A Beginner's Guide to Supergravity

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Abstract

This is a write-up of lectures on basic $N = 1$ supergravity in four dimensions given during a one-semester course at the Friedrich-Schiller-University Jena. Aimed at graduate students with some previous exposure to general relativity and rigid supersymmetry, we provide a self-contained derivation of the off-shell supergravity multiplet and the most general couplings of chiral multiplets to the latter.

Contents

1 Introduction

To be written ...
Numerous introductory texts on supergravity in various dimensions are available, e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].
Our conventions for $D = 4$ supersymmetry are essentially those of Wess & Bagger [2], in particular we use two-component Weyl spinors and a Minkowski metric with mostly plus signature. The only differences are certain numerical rescalings, chosen such that numerous factors of $\sqrt{2}$ are absent. The most prominent place where this has consequences is the supersymmetry algebra (2.15), which in [2] contains a factor of 2. These rescalings simplify calculations, but the price to pay are non-canonical normalizations of the gravitino kinetic terms. We feel that it is more convenient not to carry the $\sqrt{2}$'s around; if desired, canonical normalizations can be reinstated at the end of the calculations by performing the inverse rescalings. A translation table between our conventions and those of [2] can be found in appendix D.
Beyond the conventions, there are more severe differences to [2]. First of all, we avoid the superspace formalism and work with component multiplets exclusively. This was dictated in part by time constraints, but it also makes the derivation of the off-shell supergravity multiplet easier in that no Wess-Zumino-like gauge-fixing is needed. We follow here an approach developed by Brandt in [15] (see also [12]), which is a tensor calculus applicable to any irreducible gauge theory. Second, when it comes to coupling matter to supergravity, we restrict ourselves to chiral multiplets in these lectures and negect vector multiplets, again due to time constraints during the semester.

2 Brief Review of Rigid Supersymmetry

In this first chapter, we review the basics of rigid supersymmetry in four dimensions that are needed to develop supergravity. Throughout these lectures we will use Weyl spinors, whose properties we recall first before introducing supersymmetry.

2.1 Weyl Spinors

In our conventions, the Dirac γ-matrices γ^a with $a = 0, \ldots, 3$ satisfy the Clifford algebra

$$\{\gamma_a\,,\gamma_b\} = -2\,\eta_{ab}\mathbb{1}\ , \tag{2.1}$$

where $(\eta_{ab}) = \mathrm{diag}(-1, 1, 1, 1)$ is the Minowski metric. A particular representation of this algebra is the Weyl representation

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}\ . \tag{2.2}$$

Here, the σ-matrices are given by

$$\sigma^a = (-\mathbb{1}, \vec{\tau}\,)\ , \qquad \bar{\sigma}^a = (-\mathbb{1}, -\vec{\tau}\,)\ , \tag{2.3}$$

where $\vec{\tau}$ are the three Pauli matrices. In the Weyl representation, we have

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\ , \tag{2.4}$$

such that Dirac spinors Ψ_D decompose into left- and right-handed two-component spinors with respect to the projectors $P_{L/R} = \frac{1}{2}(\mathbb{1} \pm \gamma^5)$,

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}\ . \tag{2.5}$$

Dotted and undotted Greek indices from the beginning of the alphabet run from 1 to 2. The Weyl spinors χ and $\bar{\lambda}$ form irreducible (and inequivalent) representations of the universal covering $\mathrm{SL}(2, \mathbb{C})$ of the Lorentz group; infinitesimally, we have

$$\ell_{ab}\Psi_D = -\tfrac{1}{4}[\,\gamma_a\,,\gamma_b\,]\Psi_D = -\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix}\begin{pmatrix} \chi \\ \bar{\lambda} \end{pmatrix}\ . \tag{2.6}$$

Here, we have introduced matrices

$$\sigma^{ab} = \tfrac{1}{4}(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a)\ , \qquad \bar{\sigma}^{ab} = \tfrac{1}{4}(\bar{\sigma}^a\sigma^b - \bar{\sigma}^b\sigma^a)\ , \tag{2.7}$$

satisfying the same commutation relations as the Lorentz generators ℓ_{ab}, i.e.,

$$[\,\sigma_{ab}\,,\sigma_{cd}\,] = \eta_{ac}\sigma_{bd} - \eta_{bc}\sigma_{ad} + \eta_{bd}\sigma_{ac} - \eta_{ad}\sigma_{bc} \tag{2.8}$$

and analogously for $\bar{\sigma}_{ab}$. Using the $\mathrm{SL}(2, \mathbb{C})$-invariant ε-tensors

$$(\varepsilon^{\alpha\beta}) = -(\varepsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\varepsilon^{\dot{\alpha}\dot{\beta}}) = -(\varepsilon_{\dot{\alpha}\dot{\beta}})\ , \tag{2.9}$$

we can pull up and down spinor indices,

$$\chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta \ , \qquad \bar\lambda_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\beta} \ , \tag{2.10}$$

and form Lorentz invariants from two Weyl spinors of the same chirality,

$$\begin{aligned} \chi\psi &\equiv \chi^\alpha\psi_\alpha = \varepsilon^{\alpha\beta}\chi_\beta\psi_\alpha = -\varepsilon^{\alpha\beta}\chi_\alpha\psi_\beta = -\chi_\alpha\psi^\alpha = \psi^\alpha\chi_\alpha = \psi\chi \\ \bar\lambda\bar\psi &\equiv \bar\lambda_{\dot\alpha}\bar\psi^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\beta}\bar\psi^{\dot\alpha} = -\varepsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\alpha}\bar\psi^{\dot\beta} = -\bar\lambda^{\dot\alpha}\bar\psi_{\dot\alpha} = \bar\psi_{\dot\alpha}\bar\lambda^{\dot\alpha} = \bar\psi\bar\lambda \ . \end{aligned} \tag{2.11}$$

In the last steps, we have assumed that the spinors anticommute.
Complex conjugation reverses the order of fields and turns left-handed spinors into right-handed ones and vice versa; e.g.

$$(\chi\psi)^* = (\chi^\alpha\psi_\alpha)^* = (\psi_\alpha)^*(\chi^\alpha)^* = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\psi\bar\chi = \bar\chi\bar\psi \ . \tag{2.12}$$

In the Weyl representation, Majorana spinors are of the form

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \quad \text{with} \quad \bar\chi^{\dot\alpha} = (\chi^\alpha)^* \ . \tag{2.13}$$

The free action for a massive Majorana spinor then reads

$$\mathcal{L}_0 = -\tfrac{1}{2}\bar\Psi_M(\mathrm{i}\gamma^\mu\partial_\mu + m)\Psi_M = -\tfrac{\mathrm{i}}{2}\chi\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\chi - \tfrac{1}{2}m(\chi\chi + \bar\chi\bar\chi) \ , \tag{2.14}$$

where $A\overleftrightarrow{\partial}_\mu B = A\partial_\mu B - \partial_\mu A B$. Note that the γ- and σ-matrices carry Greek spacetime indices here; in flat spacetime and Cartesian coordinates the relation is $\gamma^\mu = \delta^\mu_a\gamma^a$ and analogously for σ^μ. Once we include gravity, we will have to distinguish between the two kinds of indices.
Numerous identities satisfied by the σ-matrices can be found in appendix A.

2.2 The Supersymmetry Algebra

The algebra of rigid supersymmetry transformations is generated by translation operators ∂_μ and spinorial derivatives D_α, $\bar D_{\dot\alpha}$ satisfying the commutation relations

$$\begin{aligned} \{D_\alpha\,, \bar D_{\dot\alpha}\} = -\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \ , \qquad \{D_\alpha\,, D_\beta\} = 0 \ , \qquad \{\bar D_{\dot\alpha}\,, \bar D_{\dot\beta}\} = 0 \\ [\,\partial_\mu\,, D_\alpha\,] = 0 \ , \qquad [\,\partial_\mu\,, \bar D_{\dot\alpha}\,] = 0 \ . \end{aligned} \tag{2.15}$$

$\bar D_{\dot\alpha} = (D_\alpha)^*$ is the complex conjugate[1] of D_α. These operators are graded derivations, i.e., they are linear and satisfy the graded Leibniz rule and chain rule.

[1]In general, the complex conjugate O^* of a graded operator O is defined through the relation $O^*\phi \equiv (-)^{|O||\phi|}(O\phi^*)^*$. Note that this implies that complex conjugation does not reverse the order in a product of operators: $(O_1\,O_2)^* = (-)^{|O_1||O_2|}\,O_1^*\,O_2^*$.

An infinitesimal rigid supersymmetry transformation δ_Q of a field $\phi(x)$ is given by the action of D_α and $\bar{D}_{\dot\alpha}$ on ϕ,

$$\delta_Q(\epsilon)\,\phi(x) = \big(\epsilon^\alpha D_\alpha + \bar\epsilon_{\dot\alpha}\bar{D}^{\dot\alpha}\big)\phi(x)\ . \tag{2.16}$$

The spinor parameters are constant and related by $\bar\epsilon_{\dot\alpha} = (\epsilon_\alpha)^*$, such that δ_Q is real. According to (2.15), the commutator of two supersymmetry transformations yields a translation,

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,] = \xi^\mu\partial_\mu\ , \tag{2.17}$$

with constant parameter

$$\xi^\mu(\epsilon_1,\epsilon_2) = \mathrm{i}(\epsilon_2\sigma^\mu\bar\epsilon_1 - \epsilon_1\sigma^\mu\bar\epsilon_2)\ . \tag{2.18}$$

An off-shell realization of the supersymmetry algebra usually requires the presence of non-dynamical auxiliary fields in the supersymmetry multiplets. However, as we will see, after construction of a model it is often advantageous to eliminate them in order to reveal geometrical structure, or it may be difficult to find a suitable set of auxiliary fields in the first place. Without them, the supersymmetry algebra closes only on-shell, i.e., modulo trivial symmetries of the form

$$\delta_{\mathrm{triv}}\phi^i = \mathcal{E}^{ij}(\phi,x)\,\frac{\delta S}{\delta\phi^j}\ , \qquad \mathcal{E}^{ij} = -(-)^{|\phi^i||\phi^j|}\mathcal{E}^{ji}\ . \tag{2.19}$$

These obviously leave the action S invariant,

$$\delta_{\mathrm{triv}}S[\phi] = \int \delta_{\mathrm{triv}}\phi^i\,\frac{\delta S}{\delta\phi^i} = \int \mathcal{E}^{ij}(\phi)\,\frac{\delta S}{\delta\phi^j}\,\frac{\delta S}{\delta\phi^i} = 0\ .$$

We expect the appearance of trivial symmetries only for fermions, since supersymmetry transformations are at most linear in derivatives while field equations of bosons are of second order.

2.3 Chiral Multiplets – Part 1

In supersymmetric theories, matter fields (such as quarks, leptons, and Higgs particles) are components of chiral multiplets. These can be built starting with a complex Lorentz scalar $\phi(x)$ satisfying the constraint

$$\bar{D}_{\dot\alpha}\phi = 0\ . \tag{2.20}$$

Accordingly, the complex conjugate field is anti-chiral, $D_\alpha\bar\phi = 0$. The higher components of the multiplet are obtained by successive application of D_α,

$$\phi\ , \qquad \chi_\alpha = D_\alpha\phi\ , \qquad F = -\tfrac{1}{2}D^2\phi\ , \tag{2.21}$$

and similarly for the complex conjugate components. The procedure stops with F due to the identities

$$D_\alpha D^2 = 0 \ , \qquad \bar{D}_{\dot\alpha}\bar{D}^2 = 0 \ , \qquad [\,\bar{D}_{\dot\alpha}\,, D^2\,] = 2\mathrm{i}\,\partial_{\alpha\dot\alpha}D^\alpha \ . \tag{2.22}$$

The multiplet thus contains in addition to ϕ a Weyl spinor χ_α and a complex scalar F, which has mass dimension $[\phi]+1=2$ and will turn out to be a non-dynamical auxiliary field. The supersymmetry transformations follow from the action of D_α and $\bar{D}_{\dot\alpha}$ on the components:

$$\begin{aligned} D_\alpha\phi &= \chi_\alpha & \bar{D}_{\dot\alpha}\phi &= 0 \\ D_\alpha\chi_\beta &= -\varepsilon_{\alpha\beta}F & \bar{D}_{\dot\alpha}\chi_\alpha &= -\mathrm{i}\partial_{\alpha\dot\alpha}\phi \\ D_\alpha F &= 0 & \bar{D}_{\dot\alpha}F &= -\mathrm{i}\partial_{\alpha\dot\alpha}\chi^\alpha \ . \end{aligned} \tag{2.23}$$

Fields subject to the constraint (2.20) are called chiral, even if they are not Lorentz scalars. Since the supersymmetry generators are (graded) derivations, chiral fields form a ring, i.e., sums and products of chiral fields are again chiral. Note that the second identity in (2.22) implies that $\bar{D}^2K$ is chiral for arbitrary fields K. $\bar{D}^2$ is called the chiral projector; it will receive corrections in local supersymmetry.

The most general supersymmetric action (with at most two derivatives) for a set of chiral multiplets ϕ^i is given in terms of a real function $K(\phi,\bar\phi)$ and a holomorphic function $W(\phi)$ of mass dimensions 2 and 3, respectively, by the integral

$$S[\phi,\chi,F] = -\frac{1}{2}\int d^4x\, D^2\Big(-\frac{1}{4}\,\bar{D}^2K(\phi,\bar\phi) + W(\phi)\Big) + \text{c.c.}\ . \tag{2.24}$$

K is called the Kähler potential and yields the kinetic terms; we will have a lot more to say about it in later chapters. W is called the superpotential and gives rise to mass and interaction terms. S is supersymmetric by virtue of the identities (2.22) and the chirality of ϕ^i. We can understand the invariance also in the following way: The above discussion implies that the two terms in brackets form composite chiral fields. Acting with D^2 on them produces the F-components of the respective multiplets. From (2.23) we infer that these transform into total derivatives under supersymmetry, hence the action is invariant. If one imposes renormalizability, no coupling constants of negative mass dimension must occur. This restricts the potentials to the form

$$K(\phi,\bar\phi) = \delta_{ij}\phi^i\bar\phi^j \tag{2.25}$$

$$W(\phi) = \lambda_i\phi^i + \tfrac{1}{2}m_{ij}\phi^i\phi^j + \tfrac{1}{3}g_{ijk}\phi^i\phi^j\phi^k \ , \tag{2.26}$$

with in general complex constant parameters. In K, terms linear in ϕ and $\bar\phi$ can be neglected as they would give rise to a total derivative upon application of $D^2\bar{D}^2$. Working out the Lagrangian explicitly using (2.23), we find (modulo a total derivative)

$$\mathcal{L} = -\partial^\mu\phi^i\,\partial_\mu\bar\phi^i - \tfrac{\mathrm{i}}{2}\,\chi^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\chi^i + F^i\bar{F}^i + \big(F^i\,W_i(\phi) - \tfrac{1}{2}\,\chi^i\chi^j\,W_{ij}(\phi) + \text{c.c.}\big) \ , \tag{2.27}$$

where we have introduced the abbreviation

$$W_{i_1\ldots i_r} = \frac{\partial^r W}{\partial\phi^{i_1}\ldots\partial\phi^{i_r}} \ . \tag{2.28}$$

As anticipated, the equations of motion for the fields F^i, $\bar{F}^i$ are algebraic ($\approx$ denotes on-shell equality),

$$\frac{\delta S}{\delta \bar{F}^i} = F^i + \bar{W}_i(\bar{\phi}) \approx 0 \ , \tag{2.29}$$

hence they are auxiliary and can be eliminated from the action and the transformations. This gives rise to a potential ($\mathcal{L} = \ldots - V$)

$$V(\phi,\bar{\phi}) = \sum_i |F^i(\bar{\phi})|^2 = \sum_i \left|\frac{\partial W}{\partial\phi^i}\right|^2 \ . \tag{2.30}$$

Note that V is non-negative. To read off the particle spectrum of the theory, we need to find the ground state(s) minimizing the potential. Let us denote the vacuum expectation values (*vevs*) of the scalars by ϕ_0,

$$\frac{\partial V}{\partial\phi^i}\Big|_{\phi_0} = W_{ij}(\phi_0)\,\bar{W}_j(\bar{\phi}_0) = 0 \ , \tag{2.31}$$

and expand the action in terms of fluctuations $\varphi = \phi - \phi_0$,

$$\begin{aligned}\mathcal{L} = &-\partial^\mu\varphi^i\,\partial_\mu\bar{\varphi}^i - \tfrac{1}{2}\,(\bar{\varphi}^i,\varphi^i)M^B_{ij}\begin{pmatrix}\varphi^j\\ \bar{\varphi}^j\end{pmatrix} - V_{\text{min}}\\ &- \tfrac{i}{2}\,\chi^i\sigma^\mu\overleftrightarrow{\partial}_\mu\bar{\chi}^i - \tfrac{1}{2}M^F_{ij}\,\chi^i\chi^j - \tfrac{1}{2}\bar{M}^F_{ij}\,\bar{\chi}^i\bar{\chi}^j + \ldots \ .\end{aligned} \tag{2.32}$$

The bosonic and fermionic mass matrices, respectively, are given by

$$M^B_{ij} = \begin{pmatrix}\bar{W}_{ik}W_{kj} & \bar{W}_{ijk}W_k\\ W_{ijk}\bar{W}_k & W_{ik}\bar{W}_{kj}\end{pmatrix}\Big|_{\phi_0} \ , \qquad M^F_{ij} = W_{ij}(\phi_0) \ . \tag{2.33}$$

After diagonalizing them by means of unitary rotations of the fields, we can read off the masses of the particles. If the ground state is given by the absolute minimum $V_{\text{min}} = 0$, we have $W_i(\phi_0) = 0$ according to (2.30). The off-diagonal terms in M^B_{ij} then vanish and the bosonic and fermionic mass matrices can be diagonalized by a joint unitary rotation $\varphi^i \to U^{ij}\varphi^j$, $\chi^i \to U^{ij}\chi^j$, implying that bosons and fermions have equal masses.
A non-vanishing, constant (thus preserving Poincaré invariance) *vev* of at least one of the F^i breaks supersymmetry spontaneously, as it leads to a non-vanishing *vev* of the variation $\delta_Q\chi^i$,

$$\langle 0|F^i|0\rangle = f^i \neq 0 \quad \Rightarrow \quad \langle 0|\delta_Q\chi^i|0\rangle = \epsilon f^i \neq 0 \ , \tag{2.34}$$

which is incompatible with the existence of a supersymmetric ground state $|0\rangle$. From (2.30) we infer that supersymmetry is spontaneously broken if and only if $V_{\text{min}} > 0$. In this case, (2.31) implies that there is (at least) one non-trivial eigenvector of the fermionic

mass matrix M^F_{ij} with zero eigenvalue. This is Goldstone's Theorem for supersymmetry; the corresponding massless fermion is called a goldstino.
If we insert the solution $F^i \approx -\bar{W}_i(\bar{\phi})$ to (2.29) into the transformation (2.23) of χ^i, the supersymmetry algebra closes only on-shell on the fermions,

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,]\chi^i_\alpha = \xi^\mu\partial_\mu\chi^i_\alpha + \mathcal{E}^{ij}_{\alpha\dot\alpha}\,\frac{\delta S}{\delta\bar\chi^j_{\dot\alpha}}\ , \qquad \mathcal{E}^{ij}_{\alpha\dot\alpha} = \delta^{ij}(\epsilon_{2\alpha}\,\bar\epsilon_{1\dot\alpha} - \epsilon_{1\alpha}\,\bar\epsilon_{2\dot\alpha})\ , \tag{2.35}$$

the extra term being a trivial symmetry of the form (2.19).

3 Spinors in General Relativity

As we have seen above, spinor fields are a central ingredient of supersymmetric theories. We shall now work out how to couple them to gravity. This will require an extension of the familiar formulation of general relativity in terms of a metric, which we recall first. With higher-dimensional supergravities in mind, we leave the number D of spacetime dimensions arbitrary in this chapter.
It should be borne in mind that the existence of spinors on topologically non-trivial spacetimes is not guaranteed. The mathematical criterion for the existence of a so-called spin structure is the triviality of the second Stiefel-Whitney class (see e.g. [17]). We shall always assume this to be the case in these lectures.

3.1 Review of the Standard Formalism

Under a general coordinate transformation $x \to x'(x)$, the components V^μ of vector fields and W_μ of 1-forms transform as

$$V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\,V^\nu(x)\ , \qquad W'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\,W_\nu(x)\ . \tag{3.1}$$

The matrix $(\partial x'^\mu/\partial x^\nu)$ is an element of the general linear group $\mathrm{GL}(D,\mathbb{R})$. General tensors of type (p,q) transform like the tensor product of p vectors and q 1-forms. We shall consider only infinitesimal transformations in the following, where $x' = x - \xi(x)$ with ξ small such that we can neglect terms of order $O(\xi^2)$. The transformed fields we denote with δ_P, e.g.

$$\delta_P(\xi)V^\mu \equiv V'^\mu(x) - V^\mu(x) = \xi^\nu\partial_\nu V^\mu - \partial_\nu\xi^\mu\,V^\nu\ . \tag{3.2}$$

On tensors, such infinitesimal transformations are generated by the Lie derivative

$$\mathcal{L}_\xi = \xi^\mu\partial_\mu + \partial_\nu\xi^\mu\Delta_\mu{}^\nu\ . \tag{3.3}$$

Here, the $\Delta_\mu{}^\nu$ are the D^2 generators of the Lie algebra $\mathrm{gl}(D,\mathbb{R})$. They act on vector and 1-form components as

$$\Delta_\mu{}^\nu\,V^\rho = -\delta^\rho_\mu\,V^\nu\ , \qquad \Delta_\mu{}^\nu\,W_\rho = \delta^\nu_\rho\,W_\mu\ , \tag{3.4}$$

and satisfy the Leibniz rule, i.e., they act additively on each index of a tensor. For example,

$$\begin{aligned}\mathcal{L}_\xi T^\rho{}_\sigma &= \xi^\mu \partial_\mu T^\rho{}_\sigma + \partial_\nu \xi^\mu (\Delta_\mu{}^\nu T^{\rho\leftarrow}{}_\sigma + \Delta_\mu{}^\nu T^\rho{}_{\sigma\leftarrow}) \\ &= \xi^\mu \partial_\mu T^\rho{}_\sigma - \partial_\nu \xi^\rho T^\nu{}_\sigma + \partial_\sigma \xi^\mu T^\rho{}_\mu \ .\end{aligned}$$

Note that the $\Delta_\mu{}^\nu$ "see" only open indices; contraction of an upper index with a lower index yields an invariant, e.g. $\Delta_\mu{}^\nu (V^\rho W_\rho) = 0$.

Ordinary derivatives of tensors in general do not transform as tensors under $\mathrm{GL}(D, \mathbb{R})$, since the transformation parameters are x-dependent. To compensate for the derivative of the latter, we introduce a connection $\Gamma_{\mu\nu}{}^\rho$ and form a covariant derivative

$$\nabla_\mu = \partial_\mu - \Gamma_{\mu\nu}{}^\rho \Delta_\rho{}^\nu \ . \tag{3.5}$$

The transformation of $\Gamma_{\mu\nu}{}^\rho$ we then determine such that the covariant derivative of a tensor transforms again as a tensor, i.e., we require

$$\delta_P(\xi) \nabla_\mu T \stackrel{!}{=} \mathcal{L}_\xi \nabla_\mu T \ . \tag{3.6}$$

We can write this as

$$\begin{aligned}\nabla_\mu \delta_P(\xi) T - \delta_P(\xi) \Gamma_{\mu\nu}{}^\rho \Delta_\rho{}^\nu T &= \nabla_\mu \delta_P(\xi) T + [\, \mathcal{L}_\xi \, , \nabla_\mu \,] T \\ \Rightarrow \ \ \delta_P(\xi) \Gamma_{\mu\nu}{}^\rho \Delta_\rho{}^\nu &= [\, \nabla_\mu \, , \mathcal{L}_\xi \,] \ .\end{aligned}$$

The commutator on the right yields a linear combination of $\mathrm{gl}(D, \mathbb{R})$ generators, from which we read off that[2]

$$\delta_P(\xi) \Gamma_{\mu\nu}{}^\rho = \partial_\mu \partial_\nu \xi^\rho + \mathcal{L}_\xi \Gamma_{\mu\nu}{}^\rho \ . \tag{3.7}$$

We recognize a part that looks like a tensor transformation, and an inhomogeneous term characteristic of a connection.

Of some importance is the commutator of two covariant derivatives. It yields a linear combination of a covariant derivative and a $\mathrm{GL}(D, \mathbb{R})$ transformation, with coefficients which depend on the connection,

$$[\, \nabla_\mu \, , \nabla_\nu \,] = -T_{\mu\nu}{}^\rho \nabla_\rho - R_{\mu\nu\rho}{}^\sigma \Delta_\sigma{}^\rho \ . \tag{3.8}$$

The so-called torsion is given by

$$T_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho \ , \tag{3.9}$$

while the curvature reads

$$R_{\mu\nu\rho}{}^\sigma = \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \partial_\nu \Gamma_{\mu\rho}{}^\sigma - \Gamma_{\mu\rho}{}^\lambda \Gamma_{\nu\lambda}{}^\sigma + \Gamma_{\nu\rho}{}^\lambda \Gamma_{\mu\lambda}{}^\sigma \ . \tag{3.10}$$

[2]Strictly speaking, the Lie derivative is defined only on tensors. When we write $\mathcal{L}_\xi \Gamma$, we mean the action of the right-hand side of (3.3) on Γ.

It is easily verified that they transform as tensors. Higher derivatives of the connection give rise to identities. They can be neatly summarized by the Jacobi identity

$$\sum\!\!\!\!\!\!\circlearrowright_{\mu\nu\rho} [\,\nabla_\mu\,,[\,\nabla_\nu\,,\nabla_\rho\,]\,] = 0\ , \tag{3.11}$$

where $\sum\!\!\!\!\!\circlearrowright$ denotes the cyclic sum. Inserting the expression for the commutator and collecting coefficients of ∇_σ and $\Delta_\sigma{}^\rho$, respectively, yields the two Bianchi identities

$$\sum\!\!\!\!\!\!\circlearrowright_{\mu\nu\rho} \left(R_{\mu\nu\rho}{}^\sigma - \nabla_\mu T_{\nu\rho}{}^\sigma - T_{\mu\nu}{}^\lambda T_{\lambda\rho}{}^\sigma\right) = 0 \tag{3.12}$$

$$\sum\!\!\!\!\!\!\circlearrowright_{\mu\nu\lambda} \left(\nabla_\lambda R_{\mu\nu\rho}{}^\sigma + T_{\mu\nu}{}^\kappa R_{\kappa\lambda\rho}{}^\sigma\right) = 0\ . \tag{3.13}$$

So far, we have dealt with some differential manifold not necessarily endowed with a metric. In order to measure lengths and angles, let us now introduce such a metric field $g_{\mu\nu}(x)$. It is a symmetric $\mathrm{GL}(D,\mathbb{R})$ tensor (meaning $\delta_P(\xi) g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$), in terms of which the line element is given by $ds^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu$. One can use the metric to define a scalar product (which is indefinite for Lorentzian signature) of two vectors,

$$\langle W, V\rangle \equiv W^\mu g_{\mu\nu} V^\nu\ . \tag{3.14}$$

We assume that the metric is invertible. It is common to denote the inverse with $g^{\mu\nu}$, i.e., $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$. Accordingly, $g_{\mu\nu}$ provides an isomorphism between co- and contravariant tensors by raising and lowering indices. For example, $V_\mu = g_{\mu\nu}V^\nu \Leftrightarrow V^\mu = g^{\mu\nu}V_\nu$.
Using the formula

$$\delta\,\mathrm{det}M = \mathrm{det}M\ \mathrm{tr}(M^{-1}\delta M) \tag{3.15}$$

for arbitrary variations of the determinant of a matrix M, it is easy to show that the square root of $g \equiv -\det(g_{\mu\nu}) > 0$ transforms into a total derivative, $\delta_P(\xi)\sqrt{g} = \partial_\mu(\xi^\mu\sqrt{g})$. Multiplying a coordinate scalar L with $\sqrt{g}$ then yields a density which can be integrated over spacetime to yield an invariant action, if the scalar L is built from the basic fields of the theory,

$$\delta_P \int d^Dx\,\sqrt{g}\;L = \int d^Dx\,\partial_\mu(\xi^\mu\sqrt{g}\;L) = 0\ . \tag{3.16}$$

In general relativity, it is common to impose conditions on the connection $\Gamma_{\mu\nu}{}^\rho$. First of all, one would like parallel transport not to change the lengths of vectors. This is the case if the metric is covariantly constant,

$$\nabla_\rho\, g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}{}^\sigma g_{\sigma\nu} - \Gamma_{\rho\nu}{}^\sigma g_{\mu\sigma} = 0\ . \tag{3.17}$$

This allows to express the symmetric part $\Gamma_{(\mu\nu)}{}^\rho$ of the connection in terms of derivatives of the metric and the torsion tensor (the antisymmetric part of $\Gamma_{\mu\nu}{}^\rho$). Second, one chooses the torsion to vanish, ${T_{\mu\nu}}^{\rho} = 0$. It then follows that the connection is symmetric and given by the so-called Christoffel symbols

$$ {\Gamma_{\mu\nu}}^{\rho} = \tfrac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \ . \tag{3.18} $$

This metric-compatible and torsion-free connection is unique and is called the Levi-Civita connection. Note how the Bianchi identities (3.12), (3.13) simplify for this choice:

$$ {R_{[\mu\nu\rho]}}^{\sigma} = 0 \ , \qquad \nabla_{[\lambda} {R_{\mu\nu]\rho}}^{\sigma} = 0 \ . \tag{3.19} $$

It is in fact natural to choose this connection, since it is the one that enters the covariant conservation equation of the energy-momentum tensor, see below. Also, light does not feel torsion: Consider the action for the Maxwell field $A_\mu(x)$,

$$ S[g, A] = -\frac{1}{4} \int d^D x \sqrt{g}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \ , \tag{3.20} $$

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ is the field strength. Invariance under general coordinate transformations is not entirely obvious, since $F_{\mu\nu}$ contains only ordinary derivatives, not covariant ones. As is easy to show,[3] however, if A_μ transforms as a tensor, then so does $F_{\mu\nu}$,

$$ \delta_P(\xi) A_\mu = \mathcal{L}_\xi A_\mu \quad \Rightarrow \quad \delta_P(\xi) F_{\mu\nu} = \mathcal{L}_\xi F_{\mu\nu} \ . \tag{3.21} $$

Second derivatives of ξ^μ are absent by virtue of the antisymmetry. The equation of motion for A_μ reads

$$ 0 \approx \frac{\delta S}{\delta A_\nu} = \partial_\mu (\sqrt{g} F^{\mu\nu}) = \sqrt{g}\, \nabla^{\rm LC}_\mu F^{\mu\nu} \ . \tag{3.22} $$

Here, $\nabla^{\rm LC}_\mu$ is the covariant derivative built from the Levi-Civita connection (3.18), which is symmetric and hence free of torsion. One may write $F_{\mu\nu} = 2\nabla^{\rm LC}_{[\mu} A_{\nu]}$, which makes covariance manifest.

