LOCATED SETS AND REVERSE MATHEMATICS

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Abstract. Let X be a compact metric space. A closed set $K \subseteq X$ is *located* if the distance function $d(x, K)$ exists as a continuous real-valued function on X; *weakly located* if the predicate $d(x, K) > r$ is Σ^0_1 allowing parameters. The purpose of this paper is to explore the concepts of located and weakly located subsets of a compact separable metric space in the context of subsystems of second order arithmetic such as RCA_0, WKL_0 and ACA_0. We also give some applications of these concepts by discussing some versions of the Tietze extension theorem. In particular we prove an RCA_0 version of this result for weakly located closed sets.

1. Introduction and Summary of Results

This paper is part of the program known as Reverse Mathematics. This program investigates what set existence axioms are needed in order to prove specific mathematical theorems. It consists of establishing the weakest subsystem of second order arithmetic in which a theorem of ordinary mathematics can be proved. The basic reference for this program is Simpson's monograph [17] while an overview can be found in [15].

In this paper we carry out a Reverse Mathematics study of the concept of located subsets of a compact complete separable metric space. This concept arises naturally in the context of metric spaces. Even if with a different aim, it plays a fundamental role in the work of Bishop and Bridges [1]. Bishop and Bridges proved a constructive version of the well known Tietze extension theorem for located closed sets in a compact space and uniformly continuous functions with modulus of uniform continuity. In this paper we prove an RCA_0 version of this result for weakly located closed sets. The version of Tietze's theorem for continuous functions and non-compact spaces has been studied by Brown in [2].

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The following definitions are made in RCA_0. Let X be a compact metric space, or we may take $X = [0,1]$. A set $K \subseteq X$ is *closed* if it is the complement of a sequence of open balls; *separably closed* if it is the closure of a sequence of points; *located* if the distance function $d(x,K)$ exists as a continuous real-valued function on X; *weakly located* if the predicate $d(x,K) > r$ is Σ^0_1 (allowing parameters, of course). Trivially located implies weakly located. We denote by $\mathcal{C}(K)$ the continuous real-valued functions on K which have a modulus of uniform continuity. *The strong Tietze theorem* for $K \subseteq X$ is the statement that every $f \in \mathcal{C}(K)$ extends to $F \in \mathcal{C}(X)$. Later we shall present these definitions in more detail.

The following theorems summarize the results obtained in this paper.

Theorem 1.1. *In RCA_0 we have:*

(1) *the functions in $\mathcal{C}(X)$ form a separable Banach space (with the sup norm);*
(2) *the nonempty closed located sets in X form a compact metric space $\mathcal{K}(X)$ (with the Hausdorff metric);*
(3) *closed + located $\Rightarrow$ separably closed;*
(4) *separably closed + weakly located $\Rightarrow$ closed, located;*
(5) *strong Tietze theorem for closed weakly located sets.*

Theorem 1.2. *In RCA_0 the following statements are pairwise equivalent:*

(1) ACA_0*;*
(2) *closed $\Rightarrow$ located;*
(3) *closed $\Rightarrow$ separably closed;*
(4) *separably closed $\Rightarrow$ closed;*
(5) *separably closed $\Rightarrow$ located;*
(6) *separably closed $\Rightarrow$ weakly located;*
(7) *closed + weakly located $\Rightarrow$ located;*
(8) *closed + weakly located $\Rightarrow$ separably closed.*

Theorem 1.3. *In RCA_0 the following statements are pairwise equivalent:*

(1) WKL_0*;*
(2) *closed $\Rightarrow$ weakly located;*
(3) *closed + separably closed $\Rightarrow$ located;*
(4) *closed + separably closed $\Rightarrow$ weakly located;*
(5) *strong Tietze theorem for separably closed sets.*

In particular, WKL_0 proves the strong Tietze theorem for closed sets. We conjecture the reverse, but we have only been able to show that the strong Tietze theorem for closed sets implies the DNR axiom: for all

$A \subseteq \mathbb{N}$ there exists $f : \mathbb{N} \to \mathbb{N}$ which is diagonally nonrecursive relative to A.

We now present a brief outline of the rest of this paper. In section 2 we review briefly some of the concepts and definitions which will be used in this paper. In section 3 we introduce $\mathcal{K}(X)$ and the notion of locatedness giving some basic results. In section 4 we study the connections between $\mathcal{K}(X)$ and separably closed subsets. In section 5 we introduce the concept of weakly located closed set. Section 6 is devoted to the Tietze extension theorem.

In the following, whenever we begin a definition, lemma or theorem by the name of one of the subsystems between parenthesis we mean that the definition is given, or the statement provable, within that subsystem.

Throughout this paper we work with compact complete separable metric spaces.

2. Preliminaries in Reverse Mathematics

We assume familiarity with the development of mathematics within subsystems of second order arithmetic such as RCA_0, WKL_0, and ACA_0. The basic reference is Simpson's monograph [17] while an overview can be found in [15]. The purpose of this section is to briefly review some of the concepts and definitions that we shall need.

Definition 2.1 (RCA_0)**.** A (code for a) *complete separable metric space* $\widehat{A}$ is a set $A \subseteq \mathbb{N}$ together with a function $d : A \times A \to \mathbb{R}$ such that for all $a, b, c \in A$ we have $d(a, a) = 0$, $d(a, b) = d(b, a) \geq 0$ and $d(a, b) \leq d(a, c) + d(c, b)$.

A (code for a) *point of* $\widehat{A}$ is a sequence $\langle a_n : n \in \mathbb{N} \rangle$ of elements of A such that for every n we have $d(a_n, a_{n+1}) < 2^{-n}$.

If $x = \langle a_n : n \in \mathbb{N} \rangle$ and $y = \langle b_n : n \in \mathbb{N} \rangle$ are points of $\widehat{A}$, we write $d(x, y) = \lim_n d(a_n, b_n)$, and we write $x = y$ if and only if $d(x, y) = 0$.

Definition 2.2 (RCA_0)**.** For every $x \in \widehat{A}$ and $\delta \in \mathbb{R}^+$ let $B(x, \delta)$ denote the *open ball* of center x and radius δ in $\widehat{A}$. This means that for every $y \in \widehat{A}$, $y \in B(x, \delta)$ if and only if $d(x, y) < \delta$.

Let $\overline{B}(x, \delta)$ denote the *closed ball* of center x and radius δ in $\widehat{A}$. In this case we mean that for every $y \in \widehat{A}$ we have that $y \in \overline{B}(x, \delta)$ if and only if $d(x, y) \leq \delta$

A (code for an) *open set* in $\widehat{A}$ is a sequence $U = \langle (a_n, r_n) : n \in \mathbb{N} \rangle$ of elements of $A \times \mathbb{Q}^+$. The meaning of this coding is that $U = \bigcup_{n \in \mathbb{N}} B(a_n, r_n)$ and hence $x \in U$ if and only if $\exists n\, d(x, a_n) < r_n$.

A *closed set* in $\widehat{A}$ is the complement of an open set, and thus is represented by the same code.

