COMBINATORICA 2 (2) (1982) 125—134

THE DIAMETER OF RANDOM REGULAR GRAPHS

B. BOLLOBÁS and W. FERNANDEZ DE LA VEGA

Received 4 September 1981

We give asymptotic upper and lower bounds for the diameter of almost every r-regular graph on n vertices ($n\to\infty$).

Though random graphs of various types have been investigated extensively over the last twenty years, random regular graphs have hardly been studied. The reason for this is that until recently there was no formula for the asymptotic number of labelled r-regular graphs of order n. Such a formula was given by Bender and Canfield [1]. Even more recently one of the present authors [3] gave a simpler proof of the same formula. More importantly, [3] contains a model for the set of regular graphs which can be used to study labelled random regular graphs. Our aim is to investigate the diameter: we shall show that for a fixed r most r-regular graphs of order n have about the same diameter. Our results have some bearing on certain extremal problems concerning graphs of small diameter and small maximum degree (see [2, Ch. IV]). For results about the diameter of the customary random graphs see [4], [5] and [6].

Let us start with the model mentioned above. Let $r\geqq 3$ be fixed and denote by $G(n, r\text{-reg})$ the probability space of all r-regular graphs with a fixed set of n labelled vertices. Here we assume that rn is even and any two graphs have the same probability. We shall say that *almost every* (a.e.) r-regular graph has a certain property if the probability that a member of $G(n, r\text{-reg})$ has this property tends to 1 as $n\to\infty$. Let $W_1, W_2, \ldots, W_n$ be disjoint r-element sets. A *configuration* with vertex set $W=\bigcup_1^n W_i$ is a partition of W into pairs. We call these pairs the *edges* of the configuration and we denote by Ω the set of all configurations with vertex set W. Once again we view Ω as a probability space in which all points are equiprobable. Given a configuration $F\in\Omega$ we may try to construct an r-regular graph $\varphi(F)$ with vertex set $\{W_1, W_2, \ldots, W_n\}$ as follows. Join two vertices W_i and W_j by an edge iff the configuration F contains an edge having one vertex in W_i and the other in W_j. Clearly $\varphi(F)$ is an r-regular graph if F has no edge joining two vertices of the same class W_i, nor has it two edges joining vertices in the same two classes W_i and W_j. All we need

AMS subject classification (1980): 05 C 99, 60 C 05.

from [3] is that the probability that a configuration F has these two properties is bounded away from 0. More precisely, we need the following immediate consequence of this assertion: if a.e. configuration has a certain property then a.e. r-regular graph has the corresponding property. The property we are concerned with is that of having a certain diameter.

Let F be a configuration and W_i, W_j arbitrary classes. Define the *distance between* W_i *and* W_j as the minimal k for which one can find classes $W_{i_0}=W_i$, $W_{i_1}, W_{i_2}, \ldots, W_{i_k}=W_j$, such that for every j, $0\leqq j<k$, the class W_{i_j} is joined to $W_{i_{j+1}}$ by an edge of F. If there is no such k then the distance between W_i and W_j is said to be infinite. The distance between W_i and W_j is denoted by $d(W_i, W_j)$ or $d_F(W_i, W_j)$. Note that $d(W_i, W_j)=0$ if and only if $W_i=W_j$. The *diameter of* F is the maximal distance between two classes. From what was said earlier it is clear that if a.e. configuration has diameter at least $d'(n)$ then a.e. r-regular graph has diameter at least $d'(n)$, and if a.e. configuration has diameter at most $d''(n)$ then a.e. r-regular graph has diameter at most $d''(n)$. Thus our aim is to give bounds for the diameter of a.e. configuration.

In our proofs we shall often find it convenient to work with another description of Ω, namely we shall find it convenient to select the edges of a random configuration one by one, taking those nearest to a fixed class W_1 first. To be precise, we shall construct inductively sets E_i, S_i and M_i in such a way that S_i will be the set of indices j with $d(W_1, W_j)=i$ in the final configuration (i.e. S_i will be the set of indices of the classes W_j on the sphere of radius i about W_1), E_i will be the set of edges incident with the classes at distance less than i from W_1 and M_i will be the set of vertices (members of the classes W_j) not incident with any edge in E_i. Thus E_{i+1} is obtained from E_i by adding to E_i all the edges incident with the vertices in $L_i=M_i\cap\bigcup_{j\in S_i} W_j$.