Using the Levi-Civita connection, one can write the general coordinate transformation of the metric as

$$ \delta_P(\xi) g_{\mu\nu} = \nabla^{\rm LC}_\mu \xi_\nu + \nabla^{\rm LC}_\nu \xi_\mu \ , \tag{3.23} $$

where $\xi_\mu = g_{\mu\nu}\xi^\nu$. (This holds even if the metric is not covariantly constant.) In this form, the variation looks like a gauge transformation, which is indeed the proper interpretation.

We now turn to the description of a dynamical metric field. It is governed by the Einstein equation, which can be obtained by variation of an action. This Einstein-Hilbert action is essentially given by the curvature scalar $\mathcal{R}$, which is the trace of the Ricci tensor $\mathcal{R}_{\mu\nu}$,

$$ \mathcal{R} \equiv g^{\mu\nu} \mathcal{R}_{\mu\nu} \ , \qquad \mathcal{R}_{\mu\nu} \equiv {R_{\mu\rho\nu}}^{\rho} \ . \tag{3.24} $$

[3]The Lie derivative $\mathcal{L}_\xi = \{\mathrm{d}\,, i_\xi\}$ commutes with the nilpotent exterior derivative d. Thus, $\delta_P(\xi)F = \mathrm{d}\delta_P(\xi)A = \mathrm{d}\mathcal{L}_\xi A = \mathcal{L}_\xi \mathrm{d}A = \mathcal{L}_\xi F$. Note that this holds for arbitrary form degree of A.

For the Levi-Civita connection, $\mathcal{R}_{\mu\nu}$ is symmetric and a function of the metric and its first and second derivatives. The general[4] action of some matter fields ϕ^i coupled to gravity is then of the form

$$S[g,\phi] = S_{\rm EH}[g] + S_{\rm mat}[g,\phi] \ , \tag{3.25}$$

where $S_{\rm EH}$ is given by

$$S_{\rm EH}[g] = -\frac{1}{2\kappa^2}\int d^Dx\,\sqrt{g}\,\mathcal{R} \ . \tag{3.26}$$

$\kappa^2 = 8\pi G_N$ is proportional to Newton's constant. Let us derive from $S[g,\phi]$ the Einstein equation. Upon variation of the metric, the Ricci tensor changes by $\delta\mathcal{R}_{\mu\nu} = \nabla_\mu \delta\Gamma_{\rho\nu}{}^\rho - \nabla_\rho \delta\Gamma_{\mu\nu}{}^\rho$. The difference of two connections is always a tensor, so this expression makes sense. In the variation of $S_{\rm EH}$, the covariant derivatives can be integrated by parts and yield only a boundary term, which we drop. The entire variation comes from the factor $\sqrt{g}\,g^{\mu\nu}$,

$$\delta(\sqrt{g}\,g^{\mu\nu}) = \sqrt{g}\,(\tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\delta g_{\rho\sigma} \ . \tag{3.27}$$

Without specifying the matter action, we denote its variation by

$$\frac{\delta S_{\rm mat}}{\delta g_{\mu\nu}} = \tfrac{1}{2}\sqrt{g}\,T^{\mu\nu} \ . \tag{3.28}$$

$T^{\mu\nu}(\phi, g)$ is called the energy-momentum tensor. Putting everything together, we obtain the Einstein equation

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} \approx -\kappa^2\,T_{\mu\nu} \ . \tag{3.29}$$

The left-hand side is called the Einstein tensor $G_{\mu\nu}$. Its covariant divergence vanishes identically, $\nabla_\mu G^{\mu\nu} = 0$, as can be derived from the second Bianchi identity in (3.19). For consistency, the energy-momentum tensor better be covariantly conserved,

$$\nabla_\mu T^{\mu\nu} \approx 0 \ . \tag{3.30}$$

This indeed follows from the Noether identity corresponding to general coordinate transformations, see e.g. [16].

3.2 The Graviton

Let us now examine the physical degrees of freedom described by the metric field by studying its linearized equations of motion. In particular, this implies the absence of matter fields. The latter and terms nonlinear in the metric field give rise to interactions which do not change the physical properties of the metric. We separate from $g_{\mu\nu}$ the flat Minkowski metric,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \ . \tag{3.31}$$

[4] We will later encounter actions where a scalar field appears in front of the Einstein-Hilbert term. These can be brought into the form (3.25) by means of a Weyl rescaling.

The symmetric tensor $h_{\mu\nu}$ measures deviations from the fixed spacetime background; in a quantum theory of gravity it describes the quantum fluctuations, and the corresponding particle is called the graviton. We shall use this terminology in the following, even though we consider only classical (super-) gravity. In the absence of matter, the equation of motion for the graviton reads simply

$$\mathcal{R}_{\mu\nu}(h) \approx 0 \ , \tag{3.32}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor (3.24). To lowest order in κ, we find

$$\mathcal{R}_{\mu\nu} = \tfrac{1}{2}\kappa\big(\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h^\rho{}_\rho - \partial_\mu\partial^\rho h_{\rho\nu} - \partial_\nu\partial^\rho h_{\rho\mu}\big) + O(\kappa^2) \ . \tag{3.33}$$

This expression is invariant under the linearized gauge transformations

$$\delta_P(\xi) h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ . \tag{3.34}$$

Now let us look for plane wave solutions[5] to the equations of motion with fixed momentum k^μ, i.e., we make an Ansatz $h_{\mu\nu}(x) = h_{\mu\nu}(k)\, \mathrm{e}^{\mathrm{i}k\cdot x} + \text{c.c.}$ (scalar products are formed with the flat metric $\eta_{\mu\nu}$). In order to decompose the Fourier transform $h_{\mu\nu}(k)$ into linearly independent polarization tensors, we introduce a complete set of longitudinal and transversal basis vectors

$$k^\mu = (k^0, \vec{k}\,) \ , \qquad \bar{k}^\mu = (k^0, -\vec{k}\,) \ , \qquad \varepsilon^{\mu i} \ , \quad i = 1, \ldots, D-2 \ , \tag{3.35}$$

which satisfy the relations

$$k \cdot \bar{k} < 0 \ , \qquad k \cdot \varepsilon^i = \bar{k} \cdot \varepsilon^i = 0 \ , \qquad \varepsilon^i \cdot \varepsilon^j = \delta^{ij} \ . \tag{3.36}$$

In terms of these basis vectors, the most general expression for $h_{\mu\nu}(k)$ is given by

$$h_{\mu\nu}(k) = \varepsilon^i_\mu \varepsilon^j_\nu\, a_{ij}(k) + 2\bar{k}_{(\mu}\varepsilon^i_{\nu)}\, b_i(k) + \bar{k}_\mu \bar{k}_\nu\, c(k) + 2\mathrm{i}\, k_{(\mu}\, \xi_{\nu)}(k) \ . \tag{3.37}$$

The degrees of freedom contained in $\xi_\nu(k)$ are pure gauge and not physical; they drop out of the equation of motion. The off-shell degrees of freedom reside in the coefficient functions c, b_i and a_{ij}, the latter being symmetric in its indices, whose total number is

$$\text{DOF}_{\text{off}} = 1 + (D-2) + \tfrac{1}{2}(D-2)(D-1) = \tfrac{1}{2}D(D-1) \ . \tag{3.38}$$

This can be written as $\frac{1}{2}D(D+1) - D$, which is the difference of the number of independent components of the symmetric tensor $h_{\mu\nu}$ and of the gauge parameters ξ_μ.
We now plug $h_{\mu\nu}(k)$ into the Fourier transformed equation of motion. Using (3.36), we find

$$0 \approx -k^2 h_{\mu\nu} - k_\mu k_\nu h^\rho{}_\rho + k_\mu k^\rho h_{\rho\nu} + k_\nu k^\rho h_{\rho\mu}$$

[5]These are nothing but gravitational waves. That they solve the field equations follows from the fact that the combination $\lambda_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\rho{}_\rho$ satisfies the wave equation $\partial^2 \lambda_{\mu\nu} \approx O(\kappa)$ in the so-called de Donder gauge $\partial^\mu \lambda_{\mu\nu} = 0$, which (partially) fixes the gauge freedom (3.34).

$$= -k^2\big(\varepsilon^i_\mu\varepsilon^j_\nu\, a_{ij} + 2\bar{k}_{(\mu}\varepsilon^i_{\nu)}\, b_i + \bar{k}_\mu\bar{k}_\nu\, c\big) - k_\mu k_\nu\big(a^i{}_i + \bar{k}^2 c\big)$$
$$+\, 2\, k\cdot\bar{k}\, k_{(\mu}\big(\varepsilon^i_{\nu)}\, b_i + \bar{k}_{\nu)}\, c\big)\ . \tag{3.39}$$

Contraction with $\varepsilon^{\mu i}$ then yields

$$0 \approx -k^2\varepsilon^j_\nu\, a_{ij} - k^2\,\bar{k}_\nu\, b_i + k\cdot\bar{k}\, k_\nu\, b_i\ . \tag{3.40}$$

Due to linear independence of the basis vectors each term in this sum has to vanish separately. Since $k\cdot\bar{k}\neq 0$, we obtain

$$b_i(k)\approx 0\ , \qquad k^2 a_{ij}(k)\approx 0\ . \tag{3.41}$$

Now contract the equation of motion with $\bar{k}^\mu$; this results in

$$0\approx -k\cdot\bar{k}\, k_\nu\, a^i{}_i + \big[(k\cdot\bar{k})^2 - k^2\bar{k}^2\,\big]\bar{k}_\nu\, c\ . \tag{3.42}$$

Again, the two terms are independent, thus

$$c(k)\approx 0\ , \qquad a^i{}_i(k)\approx 0\ . \tag{3.43}$$

This solves the complete equation of motion. We conclude that the on-shell degrees of freedom are transversal and reside in the symmetric traceless $\mathrm{SO}(D-2)$ tensor a_{ij},

$$h_{\mu\nu}(k)\approx \varepsilon^i_\mu\varepsilon^j_\nu\,\delta(k^2)\Big(\hat{a}_{ij}(k) - \frac{1}{D-2}\,\delta_{ij}\,\hat{a}^l{}_l(k)\Big) + 2\mathrm{i}\, k_{(\mu}\,\xi_{\nu)}(k)\ . \tag{3.44}$$

The delta function enforces the dispersion relation $(k^0)^2\approx\vec{k}^2$ of a massless particle. The number of on-shell degrees of freedom contained in a_{ij} is

$$\mathrm{DOF}_{\mathrm{on}} = \tfrac{1}{2}(D-2)(D-1) - 1 = \tfrac{1}{2}D(D-3)\ . \tag{3.45}$$

Note that $D=4$ is the lowest dimension in which the graviton has non-trivial dynamics.

3.3 Vielbein and Spin Connection

Bosonic matter fields form representations of $\mathrm{GL}(D,\mathbb{R})$. In the above example of the Maxwell field we have already seen how bosonic matter couples to gravity: $\mathrm{GL}(D,\mathbb{R})$ indices are contracted with the metric and the result is multiplied by a factor of $\sqrt{g}$ to give a scalar density that can be integrated over. While that recipe is still applicable for spinor fields, the problem with the latter is that the relevant symmetry group is not $\mathrm{GL}(D,\mathbb{R})$, but rather the Lorentz group $\mathrm{SO}(1,D-1)$, the structure group of the tangent spaces T_xM to the spacetime manifold M. For each $x\in M$, T_xM is a copy of flat Minkowski spacetime, with metric $(\eta_{ab}) = \mathrm{diag}(-1,1,\ldots,1)$.
We now introduce in each T_xM a set of D orthonormal vectors with components $E_a{}^\mu(x)$. Orthonormality means

$$E_a{}^\mu(x)\, E_b{}^\nu(x)\, g_{\mu\nu}(x) = \eta_{ab}\ . \tag{3.46}$$

As with the metric, we assume the matrix $E_a{}^\mu$ to be invertible. The inverse matrix we denote with $e_\mu{}^a$, i.e.,

$$e_\mu{}^a E_a{}^\nu = \delta_\mu^\nu \ , \qquad E_a{}^\mu e_\mu{}^b = \delta_a^b \ . \tag{3.47}$$

It is called the vielbein and satisfies

$$e_\mu{}^a(x)\, e_\nu{}^b(x)\, \eta_{ab} = g_{\mu\nu}(x) \ , \tag{3.48}$$

which is why one says it is the "square root" of the metric. In the vielbein formalism, the metric therefore is a composite field built from the fundamental vielbein field. We can use the latter to convert $\mathrm{GL}(D, \mathbb{R})$ indices into Lorentz indices and vice versa,

$$V^a = V^\mu e_\mu{}^a \quad \Leftrightarrow \quad V^\mu = V^a E_a{}^\mu \ . \tag{3.49}$$

A crucial observation is that for a given metric the vielbein is not uniquely determined, but rather subject to local Lorentz transformations

$$e_\mu{}^a(x) \to e_\mu{}^b(x)\, \Lambda_b{}^a(x) \ , \tag{3.50}$$

where $\Lambda_a{}^c(x)\Lambda_b{}^d(x)\eta_{cd} = \eta_{ab}$. This invariance eliminates $\dim \mathrm{SO}(1, D-1) = D(D-1)/2$ degrees of freedom from the D^2 components of $e_\mu{}^a$, leaving precisely the $D(D+1)/2$ independent components of the metric. The physics should of course be independent of the choice of vielbein, so we require invariance under local Lorentz transformations of the theory under consideration. In fact, this Lorentz invariance is needed for the theory to be well-defined globally, i.e., on the entire spacetime: If the topology of spacetime is non-trivial, we cannot introduce nowhere vanishing vector fields $E_a{}^\mu$. The best we can do in general is to cover spacetime with sufficiently small neighborhoods, in each of which we introduce a vielbein, and then glue them together by requiring the vielbeins in the overlap of two neighborhoods to be related by a local Lorentz transformation. This construction is called a frame bundle.

Given a Lorentz tensor, its derivative in general does not transform as a tensor. Covariance can be achieved by introducing a so-called spin connection $\omega_{\mu a}{}^b$ with an appropriate Lorentz transformation, and a covariant derivative

$$D_\mu = \partial_\mu - \tfrac{1}{2}\omega_\mu{}^{ab}\ell_{ab} \ . \tag{3.51}$$

The $\ell_{ab} = -\ell_{ba}$ are generators of Lorentz transformations δ_L $(= \frac{1}{2}\varepsilon^{ab}(x)\, \ell_{ab}$ on tensors) and satisfy the commutation relations

$$[\, \ell_{ab} \,, \ell_{cd} \,] = \eta_{ac}\, \ell_{bd} - \eta_{bc}\, \ell_{ad} + \eta_{bd}\, \ell_{ac} - \eta_{ad}\, \ell_{bc} \ . \tag{3.52}$$

Their action on Lorentz vectors is given by

$$\ell_{ab}\, V^c = \delta_a^c\, V_b - \delta_b^c\, V_a \ . \tag{3.53}$$

We can now also consider local Lorentz transformations of spinor fields. On Dirac spinors the ℓ_{ab} are represented by the commutator of the γ-matrices,

$$\ell_{ab}\Psi = -\tfrac{1}{4}[\,\gamma_a\,,\gamma_b\,]\Psi \ , \qquad (3.54)$$

while on Weyl spinors in four dimensions their action reads

$$\ell_{ab}\psi_\alpha = -\sigma_{ab\,\alpha}{}^\beta\psi_\beta \ , \qquad \ell_{ab}\bar\psi^{\dot\alpha} = -\bar\sigma_{ab}{}^{\dot\alpha}{}_{\dot\beta}\bar\psi^{\dot\beta} \ . \qquad (3.55)$$

Again, in general we cannot introduce fermions globally on the entire spacetime, but need to patch together local neighborhoods using such Lorentz transformations. For fermions, an obstruction can occur[6] in triple overlaps, where three successive transformations from one patch to the next need to yield the identity. As mentioned in the introduction, this can only be arranged if the second Stiefel-Whitney class of the frame bundle is trivial. Later, we will encounter, and discuss in more detail, a similar obstruction when considering fermions and Kähler manifolds.

The formalism is now completely analogous to Yang-Mills theory with internal symmetry group $\mathrm{SO}(1, D-1)$. An infinitesimal gauge transformation of the spin connection with parameters $\varepsilon_a{}^b(x)$ is given by

$$\delta_L(\varepsilon)\omega_{\mu a}{}^b = \partial_\mu\varepsilon_a{}^b - \omega_{\mu a}{}^c\varepsilon_c{}^b + \varepsilon_a{}^c\omega_{\mu c}{}^b = D_\mu\varepsilon_a{}^b \ . \qquad (3.56)$$

The curvature is obtained from the commutator of two Lorentz-covariant derivatives,

$$[\,D_\mu\,, D_\nu\,] = -\tfrac{1}{2}R_{\mu\nu}{}^{ab}\ell_{ab} \ , \qquad (3.57)$$

where

$$R_{\mu\nu a}{}^b = \partial_\mu\omega_{\nu a}{}^b - \partial_\nu\omega_{\mu a}{}^b - \omega_{\mu a}{}^c\,\omega_{\nu c}{}^b + \omega_{\nu a}{}^c\,\omega_{\mu c}{}^b \ . \qquad (3.58)$$

It transforms as a tensor in the adjoint representation,

$$\delta_L(\varepsilon)R_{\mu\nu a}{}^b = \varepsilon_a{}^c R_{\mu\nu c}{}^b - R_{\mu\nu a}{}^c\varepsilon_c{}^b \ . \qquad (3.59)$$

Using the vielbein, we can now introduce γ-matrices in curved spacetime, $\gamma_\mu(x) = e_\mu{}^a(x)\gamma_a$. These are field-dependent and satisfy the Clifford algebra with metric $g_{\mu\nu}$,

$$\{\gamma_\mu\,,\gamma_\nu\} = -2\,g_{\mu\nu}\mathbb{1} \ . \qquad (3.60)$$

Similarly, in four dimensions we introduce curved σ-matrices via $\sigma^\mu = \sigma^a E_a{}^\mu$, and analogously for $\bar\sigma^\mu$. It is then easy to couple a spin 1/2 field to gravity: Take the action for flat spacetime, replace ordinary derivatives with Lorentz-covariant ones, insert an (inverse)

[6] This is because, strictly speaking, fermions transform under the universal covering group of the Lorentz group, and the map between the two is not one-to-one.

vielbein for each γ-matrix, and finally multiply with $\sqrt{g}$ to obtain a density. For example, for a Dirac spinor we have

$$S[e,\Psi] = -\int d^D x\, e\, \bar{\Psi}(\mathrm{i}\gamma^\mu D_\mu + m)\Psi \ . \tag{3.61}$$

Here, we denote

$$e \equiv \det(e_\mu{}^a) = \sqrt{g} \ . \tag{3.62}$$

We should point out that in flat spacetime this action is real only modulo a boundary term. In curved spacetime, when partially integrating the covariant derivative after complex conjugation, we encounter a term $D_\mu(eE_a{}^\mu)$. We will show below that it vanishes for a suitably chosen spin connection.

3.4 Palatini Formulation of Gravity

In the above, the spin connection occured as an independent field, which would extend the minimal field content of general relativity. However, just like for the connection $\Gamma_{\mu\nu}{}^\rho$, we can impose a reasonable constraint which allows to express $\omega_{\mu a}{}^b$ in terms of the vielbein and its derivative: As explained above, we may work either with $\mathrm{GL}(D,\mathbb{R})$ tensors or with Lorentz tensors. The two notions of a covariant derivative, ∇_μ and D_μ, should then be equivalent, in the sense that $\nabla_\mu V_\nu = e_\nu{}^a D_\mu V_a$. This equation holds iff the vielbein is fully covariantly constant,

$$\partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a + \omega_{\mu b}{}^a e_\nu{}^b = 0 \ . \tag{3.63}$$

Note that this so-called tetrad postulate is compatible with the previous constraint (3.17) of covariant constancy of the metric. Vanishing torsion now implies that

$$D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = (\Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho)e_\rho{}^a = T_{\mu\nu}{}^\rho e_\rho{}^a = 0 \ . \tag{3.64}$$

As we show below, it is possible to solve this equation for $\omega_{\mu a}{}^b$. The equivalence of the two covariant derivatives implies a relation between the corresponding curvature tensors,

$$R_{\mu\nu a}{}^b(\omega) = E_a{}^\rho R_{\mu\nu\rho}{}^\sigma(\Gamma)\, e_\sigma{}^b \ , \tag{3.65}$$

which is easily verified by computing the commutator $[\, D_\mu\, , D_\nu\,]V_a$ using $D_\mu V_a = E_a{}^\nu \nabla_\mu V_\nu$. Accordingly, the Einstein-Hilbert action (3.26) can be written in terms of the Lorentz curvature, as $\mathcal{R} = R_{ab}{}^{ab}$.

An important observation now is that if one expresses the Einstein-Hilbert action in terms of the Lorentz curvature, the derivative term is linear in the spin connection. Hence, if we do not impose (3.63) but treat $\omega_{\mu a}{}^b$ as an independent field, with a variation independent of that of the vielbein, its equation of motion is purely algebraic. We may then solve this equation for $\omega_{\mu a}{}^b$ and insert it back into the action to derive a functional of the vielbein only. As it turns out, in the absence of matter the solution $\omega_{\mu a}{}^b(e)$ is exactly the same as the one following from the tetrad postulate. This gives an alternative version of Einstein gravity, known as the first-order or Palatini formulation: Start with the action

$$S_{\rm P}[e,\omega] = -\frac{1}{2\kappa^2}\int d^Dx\, e\, E_a{}^\mu E_b{}^\nu R_{\mu\nu}{}^{ab}(\omega)\ , \tag{3.66}$$

which is a functional of $e_\mu{}^a$ and $\omega_{\mu a}{}^b$. Then eliminate $\omega_{\mu a}{}^b$ by means of its equation of motion to obtain the Einstein-Hilbert action,

$$S_{\rm P}[e,\omega(e)] = S_{\rm EH}[e]\ . \tag{3.67}$$

The latter is being referred to as the second-order formulation.
It is the Palatini formulation which is being used in the various supergravity theories, as it significantly simplifies the variation of the action. This is due to a trick called the "1.5 order formalism," which can be employed for second-order actions which derive from first-order ones: Consider an action $S_1[\phi,U]$ which is a functional of fields ϕ^i and U^A, where the equations of motion for U^A are algebraic and can be solved for $U^A(\phi)$ as functions of ϕ^i. The change of the second-order action $S_2[\phi]=S_1[\phi,U(\phi)]$ upon variation of the ϕ^i is then given by

$$\delta S_2[\phi] = \int\left(\delta\phi^i\,\frac{\delta S_1}{\delta\phi^i} + \delta U^A(\phi)\,\frac{\delta S_1}{\delta U^A}\right)_{U=U(\phi)} = \int\left(\delta\phi^i\,\frac{\delta S_1}{\delta\phi^i}\right)_{U=U(\phi)}\ . \tag{3.68}$$

The second term vanishes since the $U^A(\phi)$ solve their equations of motion. Thus, it is sufficient to vary only the fields ϕ^i in the first-order action and then insert the solutions for U^A. Note that this trick can be used also for chiral multiplets with auxiliary fields F solving their algebraic equations of motion.
As a preparation for things to come and to make the above arguments explicit, let us now consider matter coupled to gravity in the Palatini formulation and work out the solution for the spin connection. It enters the matter action via a term of the form $\frac{1}{2}e\,\omega_\mu{}^{ab}J_{ab}{}^\mu$, where $J_{ab}{}^\mu$ is the current of rigid Lorentz transformations. For example, for the action (3.61) the current is given by $J_{ab}{}^\mu = -\frac{\rm i}{4}\bar\Psi\{\gamma_{ab}\,,\gamma^\mu\}\Psi$. Varying $\omega_\mu{}^{ab}$ in $S_P+S_{\rm mat}$ and integrating by parts the covariant derivative in $\delta R_{\mu\nu}{}^{ab} = 2D_{[\mu}\delta\omega_{\nu]}{}^{ab}$ then gives rise to the equation of motion

$$D_\nu(eE_{[a}{}^\mu E_{b]}{}^\nu) = \tfrac{1}{2}\kappa^2 e\, J_{ab}{}^\mu\ . \tag{3.69}$$

This is a linear equation for $\omega_{\mu a}{}^b$. Contracting it with $e_\mu{}^a$ yields an expression for $\omega_{ab}{}^a$. A useful corollary is the identity[7]

$$D_\mu(eE_b{}^\mu) = \frac{\kappa^2}{D-2}\,e\,J_{ab}{}^a\ . \tag{3.70}$$

[7]For the spin connection that follows from the tetrad postulate (3.63), the right-hand side vanishes. This implies that the action (3.61) is real.

Inserting the result back into (3.69) then allows to solve for $\omega_{[\mu\nu]}{}^a$,

$$\omega_{[\mu\nu]}{}^a = -\partial_{[\mu}e_{\nu]}{}^a + \frac{\kappa^2}{2}J_{\mu\nu}{}^a + \frac{\kappa^2}{D-2}J_{\rho[\mu}{}^\rho e_{\nu]}{}^a\ . \tag{3.71}$$

From this, we can finally derive the solution for $\omega_{\mu a}{}^b$ via the identity

$$\omega_{\mu ab} = E_a{}^\nu E_b{}^\rho(\omega_{[\mu\nu]\rho} - \omega_{[\nu\rho]\mu} + \omega_{[\rho\mu]\nu})\ . \tag{3.72}$$

The precise expression will not be needed. From (3.71) we infer that in the presence of matter that couples to the spin connection, such as fermions, the torsion does not vanish in the Palatini formulation, but is given by

$$D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 2(\partial_{[\mu}e_{\nu]}{}^a + \omega_{[\mu\nu]}{}^a) = \kappa^2 J_{\mu\nu}{}^a + \frac{2\kappa^2}{D-2}J_{\rho[\mu}{}^\rho e_{\nu]}{}^a\ . \tag{3.73}$$

For $J_{ab}{}^\mu = 0$, this is equation (3.64).
Finally, we remark that in the standard supergravity theories the $\mathrm{GL}(D,\mathbb{R})$ connection $\Gamma_{\mu\nu}{}^\rho$ is never used, only the spin connection. That this is possible is due to the field content of these theories; all fields are p-forms, i.e., antisymmetric tensors of rank p (this includes the fermions, which come as 0- and 1-forms, and the gravity sector, if we use the vielbein instead of the metric). Moreover, for $p > 0$ all fields are subject to gauge transformations. Gauge invariance then requires their derivatives to occur only through their field strengths, i.e., totally antisymmetrized partial derivatives of the fields. These can be shown to behave as tensors under general coordinate transformations (see footnote 3), just as we did above for the Maxwell field, which is a 1-form. Hence, $\mathrm{GL}(D,\mathbb{R})$-covariant derivatives are not needed.

4 Local Supersymmetry

In this chapter we introduce the simplest supergravity theory in four dimensions, which describes a coupled system of the vielbein and one real spin 3/2 field, the gravitino.

4.1 Noether Method

Let us first demonstrate how gauging supersymmetry, i.e., promoting it to a local symmetry, necessarily introduces gravity. Toward this end, we consider as an example the free massless Wess-Zumino model in four dimensions. This is just a single chiral multiplet with Lagrangian

$$\mathcal{L}_0 = -\partial^\mu\bar\phi\,\partial_\mu\phi - \tfrac{\mathrm{i}}{2}\chi\sigma^\mu\overleftrightarrow{\partial}_\mu\bar\chi\ , \tag{4.1}$$

where we omit the auxiliary field F, which vanishes on-shell, and indices are contracted with the (inverse) Minkowski metric $\eta^{\mu\nu}$. As we found in section 2.3, the corresponding action is invariant under supersymmetry transformations

$$\delta_Q(\epsilon)\phi = \epsilon\chi \qquad\qquad \delta_Q(\epsilon)\bar\phi = \bar\epsilon\bar\chi$$

$$\delta_Q(\epsilon)\chi = \mathrm{i}\sigma^\mu\bar{\epsilon}\,\partial_\mu\phi \qquad \delta_Q(\epsilon)\bar{\chi} = \mathrm{i}\bar{\sigma}^\mu\epsilon\,\partial_\mu\bar{\phi} \tag{4.2}$$

with constant Weyl spinor parameter ϵ. If we consider a local parameter $\epsilon(x)$ instead,[8] invariance is spoiled by a term containing its derivative ($\simeq$ denotes equality modulo a total derivative)

$$\delta_Q(\epsilon(x))\,\mathcal{L}_0 \simeq \partial_\nu\bar{\phi}\,\chi\sigma^\mu\bar{\sigma}^\nu\partial_\mu\epsilon + \text{c.c.}\ . \tag{4.3}$$

To restore supersymmetry, we thus have to introduce a connection ψ_μ with an inhomogeneous transformation law

$$\delta_Q(\epsilon)\psi_\mu = \kappa^{-1}\partial_\mu\epsilon + \dots\ , \qquad \delta_Q(\epsilon)\bar{\psi}_\mu = \kappa^{-1}\partial_\mu\bar{\epsilon} + \dots\ . \tag{4.4}$$

This gauge potential is a Weyl spinor with an additional covector index (i.e., a 1-form), as is determined by the right-hand side of the equation. The highest spin contained in $\psi_{\mu\alpha} \sim \psi_{\beta\dot{\beta}\alpha}$ is therefore $3/2$, as can be seen by decomposing it into irreducible components:

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},0\right) = \left(1,\tfrac{1}{2}\right) \oplus \left(0,\tfrac{1}{2}\right)\ . \tag{4.5}$$

Since ϵ has mass dimension $-1/2$ and we would like to assign to ψ_μ the canonical dimension $3/2$ of a spinor field (required if its kinetic term is to be quadratic in the fields and linear in derivatives), we have introduced a constant κ of dimension -1. This already suggests to identify it with the square root of Newton's constant (up to a numerical factor). A more compelling reason will emerge shortly. Given this new field, we can add to $\mathcal{L}_0$ an interaction term

$$\mathcal{L}_1^\psi = -\kappa\,\partial_\nu\bar{\phi}\,\chi\sigma^\mu\bar{\sigma}^\nu\psi_\mu + \text{c.c.}\ , \tag{4.6}$$

such that the sum $\mathcal{L}_0 + \mathcal{L}_1^\psi$ is invariant under local supersymmetry up to order κ^0,

$$\delta_Q(\epsilon)\left(\mathcal{L}_0 + \mathcal{L}_1^\psi\right) \simeq O(\kappa)\ . \tag{4.7}$$

$\mathcal{L}_1^\psi$ couples the gauge potential ψ_μ to the Noether current J^μ of rigid supersymmetry.[9] The procedure of deforming iteratively the action and the transformation laws in order to gauge symmetries (thereby introducing interactions) is known as the Noether method. The deformation parameter, with respect to which we can decompose the action and transformations into terms of definite order, acts as a coupling constant. In the case at hand it is given by κ.