We recall that the notation $B_0 < B_1$ where $B_i = B(a_i, r_i)$ for $i < 2$, means $d(a_0, a_1) + r_0 < r_1$.

The following results proved in [17] are basic facts about open sets in complete separable metric spaces.

Lemma 2.3 (RCA_0)**.** *Let $\varphi(x)$ be a Σ^0_1 formula such that $x, y \in \widehat{A}$ and $x = y$ imply $\varphi(x) \longleftrightarrow \varphi(y)$. Then there exists an open set U in $\widehat{A}$ such that $x \in U$ if and only if $\varphi(x)$ holds.*

Lemma 2.4 (RCA_0)**.** *Let $\varphi(n)$ be a Σ^0_1 formula in which X and f appear. Either there exists a finite set X such that $\forall n\ (n \in X \longleftrightarrow \varphi(n))$ or there exists a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \longleftrightarrow \exists m\ f(m) = n$.*

Definition 2.5 (RCA_0)**.** A complete separable metric space $X = \widehat{A}$ is *compact* if there exists an infinite sequence of finite sequences of points of X $\langle\langle x_{n,m} : m \leq i_n\rangle : n \in \mathbb{N}\rangle$ such that

$$\forall x \in X\ \forall n \in \mathbb{N}\ \exists m \leq i_n\ d(x, x_{n,m}) < 2^{-n}.$$

Definition 2.6 (RCA_0)**.** Let X be a compact complete separable metric space and let $\langle\langle x_{n,m} : m \leq i_n\rangle : n \in \mathbb{N}\rangle$ witness the compactness of X. Let $B_{n,m} = B(x_{n,m}, 2^{-n})$ for $m \leq i_n$.

We say that the finite sequence of balls

$$\langle B_{n,m} : m \leq i_n\rangle$$

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We say that the sequence of finite sequences of balls

$$\langle\langle B_{n,m} : m \leq i_n\rangle : n \in \mathbb{N}\rangle$$

is the *net*.

Sometimes in this paper we will use the terminology of definition 2.6 to indicate the centers of such balls (i.e. the points $x_{n,m}$); it will be clear from the context the object which we are referring to. Notice that for each $n \in \mathbb{N}$ the n-net is a covering of the space X.

Continuous functions are coded in second order arithmetic as follows (see [4, 17]).

Definition 2.7 (RCA_0)**.** Let $\widehat{A}$ and $\widehat{B}$ be two complete separable metric spaces. A (code for a) *continuous function from $\widehat{A}$ to $\widehat{B}$* is a set $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ such that, if we denote by $(a, r)\Phi(b, s)$ the formula $\exists n\ (n, a, r, b, s) \in \Phi$, the following properties hold:

(1) $(a,r)\Phi(b,s) \wedge (a,r)\Phi(b',s') \longrightarrow d(b,b') < s+s'$;
(2) $(a,r)\Phi(b,s) \wedge d(b,b') + s \leq s' \longrightarrow (a,r)\Phi(b',s')$;
(3) $(a,r)\Phi(b,s) \wedge d(a,a') + r' \leq r \longrightarrow (a',r')\Phi(b,s)$;
(4) $\forall x \in \widehat{A}\, \forall q \in \mathbb{Q}^+ \exists (a,r,b,s)((a,r)\Phi(b,s) \wedge d(x,a) < r \wedge s < q)$.

In this situation for every $x \in \widehat{A}$ there exists a unique $y \in \widehat{B}$ (unique up to $=$ on $\widehat{B}$) such that $d(y,b) \leq s$ whenever $d(x,a) < r$ and $(a,r)\Phi(b,s)$. This y is denoted by $f(x)$ and is the image of x under the function f coded by Φ.

Sometimes we shall need to consider continuous functions which are defined only on a subset of $\widehat{A}$. These can be coded omitting clause 4 in the above definition: their domain consists precisely of those $x \in \widehat{A}$ for which

$$\forall q \in \mathbb{Q}^+ \exists (a,r,b,s)((a,r)\Phi(b,s) \wedge d(x,a) < r \wedge s < q).$$

Definition 2.8 (RCA_0)**.** Let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces, and let f be a continuous function from $\widehat{A}$ into $\widehat{B}$. A *modulus of uniform continuity* for f is a function $h : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all x and y in $\widehat{A}$, if $d(x,y) < 2^{-h(n)}$ then $d(f(x), f(y)) < 2^{-n}$. In this case we say that f is a uniformly continuous function with modulus of uniform continuity. Without loss of generality we assume in this paper that the modulus of uniform continuity is a strictly increasing function.

If f is defined only on a subset of $\widehat{A}$, we say that f is uniformly continuous with modulus of uniform continuity if the property above holds for the points in the domain of f.

The following result can be found in [2] and [17].

Theorem 2.9 (RCA_0)**.** *The following are pairwise equivalent:*

(1) WKL_0.
(2) *Every continuous function defined on a compact complete separable metric space is uniformly continuous with modulus of uniform continuity.*
(3) *Every continuous function defined on* $[0,1]$ *is uniformly continuous with modulus of uniform continuity.*

Within RCA_0, let X be a compact complete separable metric space. We define $\mathcal{C}(X) = \widehat{A}$, the completion of A, where A is the vector space of rational "polynomials" over X under the sup-norm, $\|f\| = \sup_{x\in X} |f(x)|$. Thus $\mathcal{C}(X)$ is a separable Banach space. For the precise definitions within RCA_0, see [20] and Brown's thesis [2, section III.E]. The construction of $\mathcal{C}(X)$ within RCA_0 is inspired by the constructive Stone–Weierstrass theorem in the work by Bishop and Bridges [1, section 4.5]. It is provable in RCA_0 that there is a natural one-to-one correspondence between points of $\mathcal{C}(X)$ and continuous functions $f : X \to \mathbb{R}$ which have a modulus of uniform continuity.

Lemma 2.10 (RCA_0). *Let φ be a Σ^0_1 formula. Then there exists a Σ^0_1 formula $\widehat{\varphi}$ which is a uniformization of φ, namely the following properties hold:*

(1) $\forall n \forall m\, [\widehat{\varphi}(n,m) \longrightarrow \varphi(n,m)]$.
(2) $\forall n\, [\exists m\, \varphi(n,m) \longrightarrow \exists m\, \widehat{\varphi}(n,m)]$.
(3) $\forall n \forall m \forall m'\, [\widehat{\varphi}(n,m) \wedge \widehat{\varphi}(n,m') \longrightarrow m = m']$.

