The Sagnac Effect and the Postulates on the Velocity of Light

Parry Moon, Domina Eberle Spencer, and Shama Y. Uma

Abstract

With uniform translation absolute velocity is meaningless; only relative velocity is significant. On the other hand, absolute rotational velocity can be measured, as was demonstrated by Sagnac in 1913. If the original Einstein postulate of 1905 is employed in a coordinate system attached to a rotating turntable, the Sagnac effect is not predicted. However, if the modified postulates, Postulate I and III*, which were developed to explain the Michelson-Gale effect, are applied to the Sagnac experiment in suitable nonrotating coordinate systems, the fringe shift found by Sagnac is predicted.*

Key words: velocity of light, interferometry, rotation of Earth, Sagnac experiment, rotating coordinate systems

1. INTRODUCTION

The fundamental distinction between translation and rotation was pointed out by Newton and was emphasized in the nineteenth century by Mach and Hertz. With uniform translation absolute velocity is meaningless; only relative translational velocity is significant. With rotation absolute rotation is significant; relative rotational velocity is not physically interesting. Mechanical effects are dictated by a fixed frame of reference determined by the axis of rotation. Sagnac's experiment[1] indicates that similar considerations apply to light.

A previous paper[2] has shown that the postulates on the velocity of light suggested by Einstein[3] and Moon and Spencer[4] must be restricted to nonrotating coordinate systems in order to explain the Michelson-Gale experiment[5]. In this paper we will show that the same modified postulates are adequate to explain the Sagnac effect.

2. THE SAGNAC EXPERIMENT ACCORDING TO POSTULATE I*

The first optical experiment that demonstrates absolute rotation was performed by Sagnac[1] in 1913. Since this time many similar experiments have been performed, including recent experiments employing ring lasers. An excellent summary of this work has been given by Post.[6] In all the Sagnac-type experiments two beams of light travel in opposite directions about a closed path on a turntable. When the turntable rotates at angular velocity ω, a fringe shift is observed which is directly proportional to the angular velocity.

For simplicity, in this paper we will assume that the closed path is a regular n-sided polygon, Fig. 1. Let the coordinates (x, y) be attached to the laboratory with origin at the center of a rotating turntable. A source S is placed at radius R from the center at one of the vertices of the polygon. At the other vertices of the polygon $(n-1)$ mirrors are equally spaced at a distance d from one another. The mirrors are all perpendicular to the radius. In the nonrotating coordinate system (x, y) the coordinates of the source at time t are

$$x_S = R\cos\omega t, \qquad y_S = R\sin\omega t. \tag{1}$$

The coordinates of the $(i-1)$st mirror at time t are

$$\begin{aligned} x_{i-1} &= R\cos\left[\omega t + (i-1)(2\pi/n)\right], \\ y_{i-1} &= R\sin\left[\omega t + (i-1)(2\pi/n)\right]. \end{aligned} \tag{2}$$

For the ith mirror at time t,

$$\begin{aligned} x_i &= R\cos\left[\omega t + i(2\pi/n)\right], \\ y_i &= R\sin\left[\omega t + i(2\pi/n)\right]. \end{aligned} \tag{3}$$

Light leaves the sources S at time $t = 0$. It is reflected from the $(i-1)$st mirror at time t_{i-1} and is reflected from the ith mirror at time t_i. The

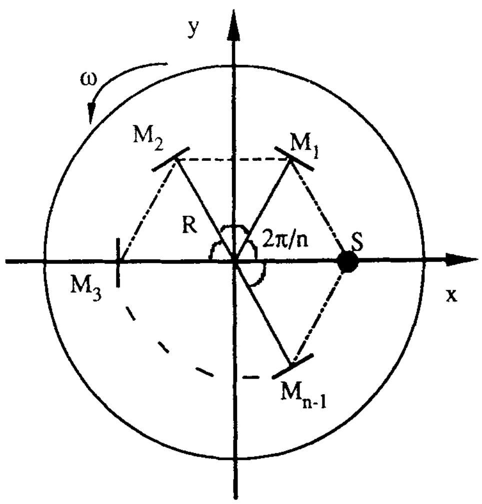

Figure 1. The Sagnac experiment in a nonrotating coordinate system (x, y) attached to the axis of rotation.

distance from the $(i-1)$st mirror at time t_{i-1}, where position is defined by

$$x_{i-1} = R\cos\left[\omega t_{i-1} + (i-1)(2\pi/n)\right],$$
$$y_{i-1} = R\sin\left[\omega t_{i-1} + (i-1)(2\pi/n)\right], \qquad (4)$$

to the ith mirror at time t_i, where position is given by

$$x_i = R\cos\left[\omega t_i + i(2\pi/n)\right],$$
$$y_i = R\sin\left[\omega t_i + i(2\pi/n)\right], \qquad (5)$$

is

$$d_i = R(2\{1 - \cos[\omega(t_i - t_{i-1}) + (2\pi/n)]\})^{1/2}. \qquad (6)$$

Since $\omega(t_i - t_{i-1})$ is a very small angle,

$$d_i \approx R\{2[1 - \cos(2\pi/n) + \omega(t_i - t_{i-1})\sin(2\pi/n)]\}^{1/2}. \qquad (7)$$