Our new action is of course still not completely invariant under the above local transformations. Let us go one step further, by inspecting the right-hand side of (4.7),

$$\delta_Q(\epsilon)\left(\mathcal{L}_0 + \mathcal{L}_1^\psi\right) \simeq \mathrm{i}\kappa\,\partial_\nu\bar{\phi}\,\partial_\rho\phi\,\bar{\epsilon}\bar{\sigma}^\rho\sigma^\mu\bar{\sigma}^\nu\psi_\mu - \kappa\partial_\nu(\bar{\epsilon}\bar{\chi})\,\chi\sigma^\mu\bar{\sigma}^\nu\psi_\mu + \text{c.c.}\ . \tag{4.8}$$

[8] At this point, we have to decide whether to put ϵ under the derivative in $\delta_Q\chi$. It is natural not to do so, as only gauge fields should contain derivatives of transformation parameters.

[9] Whenever an action $S[\phi]$ with Lagrangian $\mathcal{L}$ is invariant under rigid transformations $\delta_\xi\phi^i$, one has for the corresponding local transformations $\delta_{\xi(x)}\mathcal{L} \simeq \partial_\mu\xi^I J_I^\mu$, and the J_I^μ are conserved currents: $\partial_\mu J_I^\mu = -\partial(\delta_\xi\phi^i)/\partial\xi^I\,\delta S/\delta\phi^i \approx 0$, which can be easily shown by applying the Euler-Lagrange derivative w.r.t. ξ^I to $\delta_{\xi(x)}\mathcal{L}$.

We disregard the pure fermion terms in the following and concentrate on the terms containing $\partial_\mu\phi$. They can be written as

$$\delta_Q(\epsilon)\left(\mathcal{L}_0+\mathcal{L}_1^\psi\right) \simeq -\mathrm{i}\kappa\,(\epsilon\sigma_\nu\bar\psi_\mu-\psi_\mu\sigma_\nu\bar\epsilon)\,(\partial^\mu\bar\phi\,\partial^\nu\phi+\partial^\nu\bar\phi\,\partial^\mu\phi-\eta^{\mu\nu}\partial^\rho\bar\phi\,\partial_\rho\phi) + \kappa\,\epsilon^{\mu\nu\rho\sigma}\partial_\nu\bar\phi\,\partial_\mu\phi\,(\epsilon\sigma_\rho\bar\psi_\sigma-\psi_\sigma\sigma_\rho\bar\epsilon)+\dots\;. \tag{4.9}$$

In the first line we recognize the energy-momentum tensor of the free complex scalar ϕ, i.e., the Noether current of translations. This term can only be canceled by introducing another new field, a bosonic symmetric tensor $h_{\mu\nu}$, which couples to the energy-momentum tensor in the action,

$$\mathcal{L}_1^h=\tfrac{1}{2}\kappa\,h_{\mu\nu}T^{\mu\nu}(\phi)\;, \tag{4.10}$$

and transforms under local supersymmetry as

$$\delta_Q(\epsilon)h_{\mu\nu}=2\mathrm{i}\,(\epsilon\sigma_{(\mu}\bar\psi_{\nu)}-\psi_{(\mu}\sigma_{\nu)}\bar\epsilon)\;. \tag{4.11}$$

The coupling constant κ should appear in the action and not the transformation since bosonic fields have canonical dimension 1. The bosonic terms in the resulting Lagrangian up to order κ now add up to

$$\mathcal{L}_0+\mathcal{L}_1^\psi+\mathcal{L}_1^h=-(1+\tfrac{1}{2}\kappa h^\rho{}_\rho)(\eta^{\mu\nu}-\kappa h^{\mu\nu})\,\partial_\mu\bar\phi\,\partial_\nu\phi+\dots\;. \tag{4.12}$$

If we introduce a metric $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$, we have

$$g^{\mu\nu}=\eta^{\mu\nu}-\kappa h^{\mu\nu}+O(\kappa^2)\;,\qquad \sqrt{g}=1+\tfrac{1}{2}\kappa\,h^\rho{}_\rho+O(\kappa^2)\;, \tag{4.13}$$

so we see the covariantized kinetic term of ϕ emerging. It is clear how to complete the Noether coupling $\mathcal{L}_1^h$ to all orders; contract the indices of $\partial_\mu\bar\phi\,\partial_\nu\phi$ with $g^{\mu\nu}$ and multiply with $\sqrt{g}$. Moreover, this confirms that κ is indeed the gravitational coupling constant.

We have arrived at the important result that gauging supersymmetry automatically gives rise to gravity. While we have only considered an example, this actually holds in general: Gauging supersymmetry always begins with coupling a spinor connection to the Noether current of rigid supersymmetry. It is well-known that the latter forms a supersymmetry multiplet with the energy-momentum tensor, which therefore occurs in the variation of the Noether coupling. As above, this forces one to introduce a metric field as a connection (recall (3.23)) for gauging the translations. In fact, that this would be necessary was to be expected since the commutator of two supersymmetry transformations produces a translation. If the parameters of the former are local functions of spacetime, then so will be the composite parameter of the translation. We are thus led to consider general coordinate transformations also from this point of view.

ψ_μ is the supersymmetric partner of the graviton; it is called the gravitino. We could have considered several supersymmetries with parameters ϵ^i, $i=1,\dots,N$. Each gives rise to a supersymmetry current, which would have to be coupled to a connection. Accordingly, we need a gravitino ψ_μ^i for each supersymmetry.[10] One then speaks of N-extended supergravity. In these lectures, we confine ourselves to $N = 1$.

What about the second line in (4.9)? After writing it as

$$2\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\tfrac{1}{4}\kappa\,\phi\overleftrightarrow{\partial}_\nu\bar{\phi}\right)(\epsilon\sigma_\rho\bar{\psi}_\sigma - \psi_\sigma\sigma_\rho\bar{\epsilon})\ ,$$

it can be seen that the only way of canceling it is to add to $\delta_Q\psi_\nu$ the matter term $-\frac{1}{4}\kappa\epsilon\,\phi\overleftrightarrow{\partial}_\nu\bar{\phi}$ (which we will later interpret as a Kähler connection) and to introduce the following kinetic term for the gravitino:

$$\mathcal{L}_0^\psi = \epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu\psi_\nu\sigma_\rho\bar{\psi}_\sigma + \psi_\sigma\sigma_\rho\partial_\mu\bar{\psi}_\nu\right)\ . \tag{4.14}$$

Observe that the corresponding action is invariant under the gauge transformation (4.4). It is called the Rarita-Schwinger action, who have shown it to be the unique physically acceptable action for a free spin 3/2 field.

Using the gravitino, we can define a supercovariant derivative

$$\mathcal{D}_\mu = \partial_\mu - \delta_Q(\kappa\psi_\mu)\ . \tag{4.15}$$

This indeed maps tensors (w.r.t supersymmetry) into tensors; e.g., for the chiral scalar ϕ the variation of

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - \kappa\psi_\mu\chi \tag{4.16}$$

does not contain derivatives of ϵ. This covariant derivative actually emerges from the above Noether method: (4.6) contains terms $\kappa\,\partial^\mu\bar{\phi}\,\psi_\mu\chi$ + c.c. which are just the mixed terms of the covariantized kinetic term $-\mathcal{D}^\mu\bar{\phi}\,\mathcal{D}_\mu\phi$.

We shall not complete the coupling of the Wess-Zumino model to supergravity in this chapter. For $N = 1$ supergravity in four dimensions one can use a more powerful tensor calculus instead of the Noether method. We have employed the latter merely as a motivation for the next sections.

4.2 The Gravitino

Before we turn to the formulation of supergravity in four dimensions, let us first have a closer look at some properties of the gravitino. In particular, we should count the number of degrees of freedom described by it, so that we can compare with the number of degrees of freedom of its bosonic superpartners (they should of course match). To be as general as possible, let us for the time being consider a gravitino in arbitrary dimensions $D \geq 3$. Its essential physical properties, such as the number of degrees of freedom, follow from

[10] To match the numbers of bosonic and fermionic degrees of freedom for $N > 1$, one also needs additional fields with spin less than 3/2 in the supergravity multiplet.

the linearized equations of motion. The free action (4.14), however, is specific to Weyl spinors in four dimensions. The equivalent expression

$$\mathcal{L}_0 = \mathrm{i}\big(\partial_\mu \psi_\nu \sigma^{[\mu}\bar{\sigma}^\nu \sigma^{\rho]}\bar{\psi}_\rho + \psi_\rho \sigma^{[\mu}\bar{\sigma}^\nu \sigma^{\rho]}\partial_\mu \bar{\psi}_\nu\big) \tag{4.17}$$

suggests how to generalize it to arbitrary dimensions:

$$\mathcal{L}_0 = \mathcal{N}\,\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho \ , \tag{4.18}$$

where

$$\gamma^{\mu_1\cdots\mu_p} \equiv \gamma^{[\mu_1}\ldots\gamma^{\mu_p]} \ , \tag{4.19}$$

γ^μ with $\mu = 0,\ldots,D-1$ are $2^{[D/2]}$-dimensional gamma matrices, and $\mathcal{N}$ is a dimensionless normalization factor that depends on whether the spinors Ψ_μ satisfy constraints such as Majorana and/or Weyl conditions. Up to a total derivative, $\mathcal{L}_0$ is invariant under the gauge transformations[11]

$$\delta\Psi_\mu = \kappa^{-1}\partial_\mu \epsilon \ . \tag{4.20}$$

This is due to the Bianchi identity satisfied by the gauge-invariant gravitino field strength $\Psi_{\mu\nu} \equiv 2\,\partial_{[\mu}\Psi_{\nu]}$,

$$\partial_{[\mu}\Psi_{\nu\rho]} = 0 \ . \tag{4.21}$$

The equation of motion following from $\mathcal{L}_0$ reads (recall that $\approx$ denotes on-shell equality)

$$\gamma^{\mu\nu\rho}\Psi_{\nu\rho} \approx 0 \ . \tag{4.22}$$

This can be simplified by contracting with $\gamma_\mu\gamma_\sigma$,

$$\gamma_\mu\gamma_\sigma\gamma^{\mu\nu\rho}\Psi_{\nu\rho} = 2(D-2)\,\gamma^\mu\Psi_{\mu\sigma} + (D-4)\,\gamma_\sigma{}^{\nu\rho}\Psi_{\nu\rho} \ ,$$

which yields

$$\gamma^\mu\Psi_{\mu\nu} \approx 0 \ . \tag{4.23}$$

Now, let us look for plane wave solutions with fixed momentum k^μ, like we did for the graviton. We decompose the Fourier transform $\Psi_{\mu\nu}(k)$ of the field strength $\Psi_{\mu\nu}(x)$ into linearly independent polarization tensors built from the basis vectors (3.35),

$$\Psi_{\mu\nu}(k) = k_{[\mu}\varepsilon^i_{\nu]}\,a_i(k) + \bar{k}_{[\mu}\varepsilon^i_{\nu]}\,b_i(k) + k_{[\mu}\bar{k}_{\nu]}\,c(k) + \varepsilon^i_{[\mu}\varepsilon^j_{\nu]}\,d_{ij}(k) \ . \tag{4.24}$$

Here, the coefficient functions are spinors in the same representation as Ψ_μ with μ fixed. The Fourier transformed Bianchi identity (4.21) now implies

$$0 = k_{[\mu}\Psi_{\nu\rho]}(k) = k_{[\mu}\bar{k}_\nu\varepsilon^i_{\rho]}\,b_i + k_{[\mu}\varepsilon^i_\nu\varepsilon^j_{\rho]}\,d_{ij} \ . \tag{4.25}$$

[11] As above, the power of κ is determined by the fact that its mass dimension is $(2-D)/2$, while that of Ψ_μ is $(D-1)/2$.

a_i and c drop out by virtue of the identity $k_{[\mu}k_{\nu]} = 0$. Since the two polarization tensors appearing in the equation are linearly independent, b_i and d_{ij} have to vanish separately,

$$b_i(k) = 0 \ , \qquad d_{ij}(k) = 0 \ . \tag{4.26}$$

Hence, the off-shell degrees of freedom are contained in the $D-1$ spinors a_i and c. This amounts to the number

$$\text{DOF}_{\text{off}} = (D-1)f \ , \tag{4.27}$$

where f counts the independent real components of the spinors Ψ_μ with μ fixed,

$$f = 2^{[D/2]} \times \begin{cases} 2 & \text{Dirac} \\ 1 & \text{Majorana / Weyl} \\ 1/2 & \text{Majorana-Weyl} \end{cases} . \tag{4.28}$$

a_i and c parametrize the ε^i_μ and $\bar{k}_\mu$ polarizations of Ψ_μ, respectively; a k_μ polarization is pure gauge and thus carries no degrees of freedom.
The equation of motion (4.23) imposes further constraints; using $b_i = d_{ij} = 0$, we find

$$0 \approx 2\,\gamma^\mu \Psi_{\mu\nu}(k) = \varepsilon^i_\nu\, \not{k} a_i + \bar{k}_\nu\, \not{k} c - k_\nu\, (\not{\varepsilon}^i a_i + \not{\bar{k}} c) \ , \tag{4.29}$$

from which we infer that $\not{k} a_i \approx 0$, $\not{k} c \approx 0$, and $\not{\varepsilon}^i a_i \approx -\not{\bar{k}} c$. These conditions allow to derive the following one:

$$2\,(k\cdot\bar{k})\, c = -\{\not{k}\ , \not{\bar{k}}\,\}\, c \approx -\not{k}\,\not{\bar{k}} c \approx \not{k}\,\not{\varepsilon}^i a_i \approx \{\not{k}\ , \not{\varepsilon}^i\}\, a_i = -2\,(k\cdot\varepsilon^i)\, a_i = 0 \ . \tag{4.30}$$

Since $k\cdot\bar{k} \neq 0$, we conclude that

$$c(k) \approx 0 \ . \tag{4.31}$$

Thus, all on-shell degrees of freedom reside in the $D-2$ spinors $a_i(k)$, which are subject to the constraints

$$\not{k} a_i(k) \approx 0 \ , \qquad \not{\varepsilon}^i a_i(k) \approx 0 \ . \tag{4.32}$$

The first is just the massless Dirac equation in momentum space and implies

$$k^2 a_i(k) \approx 0 \quad \Rightarrow \quad a_i(k) \approx \delta(k^2)\, \hat{a}_i(k) \ . \tag{4.33}$$

We find again the zero mass shell condition $k^2 \approx 0$. Moreover, the Dirac equation halves[12] the number of degrees of freedom contained in a_i. By multiplication with $\not{\varepsilon}^j$ the second constraint yields

$$a_i \approx \varepsilon^i_\mu \varepsilon^j_\nu\, \gamma^{\mu\nu} a_j = \sum_{j\neq i} \varepsilon^i_\mu \varepsilon^j_\nu\, \gamma^{\mu\nu} a_j \ . \tag{4.34}$$

The latter equality holds by virtue of the antisymmetry of $\gamma^{\mu\nu}$. Hence, one of the a_i can be expressed in terms of the others, leaving $D-3$ independent spinors satisfying the Dirac equation. We conclude that a gravitino describes

$$\text{DOF}_{\text{on}} = \tfrac{1}{2}(D-3)f \tag{4.35}$$

on-shell degrees of freedom.

[12]For $k^2 = 0$ the operator $-\gamma^0 \not{k}/2k^0$ projects out half of the spinor components.

4.3 On-shell Supergravity

Let us now consider only the $D = 4$, $N = 1$ graviton-gravitino multiplet, without any additional matter. Note that on-shell the number of bosonic and fermionic degrees of freedom is equal, namely two. Off the mass shell, however, we have 12 fermionic degrees of freedom, but only 6 bosonic ones. For an off-shell formulation we have to find a suitable set of bosonic auxiliary fields. We postpone this issue until the next chapter and confine ourselves to an on-shell formulation of supergravity for the time being.

Above we already found the free action for the gravitino. It is natural to replace the partial derivatives with Lorentz-covariant ones. A connection $\Gamma_{\mu\nu}{}^{\rho}$ for the covector index of ψ_μ is not needed, as we had argued above that antisymmetrized derivatives are $\mathrm{GL}(D,\mathbb{R})$-covariant even without a connection. The kinetic term for the graviton can only be provided by the Einstein-Hilbert action. We thus consider the tentative Lagrangian

$$\mathcal{L}(e,\psi) = -\frac{e}{2\kappa^2}\, E_a{}^{\mu} E_b{}^{\nu} R_{\mu\nu}{}^{ab}(\omega) + \epsilon^{\mu\nu\rho\sigma}\left(D_\mu\psi_\nu\sigma_\rho\bar{\psi}_\sigma + \psi_\sigma\sigma_\rho D_\mu\bar{\psi}_\nu\right) , \tag{4.36}$$

where for the spin connection we take the solution to its algebraic equation of motion which follows from $\mathcal{L}$ considered as a first-order Lagrangian with independent $\omega_\mu{}^{ab}$, see below.

Observe that the first term in $\mathcal{L}$ contains a factor e, making it a density, but not the second. The reason is that the constant antisymmetric ϵ-symbol is a density itself, so no e is needed. We can construct a proper tensor from it by multiplication with e^{-1},

$$\varepsilon^{\mu\nu\rho\sigma} = e^{-1}\epsilon^{\mu\nu\rho\sigma} , \qquad \varepsilon_{\mu\nu\rho\sigma} = e\,\epsilon_{\mu\nu\rho\sigma} . \tag{4.37}$$

For the second equation, the indices have been lowered with the metric $g_{\mu\nu}$, which gives a factor $\det(g_{\mu\nu}) = -e^2$. The sign cancels thanks to $\epsilon_{0123} = -\epsilon^{0123} = -1$. The ε-tensor is field-dependent and not invariant under supersymmetry; it is therefore advantageous to work with the constant ϵ-symbol instead.

We claim that the action with Lagrangian (4.36) is invariant under the supersymmetry transformations

$$\delta_Q(\epsilon)e_\mu{}^a = \mathrm{i}\kappa\,(\epsilon\sigma^a\bar{\psi}_\mu - \psi_\mu\sigma^a\bar{\epsilon}) \tag{4.38}$$

$$\delta_Q(\epsilon)\psi_\mu = \kappa^{-1}D_\mu\epsilon , \qquad \delta_Q(\epsilon)\bar{\psi}_\mu = \kappa^{-1}D_\mu\bar{\epsilon} . \tag{4.39}$$

The transformation of the vierbein implies the one of $h_{\mu\nu}$ (4.11) found in section 4.1. The converse is not quite true due to the Lorentz invariance of the metric, but (4.38) is natural in that it is linear in the fields (unlike $\epsilon\sigma_\mu\bar{\psi}^a$, which would also be possible).

Before we prove our claim, it is convenient to rewrite (4.36). Using the identity

$$4eE_{[a}{}^{\mu}E_{b]}{}^{\nu} = -\epsilon_{abcd}\,\epsilon^{\mu\nu\rho\sigma}e_\rho{}^c e_\sigma{}^d , \tag{4.40}$$

$\mathcal{L}$ can be brought into the form[13]

$$\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\Big(\frac{1}{8\kappa^2}\,\epsilon_{abcd}\,e_\rho{}^c e_\sigma{}^d R_{\mu\nu}{}^{ab}(\omega) + D_\mu\psi_\nu\sigma_\rho\bar\psi_\sigma + \psi_\sigma\sigma_\rho D_\mu\bar\psi_\nu\Big)\ , \tag{4.41}$$

which simplifies its variation a lot.

Let us demonstrate this by computing the torsion induced by the gravitino. While we could use the general formula (3.73), it is instructive and in fact easier to repeat the derivation for this special case. Upon variation of the spin connection in the first-order Lagrangian $\mathcal{L}(e,\psi,\omega)$, the curvature changes by $2D_{[\mu}\delta\omega_{\nu]}{}^{ab}$, so we find

$$\delta\mathcal{L} = -\frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\Big(\frac{1}{2\kappa^2}\,\epsilon_{abcd}\,e_\rho{}^c e_\sigma{}^d D_\nu\delta\omega_\mu{}^{ab} + \delta\omega_\mu{}^{ab}\psi_\nu(\sigma_{ab}\sigma_c + \sigma_c\bar\sigma_{ab})\bar\psi_\sigma\,e_\rho{}^c\Big)\ .$$

Next we integrate by parts the covariant derivative D_ν and use the identity

$$\sigma_{ab}\sigma_c + \sigma_c\bar\sigma_{ab} = \mathrm{i}\epsilon_{abcd}\,\sigma^d \tag{4.42}$$

to obtain

$$\delta\mathcal{L} \simeq \frac{1}{2}\,\delta\omega_\mu{}^{ab}\epsilon_{abcd}\,\epsilon^{\mu\nu\rho\sigma}e_\rho{}^c\Big(\frac{1}{\kappa^2}\,D_\nu e_\sigma{}^d - \mathrm{i}\psi_\nu\sigma^d\bar\psi_\sigma\Big)\ .$$

From this we read off the equation of motion for the spin connection, which yields the torsion relation

$$D_{[\mu}e_{\nu]}{}^a = \mathrm{i}\kappa^2\psi_{[\mu}\sigma^a\bar\psi_{\nu]}\ . \tag{4.43}$$

The second-order Lagrangian $\mathcal{L}(e,\psi)$ is obtained by solving this equation for $\omega_\mu{}^{ab}$ and substituting it in the first-order Lagrangian. We observe that this gives rise to four-gravitino terms through the ω^2 part of the curvature tensor and the covariant derivative of the gravitino. These would make variation of $\mathcal{L}(e,\psi)$ very complicated, if it wasn't for the 1.5 order formalism.

Let us now verify invariance of the action (4.36) under the supersymmetry transformations (4.38), (4.39). Using the 1.5 order formalism, we have to vary only the vierbein and gravitino, not the spin connection; afterwards, the latter is substituted by the solution to its field equation. A further simplification is to consider only the ϵ-part of the transformations (denoted with δ_+ in the following), not the $\bar\epsilon$-part (δ_-). This is sufficient because $\mathcal{L}$ is real and $\delta_-^* = \delta_+$, hence $\delta_-\mathcal{L} = (\delta_+\mathcal{L})^* \simeq 0$ iff $\delta_+\mathcal{L} \simeq 0$. We therefore do not have to vary $\bar\psi_\mu$ at all; the fields which δ_+ acts on in the 1.5 order formalism are marked with an arrow:

$$\delta_+\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\delta_+\Big(\frac{1}{8\kappa^2}\,\epsilon_{abcd}\,\overset{\downarrow}{e}_\rho{}^c\overset{\downarrow}{e}_\sigma{}^d R_{\mu\nu}{}^{ab}(\omega) + D_\mu\overset{\downarrow}{\psi}_\nu\overset{\downarrow}{\sigma}_\rho\bar\psi_\sigma + \overset{\downarrow}{\psi}_\sigma\overset{\downarrow}{\sigma}_\rho D_\mu\bar\psi_\nu\Big)\ .$$

Note that σ_ρ is field-dependent and has a non-trivial variation,

$$\delta_+\sigma_{\rho\,\alpha\dot\alpha} = \delta_+ e_\rho{}^a\,\sigma_{a\,\alpha\dot\alpha} = \mathrm{i}\kappa\,(\epsilon\sigma^a\bar\psi_\rho)\,\sigma_{a\,\alpha\dot\alpha} = -2\mathrm{i}\kappa\,\epsilon_\alpha\bar\psi_{\rho\dot\alpha}\ . \tag{4.44}$$

[13] This form of the Lagrangian appears naturally when it is written in terms of exterior forms. The ϵ-symbol in front belongs to the volume element: $dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_D} = d^Dx\,\epsilon^{\mu_1\cdots\mu_D}$.

We now calculate

$$\delta_+\mathcal{L} = \kappa^{-1}\epsilon^{\mu\nu\rho\sigma}\big(\tfrac{\mathrm{i}}{4}\,\epsilon_{abcd}\,\epsilon\sigma^d\bar{\psi}_\sigma\,e_\rho{}^c R_{\mu\nu}{}^{ab} + D_\mu D_\nu\epsilon\,\sigma_\rho\bar{\psi}_\sigma - 2\mathrm{i}\kappa^2 D_\mu\psi_\nu\epsilon\,\bar{\psi}_\rho\bar{\psi}_\sigma$$
$$+ D_\sigma\epsilon\,\sigma_\rho D_\mu\bar{\psi}_\nu - 2\mathrm{i}\kappa^2\psi_\sigma\epsilon\,\bar{\psi}_\rho D_\mu\bar{\psi}_\nu\big)\ .$$

The last term in the first line containing the symmetric expression $\bar{\psi}_\rho\bar{\psi}_\sigma$ drops out due to antisymmetrization through $\epsilon^{\mu\nu\rho\sigma}$. The covariant derivative D_σ in the second line we integrate by parts, and then we use antisymmetry to write two consecutive covariant derivatives as commutators,

$$D_{[\mu}D_{\nu]}\epsilon^\alpha = -\tfrac{1}{4}R_{\mu\nu}{}^{ab}\,(\epsilon\sigma_{ab})^\alpha\ , \qquad D_{[\sigma}D_{\mu]}\bar{\psi}_\nu^{\dot{\alpha}} = \tfrac{1}{4}R_{\sigma\mu}{}^{ab}\,(\bar{\sigma}_{ab}\bar{\psi}_\nu)^{\dot{\alpha}}\ .$$

Collecting curvature terms, this results in

$$\delta_+\mathcal{L} \simeq \kappa^{-1}\epsilon^{\mu\nu\rho\sigma}\big[\tfrac{1}{4}\,\epsilon\big(\mathrm{i}\epsilon_{abcd}\sigma^d - \sigma_{ab}\sigma_c - \sigma_c\bar{\sigma}_{ab}\big)\bar{\psi}_\sigma\,e_\rho{}^c R_{\mu\nu}{}^{ab} - 2\mathrm{i}\kappa^2\psi_\sigma\epsilon\,\bar{\psi}_\rho D_\mu\bar{\psi}_\nu$$
$$- D_\sigma e_\rho{}^a\,\epsilon\sigma_a D_\mu\bar{\psi}_\nu\big]\ .$$

The first three terms vanish thanks to (4.42). The last term, which originates from partial integration of D_σ and the field-dependence of σ_ρ, can be simplified by means of the torsion relation (4.43) (recall that in the 1.5 order formalism after variation of the action the auxiliary fields are replaced by their on-shell expressions),

$$\delta_+\mathcal{L} \simeq -\mathrm{i}\kappa\,\epsilon^{\mu\nu\rho\sigma}\psi_\sigma(2\epsilon\,\bar{\psi}_\rho + \sigma^a\bar{\psi}_\rho\,\epsilon\sigma_a)D_\mu\bar{\psi}_\nu = 0\ .$$

The last equality follows from a Fierz rearrangement similar to that in (4.44). This concludes the proof of invariance of the action; taking into account the total derivative we picked up when integrating by parts and the δ_--part of the variation, we have found

$$\delta_Q(\epsilon)\mathcal{L} = \kappa^{-1}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\big(\bar{\epsilon}\bar{\sigma}_\nu D_\rho\psi_\sigma - \epsilon\sigma_\nu D_\rho\bar{\psi}_\sigma\big)\ . \tag{4.45}$$

The 1.5 order formalism can also be used to derive the equations of motion, since in (3.68) the variations of the fields were arbitrary. For the gravitino, we obtain

$$0 \approx \frac{\delta S}{\delta\psi_\mu} = -2\epsilon^{\mu\nu\rho\sigma}\sigma_\nu D_\rho\bar{\psi}_\sigma \equiv -2e\mathcal{R}^\mu\ , \tag{4.46}$$

where we have used (4.43) again. We can derive several identities for the gravitino field strength $\bar{\psi}_{\mu\nu} \equiv 2D_{[\mu}\bar{\psi}_{\nu]}$ which will prove useful later on:

$$\varepsilon_{\mu\nu\rho\sigma}\mathcal{R}^\sigma = 3\sigma_{[\mu}\bar{\psi}_{\nu\rho]}\ , \qquad \bar{\sigma}_\mu\mathcal{R}^\mu = 2\mathrm{i}\,\bar{\sigma}^{\mu\nu}\bar{\psi}_{\mu\nu}$$
$$\sigma_\nu\bar{\sigma}_\mu\mathcal{R}^\nu = 2\mathrm{i}\,\sigma^\nu\bar{\psi}_{\mu\nu}\ , \qquad \bar{\sigma}_\rho\sigma_{\mu\nu}\mathcal{R}^\rho = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\bar{\psi}^{\rho\sigma} - \mathrm{i}\bar{\psi}_{\mu\nu}\ . \tag{4.47}$$

The last of these implies that on-shell $\bar{\psi}_{\mu\nu}$ is anti-selfdual. Accordingly, $\psi_{\mu\nu}$ is selfdual when the field equations are satisfied.