Proof. By the Normal Form Theorem for Σ^0_1 formulas there exists a Σ^0_0 formula θ such that

$$\varphi(n,m) \equiv \exists k\, \theta(n,m,k).$$

We define

$$\widehat{\varphi}(n,m) \equiv \exists k\, [\theta(n,m,k) \wedge \forall \langle m',k'\rangle < \langle m,k\rangle \neg\theta(n,m',k')].$$

It is clear that $\widehat{\varphi}$ fulfills (1), (2) and (3). □

Lemma 2.11 (RCA_0). *Let $\varphi(n,m)$ and $\psi(n)$ be Σ^0_1 formulas. Assume that*

$$\forall n\, [\psi(n) \longrightarrow \exists m\, [\psi(m) \wedge \varphi(n,m)]].$$

Then

$$\forall n\, [\psi(n) \longrightarrow \exists f\, [f(0) = n \wedge \forall k\, \psi(f(k)) \wedge \forall k\, \varphi(f(k), f(k+1))]].$$

Proof. We define

$$\begin{aligned} f(k) = m \iff & \exists s\, [s(0) = n \wedge \forall j < k\, \theta(s(j), s(j+1)) \wedge s(k) = m] \\ \iff & \forall s\, [[s(0) = n \wedge \forall j < k\, \theta(f(j), f(j+1))] \to s(k) = m] \end{aligned}$$

where $\theta(n,m)$ is a uniformization of $\varphi(n,m) \wedge \psi(m)$ and s ranges over codes for finite sequences. The equivalence follows from lemma 2.10 (see in particular 2.10(3)). Hence f is defined by Δ^0_1 comprehension. □

The uniform version 2.13 of the following lemma will be used several times throughout this paper to carry out many proofs in RCA_0. Notice that a formal proof of lemma 2.12 and 2.13 uses lemma 2.10 and 2.11.

Lemma 2.12 (RCA_0). *Let I be a finite set, let $\varphi_0, \ldots, \varphi_k$ be Σ^0_1 formulas such that $\forall m \in I\ \varphi_0(m) \vee \cdots \vee \varphi_k(m)$. Then there exist finite sets $I_0, \ldots, I_k$ such that*

(1) $I = I_0 \cup \cdots \cup I_k$
(2) $\forall j \leq k\, (m \in I_j \implies \varphi_j(m))$.

Proof. For each $j \leq k$ we start simultaneously the enumeration of the m's such that $\varphi_j(m)$ (cf. lemma 2.4). Since $\forall m \in I\ \varphi_0(m) \vee \cdots \vee \varphi_k(m)$, at least one of the enumerations, say $\varphi_0(m)$, stops. Therefore m is an element of I_0. If more than one enumeration stops at the same time we put the element m in all the corresponding I_j's. Therefore at the end of this process (which is finite, since I is finite), we get $I_0, \ldots, I_k$. It is clear that (1) and (2) hold. □

Lemma 2.13 (RCA_0)**.** *Let I be a finite set, let $\varphi_0(n), \ldots, \varphi_k(n)$ be a family of Σ^0_1 formulas depending on a parameter $n \in \mathbb{N}$ such that $\forall m \in I\ \varphi_0(n,m) \vee \cdots \vee \varphi_k(n,m)$. Then there exists an effective enumeration of finite sequences of finite sets $I_0(n), \ldots, I_k(n)$ such that for all $n \in \mathbb{N}$*

(1) $I = I_0(n) \cup \cdots \cup I_k(n)$
(2) $\forall j \leq k\ (m \in I_j(n) \implies \varphi_j(n,m))$.

3. $\mathcal{K}(X)$ and Located Sets

Let X be a compact complete separable metric space. In the literature of general topology (see e.g. [11] and [5]), the space of nonempty compact subsets of X is known as $\mathcal{K}(X)$. It is usually equipped with the Vietoris topology, generated by sets of the form $\{K \in \mathcal{K}(X) : K \subseteq U\}$ and $\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$ for U open in X. Moreover $\mathcal{K}(X)$ is usually equipped with the Hausdorff metric

$$d^*_H(K_1, K_2) = \sup\{d(x, K_2), d(K_1, y) \ : \ x \in K_1,\ y \in K_2\}.$$

Here we introduce $\mathcal{K}(X)$ in RCA_0 by providing a code for it. We shall see that the elements of $\mathcal{K}(X)$ can be identified with the closed and located subsets of X (theorem 3.7).

Definition 3.1 (RCA_0)**.** Let $X = \widehat{A}$ be a compact complete separable metric space with metric d. Let $A^* = \{K \subseteq A : K \neq \emptyset \text{ is finite}\}$. On A^* we define the metric d^* by

$$d^*(K_1, K_2) = \sup_{x \in X} |d(x, K_1) - d(x, K_2)|. \tag{1}$$

We define $\mathcal{K}(X) = \widehat{A^*}$ as the completion of A^* under the metric d^* and we equip it with the obvious extension of d^* (which we still call d^* with abuse of notation).

Remark 3.2. Notice that since the space X is compact and the function $x \mapsto d(x, K_1) - d(x, K_2)$ is clearly a uniformly continuous function, the metric d^* defined as in (1) exists in RCA_0. Hence $\mathcal{K}(X)$ is actually a complete separable metric space.

There is an interesting relationship between the elements of $\mathcal{K}(X)$ and the elements of $\mathcal{C}(X)$, as the following remark testifies.

Remark 3.3. Let $\mathcal{K}(X)$ and $\mathcal{C}(X)$ be equipped with the metrics d^* and $\| \ \|$ respectively. We show that if $x \in X$ and $K = \langle K_n : n \in \mathbb{N} \rangle \in \mathcal{K}(X)$ then $d(x, K) = \lim_{n\to\infty} d(x, K_n)$ is well defined. Indeed we prove that if $\langle K_n : n \in \mathbb{N} \rangle$ and $\langle K'_n : n \in \mathbb{N} \rangle$ are two codes for the same $K \in \mathcal{K}(X)$, then $\lim_{n\to\infty} d(x, K_n) = \lim_{n\to\infty} d(x, K'_n)$. By definition $\forall n \in \mathbb{N}$ $d^*(K_n, K'_n) \leq 2^{-n+1}$. Hence for all $x \in X$

$$|d(x, K_n) - d(x, K'_n)| \leq \sup_{x \in X} |d(x, K_n) - d(x, K'_n)| \leq 2^{-n+1}$$

and therefore we get the conclusion.

Using the fact above, it is immediate to see that from the definition of d^*, there is an isometric embedding

$$A^* \hookrightarrow \mathcal{C}(X)$$

defined by

$$K \mapsto (x \mapsto d(x, K))$$

which can be extended to an isometric embedding

$$\mathcal{K}(X) \hookrightarrow \mathcal{C}(X).$$

Since $\mathcal{K}(X) = \widehat{A^*}$ is isometrically isomorphic to $\overline{A^*}$, we can view $\mathcal{K}(X)$ as a separably closed subset of $\mathcal{C}(X)$. (See also definition 4.1 below.) Later (see lemma 3.5) we shall prove that it is also closed and compact.

Another possible way to code $\mathcal{K}(X)$ is to consider it as the completion of A^* under the Haudorff metric defined on A^*:

$$d^*_H(K_1, K_2) = \sup\{d(x, K_2), d(K_1, y) \ : \ x \in K_1, \ y \in K_2\}$$

Following such an approach, there is the disadvantage that remark 3.3 would not be so clear. However, the definition of d^*_H is more manageable. The following lemma holds and will be used in several proofs.