Set $E_0=\emptyset$, $S_0=\{1\}$ and $M_0=\bigcup_{i=2}^{n} W_i$. Suppose we have defined E_i, S_i and M_i. In order to define E_{i+1}, S_{i+1} and M_{i+1}, we pass through a number of intermediate stages, corresponding to the selection of single edges.

Set $E_i^{(0)}=E_i$, $S_i^{(0)}=\emptyset$, $M_i^{(0)}=M_i$ and $L_i^{(0)}=L_i=M_i\cap\bigcup_{l\in S_i} W_l$. Suppose we have defined $E_i^{(j)}$, $S_i^{(j)}$, $M_i^{(j)}$ and $L_i^{(j)}$. If $L_i^{(j)}=\emptyset$, put $E_{i+1}=E_i^{(j)}$, $S_{i+1}=S_i^{(j)}$, $M_{i+1}=M_i^{(j)}$ and $L_{i+1}=M_{i+1}\cap\bigcup_{j\in S_{i+1}} W_j$. Otherwise pick a vertex $x\in L_i^{(j)}$ (say take the first vertex in $L_i^{(j)}$ in some predefined order), give all vertices of $M_i^{(j)}-\{x\}$ the same probability and choose one of them, say y. Set $E_i^{(j+1)}=E_i^{(j)}\cup\{(x,y)\}$, $L_i^{(j+1)}=L_i^{(j)}-\{x,y\}$ and $M_i^{(j+1)}=M_i^{(j)}-\{x,y\}$. Finally, if $y\in W_l$ and $l\notin S_i$, set $S_i^{(j+1)}=S_i^{(j)}\cup\{l\}$, and otherwise put $S_i^{(j+1)}=S_i^{(j)}$. Since $M_i^{(j+1)}$ has two fewer vertices than $M_i^{(j)}$, the process does terminate so it does define E_{i+1}, S_{i+1} and M_{i+1}.

After a certain number of steps we must arrive at $E_{k+1}=E_k$ and so $S_{k+1}=\emptyset$. Then the set E_k is exactly the edge set of the component containing W_1 in a random configuration. If $M_k=\emptyset$, we have constructed the whole configuration. However, even if we end up with $M_k\neq\emptyset$, we have constructed the part of a random configuration nearest to W_1 in the following sense: for every $i\leqq k$ the probability of having $E_i=\tilde{E}$ is the same as the probability that $\tilde{E}$ is the set of edges incident with the classes at distance less than i from W_1. (In fact, even if $M_k\neq\emptyset$, we can construct an entire random configuration by iterating the process above, but this will not concern us.)

Now we are well prepared to prove our first theorem.

Theorem 1. *Let* $r \geqq 3$ *and* $\varepsilon > 0$ *be fixed and define* $d = d(n)$ *as the least integer satisfying*

$$(r-1)^{d-1} \geqq (2+\varepsilon)\, rn \log n.$$

Then a.e. r-regular graph of order n has diameter at most d.

Proof. As we remarked earlier, it suffices to show that a.e. configuration has diameter at most d, so we shall investigate random configurations.

Let t_1 and t_2 be integers satisfying $t_1+t_2=d-1$ and $t_1 \leqq t_2 \leqq t_1+1$. We shall show that with probability close to 1 for any two classes W_{j_1}, W_{j_2} there are many edges joining the classes at distance t_i from W_{j_i} to classes at distance t_i+1 $(i=1,2)$. Then we show that with probability close to 1 either $d(W_{j_1}, W_{j_2}) \leqq t_1+t_2$ or else these two large classes of edges have at least one edge in common, so $d(W_{j_1}, W_{j_2}) = t_1+t_2+1=d$. In order to avoid some inessential complications we shall assume that $r \geqq 5$. At the end of the proof we shall indicate how one can dispose of this condition.