Next, we have to convince us that the algebra of symmetry transformations closes, at least on-shell. This means that the commutator of two symmetry transformations evaluated on each field must give a linear combination of symmetry transformations, modulo trivial symmetries (2.19) that vanish on-shell. In particular, the commutator of two local supersymmetry transformations must be expressible in terms of a general coordinate transformation, a local Lorentz transformation, and a local supersymmetry transformation,[14] with possibly field-dependent parameters.
Let us start with the vierbein. It is straightforward to show that

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,]\,e_\mu{}^a = D_\mu \xi^a\ , \qquad \xi^a = \mathrm{i}(\epsilon_2\sigma^a\bar{\epsilon}_1 - \epsilon_1\sigma^a\bar{\epsilon}_2)\ . \tag{4.48}$$

The non-trivial task is to rewrite this in such a way that the three different symmetry transformations become manifest. Toward this end, we introduce a field-dependent vector ξ^ν via $\xi^a = \xi^\nu e_\nu{}^a$ and write the covariant derivative as

$$D_\mu \xi^a = \xi^\nu \partial_\mu e_\nu{}^a + \partial_\mu \xi^\nu e_\nu{}^a + \xi^\nu \omega_{\mu\nu}{}^a\ .$$

We can almost see a general coordinate transformation here; the first term is not quite right, but this can be rectified by adding and substracting $\xi^\nu \partial_\nu e_\mu{}^a$,

$$D_\mu \xi^a = \mathcal{L}_\xi e_\mu{}^a + 2\xi^\nu \partial_{[\mu} e_{\nu]}{}^a + \xi^\nu \omega_{\mu\nu}{}^a\ .$$

Now we use the torsion relation (4.43) for the second term,

$$D_\mu \xi^a = \mathcal{L}_\xi e_\mu{}^a + \xi^\nu \omega_{\nu\mu}{}^a + 2\mathrm{i}\kappa^2 \xi^\nu \psi_{[\mu}\sigma^a\bar{\psi}_{\nu]}\ .$$

Note the reversed order of the lower indices of the spin connection. We are done; this is a linear combination of a general coordinate transformation of $e_\mu{}^a$ with vector parameter $\xi^\nu(\epsilon_1, \epsilon_2, e)$, a Lorentz transformation with tensor parameter $\varepsilon_b{}^a = -\xi^\nu \omega_{\nu b}{}^a$, and a supersymmetry transformation with spinor parameter $\epsilon_{12} = -\kappa \xi^\nu \psi_\nu$,

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,]\,e_\mu{}^a = \delta_P(\xi)e_\mu{}^a + \delta_L(\varepsilon)e_\mu{}^a + \delta_Q(\epsilon_{12})e_\mu{}^a\ . \tag{4.49}$$

The commutator thus closes off-shell, as expected for bosonic fields. Observe that all three parameters on the right-hand side depend on the fields.
Evaluating the same commutator on the gravitino is significantly harder. First of all, we need to know the susy transformation of the spin connection, since

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,]\,\psi_\mu = \frac{1}{2\kappa}\delta_Q(\epsilon_1)\omega_\mu{}^{ab}\sigma_{ab}\epsilon_2 - (\epsilon_1 \leftrightarrow \epsilon_2)\ . \tag{4.50}$$

We thus take a short detour and determine $\delta_Q \omega_\mu{}^{ab}$. The easiest way is to apply δ_+ to the torsion relation (4.43),

$$\delta_+ D_{[\mu} e_{\nu]}{}^a = \mathrm{i}\kappa D_{[\mu}(\epsilon\sigma^a\bar{\psi}_{\nu]}) - e_{[\mu}{}^b \delta_+ \omega_{\nu]b}{}^a \stackrel{!}{=} \mathrm{i}\kappa D_{[\mu}\epsilon\,\sigma^a\bar{\psi}_{\nu]}\ ,$$

[14] And perhaps additional symmetries we haven't been aware of yet. Computing commutators of known symmetry transformations can sometimes be used to find new symmetries.

from which we read off $e_{[\mu}{}^b\delta_+\omega_{\nu]b}{}^a = \frac{\mathrm{i}}{2}\kappa\,\epsilon\sigma^a\bar{\psi}_{\mu\nu}$. We can get rid of the antisymmetrization by means of the identity (3.72), which gives

$$\delta_+\omega_{\mu ab} = \tfrac{\mathrm{i}}{2}\kappa E_a{}^\rho E_b{}^\nu\,\epsilon(\sigma_\rho\bar{\psi}_{\mu\nu} - \sigma_\nu\bar{\psi}_{\mu\rho} - \sigma_\mu\bar{\psi}_{\nu\rho})\ . \tag{4.51}$$

Note that $\omega_\mu{}^{ab}$ is supercovariant, i.e., its transformation contains no derivatives of ϵ. However, this property is just an accident and does not hold in every supergravity theory. Using the first identity in (4.47) to put the result into a more convenient form, we finally obtain

$$\delta_Q(\epsilon)\omega_{\mu ab} = \mathrm{i}\kappa\,(\epsilon\sigma_\mu\bar{\psi}_{ab} - \psi_{ab}\sigma_\mu\bar{\epsilon}) - \tfrac{\mathrm{i}}{2}\kappa\,\epsilon_{abcd}\,e_\mu{}^c(\epsilon\,\mathcal{R}^d - \bar{\epsilon}\,\bar{\mathcal{R}}^d)\ . \tag{4.52}$$

On-shell, the second bracket vanishes. The first bracket corresponds to a decomposition into anti-selfdual and selfdual parts respectively (again modulo field equations). Since in the above commutator they get contracted with the selfdual σ^{ab}-matrix, $\bar{\psi}_{ab}$ yields only terms containing $\mathcal{R}^\mu$. Keeping track of all on-shell vanishing terms is quite a lot of work, so we will drop them from now on as they are not particularly illuminating.[15]

We now continue with the computation of the supersymmetry commutator on ψ_μ. On-shell, we have found

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,]\,\psi_\mu \approx -\tfrac{\mathrm{i}}{2}(\psi_{ab}\sigma_\mu\bar{\epsilon}_1)\,\sigma^{ab}\epsilon_2 - (\epsilon_1 \leftrightarrow \epsilon_2)\ .$$

Next, a Fierz identity helps us to rewrite the right-hand side,

$$\begin{aligned}(\psi_{\nu\rho}\sigma_\mu\bar{\epsilon}_1)\,\sigma^{\nu\rho}\epsilon_2 &= \psi_{\nu\rho}\,(\bar{\epsilon}_1\bar{\sigma}_\mu\sigma^{\nu\rho}\epsilon_2) - \sigma_\mu\bar{\epsilon}_1(\psi_{\nu\rho}\sigma^{\nu\rho}\epsilon_2)\\ &\approx (\psi_{\mu\nu} + \tfrac{\mathrm{i}}{2}\varepsilon_{\mu\nu\rho\sigma}\psi^{\rho\sigma})\,(\epsilon_2\sigma^\nu\bar{\epsilon}_1)\\ &\approx 2\psi_{\mu\nu}\,(\epsilon_2\sigma^\nu\bar{\epsilon}_1)\ .\end{aligned}$$

Here we have used (4.47) twice and dropped the $\bar{\mathcal{R}}^\mu$-terms. We thus obtain

$$\begin{aligned}[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,]\,\psi_\mu &\approx -\psi_{\mu\nu}\xi^\nu = \xi^\nu D_\nu\psi_\mu - \xi^\nu D_\mu\psi_\nu\\ &= \mathcal{L}_\xi\psi_\mu + \tfrac{1}{2}(\xi^\nu\omega_\nu{}^{ab})\sigma_{ab}\psi_\mu - D_\mu(\xi^\nu\psi_\nu)\ ,\end{aligned}$$

with the same vector $\xi^\nu(\epsilon_1,\epsilon_2,e)$ as above. We conclude that

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,]\,\psi_\mu \approx \delta_P(\xi)\psi_\mu + \delta_L(-\xi^\nu\omega_\nu)\psi_\mu + \delta_Q(-\kappa\xi^\nu\psi_\nu)\psi_\mu\ . \tag{4.53}$$

This is the same commutator that we found for the vierbein, only now we had to use the gravitino field equations.

The other commutators all close off-shell on both fields. We find

$$[\,\delta_L(\varepsilon)\,,\delta_Q(\epsilon)\,] = \delta_Q(-\tfrac{1}{2}\varepsilon^{ab}\epsilon\sigma_{ab})$$

[15] They are important, however, when the on-shell theory is quantized in the Batalin-Vilkovisky approach, where the non-closure functions enter the BRST operator.

$$
\begin{aligned}
[\,\delta_P(\xi)\,,\delta_Q(\epsilon)\,] &= \delta_Q(-\xi^\mu\partial_\mu\epsilon)\\
[\,\delta_P(\xi)\,,\delta_L(\varepsilon)\,] &= \delta_L(-\xi^\mu\partial_\mu\varepsilon)\\
[\,\delta_P(\xi_1)\,,\delta_P(\xi_2)\,] &= \delta_P(-\xi_1^\mu\partial_\mu\xi_2+\xi_2^\mu\partial_\mu\xi_1)\\
[\,\delta_L(\varepsilon_1)\,,\delta_L(\varepsilon_2)\,] &= \delta_L(-[\,\varepsilon_1\,,\varepsilon_2\,])\ .
\end{aligned}
\tag{4.54}
$$

In order to achieve off-shell closure of the gauge algebra, we have to amend the supergravity multiplet $\big(e_\mu{}^a,\,\psi_\mu,\,\bar\psi_\mu\big)$ by a suitable set of auxiliary fields, just like for chiral or vector multiplets. In the component formalism we have employed so far, it is rather difficult to find such a set, even though there are several possible choices.
In the following, we set up a general tensor calculus for gauge theories, which we will then apply to $N=1$ supergravity, and which will allow us to derive the field content of the multiplet, its supersymmetry transformations, and the off-shell commutation relations from a few basic constraints on the geometry of superspace.

5 Off-shell Formulation of N=1 Supergravity

5.1 Tensor Calculus

Tensor fields, in the following collectively denoted by T, are characterized by gauge transformations that are homogeneous and contain only undifferentiated parameters ξ^M. The transformed tensors are assumed to be tensors again. Infinitesimally, we can write

$$
\delta_{\rm g}T=\xi^M\Delta_M T\ . \tag{5.1}
$$

The generators Δ_M of the gauge transformations map tensors into tensors and inherit a grading from the parameters, which can be even or odd ($\delta_{\rm g}$ is always even),

$$
|\xi^M|=|\Delta_M|=|M|\in\{0,1\}\ . \tag{5.2}
$$

Since $\delta_{\rm g}$ is an infinitesimal transformation and the product of two tensors is a tensor, it follows that Δ_M are derivations, i.e., they satisfy the (graded) Leibniz rule

$$
\Delta_M(T_1T_2)=\Delta_M T_1\,T_2+(-)^{|M||T_1|}T_1\,\Delta_M T_2\ . \tag{5.3}
$$

Gauge fields are introduced by expressing partial derivatives in terms of the covariant operators Δ_M,

$$
\partial_\mu T=A_\mu^M\Delta_M T\ . \tag{5.4}
$$

The grading of A_μ^M coincides with that of Δ_M,

$$
|A_\mu^M|=|M|\ . \tag{5.5}
$$

In order to obtain an off-shell formulation of the gauge theory under consideration, we require that the algebra of gauge transformations closes, i.e., the commutator of two gauge transformations (5.1) must give another gauge transformation, with possibly field-dependent parameters,

$$[\,\delta_{\rm g}(\xi_1)\,,\delta_{\rm g}(\xi_2)\,]T \overset{!}{=} \delta_{\rm g}(\xi_{12})T\ . \tag{5.6}$$

Using (5.1), this amounts to

$$\xi_2^N\xi_1^M[\,\Delta_M\,,\Delta_N\}T = \xi_{12}^P\,\Delta_P T\ , \tag{5.7}$$

which involves the graded commutator

$$[\,\Delta_M\,,\Delta_N\} = \Delta_M\Delta_N - (-)^{|M||N|}\Delta_N\Delta_M\ . \tag{5.8}$$

Since the right-hand side of (5.7) does not contain derivatives of the parameters ξ_1, ξ_2, we can write $\xi_{12}^P = -\xi_2^N\xi_1^M\mathcal{F}_{MN}{}^P$. Accordingly, the coefficients $\mathcal{F}_{MN}{}^P$ serve as structure functions of the algebra of the covariant operators Δ_M,

$$[\,\Delta_M\,,\Delta_N\} = -\mathcal{F}_{MN}{}^P\Delta_P\ . \tag{5.9}$$

Depending on the nature of the gauge transformations, they may be constant, but in general they are field-dependent. Their symmetry properties are the same as those of the graded commutators,

$$\mathcal{F}_{MN}{}^P = -(-)^{|M||N|}\mathcal{F}_{NM}{}^P\ . \tag{5.10}$$

Note that the $\mathcal{F}_{MN}{}^P$ transform as tensors.
The graded Jacobi identity

$$\sum_{MNP}(-)^{|M||P|}\,[\,\Delta_M\,,[\,\Delta_N\,,\Delta_P\}\} = 0 \tag{5.11}$$

and the assumed linear independence of the Δ_M implies so-called Bianchi identities (in the following abbreviated by BIs) for the structure functions:

$$\sum_{MNP}(-)^{|M||P|}\left(\Delta_M\mathcal{F}_{NP}{}^Q + \mathcal{F}_{MN}{}^R\mathcal{F}_{RP}{}^Q\right) = 0\ . \tag{5.12}$$

Later, we will impose constraints on some of the $\mathcal{F}_{MN}{}^P$, which then turns the BIs into non-trivial equations.
There are more consistency conditions. First, we require gauge transformations to commute with differentiation. Evaluating the commutator $[\,\partial_\mu\,,\delta_{\rm g}\,] = 0$ on tensors T using (5.1) and (5.4) yields the transformation law of the gauge fields,

$$\delta_{\rm g}A_\mu^M = \partial_\mu\xi^M + A_\mu^P\xi^N\mathcal{F}_{NP}{}^M\ , \tag{5.13}$$

with its characteristic inhomogeneous piece. The second consistency condition derives from the fact that two partial derivatives commute ($\mathrm{d}^2 = 0$). When evaluated on tensors, we find that this implies the identity

$$\partial_\mu A_\nu^M - \partial_\nu A_\mu^M + A_\mu^P A_\nu^N\mathcal{F}_{NP}{}^M = 0\ . \tag{5.14}$$

Making use of the latter and the BIs (5.12), it can be shown that the consistency conditions are satisfied automatically on the A_μ^M. Moreover, it follows that the commutator of two gauge symmetries (5.13) closes on the gauge potentials:

$$[\,\delta_{\rm g}(\xi_1)\,,\delta_{\rm g}(\xi_2)\,]A_\mu^M = \delta_{\rm g}(\xi_{12})A_\mu^M \ , \tag{5.15}$$

where $\xi_{12}^M = -\xi_2^P\xi_1^N\mathcal{F}_{NP}{}^M$. We leave the verification as an excercise to the reader.
We now assume that among the A_μ^M there is a field $e_\mu{}^a$ whose components form an invertible matrix. As the notation suggests, it can be identified with the vielbein. The remaining gauge fields we denote with $A_\mu^{\hat{M}}$,

$$A_\mu^M = \big(e_\mu{}^a\,, A_\mu^{\hat{M}}\big) \ . \tag{5.16}$$

The latter will include among others the gravitino (for $\hat{M} = \alpha, \dot{\alpha}$) in applications of the general formalism to supergravity. We can now solve (5.4) for the operators $\Delta_a \equiv \mathcal{D}_a$ corresponding to the vielbein,

$$\mathcal{D}_a T = E_a{}^\mu\big(\partial_\mu - A_\mu^{\hat{M}}\Delta_{\hat{M}}\big)T \ . \tag{5.17}$$

On the right-hand side we recognize the familiar form of a covariant derivative. Its field strength can be obtained from (5.14),

$$\mathcal{F}_{ab}{}^M = E_a{}^\mu E_b{}^\nu\big(\partial_\mu A_\nu^M - \partial_\nu A_\mu^M + A_\mu^{\hat{P}}A_\nu^{\hat{N}}\mathcal{F}_{\hat{N}\hat{P}}{}^M\big) + A_\mu^{\hat{N}}\big(E_a{}^\mu\mathcal{F}_{b\hat{N}}{}^M - E_b{}^\mu\mathcal{F}_{a\hat{N}}{}^M\big) \ . \tag{5.18}$$

In order to make contact with our previous formulation of supergravity, it is convenient to choose a different basis for the generators $\Delta_M = \big(\Delta_a\,, \Delta_{\hat{M}}\big)$, where Δ_a is replaced by ∂_μ. This is achieved by the following redefinition of the transformation parameters:

$$\xi^\mu = \xi^a E_a{}^\mu \ , \qquad \epsilon^{\hat{M}} = \xi^{\hat{M}} - \xi^\mu A_\mu^{\hat{M}} \ . \tag{5.19}$$

In terms of the new basis, the transformation laws of tensors and gauge fields read

$$\delta_{\rm g}T = \xi^\nu\partial_\nu T + \epsilon^{\hat{M}}\Delta_{\hat{M}}T \tag{5.20}$$

$$\delta_{\rm g}e_\mu{}^a = \xi^\nu\partial_\nu e_\mu{}^a + \partial_\mu\xi^\nu e_\nu{}^a + A_\mu^P\epsilon^{\hat{N}}\mathcal{F}_{\hat{N}P}{}^a \tag{5.21}$$

$$\delta_{\rm g}A_\mu^{\hat{M}} = \xi^\nu\partial_\nu A_\mu^{\hat{M}} + \partial_\mu\xi^\nu A_\nu^{\hat{M}} + \partial_\mu\epsilon^{\hat{M}} + A_\mu^P\epsilon^{\hat{N}}\mathcal{F}_{\hat{N}P}{}^{\hat{M}} \ . \tag{5.22}$$

Note that the ξ^μ-transformations are precisely generated by the Lie derivative (3.3), hence they correspond to general coordinate transformations.
A gauge theory with given field content and symmetries is now specified by a choice of structure functions $\mathcal{F}_{MN}{}^P$. The possible choices are restricted by the Bianchi identities. Once a consistent set of structure functions has been found, the gauge symmetries of both tensors and gauge fields are completely determined and the symmetry algebra closes by construction.

5.1.1 Example: Yang-Mills Theory

The above formalism applies in particular to standard gauge theories of the Yang-Mills type. In these cases, the tensors T transform in some matrix representation of the gauge group G. Infinitesimally, one has

$$\Delta_I T^i = -t_I{}^i{}_j T^j \ , \tag{5.23}$$

where $I = 1, \ldots, \dim G$. The generators Δ_I of gauge transformations form a Lie algebra with structure constants $-\mathcal{F}_{IJ}{}^K = f_{IJ}{}^K$, and so do the representation matrices t_I,

$$[\,\Delta_I\,,\Delta_J\,] = f_{IJ}{}^K \Delta_K \ , \qquad [\,t_I\,,t_J\,] = f_{IJ}{}^K t_K \ . \tag{5.24}$$

The Bianchi identity reads

$$f_{IJ}{}^L f_{LK}{}^M + f_{KI}{}^L f_{LJ}{}^M + f_{JK}{}^L f_{LI}{}^M = 0 \ . \tag{5.25}$$

In flat space and Cartesian coordinates, the vielbein is constant: $e_\mu{}^a = \delta^a_\mu$. The corresponding transformations generate global translations with constant parameters ξ^μ. The gauge-covariant derivative follows from (5.17),

$$\mathcal{D}_\mu T = (\partial_\mu - A^I_\mu \Delta_I) T = (\partial_\mu + A^I_\mu t_I) T \ . \tag{5.26}$$

The Yang-Mills field strength for the gauge potentials A^I_μ can be read off from (5.18) (where $\mathcal{F}_{aI}{}^M = 0$)

$$F_{\mu\nu}{}^I = \delta^a_\mu \delta^b_\nu \, \mathcal{F}_{ab}{}^I = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + A^J_\mu A^K_\nu f_{JK}{}^I \ . \tag{5.27}$$

Finally, the action of global translations and local gauge transformations with parameters ϵ^I is given by

$$\delta_{\rm g} T = \xi^\nu \partial_\nu T - \epsilon^I t_I T \tag{5.28}$$

$$\delta_{\rm g} A^I_\mu = \xi^\nu \partial_\nu A^I_\mu + \partial_\mu \epsilon^I + A^J_\mu \epsilon^K f_{JK}{}^I \ . \tag{5.29}$$

5.2 Bianchi Identities

Let us now apply the tensor calculus to $N = 1$ supergravity. The gauge fields A^M_μ then comprise in addition to the vierbein the gravitino and the spin connection,

$$A^M_\mu = \left(e_\mu{}^a \, , \, \kappa\psi^\alpha_\mu \, , \, \kappa\bar\psi_{\mu\dot\alpha} \, , \, \omega_\mu{}^{ab}\right) \ . \tag{5.30}$$

The constant κ appears here on dimensional grounds. The corresponding gauge transformations are general coordinate transformations, local supersymmetry transformations, and local Lorentz transformations. The generators of the first two we collectively denote with $\mathcal{D}_A$,

$$\Delta_M = \left(\mathcal{D}_A \, , \, \ell_{ab}\right) \ , \qquad \mathcal{D}_A = \left(\mathcal{D}_a \, , \, \mathcal{D}_\alpha \, , \, \bar{\mathcal{D}}^{\dot\alpha}\right) \ . \tag{5.31}$$

Thus, capital letters from the beginning of the alphabet exclude the Lorentz double-index $[ab]$. Note the position of the dotted spinor index in $\mathcal{D}_A$ and A_μ^M. Our summation convention is the following:

$$X^M Y_M = X^A Y_A + \tfrac{1}{2} X^{[ab]} Y_{[ab]}$$
$$X^A Y_A = X^a Y_a + X^{\underline{\alpha}} Y_{\underline{\alpha}} \ , \qquad X^{\underline{\alpha}} Y_{\underline{\alpha}} = X^\alpha Y_\alpha + X_{\dot\alpha} Y^{\dot\alpha} \ . \tag{5.32}$$

This convention is chosen such that $X^{\underline{\alpha}} Y_{\underline{\alpha}}$ is real if $(X^\alpha)^* = X^{\dot\alpha}$ and $(Y^\alpha)^* = Y^{\dot\alpha}$ (where we assume that the grading corresponds to the index picture). Eq. (5.17) then reads

$$\mathcal{D}_a = E_a{}^\mu \big(\partial_\mu - \kappa \psi_\mu^\alpha \mathcal{D}_\alpha - \kappa \bar\psi_{\mu\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} - \tfrac{1}{2} \omega_\mu{}^{ab} \ell_{ab}\big) = E_a{}^\mu \big(D_\mu - \kappa \psi_\mu^{\underline{\alpha}} \mathcal{D}_{\underline{\alpha}}\big) \ , \tag{5.33}$$

where D_μ is the Lorentz-covariant derivative (3.51). We shall refer to $\mathcal{D}_a$ as the supercovariant derivative (compare with (4.15)).

In the remainder of these lectures, we will set the gravitational coupling constant $\kappa = 1$ for simplicity. κ can easily be reinstated when needed; since it carries mass dimension -1, simply insert appropriate powers of κ into each term of an equation until the dimensions match.

In order to obtain a supersymmetry algebra, we have to impose certain restrictions on the structure functions $\mathcal{F}_{MN}{}^P$. First of all, the $\mathcal{F}_{[ab]N}{}^P$ occur in the commutators of ℓ_{ab} with the other generators and itself and should therefore form representation matrices of Lorentz transformations. In particular, this means they must be constant and non-vanishing only if N and P are of the same type. We thus set $\mathcal{F}_{[ab]N}{}^P = 0$ except for

$$\mathcal{F}_{[ab]c}{}^d = -(\eta_{ac}\, \delta_b^d - \eta_{bc}\, \delta_a^d) \qquad \text{(vector)}$$
$$\mathcal{F}_{[ab]\alpha}{}^\beta = \sigma_{ab\,\alpha}{}^\beta \ , \qquad \mathcal{F}_{[ab]}{}^{\dot\alpha}{}_{\dot\beta} = \bar\sigma_{ab}{}^{\dot\alpha}{}_{\dot\beta} \qquad \text{(spinor)}$$
$$\mathcal{F}_{[ab][cd]}{}^{[ef]} = -2\big(\eta_{ac}\, \delta_b^{[e}\, \delta_d^{f]} - \eta_{bc}\, \delta_a^{[e}\, \delta_d^{f]} + \eta_{bd}\, \delta_a^{[e}\, \delta_c^{f]} - \eta_{ad}\, \delta_b^{[e}\, \delta_c^{f]}\big) \ . \quad \text{(adjoint)} \tag{5.34}$$

The latter coincide with the (negative) structure constants of the Lorentz algebra (3.52). The remaining $\mathcal{F}_{AB}{}^P$ we identify as torsion and curvature of the supersymmetry algebra,

$$\mathcal{F}_{AB}{}^C = T_{AB}{}^C \ , \qquad \mathcal{F}_{AB}{}^{[cd]} = R_{AB}{}^{cd} \ . \tag{5.35}$$

In contrast to pure gravity, superspace torsion and curvature have not only purely bosonic components, but also fermionic ones. In fact, even in flat space (where $E_a{}^\mu = \delta_a^\mu$) there is torsion; as can be read off from (2.15), we have in this case that $T_{\alpha\dot\beta}{}^c = \mathrm{i}\sigma^c_{\alpha\dot\beta}$.

With the above identifications, the supersymmetry commutation relations now read

$$[\, \mathcal{D}_A \,, \mathcal{D}_B \} = -T_{AB}{}^C \mathcal{D}_C - \tfrac{1}{2} R_{AB}{}^{cd} \ell_{cd} \tag{5.36}$$

$$[\, \ell_{ab} \,, \mathcal{D}_c \,] = \eta_{ac} \mathcal{D}_b - \eta_{bc} \mathcal{D}_a \tag{5.37}$$

$$[\, \ell_{ab} \,, \mathcal{D}_\alpha \,] = -\sigma_{ab\,\alpha}{}^\beta \mathcal{D}_\beta \ , \qquad [\, \ell_{ab} \,, \bar{\mathcal{D}}^{\dot\alpha} \,] = -\bar\sigma_{ab}{}^{\dot\alpha}{}_{\dot\beta} \bar{\mathcal{D}}^{\dot\beta} \ , \tag{5.38}$$

supplemented by (3.52). When we refer to the supersymmetry algebra in the following, we mostly mean (5.36). Recall that it applies to tensor fields, but not to gauge fields.

The torsion and curvature tensors contain many more components than required for supergravity. We need to impose further constraints to reduce their number as much as possible. In doing so, we have to make sure these constraints on $\mathcal{F}_{MN}{}^P$ are consistent, i.e., they must be compatible with the BIs (5.12). There are many of them; if one of the indices in the cyclic sum is a Lorentz index pair $[ab]$, the equation is satisfied identically as a result of the $\mathcal{F}_{[ab]N}{}^P$ being representation matrices of the Lorentz algebra and the torsion and curvature being Lorentz tensors (which we have to respect in choosing constraints). For instance, for $MNP = [ab]AB$, the torsion BI reads

$$\ell_{ab} T_{AB}{}^C = -\mathcal{F}_{[ab]A}{}^D T_{DB}{}^C - \mathcal{F}_{[ab]B}{}^D T_{AD}{}^C + T_{AB}{}^D \mathcal{F}_{[ab]D}{}^C \ , \tag{5.39}$$

and likewise for the curvature. Non-trivial restrictions follow from the BIs

$$\sum_{ABC} (-)^{|A||C|} \left(\mathcal{D}_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D - R_{ABC}{}^D \right) = 0 \tag{5.40}$$

$$\sum_{ABC} (-)^{|A||C|} \left(\mathcal{D}_A R_{BC}{}^{ef} + T_{AB}{}^D R_{DC}{}^{ef} \right) = 0 \ . \tag{5.41}$$

Here,

$$R_{ABC}{}^D \equiv -\tfrac{1}{2} R_{AB}{}^{ef} \mathcal{F}_{[ef]C}{}^D \tag{5.42}$$

is the matrix-valued curvature in the representation of the Lorentz algebra determined by its indices C and D. For the spinor representations we have

$$R_{AB\gamma}{}^{\delta} = -\tfrac{1}{2} R_{AB}{}^{cd} \sigma_{cd\gamma}{}^{\delta} \ , \qquad R_{AB}{}^{\dot\gamma}{}_{\dot\delta} = -\tfrac{1}{2} R_{AB}{}^{cd} \bar\sigma_{cd}{}^{\dot\gamma}{}_{\dot\delta} \ . \tag{5.43}$$

This corresponds to a decompostion of $R_{AB}{}^{cd}$ into selfdual and anti-selfdual parts, respectively; the inverse relation reads

$$R_{AB}{}^{cd} = R_{AB\gamma}{}^{\delta} \sigma^{cd}{}_{\delta}{}^{\gamma} + R_{AB}{}^{\dot\gamma}{}_{\dot\delta} \bar\sigma^{cd\dot\delta}{}_{\dot\gamma} \ . \tag{5.44}$$

For a given set of constraints, we have to solve (5.40) and (5.41) for the torsion and curvature tensors in terms of a minimal number of independent irreducible components. Recall, however, that the torsions $T_{ab}{}^C$ and the curvature $R_{ab}{}^{cd}$ can be expressed in terms of the other structure functions and the gauge fields according to (5.18). Moreover, it is a great relief that it suffices to solve only the torsion BIs, for it was shown in [18] that a solution to the latter automatically solves the curvature BIs as well.

Finding the proper constraints and solving the BIs is rather tedious, and we shall not present the analyis in full detail; let us just sketch how it is done. First of all, observe that we have a certain freedom of redefining the fields: We may introduce new operators Δ'_M related to the old ones through

$$\Delta'_M = X_M{}^N \Delta_N \ , \tag{5.45}$$

where ${X_M}^N$ is an invertible matrix depending locally on the tensor fields. According to (5.1) and (5.4), this corresponds to a redefinition of the transformation parameters and gauge fields,

$$\xi'^M = \xi^N (X^{-1})_N{}^M \ , \qquad A'^M_\mu = A^N_\mu (X^{-1})_N{}^M \ . \tag{5.46}$$

The structure functions then transform inhomogeneously:

$$\begin{aligned} \mathcal{F}'_{MN}{}^P = -\big[&(-)^{|S||N|} X_M{}^S X_N{}^R \mathcal{F}_{RS}{}^Q \\ &+ X_M{}^R \Delta_R X_N{}^Q - (-)^{|M||N|} X_N{}^R \Delta_R X_M{}^Q\big] (X^{-1})_Q{}^P \ . \end{aligned} \tag{5.47}$$

Two theories that differ only by such a local redefinition of the fields are equivalent. We thus can employ this freedom to simplify the algebra of the Δ_M by making as many structure functions vanish as possible. It can be shown (see e.g. [15]) that it is always admissible to choose

$$\begin{aligned} T_{\alpha\dot\beta}{}^c = T_{\dot\beta\alpha}{}^c = \mathrm{i}\sigma^c_{\alpha\dot\beta} \ , \qquad T_{ab}{}^c = 0 \\ T_{\alpha\beta}{}^\gamma = T_{\dot\alpha\dot\beta}{}^{\dot\gamma} = 0 \ , \qquad T_{\alpha\dot\beta}{}^{\underline{\gamma}} = T_{\dot\beta\alpha}{}^{\underline{\gamma}} = 0 \ . \end{aligned} \tag{5.48}$$

Given these so-called conventional constraints, different versions of $N = 1$ supergravity are obtained by imposing further restrictions. Minimal supergravity satisfies in addition

$$T_{\alpha\beta}{}^c = T_{\dot\alpha\dot\beta}{}^c = 0 \ . \tag{5.49}$$

The purpose of this constraint is to allow for a consistent definition of chiral fields, see below. Let us investigate the consequences of these constraints. The BI with index picture ${}_{\alpha\beta\gamma}{}^d$ reduces to

$$\sigma^d_{\alpha\dot\delta} T_{\beta\gamma}{}^{\dot\delta} + \sigma^d_{\beta\dot\delta} T_{\gamma\alpha}{}^{\dot\delta} + \sigma^d_{\gamma\dot\delta} T_{\alpha\beta}{}^{\dot\delta} = 0 \ ,$$

all other terms vanish thanks to the constraints. Contracting this equation with $\bar\sigma_d^{\dot\gamma\gamma}$ gives

$$T_{\alpha\beta}{}^{\dot\gamma} = 0 \quad \Leftrightarrow \quad T_{\dot\alpha\dot\beta}{}^\gamma = 0 \ . \tag{5.50}$$

Likewise, the BI with index picture ${}_{\alpha\beta\dot\gamma}{}^d$ implies

$$T_{a(\alpha}{}^d \sigma^a_{\beta)\dot\gamma} = 0 \ .$$

Adopting the conventional constraints does not completely fix the freedom of redefintions, and in fact one can find a further field redefinition that yields the stronger constraint[16]

$$T_{\underline{\alpha}b}{}^c = -T_{b\underline{\alpha}}{}^c = 0 \ . \tag{5.51}$$

Among the theories obtained by imposing the conventional constraints, minimal supergravity is distinguished through (5.49) and (5.51).

[16]If the structure group includes either R-transformations or dilatations in addition to the Lorentz group, this condition can actually be included among the conventional constraints.