Lemma 3.4 (RCA_0)**.** *$d^* = d^*_H$ on A^*.*

Proof. First we prove that $d^* \leq d^*_H$. Let $K_1, \ K_2 \in A^*$. There exists $y \in K_2$ such that $d(x, K_2) = d(x, y)$. Moreover as consequence of the triangular inequality, $\forall x \in X$,

$$d(x, K_1) \leq d(x, y) + d(y, K_1).$$

Hence

$$\begin{aligned} d(x, K_1) - d(x, K_2) &\leq d(x, y) + d(y, K_1) - d(x, y) \\ &= d(y, K_1) \\ &\leq \sup\{d(x, K_2), d(K_1, y) : x \in K_1, \ y \in K_2\}. \end{aligned}$$

Similarly

$$d(x, K_2) - d(x, K_1) \leq \sup\{d(x, K_2), d(K_1, y) : x \in K_1, \ y \in K_2\}.$$

Now we prove that $d_H^* \le d^*$. We may assume that there exists $x \in K_1$ such that

$$d_H^*(K_1, K_2) = d(x, K_2).$$

Since $d(x, K_1) = 0$,

$$d_H^*(K_1, K_2) \le d(x, K_2) - d(x, K_1) \le \max_{x \in X} |d(x, K_1) - d(x, K_2)|.$$

Therefore $d_H^* \le d^*$. □

Lemma 3.5. *It is provable in* RCA_0 *that* $\mathcal{K}(X)$ *is compact.*

Proof. Let $\langle\langle x_{n,m} : m \le i_n\rangle : n \in \mathbb{N}\rangle$ witness the compactness of X. For all $n \in \mathbb{N}$ define $F_n = \{x_{n,m} : m \le i_n\}$. Let $\langle S_{n,k} : k \le j_n\rangle$ be an enumeration of the nonempty subsets of F_{n+1}. We prove that the sequence

$$\langle\langle S_{n,k} : k \le j_n\rangle : n \in \mathbb{N}\rangle \tag{2}$$

witnesses the compactness of $\mathcal{K}(X)$. To this purpose it is enough to show that fixed $n \in \mathbb{N}$ and given any $K \in A^*$, there exists an element S in the sequence (2) such that $d_H^*(S, K) < 2^{-n}$ (see lemma 3.4). To prove this, for each $m \le i_{n+1}$ consider the following Σ_1^0 formulas:

- $\varphi_0(n, m)$: $d(x_{n+1,m}, K) < 2^{-n}$.
- $\varphi_1(n, m)$: $d(x_{n+1,m}, K) > 2^{-n-1}$.

By lemma 2.13 we get two finite sets of indices $I_0(n)$ and $I_1(n)$ such that:

- $\{m : m \le i_{n+1}\} = I_0(n) \cup I_1(n)$.
- $\forall j < 2\ m \in I_j(n) \implies \varphi_j(n, m)$ holds.

Let

$$S = \{x_{n+1,m} : m \in I_0(n)\}.$$

Clearly $S \neq \emptyset$ and we claim that S is the desired subset of F_{n+1}. Indeed, by definition,

$$d_H^*(S, K) = \sup\{d(x, K), d(S, y) \ : \ x \in S,\ y \in K\}.$$

On one hand, let us consider $d(x, K)$. Since $\varphi_0(n, m)$ holds, for all $x \in S$ we have $d(x, K) < 2^{-n}$. On the other hand, consider $d(S, y)$. By definition $d(S, y) = \min_{x \in S} d(x, y)$. Fix any $y \in K$. There exists an element $x_{n+1,m} \in S$ such that $d(x_{n+1,m}, y) < 2^{-n-1}$. Hence $\min_{x \in S} d(x, y) \le 2^{-n-1}$. Therefore $\forall y \in K\ d(S, y) \le 2^{-n-1}$. Therefore, $d^*(S, K) = d_H^*(S, K) < 2^{-n}$ and the proof is complete. □

Definition 3.6 (RCA_0)**.** Let (X, d) be a complete separable metric space. Let C be a closed or a separably closed subset of X. We say that C is *located* if there exists (a code for) the continuous function $f : X \to \mathbb{R}$ such that $f(x) = d(x, C) = \inf\{d(x, y) : y \in C\}$. f is called *distance function.*

Notice that definition 3.6 describes a quite strong property from the point of view of Reverse Mathematics: in fact we shall prove (see theorem 3.8) that ACA_0 is equivalent to the statement "in a compact complete separable metric space every closed set is located"

The following theorem 3.7 allows us to think of the points of $\mathcal{K}(X)$ as the closed and located subsets of X.

Theorem 3.7 (RCA_0). *Let X be a compact complete separable metric space. The elements of $\mathcal{K}(X)$ are in one-to-one correspondence with the closed and located subsets of X. Moreover, if $K = \langle K_n : n \in \mathbb{N}\rangle \in \mathcal{K}(X)$ corresponds to the closed located set C, then*

$$\lim_{n\to\infty} d(x, K_n) = d(x, K) = d(x, C) = \inf\{d(x, y) : y \in C\}.$$

Proof. Let C be a closed and located subset of X and let d be the metric on X. We prove that there exists a code $\langle K_n : n \in \mathbb{N}\rangle$ for an element $K \in \mathcal{K}(X)$ such that

$$d(x, C) = 0 \iff \lim_{n\to\infty} d(x, K_n) = 0. \tag{3}$$

Let $\langle\langle x_{n,m} : m \le i_n\rangle : n \in \mathbb{N}\rangle$ witness the compactness of X. Since C is located, the distance function exists and is continuous; for each $m \le i_{n+1}$ and $n \in \mathbb{N}$ consider the following Σ^0_1 formulas:

- $\varphi_0(n, m)$: $d(x_{n+1,m}, C) < 2^{-n}$.
- $\varphi_1(n, m)$: $d(x_{n+1,m}, C) > 2^{-n-1}$.

Using lemma 2.13 we get a sequence of finite sets of indices $I_0(n)$ and $I_1(n)$ such that:

- $\{m : m \le i_{n+1}\} = I_0(n) \cup I_1(n)$.
- $\forall j < 2\ m \in I_j(n) \implies \varphi_j(n, m)$ holds.

Define

$$K_n = \{x_{n+1,m} : m \in I_0(n)\}$$

We prove that the sequence $\langle K_n : n \in \mathbb{N}\rangle$ is a Cauchy sequence in the metric d^*_H (lemma 3.4 assures that $d^* = d^*_H$). We prove that $d^*_H(K_n, K_{n+1}) < 2^{-n}$. Assume that $\exists x \in K_n\ d^*_H(K_n, K_{n+1}) = d(x, K_{n+1})$. There exists a point $x_{n+2,m} \in K_{n+1}$ such that $d(x, x_{n+2,m}) < 2^{-n-2}$ and hence $d(x, K_{n+1}) = \min_{y\in K_{n+1}} d(x, y) < 2^{-n-2}$. If $d^*_H(K_n, K_{n+1}) = d(K_n, y)$ for $y \in K_{n+1}$, the same argument proves that $d(K_n, y) \le 2^{-n-1}$. Therefore, since $d^*(K_n, K_{n+1}) < 2^{-n}$, $\langle K_n : n \in \mathbb{N}\rangle$ is a Cauchy sequence of elements of A^* which defines an element $K \in \mathcal{K}(X)$.