Let W_1 be an arbitrary fixed class. Consider the inductive construction of a random configuration, as described above, using the sets E_i, S_i, M_i and $L_i = M_i \cap \bigcup_{j \in S_i} W_j$. We claim that with probability $1-o(n^{-2})$ we have

$$l_i = |L_i| \geqq (1-n^{-1/16})(r-1)^i \tag{1}$$

for every $i \leqq t_0$ where t_0 is the maximal integer with

$$(r-1)^{t_0} \leqq n^{1/2} \log n.$$

Roughly speaking this inequality states that the first few edges $(\sim n^{2/3})$ reach almost as many classes as possible. Suppose an edge xy is selected as the edge incident with $x \in L_i^{(j)}$. We call this edge *dispensable* if $S_i^{(j+1)} = S_i^{(j)}$. Thus xy is indispensable if it is the first edge that insures that another class W_m is at a distance $i+1$ from W_1. What is the probability that the kth edge we select is dispensable? As the $k-1$ edges selected so far are incident with vertices in at most k classes, this probability is clearly at most

$$\frac{k(r-1)}{(n-k)r}. \tag{2}$$

Therefore the probability that more than 2 of the first $k = o(n^{1/6})$ edges are dispensable is at most

$$\binom{k}{3}\left(\frac{k}{n-k}\right)^3 = o(n^{-2}). \tag{3}$$

Note that with $s_i = |S_i|$, $l_i = |L_i|$ we have $s_0 = 1$, $l_0 = r$ and

$$s_i \leqq l_{i-1}, \quad l_i \leqq (r-1)s_i$$

for $i \geqq 1$. Hence

$$s_i \leqq r(r-1)^{i-1} \quad \text{and} \quad l_i \leqq r(r-1)^i.$$

Consequently the edges incident with vertices in $\bigcup_{l=0}^{i} L_l$ are among the first $r \sum_{l=0}^{i} (r-1)^l$

edges we select. By (2) the probability that an edge incident with a vertex in $\bigcup_{l=0}^{i} L_l$, $i \leqq \frac{2}{3} \log n/\log(r-1)$, is dispensable is at most

$$\left\{r \sum_{l=0}^{i}(r-1)^l\right\}(r-1)/\left\{\left(n-r\sum_{l=0}^{i}(r-1)^l\right)r\right\} \leqq \frac{3(r-1)^{i+1}}{n}, \tag{4}$$

if n is sufficiently large. Furthermore, if there are w_i dispensable edges (i.e. 'edges we can do without') among the edges incident with the vertices in L_i then clearly

$$l_{i+1} \geqq (l_i - 2w_i)(r-1). \tag{5}$$

Let i_0 be the minimal integer with

$$(r-1)^{i_0} \geqq n^{1/7}.$$

Then the edges incident with vertices in $\bigcup_{l=0}^{i_0} L_l$ are among the first $2r(r-1)^{i_0} = o(n^{1/6})$ edges to be selected. Hence by (3) and (5) with probability $1-o(n^{-2})$ we have $l_1 \geqq r-1$, $l_2 \geqq (r-1)^2$, and so on,

$$l_{i_0} \geqq (r-1)^{i_0} \geqq n^{1/7}. \tag{6}$$

Now suppose that $i_0 \leqq i \leqq t_0$. In order to prove (1) we shall show that the probability that l_{i+1} is large, conditional on l_i being large, is close to 1. By (4), we can define on Ω a random variable w_j' distributed as a binomial $B(m, p)$ with parameters $m = r(r-1)^i$ and $p = \frac{3(r-1)^{i+1}}{n}$, such that $w_i \leqq w_i'$ holds a.s. Therefore, using the definition of t_0 and a standard estimate, the expected number of dispensable edges incident with L_i is at most

$$r(r-1)^i \frac{3(r-1)^{i+1}}{n} \leqq (\log n)^3.$$

Therefore by a standard estimate of the probability in the tail of the binomial distribution (see [3, Lemma 1(ii)]) we have

$$P(w_i \geqq n^{1/12} | l_i = l_i^*) \leqq \exp\{-n^{1/12}\}$$

for every fixed integer l_i^*. Hence by (5) for $i_0 \leqq i \leqq t_0$ we have

$$\begin{aligned} &P\left(l_{i+1} \leqq (1-n^{-1/15}) l_i | l_i \geqq (1-n^{-1/15})^i (r-1)^i\right) \leqq \\ &\leqq P\left(w_i \geqq \tfrac{1}{2} n^{-1/15}(1-n^{-1/15})^i (r-1)^i | l_i \geqq (1-n^{-1/15})^i (r-1)^i\right) \leqq \\ &\leqq P\left(w_i \geqq n^{1/15} | l_i \geqq (1-n^{-1/15})^i (r-1)^i\right) \leqq \exp(-n^{1/15}). \end{aligned} \tag{7}$$

Finally, (6) and (7) show that

$$\begin{aligned} &P\left(l_i \geqq (1-n^{-1/16})(r-1)^i\right) \geqq P\left(l_i \geqq (1-n^{-1/15})^{t_0}(r-1)^i\right) \geqq \\ &\geqq 1 - o(n^{-2}) - t_0 \exp(-n^{1/15}) = 1 - o(n^{-2}) \end{aligned}$$

for every i, $i \leqq t_0$, proving (1).