Taking into account the above constraints, the following independent BIs (5.40) (and their complex conjugates) still need to be solved:

$$\begin{aligned}
{}_{\alpha\beta\gamma}{}^{\delta}:&\quad 0 = R_{\alpha\beta\gamma}{}^{\delta} + R_{\beta\gamma\alpha}{}^{\delta} + R_{\gamma\alpha\beta}{}^{\delta} && \text{(BI 1)}\\
{}_{\alpha\dot\beta\gamma}{}^{\delta}:&\quad 0 = R_{\alpha\dot\beta\gamma}{}^{\delta} + R_{\gamma\dot\beta\alpha}{}^{\delta} - \mathrm{i}\sigma^a_{\alpha\dot\beta}T_{a\gamma}{}^{\delta} - \mathrm{i}\sigma^a_{\gamma\dot\beta}T_{a\alpha}{}^{\delta} && \text{(BI 2)}\\
{}_{\dot\alpha\dot\beta\gamma}{}^{\delta}:&\quad 0 = R_{\dot\alpha\dot\beta\gamma}{}^{\delta} - \mathrm{i}\sigma^a_{\gamma\dot\alpha}T_{a\dot\beta}{}^{\delta} - \mathrm{i}\sigma^a_{\gamma\dot\beta}T_{a\dot\alpha}{}^{\delta} && \text{(BI 3)}\\
{}_{a\beta\gamma}{}^{\delta}:&\quad 0 = R_{a\beta\gamma}{}^{\delta} + R_{a\gamma\beta}{}^{\delta} + \mathcal{D}_\beta T_{a\gamma}{}^{\delta} + \mathcal{D}_\gamma T_{a\beta}{}^{\delta} && \text{(BI 4)}\\
{}_{a\dot\beta\gamma}{}^{\delta}:&\quad 0 = R_{a\dot\beta\gamma}{}^{\delta} + \bar{\mathcal{D}}_{\dot\beta}T_{a\gamma}{}^{\delta} + \mathcal{D}_\gamma T_{a\dot\beta}{}^{\delta} + \mathrm{i}\sigma^b_{\gamma\dot\beta}T_{ab}{}^{\delta} && \text{(BI 5)}\\
{}_{a\dot\beta\dot\gamma}{}^{\delta}:&\quad 0 = \bar{\mathcal{D}}_{\dot\beta}T_{a\dot\gamma}{}^{\delta} + \bar{\mathcal{D}}_{\dot\gamma}T_{a\dot\beta}{}^{\delta} && \text{(BI 6)}\\
{}_{\alpha\beta c}{}^{d}:&\quad 0 = R_{\alpha\beta c}{}^{d} + \mathrm{i}\sigma^d_{\alpha\dot\gamma}T_{c\beta}{}^{\dot\gamma} + \mathrm{i}\sigma^d_{\beta\dot\gamma}T_{c\alpha}{}^{\dot\gamma} && \text{(BI 7)}\\
{}_{\alpha\dot\beta c}{}^{d}:&\quad 0 = R_{\alpha\dot\beta c}{}^{d} - \mathrm{i}\sigma^d_{\beta\dot\beta}T_{c\alpha}{}^{\beta} + \mathrm{i}\sigma^d_{\alpha\dot\alpha}T_{c\dot\beta}{}^{\dot\alpha} && \text{(BI 8)}\\
{}_{ab\gamma}{}^{\delta}:&\quad 0 = R_{ab\gamma}{}^{\delta} - \mathcal{D}_a T_{b\gamma}{}^{\delta} + \mathcal{D}_b T_{a\gamma}{}^{\delta} - \mathcal{D}_\gamma T_{ab}{}^{\delta} - T_{a\gamma}{}^{\underline{\alpha}}T_{b\underline{\alpha}}{}^{\delta} + T_{b\gamma}{}^{\underline{\alpha}}T_{a\underline{\alpha}}{}^{\delta} && \text{(BI 9)}\\
{}_{ab\dot\gamma}{}^{\delta}:&\quad 0 = \mathcal{D}_a T_{b\dot\gamma}{}^{\delta} - \mathcal{D}_b T_{a\dot\gamma}{}^{\delta} + \bar{\mathcal{D}}_{\dot\gamma}T_{ab}{}^{\delta} + T_{a\dot\gamma}{}^{\underline{\alpha}}T_{b\underline{\alpha}}{}^{\delta} - T_{b\dot\gamma}{}^{\underline{\alpha}}T_{a\underline{\alpha}}{}^{\delta} && \text{(BI 10)}\\
{}_{\alpha bc}{}^{d}:&\quad 0 = R_{\alpha bc}{}^{d} - R_{\alpha cb}{}^{d} + \mathrm{i}\sigma^d_{\alpha\dot\delta}T_{bc}{}^{\dot\delta} && \text{(BI 11)}\\
{}_{abc}{}^{\delta}:&\quad 0 = \mathcal{D}_a T_{bc}{}^{\delta} - \mathcal{D}_b T_{ac}{}^{\delta} + \mathcal{D}_c T_{ab}{}^{\delta} - T_{ab}{}^{\underline{\alpha}}T_{c\underline{\alpha}}{}^{\delta} + T_{ac}{}^{\underline{\alpha}}T_{b\underline{\alpha}}{}^{\delta} - T_{bc}{}^{\underline{\alpha}}T_{a\underline{\alpha}}{}^{\delta} && \text{(BI 12)}\\
{}_{abc}{}^{d}:&\quad 0 = R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d}\ . && \text{(BI 13)}
\end{aligned}$$

A long and tedious analysis[17] then reveals that all torsions and curvatures other than $T_{ab}{}^{C}$ and $R_{ab}{}^{cd}$, which are given by the identification equations (5.18), can be expressed in terms of the latter and just two additional fields: a complex scalar M and a real vector B_a. Being components of the structure functions, these are tensors; in particular, B_a is not subject to any inhomogeneous gauge symmetry. Explicitly, one finds

$$T_{a\beta\dot\gamma} = -(T_{a\dot\beta\gamma})^* = -\tfrac{\mathrm{i}}{2}\sigma_{a\,\beta\dot\gamma}\bar{M} \tag{5.52}$$

$$T_{a\beta\gamma} = -(T_{a\dot\beta\dot\gamma})^* = -\mathrm{i}(\varepsilon_{\beta\gamma}B_a + \sigma_{ab\,\beta\gamma}B^b) \tag{5.53}$$

$$R_{\alpha\beta cd} = -(R_{\dot\alpha\dot\beta cd})^* = 2\sigma_{cd\,\alpha\beta}\,\bar{M} \tag{5.54}$$

$$R_{\alpha\dot\beta cd} = -(R_{\dot\alpha\beta cd})^* = \mathrm{i}\epsilon_{abcd}\,\sigma^a_{\alpha\dot\beta}\,B^b \tag{5.55}$$

$$R_{\alpha bcd} = (R_{\dot\alpha bcd})^* = \tfrac{\mathrm{i}}{2}\big(\sigma_{b\alpha\dot\alpha}\,T_{cd}{}^{\dot\alpha} + \sigma_{c\alpha\dot\alpha}\,T_{bd}{}^{\dot\alpha} - \sigma_{d\alpha\dot\alpha}\,T_{bc}{}^{\dot\alpha}\big)\ . \tag{5.56}$$

The BIs also yield the supersymmetry transformations of M and B_a:

$$\mathcal{D}_\alpha M = \tfrac{2}{3}\sigma^{ab}{}_{\alpha}{}^{\gamma}\,T_{ab\gamma}\ , \qquad \mathcal{D}_\alpha\bar{M} = 0 \tag{5.57}$$

$$\mathcal{D}_\alpha B_a = -\tfrac{1}{3}\sigma^b_{\alpha\dot\gamma}(T_{ab}{}^{\dot\gamma} + \tfrac{\mathrm{i}}{4}\epsilon_{abcd}\,T^{cd\,\dot\gamma})\ . \tag{5.58}$$

[17] It essentially consists of converting all vector indices into spinor indices, decomposing torsion and curvature into irreducible components completely symmetric in dotted and undotted indices, and working out the consequences for these components. See e.g. [2, 8] for details.

The action of $\bar{\mathcal{D}}_{\dot\alpha}$ on these fields can be obtained by complex conjugation. An important result, following from (BI 6), is that M is chiral.
We still need to determine the structure functions $\mathcal{F}_{ab}{}^M$. Using the torsion constraints, their identification equations read

$$T_{ab}{}^c = E_a{}^\mu E_b{}^\nu (D_\mu e_\nu{}^c - D_\nu e_\mu{}^c - \mathrm{i}\psi_\mu\sigma^c\bar\psi_\nu + \mathrm{i}\psi_\nu\sigma^c\bar\psi_\mu) \tag{5.59}$$

$$T_{ab}{}^\gamma = (T_{ab}{}^{\dot\gamma})^* = E_a{}^\mu E_b{}^\nu D_\mu\psi_\nu^\gamma + \psi_a^\beta\, T_{b\beta}{}^\gamma + T_{a\dot\beta}{}^\gamma\,\bar\psi_b^{\dot\beta} - (a\leftrightarrow b) \tag{5.60}$$

$$R_{ab}{}^{cd} = E_a{}^\mu E_b{}^\nu \mathcal{R}_{\mu\nu}{}^{cd} + \psi_a^{\underline{\alpha}}\,\psi_b^{\underline{\beta}}\,R_{\underline{\alpha}\underline{\beta}}{}^{cd} - \psi_a^{\underline{\alpha}}\,R_{\underline{\alpha}b}{}^{cd} + \psi_b^{\underline{\alpha}}\,R_{\underline{\alpha}a}{}^{cd}\ . \tag{5.61}$$

In the last equation $\mathcal{R}_{\mu\nu}{}^{cd}$ denotes the curvature tensor (3.58) of the spin connection. The conventional constraint $T_{ab}{}^c = 0$ now produces precisely the torsion relation (4.43) that we had previously obtained from the Palatini formulation of on-shell supergravity. Recall that it implies that the spin connection is composed of the vierbein and gravitino. $T_{ab}{}^\gamma$, which determines the supersymmetry transformations of the component fields M and B_a, contains the Rarita-Schwinger field strength; inserting the torsions found above, we have

$$T_{ab}{}^\gamma = (T_{ab}{}^{\dot\gamma})^* = \psi_{ab}{}^\gamma + 2\mathrm{i}\,\psi_{[a}{}^\gamma B_{b]} - 2\mathrm{i}\,(\psi_{[a}\sigma_{b]c})^\gamma B^c - \mathrm{i}(\bar\psi_{[a}\bar\sigma_{b]})^\gamma M\ . \tag{5.62}$$

We have now identified the off-shell multiplet of minimal supergravity: it consists of the vierbein $e_\mu{}^a$, the gravitino ψ_μ and $\bar\psi_\mu$, and the auxiliary fields M and B_a. The latter contribute the missing $2+4$ bosonic components that are needed to equalize the number of bosonic and fermionic degrees of freedom off-shell. The supersymmetry transformations follow from (5.20)–(5.22), (5.57), (5.58), and our result for the structure functions:

$$\delta_Q(\epsilon)e_\mu{}^a = \mathrm{i}(\epsilon\sigma^a\bar\psi_\mu - \psi_\mu\sigma^a\bar\epsilon) \tag{5.63}$$

$$\delta_Q(\epsilon)\psi_\mu = D_\mu\epsilon - \tfrac{\mathrm{i}}{2}\sigma_\mu\bar\epsilon\, M - \mathrm{i}\epsilon B_\mu - \mathrm{i}\sigma_{\mu\nu}\epsilon B^\nu \tag{5.64}$$

$$\delta_Q(\epsilon)M = \tfrac{2}{3}\epsilon\sigma^{\mu\nu}\psi_{\mu\nu} + \mathrm{i}\epsilon\psi_\mu B^\mu - \mathrm{i}\epsilon\sigma^\mu\bar\psi_\mu M \tag{5.65}$$

$$\begin{aligned}\delta_Q(\epsilon)B_a = &-\tfrac{1}{3}\epsilon\sigma^b(\bar\psi_{ab} + \tfrac{\mathrm{i}}{4}\epsilon_{abcd}\,\bar\psi^{cd}) + \tfrac{\mathrm{i}}{2}\epsilon\psi_a\bar M - \tfrac{\mathrm{i}}{2}\epsilon\sigma^b\bar\psi_b\, B_a\\ &- \tfrac{1}{6}\epsilon_{abcd}\,\epsilon\sigma^b\bar\psi^c\, B^d + \text{c.c.}\ .\end{aligned} \tag{5.66}$$

Here, $B_\mu = e_\mu{}^a B_a$ as usual. The commutator of two such transformations closes off-shell by construction. They reduce to our previous expressions if we set $M = B_a = 0$; for consistency, we then have to set their variations to zero as well, which produces the gravitino equation of motion in its various versions (4.47).
Let us spell out the algebra of $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_{\dot\alpha}$, as it holds on tensors. It reads

$$\begin{aligned}\{\mathcal{D}_\alpha\,,\bar{\mathcal{D}}_{\dot\alpha}\} &= -\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha} - \tfrac{\mathrm{i}}{2}\sigma_{a\alpha\dot\alpha}B_b\,\epsilon^{abcd}\ell_{cd}\\ \{\mathcal{D}_\alpha\,,\mathcal{D}_\beta\} &= -\bar M\sigma^{ab}{}_{\alpha\beta}\,\ell_{ab}\ ,\qquad \{\bar{\mathcal{D}}_{\dot\alpha}\,,\bar{\mathcal{D}}_{\dot\beta}\} = -M\bar\sigma^{ab}{}_{\dot\alpha\dot\beta}\,\ell_{ab}\ .\end{aligned} \tag{5.67}$$

It follows that the commutator of two local supersymmetry transformations gives a linear combination of a general coordinate transformation, a local Lorentz transformation, and another local supersymmetry transformation,

$$[\,\delta_Q(\epsilon_1)\,,\delta_Q(\epsilon_2)\,] = \delta_P(\xi_{12}) + \delta_L(\varepsilon_{12}) + \delta_Q(\epsilon_{12}) \ , \tag{5.68}$$

with parameters

$$\begin{aligned} \xi_{12}^\mu = \mathrm{i}(\epsilon_2\sigma^\mu\bar\epsilon_1 - \epsilon_1\sigma^\mu\bar\epsilon_2) \ , \qquad \epsilon_{12}^\alpha = -\xi_{12}^\mu\,\psi_\mu^\alpha \\ \varepsilon_{12}^{ab} = \epsilon^{abcd}\xi_{12c}B_d - \xi_{12}^\mu\,\omega_\mu{}^{ab} - 2(\bar M\epsilon_1\sigma^{ab}\epsilon_2 + M\bar\epsilon_1\bar\sigma^{ab}\bar\epsilon_2) \ . \end{aligned} \tag{5.69}$$

The same commutator of course holds on the gauge fields $e_\mu{}^a$ and ψ_μ. Indeed, for $M = B_a = 0$ it reduces to the one we found in section 4.3. Note that only the Lorentz transformations receive contributions from the auxiliary fields.

5.3 Chiral Multiplets – Part 2

As we had seen in section 2.3, in rigid supersymmetry chiral multiplets not only describe matter fields, but also allow to formulate an action rule that yields supersymmetric invariants. This is still the case for local supersymmetry. In particular, we will derive the off-shell action for pure $N = 1$ supergravity itself from a generalized F-term invariant.
Unlike in flat space, two spinor derivatives of the same chirality do not anticommute anymore. However, the undotted anticommutator still vanishes on tensors without undotted spinor indices.[18] This is obvious for Lorentz scalars and holds on purely right-handed (multi-) spinors by virtue of the identity $\sigma^{ab}\otimes\bar\sigma_{ab} = 0$. Likewise, the dotted anticommutator vanishes on tensors without dotted indices. It is therefore still consistent to define chiral fields, as long as they do not carry dotted spinor indices. Note that a torsion $T_{\alpha\beta}{}^c \neq 0$ is an obstruction to the existence of non-trivial chiral fields.
We can introduce $\mathrm{SL}(2,\mathbb{C})$ generators

$$\ell_{\alpha\beta} = \sigma^{ab}{}_{\alpha\beta}\,\ell_{ab} \ , \qquad \ell_{\dot\alpha\dot\beta} = -\bar\sigma^{ab}{}_{\dot\alpha\dot\beta}\,\ell_{ab} \ , \tag{5.70}$$

which commute, $[\,\ell_{\alpha\beta}\,,\ell_{\dot\alpha\dot\beta}\,] = 0$, and act on spinors according to

$$\begin{aligned} \ell_{\alpha\beta}\,\chi_\gamma = \varepsilon_{\alpha\gamma}\chi_\beta + \varepsilon_{\beta\gamma}\chi_\alpha \ , \qquad \ell_{\dot\alpha\dot\beta}\,\chi_\alpha = 0 \\ \ell_{\dot\alpha\dot\beta}\,\bar\chi_{\dot\gamma} = \varepsilon_{\dot\alpha\dot\gamma}\bar\chi_{\dot\beta} + \varepsilon_{\dot\beta\dot\gamma}\bar\chi_{\dot\alpha} \ , \qquad \ell_{\alpha\beta}\,\bar\chi_{\dot\alpha} = 0 \ . \end{aligned} \tag{5.71}$$

A chiral field must then be invariant under $\ell_{\dot\alpha\dot\beta}$.
We define a chiral scalar through the constraints

$$\ell_{ab}\phi = 0 \ , \qquad \bar{\mathcal{D}}_{\dot\alpha}\phi = 0 \ . \tag{5.72}$$

The higher components of the multiplet and their transformation laws we construct by implementing the commutation relations (5.67). Since $\mathcal{D}_\alpha\phi$ is undetermined by the latter,

$$\chi_\alpha = \mathcal{D}_\alpha\phi \tag{5.73}$$

[18] This excludes fields with vector indices $a \sim \alpha\dot\alpha$.

is the next component field. We now have to make sure that the algebra holds on ϕ. The mixed anticommutator gives

$$\{\mathcal{D}_\alpha\,,\bar{\mathcal{D}}_{\dot\alpha}\}\phi = \bar{\mathcal{D}}_{\dot\alpha}\chi_\alpha \stackrel{!}{=} -\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\phi\ , \tag{5.74}$$

from which we read off the action of $\bar{\mathcal{D}}_{\dot\alpha}$ on χ_α. The undotted anticommutator requires

$$\{\mathcal{D}_\alpha\,,\mathcal{D}_\beta\}\phi = 2\,\mathcal{D}_{(\alpha}\,\chi_{\beta)} \stackrel{!}{=} 0\ ,$$

which implies that $\mathcal{D}_\alpha\chi_\beta$ is antisymmetric. The undetermined coefficient is a new field,

$$\mathcal{D}_\alpha\chi_\beta = -\varepsilon_{\alpha\beta}F\ . \tag{5.75}$$

In terms of ϕ, we have

$$F = -\tfrac{1}{2}\mathcal{D}^2\phi\ . \tag{5.76}$$

The dotted anticommutator is satisfied trivially.
Next, we have to implement the algebra on χ_α. We start with

$$\{\mathcal{D}_\alpha\,,\mathcal{D}_\beta\}\chi_\gamma = 2\,\varepsilon_{\gamma(\alpha}\mathcal{D}_{\beta)}F \stackrel{!}{=} -\bar{M}\ell_{\alpha\beta}\,\chi_\gamma = 2\bar{M}\varepsilon_{\gamma(\alpha}\,\chi_{\beta)}\ ,$$

which implies

$$\mathcal{D}_\alpha F = \bar{M}\chi_\alpha\ . \tag{5.77}$$

Taking into account that $\bar{M}$ is anti-chiral (5.57), we can write this as

$$\mathcal{D}_\alpha(\mathcal{D}^2 + 2\bar{M})\phi = 0\ . \tag{5.78}$$

Note that the only property of ϕ that we have used to derive this identity is that it does not carry undotted spinor indices, $\ell_{\alpha\beta}\phi = 0$. Hence, the whole analysis leading to (5.78) goes through unchanged if we replace ϕ by any unconstrained tensor field that is invariant under $\ell_{\alpha\beta}$. After complex conjugation we arrive at the important lemma

$$\ell_{\dot\alpha\dot\beta}K = 0 \quad \Rightarrow \quad \bar{\mathcal{D}}_{\dot\alpha}(\bar{\mathcal{D}}^2 + 2M)K = 0\ . \tag{5.79}$$

$\bar{\mathcal{D}}^2 + 2M$ is the chiral projector in supergravity.
Let us now continue with the evaluation of the algebra on χ. We consider next the anticommutator

$$\{\mathcal{D}_\alpha\,,\bar{\mathcal{D}}_{\dot\alpha}\}\chi_\beta = -\mathrm{i}\,\mathcal{D}_\alpha\mathcal{D}_{\beta\dot\alpha}\phi - \varepsilon_{\alpha\beta}\,\bar{\mathcal{D}}_{\dot\alpha}F \stackrel{!}{=} -\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\chi_\beta - \tfrac{\mathrm{i}}{2}\sigma_{a\alpha\dot\alpha}B_b\,\epsilon^{abcd}\ell_{cd}\chi_\beta\ ,$$

which can be solved for $\bar{\mathcal{D}}_{\dot\alpha}F$,

$$\begin{aligned}\varepsilon_{\alpha\beta}\,\bar{\mathcal{D}}_{\dot\alpha}F &= \mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\chi_\beta - \mathrm{i}\,\mathcal{D}_{\beta\dot\alpha}\chi_\alpha - \mathrm{i}[\,\mathcal{D}_\alpha\,,\mathcal{D}_{\beta\dot\alpha}\,]\phi - \sigma_{a\alpha\dot\alpha}(\sigma^{ab}\chi)_\beta B_b\\ &= -\mathrm{i}\varepsilon_{\alpha\beta}\,\mathcal{D}_{\gamma\dot\alpha}\chi^\gamma - \mathrm{i}\sigma^a_{\beta\dot\alpha}\,T_{a\alpha}{}^\gamma\,\mathcal{D}_\gamma\phi - \tfrac{1}{2}\chi_\alpha B_{\beta\dot\alpha} - \tfrac{1}{2}\varepsilon_{\alpha\beta}\,\chi^\gamma B_{\gamma\dot\alpha}\\ &= \varepsilon_{\alpha\beta}\big(-\mathrm{i}\,\mathcal{D}_{\gamma\dot\alpha}\chi^\gamma + \tfrac{1}{2}\chi^\gamma B_{\gamma\dot\alpha}\big)\ .\end{aligned}$$

Here we have assumed that the commutation relation for $[\mathcal{D}_\alpha\,,\mathcal{D}_a]$ holds on ϕ. We conclude that

$$\bar{\mathcal{D}}_{\dot\alpha}F=-\mathrm{i}(\mathcal{D}_a\chi\sigma^a)_{\dot\alpha}+\tfrac{1}{2}(\chi\sigma^a)_{\dot\alpha}B_a\ . \tag{5.80}$$

It requires some work to check the remaining anticommutator on χ and to show that the algebra is satsified on F without the need to introduce any more fields. We leave this as an exercise to the reader. Summarizing, we have found the following supersymmetry transformations:

$$\delta_Q(\epsilon)\phi=\epsilon\chi \tag{5.81}$$

$$\delta_Q(\epsilon)\chi=\epsilon F+\mathrm{i}\sigma^\mu\bar\epsilon\,\mathcal{D}_\mu\phi \tag{5.82}$$

$$\delta_Q(\epsilon)F=\mathrm{i}\,\bar\epsilon\bar\sigma^\mu\mathcal{D}_\mu\chi+\epsilon\chi\bar{M}+\tfrac{1}{2}\chi\sigma^a\bar\epsilon\,B_a\ . \tag{5.83}$$

The supercovariant derivatives appearing here are given by

$$\mathcal{D}_\mu\phi=\partial_\mu\phi-\psi_\mu\chi\ ,\qquad \mathcal{D}_\mu\chi=D_\mu\chi-\psi_\mu F-\mathrm{i}\sigma^\nu\bar\psi_\mu\mathcal{D}_\nu\phi\ . \tag{5.84}$$

Recall that in rigid supersymmetry F transforms into a total derivative. This was the central ingredient in the construction of supersymmetric actions. Now, not only is this not the case anymore, we would want to multiply with a factor e anyway to obtain a density invariant under general coordinate transformations. But as it turns out, one can add suitable terms to eF such that its supersymmetry variation is indeed a total derivative! In appendix B we derive the crucial identity

$$\delta_Q(\epsilon)\big(eF+\mathrm{i}e\,\bar\psi_\mu\bar\sigma^\mu\chi-3e\bar{M}\phi-2e\,\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu\,\phi\big)=\partial_\mu\big(\mathrm{i}e\,\bar\epsilon\bar\sigma^\mu\chi-4e\,\bar\epsilon\bar\sigma^{\mu\nu}\bar\psi_\nu\,\phi\big)\ . \tag{5.85}$$

This provides the action formula for local supersymmetry: Apply the operator

$$\Delta_F=-\tfrac{1}{2}\mathcal{D}^2+\mathrm{i}\bar\psi_\mu\bar\sigma^\mu\mathcal{D}-3\bar{M}-2\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu \tag{5.86}$$

to any (composite) chiral scalar ϕ and multiply by e to obtain an invariant upon integration over spacetime:

$$\delta_Q(\epsilon)\big(e\Delta_F\phi\big)=\partial_\mu K^\mu\ . \tag{5.87}$$

5.4 Off-shell Actions

As a first application, we now use the above rule to derive the off-shell action for pure supergravity. We had found that the auxiliary scalar M is chiral. It is in fact just the right field that gives the supersymmetric extension of the Einstein-Hilbert term:

$$\mathcal{L}_{\text{sugra}}=3e\,\mathrm{Re}(\Delta_F M)\ . \tag{5.88}$$

To evaluate this expression, we have to calculate $\mathcal{D}^2M$. Using (5.57), (BI 9), and the torsion constraints, we find

$$3\mathcal{D}^2M=2\sigma^{ab}{}_\delta{}^\gamma\,\mathcal{D}_\gamma T_{ab}{}^\delta=2\sigma^{ab}{}_\delta{}^\gamma\big(R_{ab\gamma}{}^\delta-2\,\mathcal{D}_aT_{b\gamma}{}^\delta-2\,T_{a\gamma}{}^{\underline{\alpha}}T_{b\underline{\alpha}}{}^\delta\big)\ . \tag{5.89}$$

The curvature term simplifies to

$$2\sigma^{ab}{}_\delta{}^\gamma R_{ab\gamma}{}^\delta = -\operatorname{tr}(\sigma^{ab}\sigma^{cd})R_{abcd} = R_{ab}{}^{ab} + \tfrac{\mathrm{i}}{2}\,\epsilon^{abcd}R_{abcd} = R_{ab}{}^{ab}\ , \tag{5.90}$$

where the last equality is due to (BI 13). Inserting the expressions we found for the torsion tensors, it follows that

$$\begin{aligned} 3\mathcal{D}^2 M &= R_{ab}{}^{ab} + 4\mathrm{i}\operatorname{tr}(\sigma^{ab}\sigma_{bc})\,\mathcal{D}_a B^c + 4\operatorname{tr}\big[\sigma^{ab}(B_a - \sigma_{ac}B^c)(B_b - \sigma_{bd}B^d)\big] \\ &\quad + \operatorname{tr}(\sigma^{ab}\sigma_a\bar\sigma_b)M\bar M \\ &= R_{ab}{}^{ab} + 6\mathrm{i}\,\mathcal{D}_a B^a - 6B_a B^a - 12M\bar M\ . \end{aligned} \tag{5.91}$$

Next, we use the identification equation (5.61) for $R_{ab}{}^{cd}$. Substituting the curvatures listed in (5.54)–(5.56) gives

$$R_{ab}{}^{ab} = \tfrac12\mathcal{R}(\omega) - 2\bar\psi_a\bar\sigma^{ab}\bar\psi_b M + \mathrm{i}\epsilon^{abcd}\,\psi_a\sigma_b\bar\psi_c\,B_d - 2\mathrm{i}\,T_{ab}{}^\gamma(\sigma^a\bar\psi^b)_\gamma + \text{c.c.}\ , \tag{5.92}$$

with $\mathcal{R}(\omega)$ the Lorentz curvature scalar. We now calculate

$$\begin{aligned} 3\Delta_F M &= -\tfrac32\,\mathcal{D}^2 M + 3\mathrm{i}\,\bar\psi_a\bar\sigma^a\mathcal{D}M - 9M\bar M - 6\bar\psi_a\bar\sigma^{ab}\bar\psi_b M \\ &= -\tfrac12\mathcal{R}(\omega) + 3B_aB^a - 3M\bar M - 3\mathrm{i}\,\mathcal{D}_a B^a + \epsilon^{abcd}\big(T_{ab}{}^\gamma - \mathrm{i}\psi_a^\gamma B_b\big)(\sigma_c\bar\psi_d)_\gamma \\ &\quad + \psi_a\sigma^{ab}\psi_b\bar M - 5\bar\psi_a\bar\sigma^{ab}\bar\psi_b M \\ &= -\tfrac12\mathcal{R}(\omega) + 3B_aB^a - 3M\bar M + \epsilon^{abcd}\,\psi_{ab}\sigma_c\bar\psi_d \\ &\quad - 3\mathrm{i}\,\mathcal{D}_a B^a + \psi_a\sigma^{ab}\psi_b\bar M - \bar\psi_a\bar\sigma^{ab}\bar\psi_b M + 2(\psi_b\sigma^a\bar\psi_a - \psi_a\sigma^a\bar\psi_b)B^b\ . \end{aligned} \tag{5.93}$$

Note that the sum of the terms in the second line is imaginary,[19] so they drop out of (5.88). The final result reads

$$e^{-1}\mathcal{L}_{\text{sugra}} = -\tfrac12\mathcal{R}(\omega) + \varepsilon^{\mu\nu\rho\sigma}\big(D_\mu\psi_\nu\sigma_\rho\bar\psi_\sigma + \psi_\sigma\sigma_\rho D_\mu\bar\psi_\nu\big) - 3M\bar M + 3B^aB_a\ . \tag{5.94}$$

The tensor fields M and B_a evidently are auxiliary fields; in the absence of matter couplings or other extensions they vanish on-shell. The action then coincides with what we had postulated in section 4.3. There we had to explicitly verify its supersymmetry, whereas here it is realized by construction.

There is an even more obvious chiral scalar than M that can be used to construct an invariant action: an arbitrary constant $\lambda \in \mathbb{C}$, which is trivially chiral. Nevertheless, it does give rise to a non-trivial Lagrangian:

$$\mathcal{L}_\lambda = e\Delta_F\lambda + \text{c.c.} = -e\lambda\,(3\bar M + 2\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu) + \text{c.c.}\ . \tag{5.95}$$

[19] The imaginary part of $\Delta_F M$ must also give an invariant, of course. However, it turns out to be a total derivative only, which is trivially supersymmetric.