It remains to prove (3). Let $x \in C$. For every $n \in \mathbb{N}$ there exists $m \in I_0(n)$ such that $x_{n+1,m} \in K_n$ and $d(x, K_n) \le d(x, x_{n+1,m}) < 2^{-n-1}$. Therefore $\lim_{n\to\infty} d(x, K_n) = 0$. On the other hand, assume that $\lim_{n\to\infty} d(x, K_n) = 0$. We prove that $d(x, C) = 0$. For all $n \in \mathbb{N}$

there exists $m \in I_0(n)$ such that $x_{n+1,m} \in K_n$ and $d(x, x_{n+1,m}) < 2^{-n-1}$. Since $m \in I_0(n)$, $d(x_{n+1,m}, C) < 2^{-n-1}$ and hence for some $y \in C$ we have $d(x_{n+1,m}, y) < 2^{-n-1}$. Then

$$d(x, C) \leq d(x, y) \leq d(x, x_{n+1,m}) + d(x_{n+1,m}, y) < 2^{-n}.$$

Therefore $d(x, C) = 0$ and the first part of the proof is complete.

Let $K \in \mathcal{K}(X)$. We show that there exists C located and closed subset of X such that (3) holds. A code for K is a sequence $\langle K_n : n \in \mathbb{N} \rangle$ of elements of A^* such that $\forall n \ \forall i \ d^*(K_n, K_{n+i}) < 2^{-n}$. We denote $f(x) = \lim_{n\to\infty} d(x, K_n)$. Notice that $f \in \mathcal{C}(X)$. Let us define $C = \{y : f(y) = 0\}$. It is clear that C is closed and (3) holds. To prove that C is located and to complete the proof we show that

$$d(x, C) = f(x).$$

First we prove $d(x, C) \leq f(x)$. Given any $x \in X$ and $n \in \mathbb{N}$ we show that $d(x, C) \leq f(x) + 2^{-n+1}$. Begin with $y_0 \in K_{n+1}$ such that $d(x, y_0) = d(x, K_{n+1})$. Since $d^*(K_{n+1}, K_{n+2}) < 2^{-n-1}$ we can find $y_1 \in K_{n+2}$ such that $d(y_0, y_1) < 2^{-n-1}$. Similarly we can find $y_2 \in K_{n+3}$ such that $d(y_1, y_2) < 2^{-n-2}$. Continuing recursively we find a point $y = \langle y_k : k \in \mathbb{N} \rangle \in X$, $y_k \in K_{n+k+1}$ and such that $d(y_k, y_{k+1}) < 2^{-n-k-1}$. Hence $f(y) = 0$ and hence $y \in C$. Thus

$$\begin{aligned} d(x, y) &\leq d(x, y_0) + d(y_0, y) \\ &\leq d(x, K_{n+1}) + 2^{-n} \\ &\leq f(x) + 2^{-n+1} \end{aligned}$$

Thus $d(x, C) \leq f(x) + 2^{-n+1}$ for all n and hence $d(x, C) \leq f(x)$.

Now we prove that $d(x, C) \leq f(x)$. We recall that since K_n's are elements of A^*, they are finite sets of points $a \in A$. Fix $y \in C$. Since the predicate $d(z, y) < 2^{-n}$ is Σ^0_1, we can find a sequence of points $a_n \in K_n$, $n \in \mathbb{N}$ such that $d(a_n, y) < 2^{-n}$. Hence

$$d(x, y) = \lim_{n\to\infty} d(x, a_n) \geq \lim_{n\to\infty} d(x, K_n) = f(x).$$

And taking the infimum for $y \in C$ we get $d(x, C) \geq f(x)$.

Therefore $d(x, C) = f(x)$. □

Theorem 3.8 (RCA_0)**.** *The following are equivalent:*

(1) ACA_0.
(2) *Every closed subset C of a compact complete separable metric space X is located.*
(3) *Every closed set in $[0, 1]$ is located.*

Proof. (1) $\Longrightarrow$ (2). Since X is compact, Brown [3] (see also theorem 4.2 below) assures that in ACA_0 the notions of closed and separably closed set coincide. Therefore we may assume that C is a separably closed subset of X. Therefore by [8, theorem 7.3] we obtain the result.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). We shall prove that (3) implies the statement "every bounded increasing sequence of reals has a supremum", which is equivalent to ACA_0 (see [17]). Let $\langle a_n : n \in \mathbb{N} \rangle$ be an increasing sequence of reals in [0,1]. For all $n \in \mathbb{N}$ let $U_n = [0, a_n)$ and consider $\langle U_n : n \in \mathbb{N} \rangle$. Let C be the closed set $[0,1] \setminus \bigcup_{n \in \mathbb{N}} U_n$. By (3) C is located and in particular we have:

$$d(0, C) = \sup_{n \in \mathbb{N}} d(0, a_n) = \sup_{n \in \mathbb{N}} a_n.$$

Therefore $d(0, C)$ exists if and only if the supremum of the sequence exists. □

4. $\mathcal{K}(X)$ and separably closed sets

The notion of separably closed set has been studied in [3] and [2]. In this section we investigate the relationship between $\mathcal{K}(X)$ and the notion of separably closed set in compact complete separable metric spaces.

Definition 4.1 (RCA_0)**.** Let $X = \widehat{A}$ be a complete separable metric space. A code for a *separably closed set* in X is a sequence $C = \langle x_n : n \in \mathbb{N} \rangle$ of points of X. The separably closed set is then denoted by $\overline{C}$, and $x \in \overline{C}$ if and only if $\forall q \in \mathbb{Q}^+ \ \exists n \ d(x, x_n) < q$.

Working with compact spaces in ACA_0, the notions of "closed" and "separably closed" coincide, as the following theorem (see [3, page 49] and [2, page 116]) testifies.

Theorem 4.2 (RCA_0)**.** *The following are pairwise equivalent:*

(1) ACA_0.
(2) *In a compact complete separable metric space every closed subset is separably closed.*
(3) *In a compact complete separable metric space every separably closed subset is closed.*
(4) *In* $[0,1]$ *every closed set is separably closed.*
(5) *In* $[0,1]$ *every separably closed set is closed.*

In order to prove theorem 4.5, we need the following lemma.

Lemma 4.3 (RCA_0)**.** *Let* $X = \widehat{A}$ *be a complete separable metric space, let* $C \subseteq X$ *be a closed and located subset of* X *and let* $B = \overline{B}(a, r)$ *be a closed ball,* $a \in A$, $r \in \mathbb{Q}^+$, *such that* $d(a, C) < r$. *Then in* RCA_0 *we can effectively find a point* $x \in X$ *such that* $d(x, C) = 0$ *and* $d(x, a) < r$.