The proof of Theorem 1 is almost complete. Let W_{j_1} and W_{j_2} be arbitrary classes. Let $\tilde{L}_{j_i}$ be the set corresponding to L_{t_i} if V_1 is replaced by W_{j_i}. Let us estimate the conditional probability

$$P\big(d(W_{j_1}, W_{j_2}) > t_1+t_2+1 \,|\, d(W_{j_1}, W_{j_2}) > t_1+t_2\big). \tag{8}$$

Since $t_i < t_0$, relation (1) shows that with probability $1-o(n^{-2})$ there are at least

$$(1-n^{-1/16})(r-1)^{t_i}$$

edges with one endvertex in $\tilde{L}_{j_i}$ and the other outside it. The probability that such an edge does not join $\tilde{L}_{j_1}$ to $\tilde{L}_{j_2}$ is at most

$$1-|\tilde{L}_{j_2}|/(rn).$$

Hence the probability that none of these edges joins $\tilde{L}_{j_1}$ to $\tilde{L}_{j_2}$ is at most

$$\big(1-(1-n^{1/16})(r-1)^{t_2}/(rn)\big)^{(1-n^{-1/16})(r-1)^{t_1}} \leqq$$
$$\leqq \exp\{-(1-2n^{-1/16})(r-1)^{t_1+t_2}/(rn)\} \leqq$$
$$\leqq \exp\{-(1-2n^{-1/16})(2+\varepsilon)\log n\} = o(n^{-2}).$$

This shows that the conditional probability given by (8) is $o(n^{-2})$, so

$$P\big(d(W_{j_1}, W_{j_2}) > t_1+t_2+1\big) = o(n^{-2}).$$

Finally,

$$P(\operatorname{diam} F > d) = P\big(\max_{j_1, j_2} d(W_{j_1}, W_{j_2}) > t_1+t_2+1\big) \leqq$$
$$\leqq \binom{n}{2} P\big(d(W_1, W_2) > t_1+t_2+1\big) = o(1),$$

completing the proof of our theorem. ∎

The condition $r \geqq 5$ can be dispensed with fairly easily though at the price of some inconvenience. If $r=3$ or 4, we cannot claim that $l_1 \geqq r-1$ and $l_2 \geqq (r-1)^2$ with probability $1-o(n^{-2})$. If $r=3$ then with probability $c_3 n^{-2}$ we have $l_2=0$ and if $r=4$ then with probability $c_4 n^{-2}$ we have $l_1=0$. However, what we can say is that with probability $1-O(n^{-1})$ we have $l_1(W_i) \geqq r-1$ and $l_2(W_i) \geqq (r-1)^2$ for every class W_i simultaneously, where $l_j(W_i)$ denotes the number corresponding to l_j if W_1 is replaced by W_i. Therefore we can use the same argument we used if we take everything conditional on this event of probability $1-O(n^{-1})$.

Remark. By carrying out the argument in the probability space of all configurations without short cycles (see [3] for the distribution of short cycles) one can obtain the following result.

Let $r \geqq 3$ and $\varepsilon > 0$ be fixed and let $d_0 = d_0(n)$ be the least integer satisfying

$$r(r-1)^{d-1} \geqq (2+\varepsilon) n \log n.$$

Then if n is sufficiently large, there is an r-regular graph of order n with diameter at most d_0.

As an indication of the proof we say only that for every $g \in \mathbf{N}$ the probability that (the graph associated with) a configuration has girth at least g is bounded away from 0. Furthermore, given a fixed k, we have $l_i(W_j) \geqq r(r-1)^i$ for every class W_j and every integer i, $1 \leqq i \leqq k$, provided the configuration has girth at least $2k+1$.

Now let us turn to our lower bound for the diameter of a.e. r-regular graph. In this case our aim is to show that the number of classes of a configuration at distance greater then k from a fixed class W_1 does not decrease too fast as k increases. Our result will not only show that Theorem 1 is essentially best possible but, in conjunction with Theorem 1, it will locate with fair accuracy the diameter of a.e. r-regular graph. The proof of our theorem relies on a technical lemma. In this lemma a *configuration F with vertex set S* means a partition of S into pairs, called the *edges* of F.