When added to the pure supergravtiy action, it leads to a cosmological constant and gravitino mass terms (which we will discuss in the next chapter):

$$
\begin{aligned}
e^{-1}(\mathcal{L}_{\text{sugra}} + \mathcal{L}_\lambda) = &-\tfrac{1}{2}\mathcal{R} + \varepsilon^{\mu\nu\rho\sigma}\big(D_\mu\psi_\nu\sigma_\rho\bar\psi_\sigma + \psi_\sigma\sigma_\rho D_\mu\bar\psi_\nu\big) \\
&- 2\bar\lambda\,\psi_\mu\sigma^{\mu\nu}\psi_\nu - 2\lambda\,\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu + 3|\lambda|^2 \\
&- 3|M+\lambda|^2 + 3B^aB_a\ .
\end{aligned}
\tag{5.96}
$$

The last line vanishes after elimination of the auxiliary fields. We observe that the cosmological constant $\Lambda = -3|\lambda|^2$ is negative; in the absence of matter this gives anti-de Sitter space as the maximally symmetric vacuum solution, instead of Minkowski space ($\lambda = 0$). The new terms in the action require an extra piece in the gravitino transformation as compared to the on-shell expression,

$$
\delta_Q(\epsilon)\psi_\mu = D_\mu\epsilon + \tfrac{\mathrm{i}}{2}\lambda\,\sigma_\mu\bar\epsilon\ , \tag{5.97}
$$

which follows from substituting $M = -\lambda$ and $B_a = 0$ in (5.64).

6 Matter Couplings

6.1 Kähler Geometry

Recall the general formula (2.24) for globally supersymmetric actions of chiral multiplets. In its evaluation in terms of component fields we restricted ourselves to such Kähler and superpotentials that yield renormalizable actions only. Since gravity itself is not renormalizable, there is no good reason anymore to impose such constraints when it comes to coupling chiral multiplets to supergravity. Before we turn to this issue, let us examine the consequences of allowing for arbitrary real $K(\phi,\bar\phi)$ and holomorphic $W(\phi)$ in rigid supersymmetry. Eq. (2.27) is already valid for general W, but relaxing the constraint on K results in a number of extra terms. For instance, since there are four spinor derivatives acting on K, we find four-fermi terms not present in renormalizable models (the coupling constants in front of such terms have mass dimension -2). A somewhat lengthy but straightforward calculation yields the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{2}\,D^2\big[-\tfrac{1}{4}\,\bar D^2K(\phi,\bar\phi) + W(\phi)\big] + \text{c.c.} \\
&\simeq -K_{i\bar\jmath}\,\partial^\mu\phi^i\,\partial_\mu\bar\phi^{\bar\jmath} + K_{i\bar\jmath}\,F^i\bar F^{\bar\jmath} + F^i\,W_i + \bar F^{\bar\imath}\,\bar W_{\bar\imath} \\
&\quad - \tfrac{\mathrm{i}}{2}K_{i\bar\jmath}\,\chi^i\sigma^\mu\hat\nabla_\mu\bar\chi^{\bar\jmath} + \tfrac{\mathrm{i}}{2}K_{i\bar\jmath}\,\hat\nabla_\mu\chi^i\sigma^\mu\bar\chi^{\bar\jmath} - \tfrac{1}{2}W_{ij}\,\chi^i\chi^j - \tfrac{1}{2}\bar W_{\bar\imath\bar\jmath}\,\bar\chi^{\bar\imath}\bar\chi^{\bar\jmath} \\
&\quad - \tfrac{1}{2}K_{i\bar\jmath}\,\Gamma_{k\ell}{}^i\chi^k\chi^\ell\,\bar F^{\bar\jmath} - \tfrac{1}{2}K_{i\bar\jmath}\,\Gamma_{\bar k\bar\ell}{}^{\bar\jmath}\,\bar\chi^{\bar k}\bar\chi^{\bar\ell}\,F^i + \tfrac{1}{4}K_{ij\bar k\bar\ell}\,\chi^i\chi^j\,\bar\chi^{\bar k}\bar\chi^{\bar\ell}\ ,
\end{aligned}
\tag{6.1}
$$

where the total derivative on the right-hand side of the identity

$$
\tfrac{1}{2}K_{\bar\imath}\,\partial^2\bar\phi^{\bar\imath} + \tfrac{1}{2}K_{\bar\imath\bar\jmath}\,\partial^\mu\bar\phi^{\bar\imath}\,\partial_\mu\bar\phi^{\bar\jmath} + \text{c.c.} = -K_{i\bar\jmath}\,\partial^\mu\phi^i\,\partial_\mu\bar\phi^{\bar\jmath} + \tfrac{1}{2}\partial^2K
$$

was dropped. Here, as for W, subscripts on K denote differentiations with respect to ϕ^i and $\bar{\phi}^{\bar{\jmath}}$:

$$K_{i_1\ldots i_r\,\bar{\jmath}_1\ldots\bar{\jmath}_s} = \frac{\partial^{r+s}K}{\partial\phi^{i_1}\ldots\partial\phi^{i_r}\,\partial\bar{\phi}^{\bar{\jmath}_1}\ldots\partial\bar{\phi}^{\bar{\jmath}_s}}\ , \tag{6.2}$$

and we write for the third derivative of K

$$\Gamma_{ij}{}^{k} = K_{ij\bar{\ell}}\,K^{k\bar{\ell}}\ , \tag{6.3}$$

where $K^{i\bar{\jmath}}$ denotes the inverse of $K_{i\bar{\jmath}}$. Only such K for which $K_{i\bar{\jmath}}$ is positive-definite and thus invertible give well-defined kinetic terms for the ϕ^i and χ^i. Using $\Gamma_{ij}{}^{k}$ we have introduced a covariant (in what sense will be explained below) derivative acting on the fermions:

$$\hat{\nabla}_\mu\chi^i = \partial_\mu\chi^i + \partial_\mu\phi^j\,\Gamma_{jk}{}^{i}\chi^k\ . \tag{6.4}$$

Note that even for $W = 0$ we have an interacting theory, if K contains terms that are at least trilinear in the fields. Different Kähler potentials do not necessarily give rise to different models; since the Lagrangian depends on K only through $K_{i\bar{\jmath}}$ and its derivatives, it is invariant under so-called Kähler transformations

$$K(\phi,\bar{\phi}) \to K(\phi,\bar{\phi}) + f(\phi) + \bar{f}(\bar{\phi}) \tag{6.5}$$

with arbitrary holomorphic functions f. Clearly, under such transformations $K_{i\bar{\jmath}} \to K_{i\bar{\jmath}}$. The various quantities in the action have an interpretation in complex geometry [19]. However, the full structure becomes visible only after elimination of the auxiliary fields. Their equations of motion are solved by

$$F^i \approx -K^{i\bar{\jmath}}\,\bar{W}_{\bar{\jmath}} + \tfrac{1}{2}\Gamma_{k\ell}{}^{i}\chi^k\chi^\ell\ , \tag{6.6}$$

After insertion of F^i into the Lagrangian we obtain

$$\begin{aligned}\mathcal{L} = &-g_{i\bar{\jmath}}\,\partial^\mu\phi^i\,\partial_\mu\bar{\phi}^{\bar{\jmath}} - g^{i\bar{\jmath}}\,W_i\,\bar{W}_{\bar{\jmath}} - \tfrac{i}{2}g_{i\bar{\jmath}}\,\chi^i\sigma^\mu\hat{\nabla}_\mu\bar{\chi}^{\bar{\jmath}} + \tfrac{i}{2}g_{i\bar{\jmath}}\,\hat{\nabla}_\mu\chi^i\sigma^\mu\bar{\chi}^{\bar{\jmath}}\\ &- \tfrac{1}{2}\nabla_iW_j\,\chi^i\chi^j - \tfrac{1}{2}\nabla_{\bar{\imath}}\bar{W}_{\bar{\jmath}}\,\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}} - \tfrac{1}{4}R_{i\bar{k}j\bar{\ell}}\,\chi^i\chi^j\,\bar{\chi}^{\bar{k}}\bar{\chi}^{\bar{\ell}}\ .\end{aligned} \tag{6.7}$$

Some more notation has been introduced here; we write $g_{i\bar{\jmath}} = K_{i\bar{\jmath}}$ in anticipation of its geometrical interpretation, the covariant derivative of the gradient W_j is defined as $(\partial_i = \partial/\partial\phi^i)$

$$\nabla_iW_j = \partial_iW_j - \Gamma_{ij}{}^{k}W_k\ , \tag{6.8}$$

and the four-fermi tensor is given by

$$R_{i\bar{k}j\bar{\ell}} = -K_{ij\bar{k}\bar{\ell}} + g_{m\bar{n}}\,\Gamma_{ij}{}^{m}\,\Gamma_{\bar{k}\bar{\ell}}{}^{\bar{n}}\ . \tag{6.9}$$

What we have obtained is the supersymmetric extension of a nonlinear sigma model (NLSM). In general, NLSMs are scalar field theories of the form

$$\mathcal{L}_\sigma = -\tfrac{1}{2}\,g_{IJ}(X)\,\partial^\mu X^I\,\partial_\mu X^J\ , \tag{6.10}$$

where g_{IJ} is a positive-definite symmetric matrix. $\mathcal{L}_\sigma$ is invariant under reparametrizations

$$X^I \to X'^I(X) \ , \qquad g_{IJ}(X) \to g'_{IJ}(X') = \frac{\partial X^K}{\partial X'^I}\frac{\partial X^L}{\partial X'^J}\, g_{KL}(X) \ . \tag{6.11}$$

The scalar fields $X^I(x)$ can be regarded as local coordinates on some internal manifold $\mathcal{M}$, mapping spacetime into a coordinate patch over $\mathcal{M}$,

$$X^I\colon\, \mathbb{R}^{1,D-1} \to \mathcal{M} \to \mathbb{R}^m \ , \tag{6.12}$$

with $m = \delta^I_I$ the number of scalars.[20] Its transformation law identifies g_{IJ} as a tensor; as such it provides a metric on $\mathcal{M}$.

For chiral multiplets the internal manifold is complex (in particular, its real dimension $2n$ is even) and of special type, namely it is Kähler. General complex manifolds with coordinates z^i, $\bar{z}^{\bar{\jmath}}$ have metrics of the form

$$ds^2 = g_{ij}(z,\bar{z})\, dz^i\, dz^j + 2g_{i\bar{\jmath}}(z,\bar{z})\, dz^i\, d\bar{z}^{\bar{\jmath}} + g_{\bar{\imath}\bar{\jmath}}(z,\bar{z})\, d\bar{z}^{\bar{\imath}}\, d\bar{z}^{\bar{\jmath}} \tag{6.13}$$

with $g_{i\bar{\jmath}} = (g_{j\bar{\imath}})^*$ and $g_{ij} = g_{ji} = (g_{\bar{\imath}\bar{\jmath}})^*$. They are called Hermitian if $g_{ij} = g_{\bar{\imath}\bar{\jmath}} = 0$. Under holomorphic coordinate transformations

$$z^i \to z'^i(z) \ , \qquad \bar{z}^{\bar{\imath}} \to \bar{z}'^{\bar{\imath}}(\bar{z}) \ , \tag{6.14}$$

$g_{i\bar{\jmath}}$ transforms as

$$g'_{i\bar{\jmath}}(z',\bar{z}') = \frac{\partial z^k}{\partial z'^i}\frac{\partial \bar{z}^{\bar{\ell}}}{\partial \bar{z}'^{\bar{\jmath}}}\, g_{k\bar{\ell}}(z,\bar{z}) \ , \tag{6.15}$$

while hermiticity is preserved. A complex manifold with Hermitian metric is Kähler if $g_{i\bar{\jmath}}$ satisfies the conditions

$$\partial_i\, g_{j\bar{k}} - \partial_j\, g_{i\bar{k}} = 0 \ , \qquad \partial_{\bar{\imath}}\, g_{k\bar{\jmath}} - \partial_{\bar{\jmath}}\, g_{k\bar{\imath}} = 0 \ . \tag{6.16}$$

Locally, these can be solved in terms of a Kähler potential [20]:

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}}\, K(z,\bar{z}) \ . \tag{6.17}$$

In general, K does not transform as a scalar under (6.14); for $\partial_i\partial_{\bar{\jmath}}K$ to be a tensor it is sufficient if K and K' are related by a Kähler transformation (6.5),

$$K'(z',\bar{z}') = K(z,\bar{z}) + f(z) + \bar{f}(\bar{z}) \ . \tag{6.18}$$

As for spacetime manifolds, we can introduce a Levi-Civita connection (3.18). If the manifold is Kähler, its components are very simple:

$$\Gamma_{ij}{}^k = \tfrac{1}{2}\, g^{k\bar{\ell}}\big(\partial_i\, g_{j\bar{\ell}} + \partial_j\, g_{i\bar{\ell}}\big) = \partial_i\, g_{j\bar{\ell}}\, g^{k\bar{\ell}}$$

[20] Note that n chiral multiplets contain $m = 2n$ real scalars.

$$\Gamma_{i\bar{\jmath}}{}^{k} = \Gamma_{\bar{\jmath}i}{}^{k} = \tfrac{1}{2}\, g^{k\bar{\ell}}\big(\partial_{\bar{\jmath}}\, g_{i\bar{\ell}} - \partial_{\bar{\ell}}\, g_{i\bar{\jmath}}\big) = 0$$
$$\Gamma_{ij}{}^{\bar{k}} = \tfrac{1}{2}\, g^{\ell\bar{k}}\big(0\big)_{ij\ell} = 0\ , \qquad (6.19)$$

and analogously for the complex conjugate expressions. $\Gamma_{ij}{}^{k}$ has the correct inhomogeneous transformation law (cf. the infinitesimal version (3.7))

$$\Gamma'_{ij}{}^{k}(z',\bar{z}') = \frac{\partial^2 z^\ell}{\partial z'^i \partial z'^j}\,\frac{\partial z'^k}{\partial z^\ell} + \frac{\partial z^m}{\partial z'^i}\,\frac{\partial z^n}{\partial z'^j}\,\frac{\partial z'^k}{\partial z^\ell}\,\Gamma_{mn}{}^{\ell}(z,\bar{z}) \qquad (6.20)$$

to render the derivative

$$\nabla_i = \partial_i - \Gamma_{ij}{}^{k}\Delta_k{}^{j} \qquad (6.21)$$

covariant under holomorphic coordinate transformations (6.14). Here, the $\Delta_k{}^j$ act just like the $\mathrm{GL}(D,\mathbb{R})$ generators in section 3.1 on holomorphic indices, while leaving anti-holomorphic indices invariant; for example

$$\nabla_i V_j = \partial_i V_j - \Gamma_{ij}{}^{k} V_k\ , \qquad \nabla_i V_{\bar{\jmath}} = \partial_i V_{\bar{\jmath}}\ . \qquad (6.22)$$

Likewise, the complex conjugate generators $\Delta_{\bar{k}}{}^{\bar{\ell}}$, and thus $\nabla_{\bar{\imath}}$, act only on anti-holomorphic indices.
The curvature tensors are defined as usual through the commutators

$$[\,\nabla_i\,,\nabla_j\,] = -R_{ijk}{}^{\ell}\Delta_\ell{}^{k}\ , \qquad [\,\nabla_i\,,\nabla_{\bar{\jmath}}\,] = -R_{i\bar{\jmath}k}{}^{\ell}\Delta_\ell{}^{k} - R_{i\bar{\jmath}\bar{k}}{}^{\bar{\ell}}\Delta_{\bar{\ell}}{}^{\bar{k}}\ , \qquad (6.23)$$

with $[\,\nabla_{\bar{\imath}}\,,\nabla_{\bar{\jmath}}\,]$ following from complex conjugation of the first one. There is no curvature $R_{ij\bar{k}}{}^{\bar{\ell}}$ in the first equation because the ∇_i contain no generators $\Delta_{\bar{\ell}}{}^{\bar{k}}$, which therefore cannot appear on the right-hand side. With the connection expressed in terms of the metric, we find that only the curvatures in the mixed commutator are non-vanishing:

$$R_{ijk\bar{\ell}} = R_{ijk}{}^{m} g_{m\bar{\ell}} = 0$$
$$R_{i\bar{\jmath}k\bar{\ell}} = R_{i\bar{\jmath}k}{}^{m} g_{m\bar{\ell}} = -\partial_{\bar{\jmath}}\Gamma_{ik}{}^{m} g_{m\bar{\ell}} = -\partial_i\partial_{\bar{\jmath}}\, g_{k\bar{\ell}} + g_{m\bar{n}}\,\Gamma_{ik}{}^{m}\,\Gamma_{\bar{\jmath}\bar{\ell}}{}^{\bar{n}}\ . \qquad (6.24)$$

From the above discussion we conclude that target space manifolds of chiral multiplets are Kähler, as a consequence of the scalar kinetic terms deriving from a Kähler potential. The other terms in the Lagrangian (6.7) have a geometrical meaning as well: Under a holomorphic reparametrization of the scalars the fermions transform as

$$\chi'^{\,i}_{\alpha} = D_\alpha \phi'^i = \frac{\partial \phi'^i}{\partial \phi^j}\,\chi^j_\alpha\ , \qquad (6.25)$$

which identifies them as components of vector fields on the Kähler manifold. On functions of the scalars, the operator $\hat{\nabla}_\mu$ in (6.4) is just the pull-back of the target space covariant derivatives ∇_i and $\nabla_{\bar{\imath}}$ to spacetime,

$$\hat{\nabla}_\mu = \partial_\mu - \partial_\mu\phi^i\,\Gamma_{ij}{}^{k}\Delta_k{}^{j} - \partial_\mu\bar{\phi}^{\bar{\imath}}\,\Gamma_{\bar{\imath}\bar{\jmath}}{}^{\bar{k}}\Delta_{\bar{k}}{}^{\bar{\jmath}} = \partial_\mu\phi^i\,\nabla_i + \partial_\mu\bar{\phi}^{\bar{\imath}}\,\nabla_{\bar{\imath}}\ . \qquad (6.26)$$

Using (6.20), it is then easily verified that $\hat{\nabla}_\mu \chi^i$ indeed transforms as a tensor:

$$(\hat{\nabla}_\mu \chi^i_\alpha)' = \frac{\partial \phi'^i}{\partial \phi^j} \hat{\nabla}_\mu \chi^j_\alpha \ . \tag{6.27}$$

$W_i = \partial_i W$ are the components of a covector, which makes the covariant derivative $\nabla_i W_j$ appearing in the action well-defined. On the other hand, the auxiliary field F^i does not transform covariantly:

$$F'^i = -\tfrac{1}{2} D^2 \phi'^i = \frac{\partial \phi'^i}{\partial \phi^j} F^j - \frac{1}{2} \frac{\partial^2 \phi'^i}{\partial \phi^j \, \partial \phi^k} \chi^j \chi^k \ . \tag{6.28}$$

The inhomogeneous piece bilinear in the fermions can be canceled by substracting a term $\frac{1}{2}\Gamma_{jk}{}^i \chi^j \chi^k$, which explains its appearance in the on-shell expression (6.6). After elimination of F^i, the action corresponding to (6.7) is invariant under the covariant supersymmetry transformations

$$\delta_Q(\epsilon)\phi^i = \epsilon \chi^i \ , \qquad \delta_Q(\epsilon)\chi^i = \mathrm{i}\partial_\mu \phi^i \sigma^\mu \bar{\epsilon} - \delta_Q(\epsilon)\phi^j \, \Gamma_{jk}{}^i \chi^k - g^{i\bar{\jmath}} \, \bar{W}_{\bar{\jmath}} \, \epsilon \ . \tag{6.29}$$

6.1.1 Example: The $\mathbb{C}\mathrm{P}^n$ Model

As an example for nonlinear sigma models with Kähler manifolds as target spaces, let us consider the complex projective spaces $\mathbb{C}\mathrm{P}^n$. We can parametrize them in terms of $n+1$ homogeneous coordinates $(w^a) \in \mathbb{C}^{n+1} - \{\vec{0}\}$, where we identify two points if they are related by a non-vanishing complex scale factor: $(w^a) \simeq (\lambda w^a)$, $\lambda \in \mathbb{C}^*$. Equivalently, we can consider the w^a as constrained by the equation

$$\sum_{a=1}^{n+1} |w^a|^2 = 1 \ , \tag{6.30}$$

and mod out a phase, $(w^a) \simeq (\mathrm{e}^{\mathrm{i}\varphi} w^a)$, $\varphi \in \mathbb{R}$. The latter representation implies that

$$\mathbb{C}\mathrm{P}^n \cong \frac{S^{2n+1}}{U(1)} \cong \frac{\mathrm{U}(n+1)}{\mathrm{U}(n) \times \mathrm{U}(1)} \ , \tag{6.31}$$

which shows that $\mathbb{C}\mathrm{P}^n$ is a compact space.

The w^a are not proper coordinates, since they are defined only modulo scaling. Let us cover $\mathbb{C}\mathrm{P}^n$ with patches U_a in which $w^a \neq 0$. In each patch we can introduce n inhomogeneous coordinates

$$U_a : \quad z^i_{(a)} = \frac{w^i}{w^a} \ , \quad i = 1, .., \hat{a}, .., n+1 \ . \tag{6.32}$$

In the overlap of two patches, these coordinates are related by a holomorphic transformation:

$$U_a \cap U_b : \quad z^i_{(a)} = \frac{w^i}{w^a} = \frac{w^i}{w^b} \frac{w^b}{w^a} = z^i_{(b)} / z^a_{(b)} \ . \tag{6.33}$$

$\mathbb{C}\mathrm{P}^n$ is a Kähler manifold for each n. A metric can be obtained from the (locally defined) Kähler potential

$$U_a: \quad K_{(a)} = \mu^2 \ln\Big(\sum_c \Big|\frac{w^c}{w^a}\Big|^2\Big) = \mu^2 \ln\big(1 + z^i \bar{z}_i\big) \ , \quad \mu \in \mathbb{R} \ , \tag{6.34}$$

where we now drop the patch labels of the z^i and use the notation $\bar{z}_i = \delta_{i\bar{\jmath}}\, \bar{z}^{\bar{\jmath}}$. We find

$$g_{(a)i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K_{(a)} = \frac{\mu^2}{1 + z^k \bar{z}_k} \Big(\delta_{i\bar{\jmath}} - \frac{\bar{z}_i\, z_{\bar{\jmath}}}{1 + z^\ell \bar{z}_\ell}\Big) \ . \tag{6.35}$$

Using the Schwarz inequality, it is easy to show that this expression is positive-definite; it is known as the Fubini-Study metric. In order to verify that $g_{i\bar{\jmath}}$ is indeed a tensor, let us check that the Kähler potential transforms as in (6.18) under a holomorphic reparametrization. In the overlap of two patches we have

$$\begin{aligned} U_a \cap U_b: \quad K_{(a)} &= \mu^2 \ln\Big(\sum_c \Big|\frac{w^c}{w^a}\Big|^2\Big) = \mu^2 \ln\Big(\Big|\frac{w^b}{w^a}\Big|^2 \sum_c \Big|\frac{w^c}{w^b}\Big|^2\Big) \\ &= \mu^2 \ln\Big(\sum_c \Big|\frac{w^c}{w^b}\Big|^2\Big) + \mu^2 \ln\Big(\frac{w^b}{w^a}\Big) + \mu^2 \ln\Big(\frac{\bar{w}^b}{\bar{w}^a}\Big) \\ &= K_{(b)} + f_{(ab)} + \bar{f}_{(ab)} \ , \end{aligned} \tag{6.36}$$

where $f_{(ab)} = \mu^2 \ln\big(z^b_{(a)}\big) = -\mu^2 \ln\big(z^a_{(b)}\big)$ is indeed a holomorphic function.

6.2 Chiral Multiplets – Part 3

We now turn to the coupling of chiral multiplets to supergravity. As in rigid supersymmetry, the input is a real Kähler potential $K(\phi, \bar{\phi})$ and a holomorphic superpotential $W(\phi)$. From K we construct a composite chiral scalar using the chiral projector, and then we apply $e\Delta_F$ (5.86) to obtain an invariant action. The general Lagrangian is given by

$$\mathcal{L} = e\, \Delta_F \Big[\, \frac{3}{4}\big(\bar{\mathcal{D}}^2 + 2M\big) \exp\big\{-K(\phi, \bar{\phi})/3\big\} + W(\phi)\Big] + \text{c.c.} \ . \tag{6.37}$$

It will turn out that, up to a numerical factor, the Kähler potential is given by the logarithm of the term on which the chiral projector acts. $\mathcal{L}$ contains the pure supergravity Lagrangian, as can be seen by expanding the exponential,

$$\mathcal{L} = e\, \Delta_F \Big[\, \frac{1}{2}\,(3 - K)M - \frac{1}{4}\,\bar{\mathcal{D}}^2 K + W(\phi) + O(K^2)\Big] + \text{c.c.} \ . \tag{6.38}$$

If we reinstate the gravitational coupling constant κ, the $O(K^2)$ terms vanish in the limit $\kappa \to 0$ and we arrive at the Lagrangian for rigid supersymmetry. Since M comes multiplied with a field-dependent factor, we will find the same factor appearing in front of the supergravity Lagrangian. This will require a field-dependent rescaling of the vierbein,

accompanied by rescalings and shifts of the fermions, in order to restore the canonical normalizations of the kinetic terms.
Let us now evaluate the above Lagrangian in terms of component fields. The contributions from the superpotential are easy to compute:

$$\Delta_F W(\phi) = (F^i - \mathrm{i}\chi^i\sigma^\mu\bar{\psi}_\mu)W_i - \tfrac{1}{2}\chi^i\chi^j W_{ij} - (3\bar{M} + 2\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu)W \ . \tag{6.39}$$

The action of the chiral projector on the exponential is given by

$$\tfrac{3}{4}(\bar{\mathcal{D}}^2 + 2M)\,\mathrm{e}^{-K/3} = \tfrac{1}{2}\,\mathrm{e}^{-K/3}\big[3M + K_{\bar{\imath}}\bar{F}^{\bar{\imath}} - \tfrac{1}{2}(K_{\bar{\imath}\bar{\jmath}} - \tfrac{1}{3}K_{\bar{\imath}}K_{\bar{\jmath}})\,\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}\big] \ .$$

Next we apply Δ_F to this expression. If we pull it past $\mathrm{e}^{-K/3}$, we pick up a commutator

$$\begin{aligned}
[\,\Delta_F\,,\mathrm{e}^{-K/3}\,] &= -\tfrac{1}{2}[\,\mathcal{D}^2\,,\mathrm{e}^{-K/3}\,] + \mathrm{i}\bar{\psi}_\mu\bar{\sigma}^\mu[\,\mathcal{D}\,,\mathrm{e}^{-K/3}\,] \\
&= -\tfrac{1}{3}\,\mathrm{e}^{-K/3}\big[K_i(F^i - \mathrm{i}\chi^i\sigma^\mu\bar{\psi}_\mu) - \tfrac{1}{2}(K_{ij} - \tfrac{1}{3}K_iK_j)\,\chi^i\chi^j - K_i\,\chi^i\mathcal{D}\big] \ ,
\end{aligned}$$

which is still operator-valued. It follows that

$$\begin{aligned}
\tfrac{3}{4}\Delta_F(\bar{\mathcal{D}}^2 + 2M)\,\mathrm{e}^{-K/3} = {} & \tfrac{3}{2}\,\mathrm{e}^{-K/3}\Delta_F M + \tfrac{1}{2}\,\mathrm{e}^{-K/3}\Delta_F\big[K_{\bar{\imath}}\bar{F}^{\bar{\imath}} - \tfrac{1}{2}(K_{\bar{\imath}\bar{\jmath}} - \tfrac{1}{3}K_{\bar{\imath}}K_{\bar{\jmath}})\,\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}\big] \\
& + \tfrac{1}{6}\,\mathrm{e}^{-K/3}K_i\,\chi^i(\mathrm{i}\sigma^\mu\bar{\psi}_\mu + \mathcal{D})\big[K_{\bar{\imath}}\bar{F}^{\bar{\imath}} - \tfrac{1}{2}(K_{\bar{\imath}\bar{\jmath}} - \tfrac{1}{3}K_{\bar{\imath}}K_{\bar{\jmath}})\,\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}\big] \\
& - \tfrac{1}{6}\,\mathrm{e}^{-K/3}\big|K_iF^i - \tfrac{1}{2}(K_{ij} - \tfrac{1}{3}K_iK_j)\,\chi^i\chi^j\big|^2 \\
& - \tfrac{1}{2}\,\mathrm{e}^{-K/3}M\big[K_iF^i - \tfrac{1}{2}(K_{ij} - \tfrac{1}{3}K_iK_j)\,\chi^i\chi^j\big] \\
& + \tfrac{1}{2}\,\mathrm{e}^{-K/3}K_i\,\chi^i(\mathrm{i}\sigma^\mu\bar{\psi}_\mu + \mathcal{D})M \ .
\end{aligned} \tag{6.40}$$

