility is assumed to be 20 percent, the interest rate is 6 percent, and the dividend yield is 1 percent. The knockout occurs at 10 and 25 percent above the initial price.

TABLE 9
Comparison

Maturity	Standard	10% Knock-out	25% Knock-out
1 year	5.518	4.156	5.447
5 years	6.676	2.658	5.021
10 years	5.289	1.473	3.069
20 years	2.753	0.514	1.145

Final Additional Comments

1. It appears that the exponential approximation to mortality—as presented in Equation (28)—slightly overestimates the value of the guaranteed minimum death benefit. This is because, for the most part, the exponential "assumption" kills people when the option is more likely to have a higher value. The simplified formulas presented here can serve as a convenient and easy-to-use "upper bound" for the value of the GMDB.

2. At age 50, the up-front cost of the GMDB is less than 1 percent of premium, except for very high levels of volatility or a rising floor. Therefore, even without any deferred sales charges (read: penalties for early lapsation), a risk charge of 100 basis points would recover the initial cost in exactly one year. In fact, one could argue that, with a typical M&E risk charge of 100 basis points and a surrender charge of 5 percent for the first six years, the insurance company is absolutely assured of "recovering" the initial cost of the GMDB, regardless of the actions of the policyholder.

3. The look-back feature is not as valuable as one would expect, from a deterministic perspective. The intuition for this result is that life expectancy, for the most part, exceeds the termination date (K) of the variable annuity. This serves to reduce the probability that the option will be exercised. Indeed, if one prices a Titanic (Gompertz) look-back with (no maturity date) $K = \infty$, the initial price is of the order of magnitude of 30 percent.

4. The effect of lapses and surrenders is ambiguous for variable annuities. On the one hand, early termination of the product reduces the insurance fees and raises the possibility that the protection has not been fully paid for. In some sense, lapsation creates an incomplete market. See Milevsky and Salisbury (2000) for more information. On the other hand, the potential liabilities may be reduced as well. One therefore must distinguish between rational and irrational lapsation. Nevertheless, within the context of rational lapsation, one might be interested in the optimal time to lapse and the implications for proper valuation of the guaranty. In practice, the process of selling and buying to reestablish the basis of the contract will create a charge or load proportional to the market value of the account and is a decreasing function of time. If we assume full rationality, the insurance company might be interested in purchasing an up-and-out put option, where the knockout level is the optimal lapsation boundary. Table 9 provides some numerical estimates of the cost of a generic put option compared to the knock-out. Clearly, this type of hedge would be cheaper.

5. The Gompertz calibration is more than just a convenient assumption. It also enables one to deal with fractional ages in a parsimonious manner that is consistent with the continuous-time model employed for the underlying asset price. The authors did, however, run some discrete-time versions (with the 1994 GAM table), using both a uniform distribution of death and Balducci assumption [see Bowers et al. (1986) for details of this scheme]. In general, discretizing the mortality table and option expiration in monthly increments uniformly increases the present value of the guaranty and the "fair" risk charge. *However, the bias is always less than 1 percent of the initial put option price*. The very minor discrepancy,

although consistent in sigh, serves to justify the continuous-time assumption, which is more consistent with reality.

SUMMARY AND CONCLUSION

An article in the *Washington Post* (May 2, 1999) reported on a class action lawsuit filed against the major vendors of variable annuities *inside* tax-deferred retirement saving plans. The plaintiffs alleged deception and fraud in the sale of these products, since the implicit tax-deferral of the variable annuity was of absolutely no value inside a tax-sheltered plan. The plaintiffs further claimed that the Mortality and Expense risk charge was levied for a guaranty that had little—if any—economic value, "since few people are likely to die exactly when the account value is at a loss."

Motivated by the apparent controversy over fair pricing, this study uses modern option pricing theory to shed light on the *fair value* of the guaranteed minimum death benefit (GMDB) in a variable annuity and recently introduced death-protected mutual funds. Specifically, the authors compute the insurance risk charge that funds the embedded put option. Of course, by focusing solely on economic value, the authors abstract from reality somewhat by ignoring any reserving requirements as well as regulatory costs, agent commissions, and reasonable profits.