Lemma 2. *Given $0<\alpha<1/9$, the following assertion holds for every sufficiently large n.*

Let L and M be disjoint sets, $|L|=l$, $M=\bigcup_{i=1}^{m} W_i$, $|W_i|=r$ and $|M|=rm$ such that

$$n^{2/3} < 4m\left(\frac{rm}{rm+l}\right)^r \quad \textit{and} \quad rm+l \leqq rn.$$

Consider the probability space of all configurations F with vertex set $L\cup M$. Denote by $X=X(F)$ the number of classes W_i not joined to L by an edge of F and by $Y=Y(F)$ the number of vertices in M not joined to L. Then with probability at least $1-n^{-\alpha}$ we have

$$X = m\left(\frac{rm}{rm+l}\right)^r(1+\delta), \tag{9}$$

$$Y = m\frac{rm}{rm+l}(1+\varepsilon), \tag{10}$$

and

$$\frac{X}{Y} = \left(\frac{rm}{rm+l}\right)^{r-1}(1+\eta), \tag{11}$$

where

$$|\delta| \leqq \tfrac{1}{2}n^{-\alpha}, \quad |\varepsilon| \leqq n^{-1/3} \quad \text{and} \quad |\eta| \leqq n^{-\alpha}.$$

Proof. Denote by ε_1 the probability that a given class W_i contains an edge and by ε_2 the probability that a given pair of classes W_i, W_j contains an edge. Clearly

$$\varepsilon_1 < \frac{r(r-1)}{r(m-1)+l} < \frac{r}{m}$$

and

$$\varepsilon_2 < \frac{2r(2r-1)}{r(m-2)+l} < \frac{4r}{m}.$$

The probability p_1 that no vertex of a given class W_i is joined to a vertex of L satisfies

$$\frac{(r(m-1))_r}{(r(m-1)+l)_r}(1-\varepsilon_1) \leqq p_1 \leqq \frac{(r(m-1))_r}{(r(m-1)+l)_r}(1-\varepsilon_1)+\varepsilon_1$$

and the probability p_2 that no vertex of a given pair of classes W_i, W_j is joined to a vertex of L satisfies

$$\frac{(r(m-2))_{2r}}{(r(m-2)+l)_{2r}} \leqq p_2 \leqq \frac{(r(m-2))_{2r}}{(r(m-2)+l)_{2r}}(1-\varepsilon_1)+\varepsilon_2.$$

Hence

$$m\left(\frac{r(m-2)}{r(m-2)+l}\right)^r \leqq E(X) \leqq m\left(\frac{r(m-1)}{r(m-1)+l}\right)^r + r \tag{12}$$

and the second factorial moment $E_2(X)=E\big(X(X-1)\big)$ satisfies

$$m(m-1)\left(\frac{r(m-4)}{r(m-4)+l}\right)^{2r} \leqq E_2(X) \leqq m(m-1)\left(\frac{r(m-2)}{r(m-2)+l}\right)^{2r} + 4rm.$$

Consequently

$$\sigma^2(X) = E_2(X)+E(X)-E(X^2) \leqq 5rm$$

and so by Chebyshev's inequality

$$P\big(|X-E(X)| \geqq \tfrac{1}{2}n^{-\alpha}E(X)\big) \leqq 4n^{2\alpha}5rm/E(X)^2 < \tfrac{1}{4}n^{-\alpha}.$$

Making use of the approximation of $E(X)$ given by (12) we find that (9) holds with probability at least $1-\frac{1}{2}n^{-\alpha}$.

The probability that the neighbours of k given vertices $x_1, \ldots, x_k$ belong to a given set S of s vertices, where $\{x_1, \ldots, x_k\} \cap S = \emptyset$, is clearly

$$\frac{s(s-1)\ldots(s-k+1)}{(rm+l-1)(rm+l-3)\ldots(rm+l-2k+1)}.$$

Applying this with $k=1$ and 2 we find that

$$E(Y) = rm\frac{rm-1}{rm+l-1} \tag{13}$$

and

$$E_2(Y) = rm(rm-1)\left\{\frac{1}{rm+l-1}+\frac{(rm-2)(rm-3)}{(rm+l-1)(rm+l-3)}\right\}.$$

Hence simple calculations give

$$\sigma^2(Y) = E_2(Y)+E(Y)-E(Y)^2 \leqq 2E(Y).$$

By Chebyshev's inequality

$$P\big(|Y-E(Y)| \geqq \tfrac{1}{2}n^{-\alpha}E(Y)\big) \leqq 4n^{2\alpha}\sigma^2(G)/E(Y)^2 < \tfrac{1}{4}n^{-1/3},$$

and it is easily checked that by (13) relation (10) holds with probability at least $1-\frac{1}{2}n^{-\alpha}$.