In the further derivation of the Lagrangian we concentrate on the purely bosonic part, where all essential steps can be demonstrated without too much effort. The above expression then reduces to

$$\begin{aligned}
\tfrac{3}{4}\Delta_F(\bar{\mathcal{D}}^2 + 2M)\,\mathrm{e}^{-K/3} &= \tfrac{1}{2}\,\mathrm{e}^{-K/3}\big[3\Delta_F M - \tfrac{1}{2}\,\mathcal{D}^2K_{\bar{\imath}}\,\bar{F}^{\bar{\imath}} - \tfrac{1}{2}K_{\bar{\imath}}\,\mathcal{D}^2\bar{F}^{\bar{\imath}} - 3\bar{M}K_{\bar{\jmath}}\bar{F}^{\bar{\jmath}} \\
&\qquad + \tfrac{1}{2}(K_{\bar{\imath}\bar{\jmath}} - \tfrac{1}{3}K_{\bar{\imath}}K_{\bar{\jmath}})\,\mathcal{D}^\alpha\bar{\chi}^{\dot{\alpha}\bar{\imath}}\,\mathcal{D}_\alpha\bar{\chi}^{\bar{\jmath}}_{\dot{\alpha}} - K_iF^i(M + \tfrac{1}{3}K_{\bar{\jmath}}\bar{F}^{\bar{\jmath}}) \\
&\qquad + \ldots\big] \\
&= \tfrac{1}{2}\,\mathrm{e}^{-K/3}\big[-\tfrac{1}{2}\mathcal{R} + 3B^aB_a - 3M\bar{M} - 3\mathrm{i}\,D_aB^a + K_{i\bar{\jmath}}F^i\bar{F}^{\bar{\jmath}} \\
&\qquad - \tfrac{1}{2}K_{\bar{\imath}}(\mathrm{i}\,\mathcal{D}^\alpha\mathcal{D}_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}\bar{\imath}} + \tfrac{1}{2}B_{\alpha\dot{\alpha}}\mathcal{D}^\alpha\bar{\chi}^{\dot{\alpha}\bar{\imath}}) - 3\bar{M}K_{\bar{\jmath}}\bar{F}^{\bar{\jmath}} \\
&\qquad + (K_{\bar{\imath}\bar{\jmath}} - \tfrac{1}{3}K_{\bar{\imath}}K_{\bar{\jmath}})\,\partial^a\bar{\phi}^{\bar{\imath}}\,\partial_a\bar{\phi}^{\bar{\jmath}} - K_iF^i(M + \tfrac{1}{3}K_{\bar{\jmath}}\bar{F}^{\bar{\jmath}}) \\
&\qquad + \ldots\big] \ ,
\end{aligned}$$

where (5.93) has been used and the ellipses stand for terms containing fermions. Using the commutator algebra on $\bar{\chi}$,

$$\mathcal{D}^\alpha\mathcal{D}_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}\bar{\imath}} = \mathcal{D}_{\alpha\dot{\alpha}}\mathcal{D}^\alpha\bar{\chi}^{\dot{\alpha}\bar{\imath}} + [\,\mathcal{D}^\alpha\,,\mathcal{D}_{\alpha\dot{\alpha}}\,]\bar{\chi}^{\dot{\alpha}\bar{\imath}}$$

$$= -\mathrm{i}\,\mathcal{D}_{\alpha\dot\alpha}\mathcal{D}^{\dot\alpha\alpha}\bar\phi^{\bar\imath} + \bar\sigma^{a\dot\alpha\alpha}\big(T_{a\alpha}{}^{\beta}\,\mathcal{D}_\beta\bar\chi^{\bar\imath}_{\dot\alpha} - T_{a\alpha}{}^{\dot\beta}\,\bar{\mathcal{D}}_{\dot\beta}\bar\chi^{\bar\imath}_{\dot\alpha} + R_{a\alpha\dot\alpha\dot\beta}\,\bar\chi^{\dot\beta\bar\imath}\big)$$
$$= 2\mathrm{i}\,D^a\partial_a\bar\phi^{\bar\imath} - 5B^a\,\partial_a\bar\phi^{\bar\imath} + 4\mathrm{i}\bar M\bar F^{\bar\imath} + \dots\ ,$$

we find

$$\begin{aligned}\tfrac{3}{4}\Delta_F(\bar{\mathcal{D}}^2 + 2M)\,\mathrm{e}^{-K/3} = \tfrac{1}{2}\,\mathrm{e}^{-K/3}\big[&-\tfrac{1}{2}\mathcal{R} - 3|M + \tfrac{1}{3}K_{\bar\imath}\bar F^{\bar\imath}|^2 + K_{i\bar\jmath}F^i\bar F^{\bar\jmath} - 3\mathrm{i}\,D_aB^a \\ &- K_{i\bar\jmath}\,\partial^a\phi^i\,\partial_a\bar\phi^{\bar\jmath} - \tfrac{1}{3}K_{\bar\imath}K_{\bar\jmath}\,\partial^a\bar\phi^{\bar\imath}\,\partial_a\bar\phi^{\bar\jmath} + D^a(K_{\bar\imath}\partial_a\bar\phi^{\bar\imath}) \\ &+ 3B^aB_a + 2\mathrm{i}\,B^aK_{\bar\imath}\partial_a\bar\phi^{\bar\imath} + \dots\big]\ . \end{aligned} \tag{6.41}$$

It is convenient to introduce the shifted auxiliary field

$$\hat M = M + \tfrac{1}{3}K_{\bar\jmath}\bar F^{\bar\jmath}\ . \tag{6.42}$$

When combined with the superpotential contributions, the bosonic Lagrangian then reads

$$\begin{aligned}\mathcal{L} = e\,\mathrm{e}^{-K/3}\big[&-\tfrac{1}{2}\mathcal{R} - K_{i\bar\jmath}\,\partial^\mu\phi^i\,\partial_\mu\bar\phi^{\bar\jmath} + K_{i\bar\jmath}F^i\bar F^{\bar\jmath} - 3|\hat M|^2 + 3B^aB_a \\ &+ 2B^a\,\mathrm{Im}(K_i\partial_a\phi^i) - \tfrac{1}{6}K_iK_j\,\partial^\mu\phi^i\,\partial_\mu\phi^j - \tfrac{1}{6}K_{\bar\imath}K_{\bar\jmath}\,\partial^\mu\bar\phi^{\bar\imath}\,\partial_\mu\bar\phi^{\bar\jmath} \\ &+ \tfrac{1}{2}D^a\partial_aK + \mathrm{e}^{K/3}F^i(W_i + K_iW) + \mathrm{e}^{K/3}\bar F^{\bar\imath}(\bar W_{\bar\imath} + K_{\bar\imath}\bar W) \\ &- 3\,\mathrm{e}^{K/3}\hat{\bar M}W - 3\,\mathrm{e}^{K/3}\hat M\bar W + \dots\big]\ . \end{aligned} \tag{6.43}$$

We can now eliminate the auxiliary fields. The solutions to their algebraic equations of motion are given by

$$\begin{aligned}\hat M \approx -\mathrm{e}^{K/3}W + \dots\ , \qquad & B_a \approx -\tfrac{1}{3}\,\mathrm{Im}(K_i\partial_a\phi^i) + \dots \\ K_{i\bar\jmath}\bar F^{\bar\jmath} \approx -\mathrm{e}^{K/3}&(W_i + K_iW) + \dots\ , \end{aligned} \tag{6.44}$$

with the fermionic contributions again suppressed. A further simplification arises from integrating by parts the covariant derivative acting on ∂_aK. It is

$$\begin{aligned}\tfrac{1}{2}e\,\mathrm{e}^{-K/3}E^{a\mu}D_\mu(E_a{}^\nu\partial_\nu K) &= \tfrac{1}{2}D_\mu(e\,\mathrm{e}^{-K/3}\partial^\mu K) - \tfrac{1}{2}D_\mu(e\,\mathrm{e}^{-K/3}E^{a\mu})E_a{}^\nu\partial_\nu K \\ &\simeq \tfrac{1}{6}e\,\mathrm{e}^{-K/3}\partial^\mu K\,\partial_\mu K - \tfrac{1}{2}\,\mathrm{e}^{-K/3}D_\mu(eE_a{}^\mu)\,\partial^aK \\ &= \tfrac{1}{6}e\,\mathrm{e}^{-K/3}\partial^\mu K\,\partial_\mu K + \dots\ , \end{aligned}$$

where a total derivative was dropped and in the last step the torsion relation (4.43) was used. Substituting the auxiliary fields, we obtain the following bosonic Lagrangian:

$$\begin{aligned}\mathcal{L} = -e\,\mathrm{e}^{-K/3}\big[\, &\tfrac{1}{2}\mathcal{R} + (K_{i\bar\jmath} - \tfrac{1}{3}K_iK_{\bar\jmath})\,\partial^\mu\phi^i\,\partial_\mu\bar\phi^{\bar\jmath} + \tfrac{1}{3}\,\mathrm{Im}(K_i\partial_\mu\phi^i)^2 \\ &+ \mathrm{e}^{2K/3}(K^{i\bar\jmath}D_iW D_{\bar\jmath}\bar W - 3|W|^2) + \dots\big]\ . \end{aligned} \tag{6.45}$$

Here we have introduced the notation

$$D_iW = (\partial_i + K_i)W\ . \tag{6.46}$$

As anticipated, the kinetic terms of the graviton and the scalars are not canonically normalized; the overall factor $\mathrm{e}^{-K/3}$ acts as a field-dependent gravitational coupling "constant". This situation can be rectified by an appropriate Weyl rescaling of the vierbein. If we make the substitution

$$e_\mu{}^a \to \mathrm{e}^{K/6}\, e_\mu{}^a \quad \Rightarrow \quad e \to \mathrm{e}^{2K/3}\, e\ , \quad g^{\mu\nu} \to e^{-K/3} g^{\mu\nu}\ , \tag{6.47}$$

the Einstein-Hilbert term transforms as[21]

$$-\tfrac{1}{2} e\, \mathrm{e}^{-K/3} \mathcal{R} \to -\tfrac{1}{2} e\,(\mathcal{R} + \tfrac{1}{6}\partial^\mu K\, \partial_\mu K) - \tfrac{1}{2}\,\partial_\mu (e\, \partial^\mu K)\ . \tag{6.48}$$

Dropping the total derivative, this results in the final Lagrangian

$$e^{-1}\mathcal{L} = -\tfrac{1}{2}\mathcal{R} - g_{i\bar{\jmath}}\, \partial^\mu \phi^i\, \partial_\mu \bar{\phi}^{\bar{\jmath}} - V(\phi, \bar{\phi}) + \text{fermions}\ , \tag{6.49}$$

where the scalar potential is given by

$$V(\phi, \bar{\phi}) = \mathrm{e}^K \left(g^{i\bar{\jmath}} D_i W D_{\bar{\jmath}} \bar{W} - 3|W|^2 \right)\ . \tag{6.50}$$

The first term in V we recognize from rigid supersymmetry; it is positive-definite. The second term, which derives from elimination of the auxiliary field M (and is proportional to $\kappa^2 = 1$), is entirely new. It generalizes the cosmological constant in section 5.4 and makes a negative contribution to the potential. In supergravity, positive energy is not a necessary condition anymore for a ground state to break supersymmetry spontaneously. We have to inspect the supersymmetry transformations of the fermions to decide about the situation. We shall find that the converse is still true, i.e., a positive vacuum energy implies that supersymmetry is broken.

Another difference to the rigid case is the appearance of K and K_i in the potential, which unlike $g^{i\bar{\jmath}}$ are not invariant under Kähler transformations (6.5). However, if we assign to W the transformation law

$$W(\phi) \to \mathrm{e}^{-f(\phi)}\, W(\phi)\ , \tag{6.51}$$

which preserves the holomorphicity of W, then $D_i W$ transforms covariantly, $D_i W \to \mathrm{e}^{-f} D_i W$, and invariance of the potential is restored. In fact, for non-vanishing superpotential, V depends on K and W only through the invariant combination

$$G = K + \ln |W|^2\ ; \tag{6.52}$$

since G differs from K only by a Kähler transformation with $f = \ln W$, the Kähler metric is given by $g_{i\bar{\jmath}} = G_{i\bar{\jmath}}$, and we can write

$$V = \mathrm{e}^G \left(G_i\, G^{i\bar{\jmath}}\, G_{\bar{\jmath}} - 3 \right)\ . \tag{6.53}$$

[21] The required power of e^K can easily been found by restricting oneself to constant K first. Since the Levi-Civita connection is of the form $g^{-1}\partial g$, it does not scale. Accordingly, neither the curvature nor the Ricci tensor changes. The curvature scalar then scales like the inverse metric that is used to form the trace of the Ricci tensor.

Let us now complete the Lagrangian by including the fermions. The above Weyl rescaling must also be performed on the gravitino and matter fermions, in order to restore the canonical normalizations of their kinetic terms. Moreover, a shift of ψ_μ is required to decouple it from the χ^i. After the substitution

$$\psi_\mu \to \mathrm{e}^{K/12}\big(\psi_\mu + \tfrac{\mathrm{i}}{6} K_{\bar\imath}\sigma_\mu\bar\chi^{\bar\imath}\big) , \qquad \chi^i \to \mathrm{e}^{-K/12}\,\chi^i , \tag{6.54}$$

we arrive at the Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L} = &-\tfrac12\mathcal{R} - g_{i\bar\jmath}\,\partial^\mu\phi^i\,\partial_\mu\bar\phi^{\bar\jmath} + 2\,\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\bar\sigma_\nu\tilde D_\rho\psi_\sigma - \mathrm{i}g_{i\bar\jmath}\,\bar\chi^{\bar\jmath}\bar\sigma^\mu\tilde\nabla_\mu\chi^i \\ &- V(\phi,\bar\phi) - g_{i\bar\jmath}\,\partial_\nu\phi^i\,\bar\chi^{\bar\jmath}\bar\sigma^\mu\sigma^\nu\bar\psi_\mu - g_{i\bar\jmath}\,\partial_\nu\bar\phi^{\bar\jmath}\,\chi^i\sigma^\mu\bar\sigma^\nu\psi_\mu \\ &- \tfrac12 g_{i\bar\jmath}\,\chi^i\sigma_\mu\bar\chi^{\bar\jmath}\,(\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\psi_\nu\sigma_\rho\bar\psi_\sigma - \psi_\nu\sigma^\mu\bar\psi^\nu) - \tfrac14(R_{i\bar\jmath k\bar\ell} + \tfrac12 g_{i\bar\jmath}\,g_{k\bar\ell})\,\chi^i\chi^k\,\bar\chi^{\bar\jmath}\bar\chi^{\bar\ell} \\ &- \mathrm{e}^{K/2}\big[2\bar W\,\psi_\mu\sigma^{\mu\nu}\psi_\nu + 2W\,\bar\psi_\mu\bar\sigma^{\mu\nu}\bar\psi_\nu + \mathrm{i}D_iW\,\chi^i\sigma^\mu\bar\psi_\mu + \mathrm{i}D_{\bar\imath}\bar W\,\bar\chi^{\bar\imath}\bar\sigma^\mu\psi_\mu \\ &\qquad + \tfrac12\tilde\nabla_i D_j W\,\chi^i\chi^j + \tfrac12\tilde\nabla_{\bar\imath}D_{\bar\jmath}\bar W\,\bar\chi^{\bar\imath}\bar\chi^{\bar\jmath}\big] . \end{aligned} \tag{6.55}$$

The generalized second derivative of W,

$$\tilde\nabla_i D_j W = D_i D_j W - \Gamma_{ij}{}^k D_k W , \tag{6.56}$$

transforms covariantly under both holomorphic reparametrizations of the scalars ϕ^i and Kähler transformations (6.5), (6.51).
The derivatives in the kinetic terms of the fermions read

$$\tilde D_\mu\psi_\nu = (D_\mu + \tfrac{\mathrm{i}}{2}a_\mu)\,\psi_\nu , \qquad \tilde\nabla_\mu\chi^i = (D_\mu - \tfrac{\mathrm{i}}{2}a_\mu)\,\chi^i - \partial_\mu\phi^j\,\Gamma_{jk}{}^i\chi^k , \tag{6.57}$$

where we denote

$$a_\mu = \mathrm{Im}(K_i\partial_\mu\phi^i) . \tag{6.58}$$

Under a Kähler transformation, a_μ behaves just like an abelian gauge field:

$$a_\mu \to a_\mu + \partial_\mu\,\mathrm{Im}\,f(\phi) . \tag{6.59}$$

If the Lagrangian is supposed to be invariant, we need to compensate this by a chiral U(1) rotation of the fermions,

$$\psi_\mu \to \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}f/2}\,\psi_\mu , \qquad \chi^i \to \mathrm{e}^{\mathrm{i}\,\mathrm{Im}f/2}\,\chi^i . \tag{6.60}$$

Then $\tilde D_\mu\psi_\nu$ and $\tilde\nabla_\mu\chi^i$ transform covariantly and the phases are canceled by the transformation of the complex conjugate fields in the kinetic terms.
Finally, elimination of the auxiliary fields gives rise to the following supersymmetry variations:

$$\begin{aligned} \delta_Q(\epsilon)\phi^i &= \epsilon\chi^i \\ \delta_Q(\epsilon)e_\mu{}^a &= \mathrm{i}(\epsilon\sigma^a\bar\psi_\mu - \psi_\mu\sigma^a\bar\epsilon) \end{aligned}$$

$$\begin{aligned}\delta_Q(\epsilon)\psi_\mu &= (D_\mu + \tfrac{\mathrm{i}}{2}a_\mu)\epsilon - \tfrac{\mathrm{i}}{4}g_{i\bar{\jmath}}\,\chi^i\sigma^\nu\bar{\chi}^{\bar{\jmath}}\,\sigma_{\mu\nu}\epsilon + \tfrac{\mathrm{i}}{2}\mathrm{e}^{K/2}\,W\sigma_\mu\bar{\epsilon}\\ &\quad - \tfrac{\mathrm{i}}{2}\,\mathrm{Im}\big(K_i\delta_Q(\epsilon)\phi^i\big)\psi_\mu\\ \delta_Q(\epsilon)\chi^i &= \mathrm{i}\sigma^\mu\bar{\epsilon}\,(\partial_\mu\phi^i - \psi_\mu\chi^i) - \delta_Q(\epsilon)\phi^j\,\Gamma_{jk}{}^i\chi^k - \mathrm{e}^{K/2}\,g^{i\bar{\jmath}}D_{\bar{\jmath}}\bar{W}\epsilon\\ &\quad + \tfrac{\mathrm{i}}{2}\,\mathrm{Im}\big(K_j\delta_Q(\epsilon)\phi^j\big)\chi^i\ . \end{aligned} \tag{6.61}$$

We observe that they are invariant under holomorphic reparametrizations $\phi^i \longrightarrow \phi'^i(\phi)$ and Kähler transformations (6.5), (6.51), (6.60) accompanied by $\epsilon \longrightarrow \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}f/2}\epsilon$. The latter invariance holds for the fermion variations thanks to the presence of the $\mathrm{Im}\big(K_i\delta_Q(\epsilon)\phi^i\big)$ terms, which compensate for the supersymmetry variation of the phase factors in (6.60). Note that in general Kähler transformations do not generate a symmetry of the action. Rather, the invariance found above tells us that two models whose input K and W differs only by a Kähler transformation are physically indistinguishable, since we can redefine the fields such that the action remains unchanged if we substitute K and W. The physically relevant input is the Kähler-invariant function G defined in (6.52). For $W \neq 0$ it is possible to express the whole action in terms of G. Since it is related to K through a Kähler transformation with $f = \ln W$, we can simply take the above Lagrangian and substitute

$$K \longrightarrow G\ , \qquad W \longrightarrow 1\ , \qquad D_iW \longrightarrow G_i\ , \qquad \tilde{\nabla}_iD_jW \longrightarrow (\nabla_i + G_i)G_j\ . \tag{6.62}$$

We then obtain

$$\begin{aligned} e^{-1}\mathcal{L} = &-\tfrac{1}{2}\mathcal{R} - G_{i\bar{\jmath}}\,\partial^\mu\phi^i\,\partial_\mu\bar{\phi}^{\bar{\jmath}} + 2\,\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\tilde{D}_\rho\psi_\sigma - \mathrm{i}G_{i\bar{\jmath}}\,\bar{\chi}^{\bar{\jmath}}\bar{\sigma}^\mu\tilde{\nabla}_\mu\chi^i\\ &- V(\phi,\bar{\phi}) - G_{i\bar{\jmath}}\,\partial_\nu\phi^i\,\bar{\chi}^{\bar{\jmath}}\bar{\sigma}^\mu\sigma^\nu\bar{\psi}_\mu - G_{i\bar{\jmath}}\,\partial_\nu\bar{\phi}^{\bar{\jmath}}\,\chi^i\sigma^\mu\bar{\sigma}^\nu\psi_\mu\\ &- \tfrac{1}{2}G_{i\bar{\jmath}}\,\chi^i\sigma_\mu\bar{\chi}^{\bar{\jmath}}\,(\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\psi_\nu\sigma_\rho\bar{\psi}_\sigma - \psi_\nu\sigma^\mu\bar{\psi}^\nu) - \tfrac{1}{4}(R_{i\bar{\jmath}k\bar{\ell}} + \tfrac{1}{2}G_{i\bar{\jmath}}\,G_{k\bar{\ell}})\,\chi^i\chi^k\,\bar{\chi}^{\bar{\jmath}}\bar{\chi}^{\bar{\ell}}\\ &- \mathrm{e}^{G/2}\big[\,2\psi_\mu\sigma^{\mu\nu}\psi_\nu + 2\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu + \mathrm{i}G_i\,\chi^i\sigma^\mu\bar{\psi}_\mu + \mathrm{i}G_{\bar{\imath}}\,\bar{\chi}^{\bar{\imath}}\bar{\sigma}^\mu\psi_\mu\\ &\qquad + \tfrac{1}{2}(\nabla_iG_j + G_iG_j)\,\chi^i\chi^j + \tfrac{1}{2}(\nabla_{\bar{\imath}}G_{\bar{\jmath}} + G_{\bar{\imath}}G_{\bar{\jmath}})\,\bar{\chi}^{\bar{\imath}}\bar{\chi}^{\bar{\jmath}}\big]\ , \end{aligned} \tag{6.63}$$

where the potential is given by (6.53).

Let us now study some properties of these theories. First of all, from the transformations (6.61) we can read off the criterion for spontaneous supersymmetry breakdown. As in rigid supersymmetry, a non-vanishing vacuum expectation value (meaning, when evaluated for a scalar field configuration that minimizes the potential) of an auxiliary scalar F^i implies that the corresponding fermion χ^i transforms like a goldstino by a shift: $\delta_Q(\epsilon)\chi^i = \langle F^i\rangle\epsilon + \ldots$. Using the solution (6.44) for F^i, taking into account the Weyl rescalings (6.54), and making the substitutions (6.62), we find

$$\langle F^i\rangle = -\langle \mathrm{e}^{G/2}G^{i\bar{\jmath}}G_{\bar{\jmath}}\rangle\ . \tag{6.64}$$

Since neither the scalar metric nor the exponential (except at points where $W = 0$) can vanish, we conclude that the criterion for spontaneously broken supersymmetry is given by a non-vanishing *vev*

$$\langle G_i \rangle \neq 0 \ . \tag{6.65}$$

From the scalar potential (6.53) we infer that for $W \neq 0$, unbroken supersymmetry implies a negative cosmological constant $V_{\text{min}} = -3\langle \mathrm{e}^G \rangle$. Flat space on the other hand, which corresponds to $V_{\text{min}} = 0$, will necessarily break supersymmetry if $W \neq 0$. This is in fact a desired feature of supergravity; after all, supersymmetry is not observed at low energies and therefore must be broken. Nowadays, however, we know from astronomical observations that we live in an expanding universe governed by a tiny positive cosmological constant. It is an unsolved problem how to realize such a stable vacuum in supergravity (or in string theory, for that matter). Nevertheless, let us derive the conditions for a vacuum with vanishing cosmological constant. Using the notation $G^i = G^{i\bar{\jmath}} G_{\bar{\jmath}}$ and

$$\partial_i V = \mathrm{e}^G \big[G^j (G_{ij} - \Gamma_{ij}{}^k G_k) + G_i (G_j G^j - 2) \big] \ , \tag{6.66}$$

we find that we need

$$\langle G_i G^i \rangle = 3 \ , \qquad \langle G^j \nabla_i G_j + G_i \rangle = 0 \ . \tag{6.67}$$

The above discussion shows that, unlike in rigid supersymmetry, a non-vanishing V_{min} does not signal spontaneous symmetry breakdown. Rather, it is the *vev* $\langle F^i \rangle$ that serves as the order parameter in supergravity. Carrying mass dimension 2, it (more precisely, a suitable linear combination) sets the supersymmetry breaking scale M_s^2.

We now turn to the fermionic mass terms. They are given by the fermion bilinears in (6.63) with the scalar prefactors evaluated at the minimum of the potential:

$$e^{-1} \mathcal{L} = -\langle \mathrm{e}^{G/2} \rangle \big[2 \psi_\mu \sigma^{\mu\nu} \psi_\nu + \mathrm{i} \langle G_i \rangle \, \chi^i \sigma^\mu \bar{\psi}_\mu + \tfrac{1}{2} \langle \nabla_i G_j + G_i G_j \rangle \, \chi^i \chi^j + \text{c.c.} \big] + \ldots \ . \tag{6.68}$$

We can directly read off the gravitino mass[22]

$$m_{3/2} = \langle \mathrm{e}^{G/2} \rangle \ , \tag{6.69}$$

where the right-hand side is proportional to the reduced Planck mass $\kappa^{-1} = M_{\text{P}} / \sqrt{8\pi}$ that equals 1 in our units. For $V_{\text{min}} = 0$, we obtain from (6.64) and (6.67) the Deser-Zumino mass scale relation [21]

$$m_{3/2} = \sqrt{\frac{8\pi}{3}} \frac{M_s^2}{M_{\text{P}}} \approx 2.37 \times 10^{-19} M_s^2 / \text{GeV} \ . \tag{6.70}$$

It is surprising at first that $m_{3/2} \neq 0$ even for unbroken supersymmetry, while the graviton remains massless. However, in that case the (maximally symmetric) background is an anti-de Sitter space, in which the concept of mass is different from flat Minkowski space; it turns out that the gravitino still describes only two physical degrees of freedom.

[22] As mentioned in the introduction, canonical normalization of the gravitino terms requires a rescaling $\psi_\mu \to \psi_\mu / \sqrt{2}$.

When supersymmetry is broken, $\langle G_i \rangle \neq 0$, we observe a mixing of the gravitino and matter fermion mass terms. This is similar to the situation in Yang-Mills-Higgs theory with broken gauge symmetry. For vanishing cosmological constant, the (would-be) goldstino $\varrho = \frac{1}{3}\langle G_i \rangle \chi^i$ transforms under supersymmetry by a shift,

$$\delta_Q(\epsilon)\varrho = -\tfrac{1}{3}\,\mathrm{e}^{G/2}\langle G_i \rangle G^i \epsilon + \ldots = -m_{3/2}\,\epsilon + \ldots \ , \tag{6.71}$$

and hence can be gauged away. Indeed, choosing the supersymmetry parameter $\epsilon = \varrho/m_{3/2}$, we have for the fermions

$$\delta_Q(\epsilon)\varrho = -\varrho + \ldots \ , \qquad \delta_Q(\epsilon)\psi_\mu = \frac{1}{m_{3/2}}\,\partial_\mu \varrho + \frac{\mathrm{i}}{2}\sigma_\mu \bar{\varrho} + \ldots \ . \tag{6.72}$$

In this unitary gauge, the goldstino gets "eaten" by the gravitino, which thereby acquires two additional degrees of freedom. This is the super-Higgs effect. Alternatively, we can just redefine the gravitino in order to diagonalize the mass matrix. The replacement

$$\psi_\mu = \hat{\psi}_\mu - \frac{1}{m_{3/2}}\,\partial_\mu \varrho - \frac{\mathrm{i}}{2}\sigma_\mu \bar{\varrho} \tag{6.73}$$

yields

$$2\psi_\mu \sigma^{\mu\nu}\psi_\nu + 3\mathrm{i}\bar{\varrho}\,\bar{\sigma}^\mu \psi_\mu = 2\hat{\psi}_\mu \sigma^{\mu\nu}\hat{\psi}_\nu - 3\bar{\varrho}\bar{\varrho} + \ldots \ , \tag{6.74}$$

where we suppress the derivative terms on the right-hand side. This results in the mass terms

$$e^{-1}\mathcal{L} = -m_{3/2}\big[\,2\hat{\psi}_\mu \sigma^{\mu\nu}\hat{\psi}_\nu + \tfrac{1}{2}\langle \nabla_i G_j + \tfrac{1}{3}G_i G_j \rangle\,\chi^i \chi^j + \text{c.c.}\big] + \ldots \ . \tag{6.75}$$

If we make a decomposition

$$\chi^i = \langle G^i \rangle \varrho + \hat{\chi}^i \tag{6.76}$$

with $\langle G_i \rangle \hat{\chi}^i = 0$, then according to the conditions (6.67) the matter fermion mass matrix

$$M_{ij} = m_{3/2}\,\langle \nabla_i G_j + \tfrac{1}{3}G_i G_j \rangle \tag{6.77}$$

projects to the subspace spanned by the $\hat{\chi}^i$,

$$M_{ij}\,\chi^j = M_{ij}\,\hat{\chi}^j \ . \tag{6.78}$$

For an overview of various explicit models of supergravity, we refer to the literature, in particular [4, 5].

6.3 Hodge Manifolds

Above we found that theories of chiral matter coupled to supergravity are Kähler invariant. This invariance can also be understood in the following way: It is necessary for the theory to be globally well-defined on topologically non-trivial target space manifolds. On such spaces, the Lagrangian is defined only locally in each coordinate patch U_a (cf.

the $\mathbb{CP}^n$ model in the previous section). The latter are being glued together by relating the Lagrangians in overlaps $U_a \cap U_b \neq \emptyset$ through a holomorphic coordinate transformation $\phi_{(a)} = \phi_{(a)}(\phi_{(b)})$. The Kähler potentials in the two patches may differ by a Kähler transformation

$$K_{(a)} - K_{(b)} = f_{(ab)} + \bar{f}_{(ab)} \tag{6.79}$$

with holomorphic functions $f_{(ab)} = -f_{(ba)}$, accompanied by transformations of the superpotential and fermions (where we have restored the dimensionful coupling κ),

$$W_{(a)} = \mathrm{e}^{-\kappa^2 f_{(ab)}} W_{(b)}$$
$$\psi_{\mu(a)} = \mathrm{e}^{-\mathrm{i}\kappa^2 \, \mathrm{Im} f_{(ab)}/2} \, \psi_{\mu(b)} \ , \qquad \chi^i_{(a)} = \mathrm{e}^{\mathrm{i}\kappa^2 \, \mathrm{Im} f_{(ab)}/2} \, \frac{\partial \phi^i_{(a)}}{\partial \phi^j_{(b)}} \, \chi^j_{(b)} \ . \tag{6.80}$$

In rigid supersymmetry, W and χ^i are inert under Kähler transformations (ψ_μ doesn't appear). In supergravity, however, we see that they are not ordinary functions, but rather sections of certain bundles, with transition functions given in terms of the $f_{(ab)}$. In the case of W it is a holomorphic line bundle $\mathcal{L}$, while ψ_μ and χ^i are sections of $\mathcal{L}^{1/2}$ and $\mathcal{L}^{-1/2} \otimes T\mathcal{M}$, respectively. As we now explain, these nontrivial Kähler transformations impose restrictions on the possible manifolds that can be chosen as target spaces [22]: First of all, we observe that the curvature of the U(1) connection a_μ in (6.58) is given by

$$\mathrm{d}a = \mathrm{i} g_{i\bar{\jmath}} \mathrm{d}\phi^i \wedge \mathrm{d}\bar{\phi}^{\bar{\jmath}} \ . \tag{6.81}$$

In mathematical terms, this expresses the fact that there must exist a holomorphic line bundle $\mathcal{L}$ whose first Chern class (represented by the left-hand side of the equation) equals the Kähler class (represented by the right-hand side). Such Kähler manifolds are said to be of restricted type or Hodge manifolds. Moreover, for the fermions the bundles $\mathcal{L}^{1/2}$ and $\mathcal{L}^{-1/2}$ must exist. In particular, we have to make sure that the above transformations are consistent on triple overlap regions $U_a \cap U_b \cap U_c \neq \emptyset$. Note first that (6.79) implies the identity

$$f_{(ab)} + f_{(bc)} + f_{(ca)} = -\bar{f}_{(ab)} - \bar{f}_{(bc)} - \bar{f}_{(ca)} \ , \tag{6.82}$$

thus

$$f_{(ab)} + f_{(bc)} + f_{(ca)} = 2\pi \mathrm{i} \, C_{(abc)} \tag{6.83}$$

with the totally antisymmetric $C_{(abc)}$ being real constants. The functions $f_{(ab)}$ are not uniquely defined, but only modulo shifts

$$f_{(ab)} \longrightarrow f_{(ab)} + 2\pi \mathrm{i} \, C_{(ab)} \tag{6.84}$$

with real constants $C_{(ab)} = -C_{(ba)}$ that leave (6.79) unchanged. It follows that we can redefine

$$C_{(abc)} \longrightarrow C_{(abc)} + C_{(ab)} + C_{(bc)} + C_{(ca)} \ . \tag{6.85}$$

Now consider the gravitino transformation. On triple overlap regions we must require

$$\psi_{\mu(a)} \stackrel{!}{=} \mathrm{e}^{-\mathrm{i}\kappa^2 \,\mathrm{Im}(f_{(ab)}+f_{(bc)}+f_{(ca)})/2}\, \psi_{\mu(a)} = \mathrm{e}^{-\mathrm{i}\pi\kappa^2 C_{(abc)}}\, \psi_{\mu(a)} \ . \tag{6.86}$$

We conclude that, with an appropriate choice of the $C_{(ab)}$, the $C_{(abc)}$ must be even integer multiples of κ^{-2}:

$$\kappa^2 C_{(abc)} \in 2\mathbb{Z} \ . \tag{6.87}$$

This also guarantees the consistency of the transformations of W and χ^i. Only such Kähler manifolds are admissible in supergravity for which transition functions can be found that satisfy (6.87).