Proof. Since C is located, we can effectively find $\varepsilon > 0$ such that $d(a, C) < r - \varepsilon$. Since we are dealing with Σ^0_1 formulas for which we know in advance that there is at least one witness, we can effectively find a point $a_0 \in A$ such that $d(a_0, a) < r - \varepsilon$ and $d(a_0, C) < \varepsilon/4$. Then we can effectively find a point $a_1 \in A$ such that $d(a_1, a_0) < \varepsilon/4$ and $d(a_1, C) < \varepsilon/8$. Repeating the process, we can effectively find a point $a_2 \in A$ such that $d(a_2, a_1) < \varepsilon/8$ and $d(a_2, C) < \varepsilon/16$ and so on (the argument can be formalized precisely using lemma 2.11). Therefore we effectively find a sequence $\langle a_n : n \in \mathbb{N}\rangle$ of points of A which defines a point $x \in X$. Since by construction $d(a, x) \leq r - \varepsilon/2$, we have $x \in B(a, r)$. By construction also we have that $d(x, C) = 0$ and hence (cf. theorem 3.7) $x \in C$. □

Remark 4.4. Notice that what we really need to carry out the proof of lemma 4.3 is that the predicate $d(a_n, C) < \varepsilon/2^n$, $n \in \mathbb{N}$, is Σ^0_1 (also cf. definition 5.1).

Theorem 4.5 (RCA_0)**.** *Let X be a compact complete separable metric space. Every closed and located subset of X is separably closed.*

Proof. Let $C \subseteq X$ closed and located. Let

$$\left\langle \left\langle \overline{B}(x_{n,m}, 2^{-n}) : m \leq i_n \right\rangle : n \in \mathbb{N} \right\rangle$$

be the net of closed balls. For each $m \leq i_n$, $n \in \mathbb{N}$, consider the following Σ^0_1 formulas:

- $\varphi_0(n, m)$: $d(x_{n,m}, C) < 2^{-n+1}$.
- $\varphi_1(n, m)$: $d(x_{n,m}, C) > 2^{-n}$.

By lemma 2.13 we get two finite sets of indices $I_0(n)$ and $I_1(n)$ such that:

- $\{m : m \leq i_n\} = I_0(n) \cup I_1(n)$.
- $\forall j < 2 \; m \in I_j(n) \implies \varphi_j(n, m)$ holds.

Let $B'_{n,m} = \overline{B}(x_{n,m}, 2^{-n+1})$. Applying lemma 4.3 to each ball $B'_{n,m}$, $m \in I_0(n)$,

$$\forall m \in I_0(n) \; \exists x \in B'_{n,m} \cap C$$

effectively. Using the locatedness of C, it is easy to check in RCA_0 that this last formula is Π^0_1 (cf. [17, proof of IV.1.7]). Therefore, using Π^0_1-induction in RCA_0 and applying repeatedly the argument, we have proved that

$$\forall n \in \mathbb{N} \; \forall m \in I_0(n) \; \exists x \in B'_{n,m} \cap C. \tag{4}$$

In particular, it follows that for all $n \in \mathbb{N}$ and for all $y \in C$ there exists x as in (4) such that $d(x, y) < 2^{-n+1}$ hold. Therefore this sequence of points $x \in C$ gives the code for C as a separably closed set. □

Theorem 4.6 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0.
(2) *In a compact complete separable metric space, every closed and separably closed set is located.*
(3) *Every closed and separably closed subset of* $[0,1]$ *is located.*

Proof. (1) $\Longrightarrow$ (2). Let $\overline{C}$ be closed and separably closed subset of an arbitrary compact metric space. We prove that we can code $d(x,\overline{C})$ as a continuous function. Since $\overline{C}$ is separably closed we have

$$d(x,\overline{C}) \le \inf_{y\in C} d(x,y).$$

On the other hand, since we work in WKL_0 and $\overline{C}$ is closed,

$$\overline{B}(a,r)\cap\overline{C}=\emptyset$$

is described by a Σ^0_1 predicate (cf. [17, lemma IV.1.7]). Thus

$$d(x,\overline{C}) \ge \sup_{a\in A,\ r\in\mathbb{Q}^+}\{r-d(a,x)\ :\ \overline{B}(a,r)\cap\overline{C}=\emptyset\}.$$

Therefore we can give a code Φ for the distance function as continuous function, namely

$$\begin{aligned}(b,t)\Phi(q,s) \quad &\Longleftrightarrow \quad \exists y\in C\ (q+s>d(x,y)+t)\\ &\wedge \quad \exists (a,r)\ (\overline{B}(a,r)\cap\overline{C}=\emptyset\ \wedge\ q-s<r-d(a,x)-t)\end{aligned}$$

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). If WKL_0 fails, there exists $\langle (a_k,b_k) : k\in\mathbb{N}\rangle$, open covering of [0,1], with no finite subcovering. We may assume that for all $k\in\mathbb{N}\ -2^{-2}<a_k<b_k<1+2^{-2}$. Since WKL_0 fails, ACA_0 too fails and therefore there is a one-to-one function $f:\mathbb{N}\to\mathbb{N}$ whose range does not exists in RCA_0. For all $n\in\mathbb{N}$ let

$$I_n=\left[\frac{1}{2^{2n+1}},\frac{1}{2^{2n}}\right].$$

The linear transformation $x\mapsto (x+1)/2^{2n+1}$ transfers to I_n the covering of [0,1]. We denote the trasferred covering by $\langle (a_{n,k},b_{n,k}) : k\in\mathbb{N}\rangle$. Since we assumed for all $k\in\mathbb{N}\ -2^{-2}<a_k<b_k<1+2^{-2}$, for $n\neq m$ the coverings of I_n and I_m do not intersect. Let us define

$$C=\{0\}\cup\bigcup_{m\in\mathbb{N}}\left(\left[\frac{1}{2^{2f(m)+1}},\ \frac{1}{2^{2f(m)}}\right]\setminus\bigcup_{k<m}(a_{f(m),k},\ b_{f(m),k})\right)$$

We prove that C is closed, separably closed, but not located.

To prove that C is closed we show that its complement is open:

$$[0,1] \setminus C = \bigcup_{m\in\mathbb{N}} \bigcup_{n\notin\{f(0),\dots,f(m)\}} \bigcup_{k\geq m} (a_{n,k}, b_{n,k}) \cup \bigcup_{n\in\mathbb{N}} \left(\frac{1}{2^{2n+2}}, \frac{1}{2^{2n+1}}\right)$$

We denote

$$J_m = \left[\frac{1}{2^{2f(m)+1}}, \frac{1}{2^{2f(m)}}\right] \setminus \bigcup_{k<m} (a_{f(m),k}, b_{f(m),k}).$$

C is separably closed because for each m, J_m is finite union of known closed intervals and therefore we can enumerate a dense set of points in them (and hence in C).