It has previously been shown[2] that in order to explain the Michelson-Gale experiment, Einstein's 1905 paper must be modified to

Postulate I*: The velocity of light in free space is a constant c irrespective of the velocity of source or receiver in any coordinate system *which is not in rotation.*

Since the laboratory coordinate system (x, y) can be considered for the analysis of this experiment to be nonrotating, according to Postulate I*,

$$d_i = c(t_i - t_{i-1}). \qquad (8)$$

Substitution of Eq. (7) into Eq. (8) gives a quadratic equation in the time interval $(t_i - t_{i-1})$,

$$(t_i - t_{i-1})^2 - \frac{2R^2\omega\sin(2\pi/n)}{c^2}(t_i - t_{i-1}) - \frac{2R^2[1 - \cos(2\pi/n)]}{c^2} = 0. \qquad (9)$$

Solving for $(t_i - t_{i-1})$,

$$(t_i - t_{i-1}) = \frac{R^2\omega\sin(2\pi/n)}{c^2} \pm \frac{R}{c}\left\{\frac{R^2\omega^2\sin^2(2\pi/n)}{c^2} + 2[1 - \cos(2\pi/n)]\right\}^{1/2}. \qquad (10)$$

But $R^2\omega^2/c^2 \ll 1$ and only the positive root is significant, so

$$(t_i - t_{i-1}) \approx \frac{R^2\omega\sin(2\pi/n)}{c^2} + \frac{R}{c}\{2[1 - \cos(2\pi/n)]\}^{1/2}. \qquad (11)$$

Note that according to Postulate I*, the time required to traverse one side of the polygon is independent of i, that is, it is the same for each side of the polygon. Therefore, the time required to traverse the entire polygon counterclockwise is

$$t_{n+} \approx \frac{nR^2\omega\sin(2\pi/n)}{c^2} + \frac{nR}{c}\{2[1 - \cos(2\pi/n)]\}^{1/2}. \qquad (12)$$

Similarly, traversing the polygon in the clockwise direction,

$$t_{n-} \approx -\frac{nR^2\omega\sin(2\pi/n)}{c^2} + \frac{nR}{c}\{2[1 - \cos(2\pi/n)]\}^{1/2}. \qquad (13)$$

Thus the difference between the arrival times of the counterclockwise and clockwise beams is

$$\Delta t = t_{n+} - t_{n-} \approx [2nR^2\omega\sin(2\pi/n)]/c^2. \qquad (14)$$

The area of an n-sided polygon is

$$A = [nR^2\sin(2\pi/n)]/2. \qquad (15)$$

Substitution of Eq. (15) into Eq. (14) gives

$$\Delta t \approx 4A\omega/c^2. \qquad (16)$$

For a complete fringe shift $c\Delta t = \lambda$, so the number of fringe shifts N is

$$N = 4A\omega/\lambda c. \qquad (17)$$

This is the equation obtained experimentally by Sagnac.

3. THE SAGNAC EXPERIMENT ACCORDING TO POSTULATE III*

The Sagnac experiment can equally well be analyzed using Postulate III*, which has provided an alternative explanation of the Michelson-Gale experiment.

Postulate III*: In a coordinate system that is not moving with respect to the source *and is not in rotation*, the velocity of light in free space is a constant c.

In order to work in a coordinate system in which the velocity of light is constant, it is necessary to employ a nonrotating coordinate system (x', y'), Fig. 2, which is attached to the source S. The relation between the non-

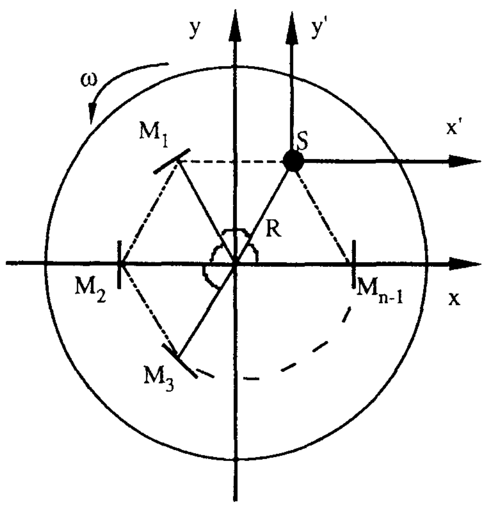

Figure 2. The Sagnac experiment in a nonrotating coordinate system (x', y') attached to the light source S.

rotating coordinates (x, y) employed with Postulate I* and the nonrotating coordinates (x', y') attached to the source S which are needed for Postulate III* is

$$x' = x - R \cos \omega t, \qquad y' = y - R \sin \omega t. \tag{18}$$

The coordinates of the $(i-1)$ st mirror at time t_{i-1} are found by substituting Eq. (4) into Eq. (18):

$$\begin{aligned} x'_{i-1} &= R \cos [\omega t_{i-1} + (i-1)(2\pi/n)] - R \cos \omega t_{i-1}, \\ y'_{i-1} &= R \sin [\omega t_{i-1} + (i-1)(2\pi/n)] - R \sin \omega t_{i-1}. \end{aligned} \tag{19}$$