Finally, note that both (9) and (10) hold with probability at least $1-n^{-\alpha}$, and if they both hold then so does (11). ∎

Theorem 3. *The diameter of a.e. r-regular graph of order n is at least*

$$[\log_{r-1} n]+\left[\log_{r-1}\log n-\log_{r-1}\frac{6r}{r-2}\right]+1.$$

Proof. Set $f=[\log_{r-1}n]-3$ so that

$$(r-1)^{f+3}\leqq n$$

and put

$$q_1 = 1-\left\{(r-1)^3-\frac{2}{r-2}\right\}^{-1}.$$

Let g be the maximal integer satisfying

$$q_1^{r(r-1)^g/(r-2)} \geqq n^{-1/3}.$$

Since

$$\log(1/q_1) \leqq \frac{2}{(r-1)^3},$$

$$g \geqq \left[\log_{r-1}\log n-\log_{r-1}\frac{6r}{r-2}\right]+3.$$

Thus to prove the theorem it suffices to show that the diameter of a.e. graph is at least $f+g+1$. In fact we shall show that for a.e. configuration there is a class at distance at least $f+g+1$ from W_1.

Given a configuration F, denote by a_k the number of classes W_j at distance greater than k from W_1. Furthermore, denote by b_k the number of vertices not in a class at distance at most $k-1$ from W_1 and not adjacent to a vertex in a class at distance at most $k-1$ from W_1. Thus, with the notation of Theorem 1, $b_k=|M_k|$ and $b_k=ra_k+|L_k|$.

Let $0<\alpha<1/9$. We claim that if n is sufficiently large then with probability at least $1-gn^{-\alpha}$ we have

(14) $$a_{f+g}\geqq n^{2/3}.$$

Since $gn^{-\alpha}=o(1)$, this will imply the assertion of the theorem.

For $0\leqq k\leqq g$ set

$$A_k = \tfrac{1}{2}nq_1^{r\{(r-1)^k-1\}/(r-2)}q_2^{r\{(r-1)^k-(r-2)k-1\}/(r-2)^2}$$

and

$$R_k = q_1^{(r-1)^k}q_2^{\{(r-1)^k-1\}/(r-2)},$$

where $q_2=1-n^{-\alpha}$. Note that $A_0=\frac{1}{2}n$ and $R_0=q_1$. Furthermore,

$$A_k R_k^r q_2 = \tfrac{1}{2}nq_1^{r\{(r-1)^k-1\}/(r-2)}q_2^{r\{(r-1)^k-(r-2)k-1\}}q_1^{(r-1)^k r}q_2^{r\{(r-1)^k-1\}/(r-2)+1} =$$

$$= \tfrac{1}{2}nq_1^{r\{(r-1)^{k+1}-1\}/(r-2)}q_2^{r\{(r-1)^{k+1}(r-2)(k+1)-1\}/(r-2)^2} = A_{k+1}$$

and

$$R_k^{r-1}q_2 = q_1^{(r-1)^{k+1}}q_2^{(r-1)\{(r-1)^{k+1}-1\}/(r-2)+1} = R_{k+1}.$$

We shall prove by induction on k that

(15) $$a_{f+k} \geqq A_k$$

and

(16) $$\frac{ra_{f+k}}{b_{f+k}} \geqq R_k$$

for all $k, 0 \leqq k \leqq g$.