Consider for example $\mathbb{C}\mathrm{P}^n$, for which we found the transition functions in (6.36),

$$f_{(ab)} = \mu^2 \ln(w^b/w^a) + 2\pi\mathrm{i}\, C_{(ab)} \ . \tag{6.88}$$

Choosing $C_{(ab)} = 0$, we have in triple overlap regions

$$f_{(ab)} + f_{(bc)} + f_{(ca)} = \mu^2 \ln\Big(\frac{w^b}{w^a}\,\frac{w^c}{w^b}\,\frac{w^a}{w^c}\Big) = \mu^2 \ln 1 = 0 \ , \tag{6.89}$$

hence condition (6.87) appears to be satisfied. Note, however, that the logarithm is a multi-valued function on each patch U_a. For the fermion transition functions in (6.80) to be single-valued requires that $(\kappa\mu)^2$ be an even integer. With $\kappa^2 = 8\pi G_N$, this amounts to a quantization of Newton's constant in units of the parameter μ^2.

A Sigma Matrices

Minkowski metric: $(\eta_{ab}) = \mathrm{diag}(-1,1,1,1)$.

$$(\varepsilon^{\alpha\beta}) = -(\varepsilon_{\alpha\beta}) = (\varepsilon^{\dot\alpha\dot\beta}) = -(\varepsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.1}$$

$$\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma \ , \qquad \varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\beta\dot\gamma} = \delta^{\dot\alpha}_{\dot\gamma} \ , \qquad \varepsilon^{\dot\alpha\dot\beta} = (\varepsilon^{\alpha\beta})^* \tag{A.2}$$

$$\sigma^a = (-\mathbb{1}, \tau^i) \ , \quad \bar\sigma^a = (-\mathbb{1}, -\tau^i) \ , \qquad \bar\sigma^{a\,\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^a_{\beta\dot\beta} \tag{A.3}$$

$$(\sigma^a_{\alpha\dot\beta})^* = \sigma^a_{\beta\dot\alpha} \ , \qquad (\bar\sigma^{a\,\dot\alpha\beta})^* = \bar\sigma^{a\,\dot\beta\alpha} \tag{A.4}$$

$$\sigma^{ab} = \tfrac{1}{2}\sigma^{[a}\bar\sigma^{b]} \ , \qquad \bar\sigma^{ab} = \tfrac{1}{2}\bar\sigma^{[a}\sigma^{b]} \ , \qquad (\sigma^{ab}{}_\alpha{}^\beta)^* = -\bar\sigma^{ab\,\dot\beta}{}_{\dot\alpha} \tag{A.5}$$

$$\sigma^{0i} = -\bar\sigma^{0i} = \tfrac{1}{2}\tau^i \ , \qquad \sigma^{ij} = \bar\sigma^{ij} = -\tfrac{\mathrm{i}}{2}\varepsilon^{ijk}\tau^k \ , \qquad \mathrm{tr}\,\sigma^{ab} = \mathrm{tr}\,\bar\sigma^{ab} = 0 \tag{A.6}$$

$$\sigma^{ab}{}_{\alpha\beta} = \varepsilon_{\beta\gamma}\,\sigma^{ab}{}_\alpha{}^\gamma = \sigma^{ab}{}_{\beta\alpha} \ , \qquad \bar\sigma^{ab}{}_{\dot\alpha\dot\beta} = \varepsilon_{\dot\alpha\dot\gamma}\,\bar\sigma^{ab\,\dot\gamma}{}_{\dot\beta} = \bar\sigma^{ab}{}_{\dot\beta\dot\alpha} \ . \tag{A.7}$$

Identities containing two σ-matrices ($\epsilon^{0123} = 1$):

$$\sigma^a_{\alpha\dot\alpha}\sigma_{a\,\beta\dot\beta} = -2\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta} \ , \qquad \sigma^a_{\alpha\dot\alpha}\,\bar\sigma_a^{\dot\beta\beta} = -2\,\delta^\beta_\alpha\,\delta^{\dot\beta}_{\dot\alpha} \tag{A.8}$$

$$(\sigma^a\bar\sigma^b)_\alpha{}^\beta = -\eta^{ab}\,\delta_\alpha^\beta + 2\,\sigma^{ab}{}_\alpha{}^\beta\ , \qquad (\bar\sigma^a\sigma^b)^{\dot\alpha}{}_{\dot\beta} = -\eta^{ab}\,\delta^{\dot\alpha}_{\dot\beta} + 2\,\bar\sigma^{ab\,\dot\alpha}{}_{\dot\beta} \tag{A.9}$$

$$\sigma^{[a}_{\alpha\dot\alpha}\sigma^{b]}_{\beta\dot\beta} = \varepsilon_{\alpha\beta}\,\bar\sigma^{ab}{}_{\dot\alpha\dot\beta} - \varepsilon_{\dot\alpha\dot\beta}\,\sigma^{ab}{}_{\alpha\beta} \tag{A.10}$$

$$\tfrac{1}{2}\epsilon^{abcd}\sigma_{cd} = -\mathrm{i}\sigma^{ab}\ , \qquad \tfrac{1}{2}\epsilon^{abcd}\bar\sigma_{cd} = \mathrm{i}\bar\sigma^{ab} \tag{A.11}$$

$$\tfrac{1}{2}\epsilon^{abcd}\sigma_{c\,\alpha\dot\alpha}\sigma_{d\,\beta\dot\beta} = \mathrm{i}(\varepsilon_{\alpha\beta}\,\bar\sigma^{ab}{}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}\,\sigma^{ab}{}_{\alpha\beta})\ . \tag{A.12}$$

Identities containing three σ-matrices:

$$\sigma^{ab}\sigma^c = \eta^{c[a}\sigma^{b]} + \tfrac{\mathrm{i}}{2}\epsilon^{abcd}\sigma_d\ , \qquad \bar\sigma^c\sigma^{ab} = -\eta^{c[a}\bar\sigma^{b]} - \tfrac{\mathrm{i}}{2}\epsilon^{abcd}\bar\sigma_d \tag{A.13}$$

$$\bar\sigma^{ab}\bar\sigma^c = \eta^{c[a}\bar\sigma^{b]} - \tfrac{\mathrm{i}}{2}\epsilon^{abcd}\bar\sigma_d\ , \qquad \sigma^c\bar\sigma^{ab} = -\eta^{c[a}\sigma^{b]} + \tfrac{\mathrm{i}}{2}\epsilon^{abcd}\sigma_d \tag{A.14}$$

$$\sigma^{ab}{}_{\alpha\beta}\,\sigma_{b\,\gamma\dot\alpha} = \varepsilon_{\gamma(\beta}\,\sigma^a_{\alpha)\dot\alpha}\ , \qquad \bar\sigma^{ab}{}_{\dot\alpha\dot\beta}\,\sigma_{b\,\alpha\dot\gamma} = \sigma^a_{\alpha(\dot\alpha}\,\varepsilon_{\dot\beta)\dot\gamma}\ . \tag{A.15}$$

Identities containing four σ-matrices:

$$\begin{aligned}\sigma^{ab}\sigma^{cd} &= -\tfrac{1}{2}(\eta^{ad}\sigma^{bc} - \eta^{ac}\sigma^{bd} + \eta^{bc}\sigma^{ad} - \eta^{bd}\sigma^{ac})\\ &\quad + \tfrac{1}{4}(\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd} - \mathrm{i}\epsilon^{abcd})\mathbb{1}\end{aligned} \tag{A.16}$$

$$\begin{aligned}\bar\sigma^{ab}\bar\sigma^{cd} &= -\tfrac{1}{2}(\eta^{ad}\bar\sigma^{bc} - \eta^{ac}\bar\sigma^{bd} + \eta^{bc}\bar\sigma^{ad} - \eta^{bd}\bar\sigma^{ac})\\ &\quad + \tfrac{1}{4}(\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd} + \mathrm{i}\epsilon^{abcd})\mathbb{1}\end{aligned} \tag{A.17}$$

$$\sigma^{ab}{}_\alpha{}^\beta\,\sigma_{ab\,\gamma}{}^\delta = \delta_\alpha^\beta\,\delta_\gamma^\delta - 2\,\delta_\alpha^\delta\,\delta_\gamma^\beta\ , \qquad \sigma^{ab}{}_\alpha{}^\beta\bar\sigma_{ab}{}^{\dot\gamma}{}_{\dot\delta} = 0 \tag{A.18}$$

$$\sigma^{ac}{}_\alpha{}^\beta\,\sigma_c{}^b{}_\gamma{}^\delta = -\tfrac{1}{2}(\delta_\gamma^\beta\sigma^{ab}{}_\alpha{}^\delta - \delta_\alpha^\delta\sigma^{ab}{}_\gamma{}^\beta) + \tfrac{1}{4}\eta^{ab}\,(\varepsilon_{\alpha\gamma}\,\varepsilon^{\beta\delta} + \delta_\alpha^\delta\,\delta_\gamma^\beta)\ . \tag{A.19}$$

B Derivation of Action Formula

In this section we are looking for an extension of eF, where F is the auxiliary scalar of a chiral multiplet, that transforms into a total derivative under local supersymmetry. We have

$$\begin{aligned}\delta_Q(\epsilon)(eF) &= e\,\delta_Q(\epsilon)F + eE_a{}^\mu\delta_Q(\epsilon)e_\mu{}^a\,F\\ &= \mathrm{i}e\,\bar\epsilon\bar\sigma^\mu(D_\mu\chi - \mathrm{i}\sigma^\nu\bar\psi_\mu\,\mathcal{D}_\nu\phi) + e\bar M\epsilon\chi + \tfrac{1}{2}e\,\chi\sigma^a\bar\epsilon\,B_a - \mathrm{i}e\,\bar\psi_\mu\bar\sigma^\mu\epsilon F\ ,\end{aligned}$$

where the ψ_μ term in the vierbein transformation has canceled the $\psi_\mu F$ term in the supercovariant derivative of χ. The remaining F can be only be canceled by the variation of χ, so we add a term $\mathrm{i}e\,\bar\psi_\mu\bar\sigma^\mu\chi$, which transforms into

$$\begin{aligned}\delta_Q(\epsilon)(\mathrm{i}e\,\bar\psi_\mu\bar\sigma^\mu\chi) &= \mathrm{i}e\,\bar\psi_\mu\sigma^\mu\epsilon F - e\,\bar\psi_\mu\bar\sigma^\mu\sigma^\nu\bar\epsilon\mathcal{D}_\nu\phi + 2e\bar M\epsilon\chi + \mathrm{i}e\,D_\mu\bar\epsilon\,\bar\sigma^\mu\chi\\ &\quad - \tfrac{1}{2}e\,\chi\sigma^a\bar\epsilon\,B_a + \mathrm{i}\delta_Q(\epsilon)(eE_a{}^\mu)\,\bar\psi_\mu\bar\sigma^a\chi\ .\end{aligned}$$

This has the added benefit of canceling the B_a term. The transformation of $eE_a{}^\mu$ is given by

$$\delta_Q(\epsilon)(eE_a{}^\mu) = 2e\,E_{[a}{}^\mu E_{b]}{}^\nu\,\delta_Q(\epsilon)e_\nu{}^b = 2\mathrm{i}e\,E_{[a}{}^\mu(\epsilon\sigma^b\bar\psi_{b]} - \psi_{b]}\sigma^b\bar\epsilon)\ .$$

Next we eliminate the $\bar{M}\chi$ terms by substracting $3e\bar{M}\phi$ from the previous two terms,

$$\delta_Q(\epsilon)(-3e\bar{M}\phi) = -3e\bar{M}\epsilon\chi + e\,(3\mathrm{i}\,\bar{\epsilon}\bar{\psi}_\mu B^\mu + 2\,\bar{\psi}_{\mu\nu}\bar{\sigma}^{\mu\nu}\bar{\epsilon})\phi - 3\mathrm{i}e\,\epsilon\sigma^\mu\bar{\psi}_\mu\bar{M}\phi\ .$$

Let us take stock of what we have found so far:

$$\begin{aligned}\delta_Q(\epsilon)(eF + \mathrm{i}e\,\bar{\psi}_\mu\bar{\sigma}^\mu\chi - 3e\bar{M}\phi) =&\\ = \mathrm{i}e\,(D_\mu\bar{\epsilon}\,\bar{\sigma}^\mu\chi + \bar{\epsilon}\bar{\sigma}^\mu D_\mu\chi) + 4e\,\bar{\psi}_\nu\bar{\sigma}^{\mu\nu}\bar{\epsilon}\,\mathcal{D}_\mu\phi + 4eD_\mu\bar{\psi}_\nu\bar{\sigma}^{\mu\nu}\bar{\epsilon}\,\phi&\\ + 3\mathrm{i}e\,(\bar{\epsilon}\bar{\psi}_\mu B^\mu - \epsilon\sigma^\mu\bar{\psi}_\mu\bar{M})\phi + 2e\,\chi\sigma^\mu\bar{\psi}_{[\mu}\,(\epsilon\sigma^\nu\bar{\psi}_{\nu]} - \psi_{\nu]}\sigma^\nu\bar{\epsilon})\ .&\end{aligned}$$

In the first line on the right-hand side we can see total derivatives emerging. The χ terms can be written as

$$\begin{aligned}\mathrm{i}e\,(D_\mu\bar{\epsilon}\,\bar{\sigma}^\mu\chi + \bar{\epsilon}\bar{\sigma}^\mu D_\mu\chi) &= \mathrm{i}\partial_\mu(e\,\bar{\epsilon}\bar{\sigma}^\mu\chi) - \mathrm{i}D_\mu(eE_a{}^\mu)\,\bar{\epsilon}\bar{\sigma}^a\chi\\ &= \mathrm{i}\partial_\mu(e\,\bar{\epsilon}\bar{\sigma}^\mu\chi) + e\,(\psi_\mu\sigma^\nu\bar{\psi}_\nu - \psi_\nu\sigma^\nu\bar{\psi}_\mu)\,\bar{\epsilon}\bar{\sigma}^\mu\chi\ ,\end{aligned}$$

where we have used the torsion equation (4.43). There are various $\psi_\mu\bar{\psi}_\nu\chi$ terms in the previous equation, in particular there is one hidden in $\bar{\psi}_\nu\mathcal{D}_\mu\phi$. Taken together, they vanish:

$$\begin{aligned}-4\,\bar{\psi}_\nu\bar{\sigma}^{\mu\nu}\bar{\epsilon}\,\psi_\mu\chi - 2\,\chi\sigma^\mu\bar{\psi}_{[\mu}\,\psi_{\nu]}\sigma^\nu\bar{\epsilon} + 2\,\psi_{[\mu}\sigma^\nu\bar{\psi}_{\nu]}\,\bar{\epsilon}\bar{\sigma}^\mu\chi =&\\ = \psi_\mu^\alpha\,\bar{\psi}_\nu^{\dot{\alpha}}\,\bar{\epsilon}^{\dot{\beta}}\chi^\beta\,(4\,\varepsilon_{\alpha\beta}\,\bar{\sigma}^{\mu\nu}{}_{\dot{\alpha}\dot{\beta}} + 2\,\sigma^{[\mu}_{\beta\dot{\beta}}\,\sigma^{\nu]}_{\alpha\dot{\alpha}} - 2\,\sigma^{[\mu}_{\alpha\dot{\beta}}\,\sigma^{\nu]}_{\beta\dot{\alpha}}) = 0\ .&\end{aligned}$$

This results in

$$\begin{aligned}\delta_Q(\epsilon)(eF + \mathrm{i}e\,\bar{\psi}_\mu\bar{\sigma}^\mu\chi - 3e\bar{M}\phi) =&\\ = \mathrm{i}\partial_\mu(e\,\bar{\epsilon}\bar{\sigma}^\mu\chi) - 4e\,\bar{\epsilon}\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\,\partial_\mu\phi - 4e\,\bar{\epsilon}\bar{\sigma}^{\mu\nu}D_\mu\bar{\psi}_\nu\,\phi&\\ + 3\mathrm{i}e\phi\,(\bar{\epsilon}\bar{\psi}_\mu B^\mu - \epsilon\sigma^\mu\bar{\psi}_\mu\bar{M}) + 2e\,\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\,\epsilon\chi\ .&\end{aligned}$$

The second line can be canceled by a final addition,

$$\begin{aligned}\delta_Q(\epsilon)(-2e\,\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\,\phi) = -2e\,\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\,\epsilon\chi - 4eD_\mu\bar{\epsilon}\,\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\,\phi - 3\mathrm{i}e\phi\,(\bar{\epsilon}\bar{\psi}_\mu B^\mu - \epsilon\sigma^\mu\bar{\psi}_\mu\bar{M})&\\ - 2\,\delta_Q(\epsilon)(eE_{[a}{}^\mu E_{b]}{}^\nu)\,\bar{\psi}_\mu\bar{\sigma}^{ab}\bar{\psi}_\nu\,\phi\ ,&\end{aligned}$$

leading to

$$\begin{aligned}\delta_Q(\epsilon)(eF + \mathrm{i}e\,\bar{\psi}_\mu\bar{\sigma}^\mu\chi - 3e\bar{M}\phi - 2e\,\bar{\psi}_\mu\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\,\phi) =&\\ = \partial_\mu(\mathrm{i}e\,\bar{\epsilon}\bar{\sigma}^\mu\chi - 4e\,\bar{\epsilon}\bar{\sigma}^{\mu\nu}\bar{\psi}_\nu\,\phi) + 2\phi\left[2\bar{\epsilon}D_\mu(e\bar{\sigma}^{\mu\nu})\bar{\psi}_\nu - \bar{\psi}_\mu\delta_Q(\epsilon)(e\bar{\sigma}^{\mu\nu})\bar{\psi}_\nu\right]\ .&\end{aligned}$$

It remains to show that the terms in square brackets cancel each other. They both originate from varying the vierbein in $e\bar{\sigma}^{\mu\nu}$. By virtue of the anti-selfduality of $\bar{\sigma}^{\mu\nu}$ we have

$$\delta(e\bar{\sigma}^{\mu\nu}) = -\tfrac{\mathrm{i}}{2}\delta(\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}) = -\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\delta e_\rho{}^a\,\bar{\sigma}_{a\sigma}\ .$$

Using (4.43) again, we now calculate

$$
\begin{aligned}
&\big[2\bar\epsilon D_\mu(e\bar\sigma^{\mu\nu}) - 2\,\bar\psi_\mu\delta_Q(\epsilon)(e\bar\sigma^{\mu\nu})\big]\bar\psi_\nu = \\
&\quad = -\mathrm{i}\epsilon^{\mu\nu\rho\sigma}\big[2\mathrm{i}\,\psi_\mu\sigma^a\bar\psi_\rho\,\bar\epsilon - \mathrm{i}(\epsilon\sigma^a\bar\psi_\rho - \psi_\rho\sigma^a\bar\epsilon)\,\bar\psi_\mu\big]\,\bar\sigma_{a\sigma}\bar\psi_\nu \\
&\quad = \epsilon^{\mu\nu\rho\sigma}\big[\psi^\alpha_\mu\,\bar\epsilon^{\dot\beta}\,\bar\psi^{\dot\alpha}_\rho\,\bar\psi^{\dot\gamma}_\nu\,(2\,\sigma^a_{\alpha\dot\alpha}\,\bar\sigma_{a\sigma\,\dot\beta\dot\gamma} + \sigma^a_{\alpha\dot\beta}\,\bar\sigma_{a\sigma\,\dot\alpha\dot\gamma}) + \epsilon^\alpha\,\bar\psi^{\dot\alpha}_\rho\,\bar\psi^{\dot\beta}_\mu\,\bar\psi^{\dot\gamma}_\nu\,\sigma^a_{\alpha\dot\alpha}\,\bar\sigma_{a\sigma\,\dot\beta\dot\gamma}\big] \\
&\quad = \epsilon^{\mu\nu\rho\sigma}\big(3\,\psi^\alpha_\mu\,\bar\epsilon^{\dot\beta}\,\bar\psi^{\dot\alpha}_\rho\,\bar\psi^{\dot\gamma}_\nu + \epsilon^\alpha\,\bar\psi^{\dot\alpha}_\rho\,\bar\psi^{\dot\beta}_\mu\,\bar\psi^{\dot\gamma}_\nu\big)\,\sigma^a_{\alpha(\dot\alpha}\,\bar\sigma_{a\sigma\,\dot\beta\dot\gamma)} \\
&\quad = 0\ ,
\end{aligned}
$$

thanks to (A.15).

C p-Forms

A new ingredient in higher-dimensional supergravity theories are p-forms (antisymmetric tensors) of rank $p > 1$. Although we do not cover such theories in these lectures, let us work out the physical degrees of freedom of p-forms for completeness.

Consider a $(p+1)$-form field strength subject to the linearized Bianchi identity and massless equation of motion (recall that $\approx$ denotes on-shell equality), respectively,

$$
\partial_{[\nu}F_{\mu_0\ldots\mu_p]} = 0\ , \qquad \partial_{\mu_0}F^{\mu_0\ldots\mu_p} \approx 0\ . \tag{C.1}
$$

Let us examine these equations like we did for the graviton and gravitino, i.e., we look for plane wave solutions with fixed momentum k^μ. The Fourier transform $F(k)$ of the field strength $F(x)$ can again be decomposed into linearly independent polarization tensors built from the basis vectors (3.35). In terms of these vectors, the most general expression for $F(k)$ is given by

$$
\begin{aligned}
F_{\mu_0\ldots\mu_p}(k) &= k_{[\mu_0}\,\varepsilon^{i_1}_{\mu_1}\ldots\varepsilon^{i_p}_{\mu_p]}\,a_{i_1\ldots i_p}(k) + \bar k_{[\mu_0}\,\varepsilon^{i_1}_{\mu_1}\ldots\varepsilon^{i_p}_{\mu_p]}\,b_{i_1\ldots i_p}(k) \\
&\quad + k_{[\mu_0}\,\varepsilon^{i_1}_{\mu_1}\ldots\varepsilon^{i_{p-1}}_{\mu_{p-1}}\,\bar k_{\mu_p]}\,c_{i_1\ldots i_{p-1}}(k) + \varepsilon^{i_0}_{[\mu_0}\ldots\varepsilon^{i_p}_{\mu_p]}\,d_{i_0\ldots i_p}(k)\ ,
\end{aligned} \tag{C.2}
$$

with coefficient functions that are totally antisymmetric in their indices and describe the degrees of freedom. The Bianchi identity now implies

$$
0 = k_{[\nu}\,F_{\mu_0\ldots\mu_p]}(k) = k_{[\nu}\,\bar k_{\mu_0}\,\varepsilon^{i_1}_{\mu_1}\ldots\varepsilon^{i_p}_{\mu_p]}\,b_{i_1\ldots i_p}(k) + k_{[\nu}\,\varepsilon^{i_0}_{\mu_0}\ldots\varepsilon^{i_p}_{\mu_p]}\,d_{i_0\ldots i_p}(k)\ . \tag{C.3}
$$

Since the two polarization tensors appearing here are linearly independent, the coefficients b and d have to vanish separately,

$$
b_{i_1\ldots i_p}(k) = 0\ , \qquad d_{i_0\ldots i_p}(k) = 0\ . \tag{C.4}
$$

This leaves the off-shell degrees of freedom contained in a and c, whose total number is

$$
\mathrm{DOF}_{\mathrm{off}}(p) = \binom{D-2}{p} + \binom{D-2}{p-1} = \binom{D-1}{p}\ . \tag{C.5}
$$

Using $b = d = 0$, the equation of motion imposes in addition the constraint

$$\begin{aligned} 0 &\approx (p+1)\, k^{\mu_0} F_{\mu_0\ldots\mu_p}(k) \\ &= \left[k^2\, \varepsilon^{i_1}_{[\mu_1} \ldots \varepsilon^{i_{p-1}}_{\mu_{p-1}}\, \bar{k}_{\mu_p]} - (k\cdot\bar{k})\, \varepsilon^{i_1}_{[\mu_1} \ldots \varepsilon^{i_{p-1}}_{\mu_{p-1}}\, k_{\mu_p]}\right] c_{i_1\ldots i_{p-1}}(k) \\ &\quad + k^2\, \varepsilon^{i_1}_{[\mu_1} \ldots \varepsilon^{i_p}_{\mu_p]}\, a_{i_1\ldots i_p}(k)\ . \end{aligned} \tag{C.6}$$

Linear independence of the polarization tensors and the fact that $k\cdot\bar{k} \neq 0$ imply that

$$c_{i_1\ldots i_{p-1}}(k) \approx 0\ . \tag{C.7}$$

The remaining on-shell degrees of freedom all reside in a,

$$a_{i_1\ldots i_p}(k) \approx \delta(k^2)\, \hat{a}_{i_1\ldots i_p}(k)\ , \tag{C.8}$$

and describe a massless particle. For $0 < p < D-2$ the degrees of freedom are transversal. Their number is

$$\mathrm{DOF}_{\mathrm{on}}(p) = \binom{D-2}{p}\ . \tag{C.9}$$

Note that on-shell, p-forms and $(D-2-p)$-forms describe the same number of degrees of freedom:

$$\binom{D-2}{p} = \binom{D-2}{D-2-p}\ . \tag{C.10}$$

This duality can be seen in (C.1): One may either regard the first equation as the Bianchi identity, in which case it is solved by a p-form A,

$$\partial_{[\nu} F_{\mu_0\ldots\mu_p]} = 0 \quad \Rightarrow \quad F_{\mu_0\ldots\mu_p} = (p+1)\, \partial_{[\mu_0} A_{\mu_1\ldots\mu_p]}\ , \tag{C.11}$$

or the second one, which can be solved in terms of a $(D-2-p)$-form $\tilde{A}$,

$$\partial_{\mu_1} F^{\mu_1\ldots\mu_{p+1}} = 0 \quad \Rightarrow \quad F^{\mu_1\ldots\mu_{p+1}} = \frac{1}{(D-2-p)!}\, \varepsilon^{\mu_1\ldots\mu_D}\, \partial_{\mu_{p+2}} \tilde{A}_{\mu_{p+3}\ldots\mu_D}\ . \tag{C.12}$$

In each case, the other equation then provides the equation of motion. Note, however, that in general this duality holds only for the free massless fields studied here.
The number of off-shell degrees of freedom can be derived also in another way. The p-form $A_{\mu_1\ldots\mu_p}$ in (C.11) is subject to gauge transformations

$$A_{\mu_1\ldots\mu_p} \rightarrow A_{\mu_1\ldots\mu_p} + \partial_{[\mu_1} \lambda^{(p-1)}_{\mu_2\ldots\mu_p]}\ , \tag{C.13}$$

which reduce the number of degrees of freedom contained in A by those in $\lambda^{(p-1)}$. For $p > 1$, $\lambda^{(p-1)}$ is itself subject to gauge transformations, i.e.,

$$\lambda^{(p-1)}_{\mu_1\ldots\mu_{p-1}} \rightarrow \lambda^{(p-1)}_{\mu_1\ldots\mu_{p-1}} + \partial_{[\mu_1} \lambda^{(p-2)}_{\mu_2\ldots\mu_{p-1}]} \tag{C.14}$$

does not change A. In this case the gauge transformation of A is called reducible. The degrees of freedom contained in $\lambda^{(p-2)}$ are not substracted from those in A. Continuing all the way down to $\lambda^{(0)}$, we find that the number of off-shell degrees of freedom is given by the alternating sum

$$\binom{D}{p} - \binom{D}{p-1} + \binom{D}{p-2} - \ldots + (-)^p \binom{D}{0} = \binom{D-1}{p} . \tag{C.15}$$

This result agrees with what we found above.
Modulo duality, the highest rank that occurs[23] in the various supergravities is $p = 3$. For these cases, we list the numbers of on-shell degrees of freedom in the following table:

p	$\mathrm{DOF}_{\mathrm{on}}$
0	1
1	$D-2$
2	$(D-2)(D-3)/2$
3	$(D-2)(D-3)(D-4)/6$

(C.16)

D Comparison with Wess & Bagger

The following relations allow to translate equations in [2] into our conventions and vice versa. The same rescalings hold for the complex conjugate expressions.

Grassmann variables:

$$\theta_\alpha^{\mathrm{WB}} = \frac{1}{\sqrt{2}}\,\theta_\alpha \ , \qquad d\theta_\alpha^{\mathrm{WB}} = \sqrt{2}\,d\theta_\alpha \ . \tag{D.1}$$

Supersymmetry generators:

$$D_\alpha^{\mathrm{WB}} = \sqrt{2}\,D_\alpha \quad \Rightarrow \quad \{D_\alpha^{\mathrm{WB}}\,, \bar{D}_{\dot\alpha}^{\mathrm{WB}}\} = -2\mathrm{i}\,\partial_{\alpha\dot\alpha} = 2\,\{D_\alpha\,, \bar{D}_{\dot\alpha}\} \ . \tag{D.2}$$

Supersymmetry parameters (note that in [2] the signs vary from chapter to chapter):

$$\xi_\alpha^{\mathrm{WB}} = \pm\zeta_\alpha^{\mathrm{WB}} = \pm\frac{1}{\sqrt{2}}\,\epsilon_\alpha \ . \tag{D.3}$$

Gravitino and gaugino:

$$\psi_{\mu\alpha}^{\mathrm{WB}} = \sqrt{2}\,\psi_{\mu\alpha} \ , \qquad \lambda_\alpha^{\mathrm{WB}} = \sqrt{2}\,\lambda_\alpha \ . \tag{D.4}$$

Gravitational auxiliary fields:

$$M^{\mathrm{WB}} = 3M \ , \qquad b_a^{\mathrm{WB}} = 3B_a \ . \tag{D.5}$$

Torsion and curvature:

$$(T^{\mathrm{WB}})_{AB}{}^{C} = (\sqrt{2}\,)^{|A|+|B|-|C|}\,T_{AB}{}^{C} \tag{D.6}$$

$$(R^{\mathrm{WB}})_{ABC}{}^{D} = (\sqrt{2}\,)^{|A|+|B|}\,R_{ABC}{}^{D} \ . \tag{D.7}$$

[23] Actually, ten-dimensional Type IIB supergravity contains a 4-form gauge potential. It is selfdual, however, and not covered by our analysis above.

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