Variable annuities are quite similar to unit-linked life insurance policies, which have been studied by a variety of authors in the literature. However, most of the previous academic research has been nonempirical in nature. Other studies attempt to precisely model a particular aspect of these contracts, as opposed to roughly value the products in their entirety. The authors attempt both theoretical and applied contributions to the literature.

On the theoretical side, the authors model the equilibrium insurance risk charge as a dividend yield in an option pricing context. The expected present value of the stochastic dividend flow must be equated to the value of the put option with a stochastic maturity. Furthermore, the authors demonstrate that the present value of the guaranteed minimum death benefit is the Laplace (Carson) transform—in time—of the Black-Scholes/Merton equation adjusted for dividends, when the force of mortality is assumed constant and the product matures at death. Conveniently, the final expression is available in closed form and serves as an upper bound for the value of the guaranty under more realistic mortality. Furthermore, when interest rates and risk charges are set to zero, the value of the Titanic put option is $(1+8\lambda/\sigma^2)^{-1/2}$, where λ is the force of mortality and σ is the volatility.

From an applied perspective, when the mortality is given a more realistic structure—and in particular Gompertz—the authors can obtain quite robust numerical values for the equilibrium insurance expense. The models are calibrated to market conditions using a volatility estimate of 20 percent per annum and the 1994 Group Annuity Mortality (Basic) table.

The authors conclude that a typical 50-year-old male (female) who purchases a variable annuity—with a simple return of premium guaranty—should be charged no more than 3.5 (2.0) basis points per year in exchange for this stochastic-maturity put option. In the event of a 5 percent rising-floor guaranty, the fair premium rises to 20 (11) basis points.

However, Morningstar indicates that the insurance industry is charging a median M&E risk fee of 115 basis points per year, which is approximately five to ten times the most optimistic estimate of the economic value of the guaranty. The authors' conclusion is invariant to any implicit lapsation assumption because (1) the initial value of the guaranty is well under 2 percent of the initial premium and (2) deferred sales charges of 5 percent for an average of six years guaranty this up-front cost will be recuperated in either event.

Presumably, the remaining "markup" on variable annuities can either be attributed to model and market imperfections or, more cynically, to an implicit payment for the tax-deferral privilege. Moreover, the recently introduced death-protected mutual funds—whose investment gains are not tax sheltered—appear to be somewhat closer to fair value, depending on age.

Further research will expand the methodology outlined in this study to deal with more complex lapsation issues as well as alternative hedging strategies for these guarantees. In addition, the authors intend to examine some of the guaranteed living benefits, such as GMIB and GMAB, from an option pricing perspective.

APPENDIX

To obtain the analytic expression for the Titanic option value—in the case of exponential mortality—we must evaluate

$$\int_0^K e^{at} \int_{-\infty}^{b\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx dt.$$

We have

$$\int_0^K e^{at} \int_{-\infty}^{b\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx dt = \int_0^K e^{at} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx dt + \int_0^K e^{at} \int_{0}^{b\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx dt$$

$$= \frac{e^{aK}-1}{2a} + \int_0^{b\sqrt{K}} \int_{(x/b)^2}^{K} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{at}\, dt dx$$

$$= \frac{e^{aK}-1}{2a} + \frac{1}{a}\int_0^{b\sqrt{K}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \left(e^{aK} - e^{a(x/b)^2}\right) dx$$

$$= -\frac{1}{a}\left(\frac{1}{2} - e^{aK}\mathrm{N}(b\sqrt{K})\right) + \frac{\mathrm{N}\left(-b\sqrt{K}\sqrt{1-2a/b^2}\right) - \frac{1}{2}}{a\sqrt{1-2a/b^2}}.$$

Finally, for $K = \infty$, the expression reduces to

$$-\frac{1}{2a}\left[1 + \frac{b}{\sqrt{b^2-2a}}\right].$$

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