Finally, C is not located in RCA_0. Indeed we show that if x is a point in [0,1], the knowledge of $d(x, C)$ gives informations about the range of f, leading to a contradiction. Assume that we can code $x \mapsto d(x, C)$ as continuous funtion in RCA_0. We claim that

$$\exists m\, f(m) = n \longleftrightarrow d(x, C) \leq \frac{1}{2^{2n+2}}$$

Let x_n be the midpoint of I_n. If $\exists m\, f(m) = n$ then, since $x_n \in I_n$ and the covering has no finite subcovering, there are points of C in I_n. Therefore $d(x_n, C) \leq 2^{-2n-2}$. On the other hand, if $d(x_n, C) \leq 2^{-2n-2}$, there must be some point of C in the interval $I_n = [x_n - 2^{-2n-2}, x_n + 2^{-2n-2}]$; since I_n is disjoint from I_{n+1} and I_{n-1} by construction, it follows that $\exists m\, f(m) = n$. Therefore by Δ^0_1 comprehension in RCA_0 the range of f exists. ☐

Remark 4.7. Theorem 4.6 implies that in REC, the model of recursive sets, there exists a closed and separably closed set $C \subseteq [0,1]$ which is not located.

5. $\mathcal{K}(X)$ and weakly located sets

In this section we introduce the concept of "weakly located" set, which is powerful enough to allow to prove in RCA_0 the strong Tietze extension theorem (see section 6).

Definition 5.1 (RCA_0)**.** Let $X = \widehat{A}$ be a complete separable metric space and let C be a closed or a separably closed subset of $\widehat{A}$. We say that C is *weakly located* if the predicate $\exists \varepsilon > 0\, B(a, r+\varepsilon) \cap C = \emptyset$ is Σ^0_1.

Lemma 5.2 (RCA_0)**.** *Every closed located set is weakly located.*

Proof. If C is a closed located set, the distance function $d(x, C)$ is continuous, hence $\exists \varepsilon > 0\, B(a, r+\varepsilon) \cap C = \emptyset$ is equivalent to the Σ^0_1 predicate $d(a, C) > r$. ☐

Remark 5.3 and 5.6 will provide examples of sets which are closed and not weakly located.

Remark 5.3. In light of definition 5.1, theorem 3.8 actually proves the equivalence between ACA_0 and the statement "Every closed and weakly located subset of X is located", because the closed set C defined in (3) $\Longrightarrow$ (1) is weakly located. Indeed

$$\exists \varepsilon > 0 \; B(a,r) \cap C = \emptyset \iff \exists \varepsilon > 0 \; \exists n \; (a + r < a_n)$$

is described by a Σ^0_1 formula. Therefore, in $\mathsf{RCA}_0 + \neg \mathsf{ACA}_0$, C is an example of closed weakly located set which is not located.

Lemma 5.4 (WKL_0)**.** *Every closed set in a compact metric space is weakly located.*

Proof. WKL_0 proves that if C is a closed subset of a compact space, the assertion "C is nonempty" is expressible by a Π^0_1 formula (see [17, theorem IV.1.7]). Therefore it follows that if $C \subseteq X$ is closed,

$$\overline{B}(a,r) \cap C = \emptyset \tag{5}$$

is described by a Σ^0_1 formula. To prove that if (5) is described by a Σ^0_1 predicate then C is weakly located, let $\varphi(a,r)$ be the Σ^0_1 formula which describes (5); we show that

$$\exists \varepsilon > 0 \, B(a, r+\varepsilon) \cap C = \emptyset \iff \exists \delta > 0 \, \varphi(a, r+\delta).$$

$\Longrightarrow$. It follows from the hypothesis that $\overline{B}(a, r+\varepsilon/2) \cap C = \emptyset$. Taking $\delta = \varepsilon/2$ we have $\varphi(a, r+\delta)$.

$\Longleftarrow$. It is enough to take $\varepsilon = \delta$. □

Theorem 5.5 (RCA_0)**.** *In a compact complete separable metric space $X = \widehat{A}$ every separably closed weakly located set is located and closed.*

Proof. We repeat the proof of (1) $\Longrightarrow$ (2) in theorem 4.6 with a slight modification: using the weak locatedness of $\overline{C}$ we set

$$d(x, \overline{C}) = \sup_{a,r} \{ r + \varepsilon - d(a,x) \; : \; \exists \varepsilon > 0 \; B(a, r+\varepsilon) \cap \overline{C} = \emptyset \}.$$

Therefore as in theorem 4.6, it is possible to give the code for d as continuous function in RCA_0. Moreover since $\overline{C} = d^{-1}(\{0\})$, $\overline{C}$ is closed. □

Remark 5.6. The closed and separably closed subset C defined in the proof of theorem 4.6 is an example of a closed subset which is not weakly located in $\mathsf{RCA}_0 + \neg \mathsf{WKL}_0$. Indeed, if C were weakly located in RCA_0, it would be located (see theorem 5.5), but this is not the case.

The following result shows that in the case of separably closed subsets the concepts of located and weakly located coincide in RCA_0. This is not true for closed subsets: indeed WKL_0 is needed to prove the equivalence (see remark 5.6 and theorem 5.8).

Theorem 5.7 (RCA_0)**.** *Let X be a compact complete separable metric space. Let $\overline{C}$ be a separably closed subset of X. Then $\overline{C}$ is weakly located if and only if it is located.*

Proof. Lemma 5.2 proves that located implies weakly located. The other implication follows from theorem 5.5. □

Working in stronger subsystems of second order arithmetic the notions of located and weakly located coincide; indeed in WKL_0 we to have a good theory of located sets. Theorem 5.8 summarizes the main results.

Theorem 5.8 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0.
(2) *Every closed set in a compact complete separable metric space X is weakly located.*
(3) *Every closed and separably closed set in a compact complete metric space is located.*
(4) *Every closed and separably closed set in a compact complete metric space is weakly located.*
(5) *Every closed subset of $[0, 1]$ is weakly located.*
(6) *Every closed and separably closed subset of $[0, 1]$ is located.*
(7) *Every closed and separably closed subset of $[0, 1]$ is weakly located.*

Proof. (1) $\implies$ (2). It follows from lemma 5.4.
(2) $\implies$ (5). Trivial.
(2) $\implies$ (7). Trivial.
(7) $\implies$ (6). It follows from theorem 5.7.
(5) $\implies$ (7). Trivial.
(6) $\implies$ (1). It is theorem 4.6.
(1) $\implies$ (3). It follows from theorem 4.6.
(3) $\implies$ (4). It follows from lemma 5.2.
(4) $\implies$ (3). It follows from theorem 5.7.
(3) $\implies$ (6). Trivial. □

Notice that theorem 5.8 gives a reversal of [17, theorem IV.1.7], which is described in the proof of lemma 5.4.

Remark 5.9. In theorem 5.8 we could also replace [0,1] with the Cantor space $2^{\mathbb{N}}$.