Let us check the inequalities for $k=0$. Given a configuration, denote by t_f the total number of classes at distance at most f from W_1 and by s_f the number of classes at distance f from W_1. Then clearly

$$t_f \leqq 1+r\sum_{i=0}^{f-1}(r-1)^i = 1+\frac{r(r-1)^f-1}{r-2} \leqq r(r-1)^f/(r-2) \leqq \frac{rn}{(r-1)^3(r-2)},$$

$$s_f \leqq r(r-1)^{f+1}$$

and

$$a_f = n-t_f, \quad b_f \leqq rn-rt_f+(r-1)s_f.$$

Hence

$$a_f \geqq n\left(1-\frac{r}{(r-1)^3(r-2)}\right) > \frac{n}{2}$$

and

$$\frac{ra_f}{b_f} \geqq 1-\frac{(r-1)s_f}{rn-rt_f+(r-1)s_f} \geqq 1-\frac{(r-1)^f}{n-r(r-1)^f/(r-2)+(r-1)^f} =$$

$$= 1-\{(r-1)^3-2/(r-2)\}^{-1} = q_1.$$

Thus (15) and (16) do hold for $k=0$.

Assume that (15) and (16) hold for some $k, 0 \leqq k < g$, with probability at least $1-kn^{-\alpha}$. It suffices to show that the probability that (15) and (16) hold for $k+1$, conditional on the same inequalities holding for k, is at least $1-n^{-\alpha}$.

Pick m and l such that

(15') $$m \geqq A_k,$$

(16') $$\frac{rm}{rm+l} \geqq R_k$$

and

$$rm+l \leqq rn.$$

By what has just been said, the proof of the induction step will be complete if we show that (15') and (16') hold for $k+1$ with probability at least $1-n^{-\alpha}$, conditional on $a_{f+k}=m$ and $b_{f+k}=rm+l$. Note now that the distribution of a_{f+k+1} and b_{f+k+1}, conditional on $a_{f+k}=m$ and $b_{f+k}=rm+l$, is identical with the distribution of X and Y in Lemma 2. As $g<\log_{r-1}\log n+1$, we have

$$q_2^{r(r-1)^g} \geqq 1-n^{-\alpha}r(r-1)^g \geqq 1/2,$$

so

$$m\left(\frac{rm}{rm+l}\right)^r \geqq A_k R_k^r > A_{k+1} \geqq A_g \geqq \tfrac{1}{4}nq_1^{r(r-1)^g/(r-2)} \geqq \tfrac{1}{4}n^{2/3}.$$

Thus the conditions of Lemma 2 are satisfied. Hence, conditional on $a_{f+k}=m$ and $b_{f+k}=rm+l$, with probability at least $1-n^{-\alpha}$ we have

$$a_{f+k+1} \geqq a_{f+k}\left(\frac{a_{f+k}}{b_{f+k}}\right)^r q_2 \geqq A_k R_k^r q_2 = A_{k+1}$$

and

$$\frac{a_{f+k+1}}{b_{f+k+1}} \geqq R_k^{r-1} q_2 = R_{k+1}.$$

This completes the proof of the induction step. Therefore (14) holds and so the proof of Theorem 2 is complete. ∎

In conclusion let us note that it is likely that with considerably more effort one could locate the diameter even more precisely. It seems to us that the following is true. Let $\varepsilon>0$ and let $d_1=d_1(n)$ be the maximal integer with

$$r(r-1)^{d_1-2} \leqq (2-\varepsilon)n \log n$$

and let d_2 be the minimal integer with

$$r(r-1)^{d_2-1} \geqq (2+\varepsilon)n \log n.$$

Then if r is sufficiently large, the diameter $d(G)$ of a.e. r-regular graph G of order n satisfies

$$d_1 \leqq d(G) \leqq d_2.$$

References

[1] E. A. BENDER and E. R. CANFIELD, The asymptotic number of labelled graphs with given degree sequences, *J. Combinatorial Theory (A)* **24** (1978) 296—307.

[2] B. BOLLOBÁS, *Extremal Graph Theory*, Academic Press, London, New York and San Francisco, 1978.

[3] B. BOLLOBÁS, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *Europ. J. Combinatorics* **1** (1980). 311—316.

[4] B. BOLLOBÁS, The diameter of random graphs, *Trans. Amer. Math. Soc.* **267** (1981) 41—52.

[5] V. KLEE and D. LARMAN, Diameters of random graphs, *Canad. J. Math.* **33** (1981) 618—640.

[6] A. D. KORSHUNOV, On the diameter of random graphs, *Soviet Mat. Doklady* **12** (1971) 302—305.

Béla Bollobás

Dept. of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge, England

W. Fernandez de la Vega

Laboratoire de Recherche en Informatique
ERA 452, Bât. 490
Université Paris Sud
91405 Orsay, France

Mailing address:
4 bis rue Wulfran Puget,
13008 Marseille, France