Similarly, the coordinates of the ith mirror at time t_i are found by substituting Eq. (3) into Eq. (18):

$$\begin{aligned} x'_i &= R \cos [\omega t_i + i(2\pi/n)] - R \cos \omega t_i, \\ y'_i &= R \sin [\omega t_i + i(2\pi/n)] - R \sin \omega t_i. \end{aligned} \tag{20}$$

The distance from the $(i-1)$ st mirror at time t_{i-1} to the mirror at time t_i is

$$d_i = \left(\left\{ \begin{matrix} R \cos (\omega t_i + i 2\pi/n) - R \cos \omega t_i \\ -R \cos [\omega t_{i-1} + (i-1) 2\pi/n] + R \cos \omega t_{i-1} \end{matrix} \right\}^2 + \left\{ \begin{matrix} R \sin (\omega t_i + i 2\pi/n) - R \sin \omega t_i \\ -R \sin [\omega t_{i-1} + (i-1) 2\pi/n] + R \sin \omega t_{i-1} \end{matrix} \right\}^2 \right)^{1/2}. \tag{21}$$

Since $\omega(t_i - t_{i-1})$ is very small,

$$d_i \approx R \left(2 \left\{ 1 - \cos \frac{2\pi}{n} + \omega(t_i - t_{i-1}) \times \left[\sin \frac{2\pi}{n} + \sin (i-1) \frac{2\pi}{n} - \sin \frac{i 2\pi}{n} \right] \right\} \right)^{1/2}. \tag{22}$$

By Postulate III*

$$d_i = c(t_i - t_{i-1}). \tag{23}$$

Combining Eqs. (22) and (23) and solving for $(t_i - t_{i-1})$,

$$(t_i - t_{i-1}) \approx \frac{R^2 \omega}{c^2} [\sin (2\pi/n) + \sin (i-1)(2\pi/n) - \sin i(2\pi/n)] + \frac{R}{c} \{2[1 - \cos (2\pi/n)]\}^{1/2}. \tag{24}$$

Unlike Eq. (11), Eq. (24) depends on i. Therefore, the phase shift is not uniformly distributed about the closed path, according to Postulate III*. No phase shift occurs in the two branches in which one of the vertices is the source. The largest phase shift occurs in the branch furthest from the source. To find the time t_{n+} to traverse the entire polygon in the counterclockwise direction, it is necessary to add the transit times for each branch:

$$t_{n+} \approx \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{nR^2 \omega \sin (2\pi/n)}{c^2} + \frac{nR}{c} \{2[1 - \cos (2\pi/n)]\}^{1/2}. \tag{25}$$

This equation is identical to Eq. (12). Consequently, the number of fringe shifts N predicted from Postulate III* is identical with that predicted by Postulate I*, Eq. (17).

4. CONCLUSIONS

The Sagnac experiment has been analyzed for two Postulates, I* and III*. In 1905 Einstein proposed Postulate I: the velocity of light is a constant c irrespective of the velocity of the source or the receiver. In 1907 he found it necessary to limit this postulate to Postulate I* which restricts the previous statement to nonrotating coordinate systems. In 1956 Moon and Spencer proposed Postulate III that the velocity of light is a constant with respect to the source. In 1990 it was found by Moon, Spencer, and Moon that in order to explain the Michelson-Gale experiment, this postulate must be modified to Postulate III* which likewise is restricted to nonrotating coordinate systems.

This paper has shown that the Sagnac experiment can be explained by both Postulates I* and III*. In fact, the equations for the fringe shift in the Sagnac experiment and in the Michelson-Gale experiment become identical if expressed in terms of the area projected onto the plane perpendicular to the axis of rotation.

A method of discriminating between Postulates I* and III* is suggested by the fact that the fringe shifts are differently distributed about the closed path. According to Postulate I*, for a regular polygonal path, the fringe shift is uniformly distributed. On the other hand, Postulate III* predicts a nonuniform fringe shift distribution with maxima in the branches furthest from the source. This would require the ability to measure the fringe shift in a single branch rather than in a closed path.

In a recent paper(7) Silvertooth and Whitney have described a modern laser version of the Michelson-Morley experiment which may shed light on this question.

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Résumé

En translation uniforme la vitesse absolue n'a aucun sense, et seule la vitesse relative importe. D'autre part, il est possible de mesurer la vitesse absolue de rotation, comme le prouva Sagnac en 1913. Si le postulat original d'Einstein de 1905 est employé dans un référenciel attaché à un tournedisque, l'effet Sagnac n'est pas prévu. Toutefois, si on applique les postulats modifiés I et III*, qui furent proposés pour expliquer l'effet Michelson-Gale, à l'expérience de Sagnac dans un référenciel approprié non en rotation, on prévoit le deplacement des franges trouvé par Sagnac.*

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Parry Moon
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

Domina Eberle Spencer
University of Connecticut
Storrs, Connecticut 06268 U.S.A.

Shama Y. Uma
Bridgewater State College
Bridgewater, Massachusetts 02325 U.S.A.