Theorem 5.10 (RCA_0). *The following are equivalent:*

(1) ACA_0.
(2) *Every separably closed set in a compact complete separable metric space X is located.*
(3) *Every separably closed set in a compact complete separable metric space X is weakly located.*
(4) *Every separably closed set in $[0, 1]$ is located.*
(5) *Every separably closed set in $[0, 1]$ is weakly located.*

Proof. (1) $\Longrightarrow$ (2). Since X is compact, theorem 4.2 assures that a separably closed subset is closed; hence by theorem 3.8 we get the conclusion.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). We prove 4.2(3) which is equivalent to ACA_0. Let $\overline{C}$ be a separably closed subset of X. By (3) $\overline{C}$ is weakly located and hence closed (see theorem 5.5).

(4) $\Longrightarrow$ (5). It follows from lemma 5.2.

(5) $\Longrightarrow$ (4). It is a special case of theorem 5.7.

(3) $\Longrightarrow$ (5). Trivial.

(5) $\Longrightarrow$ (1). As (3) $\Longrightarrow$ (1) using 4.2(5) which is equivalent to ACA_0. □

Theorem 5.11 (RCA_0). *The following are equivalent:*

(1) ACA_0.
(2) *In a compact complete separable metric space every closed and weakly located subset is separably closed.*
(3) *In $[0, 1]$ every closed and weakly located set is separably closed.*

Proof. (1) $\Longrightarrow$ (2). It is a particular case of theorem 4.2.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. We prove that its range exists. Let us consider the open set in [0,1]

$$U = \bigcup_{m=0}^{\infty} B(2^{-f(m)}, 2^{-f(m)-2})$$

Let C be the complement of U. Clearly C is closed. Moreover we prove that C is weakly located. In fact

$$\exists \varepsilon > 0 \;\; B(a, r + \varepsilon) \cap B(2^{-f(m)}, 2^{-f(m)-2}) = \emptyset \\ \iff \;\; d(a, 2^{-f(m)}) > r + 2^{-f(m)-2}$$

and since $B(2^{-f(m)}, 2^{-f(m)-2})$'s are given uniformly, we get the conclusion. Therefore, by (3), C is separably closed. We prove that we can give a Π^0_1 description of the range of f. Let $\langle c_k : k \in \mathbb{N} \rangle$ be the witnesses of the fact that C is separably closed. We have:

$$\begin{aligned} n \in \operatorname{rng}(f) &\iff B(2^{-n}, 2^{-n-2}) \subseteq U \\ &\iff \forall k\, [c_k \leq 2^{-n} - 2^{-n-2} \ \vee \ c_k \geq 2^{-n} + 2^{-n-2}] \end{aligned}$$

The equivalences above follow from the definition of U and from the fact that for all k $c_k \in C$ by definition. Therefore by Δ^0_1 comprehension the range of f exists. □

Remark 5.12. Theorem 5.11 improves theorem 4.2 because in RCA_0 there is a closed set which is not weakly located (cf. remarks 5.6 and 5.3).

We have the following characterization of the weakly located closed subsets in a compact complete separable metric space.

Corollary 5.13 (RCA_0). *Let X be a compact complete separable metric space, C a closed subset of X. The following are equivalent:*

1. *C is weakly located.*
2. *There exists a Σ^0_1 collection $\mathcal{D}$ of pairs (a, r), $a \in A$, $r \in \mathbb{Q}^+$ such that*
   (a) $(a, r) \in \mathcal{D} \implies B(a, r) \cap C = \emptyset$
   (b) $B(a, r) \cap C = \emptyset \implies (a, r/2) \in \mathcal{D}$.

Proof. (1) $\implies$ (2). Let φ be a Σ^0_1 formula such that

$$\varphi(a, r) \iff \exists \varepsilon > 0\ B(a, r + \varepsilon) \cap C = \emptyset$$

We prove that if (a, r) is such that $\varphi(a, r)$, then (a) and (b) hold. If $\varphi(a, r)$ then $B(a, r) \cap C = \emptyset$. Therefore (a) holds. If $\varphi(a, r)$, then $\varphi(a, r/2)$ and hence (b) holds.

(2) $\implies$ (1). Let $\langle \langle x_{n,m} : m \leq i_n \rangle : n \in \mathbb{N} \rangle$ witness the compactness of X. We prove that

(6)
$$\varphi(a, r) \iff \exists n\, \forall m \leq i_n\, (d(x_{n,m}, a) \leq r + 2^{-n+1} \longrightarrow (x_{n,m}, 2^{-n}) \in \mathcal{D})$$

Assume $\varphi(a, r)$, let n be such that $2^{-n} < \varepsilon/4$ and fix $m \leq i_n$. We have

$$\begin{aligned} d(x_{n,m}, a) \leq r + 2^{-n+1} &\implies B(x_{n,m}, 2^{-n+1}) \cap C = \emptyset \\ &\implies (x_{n,m}, 2^{-n}) \in \mathcal{D} \end{aligned}$$

Assume now that the right hand side of (6) holds and let $x \in B(a, r + 2^{-n})$. We prove that $x \notin C$. Since X is compact, there exists $m \leq i_n$ such that $d(x_{n,m}, x) < 2^{-n}$. By the triangular inequality we have $d(x_{n,m}, a) \leq d(x_{n,m}, x) + d(x, a) < r + 2^{-n+1}$. Hence for such m

$$(x_{n,m}, 2^{-n}) \in \mathcal{D} \implies B(x_{n,m}, 2^{-n}) \cap C = \emptyset \implies x \notin C.$$

Hence $B(a, r + 2^{-n}) \cap C = \emptyset$ and therefore $\varphi(a, r)$ for $\varepsilon = 2^{-n}$. $\square$

We might wonder under what hypothesis a closed subset is weakly located; lemma 5.14 and corollary 5.16 give an answer under some additional hypothesis either on the space or on the closed set. We use the formulation 5.13(2) of weakly locatedness.

Lemma 5.14 (RCA_0). *Let $\langle C_n : n \in \mathbb{N} \rangle$ be a sequence of weakly located closed sets in $I = [0,1]^k$ and assume that there exixts $\langle \mathcal{D}_n : n \in \mathbb{N} \rangle$, the sequence of Σ^0_1 families which witnesses the weak locatedness of the C_n's. Let $U_n = I \setminus C_n,\ n \in \mathbb{N}$ and assume that $\langle U_n : n \in \mathbb{N} \rangle$ is a sequence of pairwise disjoint open sets. Then the closed set $C = \bigcap_{n \in \mathbb{N}} C_n$ is weakly located.*

Proof. We prove that $\mathcal{D}$, the union of the Σ^0_1 families $\mathcal{D}_n$, is a Σ^0_1 family witnessing the weak locatedness of C. Indeed if $B(a, r) \cap C = \emptyset$ then $B(a, r) \subseteq \bigcup_{n \in \mathbb{N}} U_n$ and since U_n's are disjoint and we are working in I which is connected, it follows that $\exists n\ B(a, r) \subseteq U_n$. Therefore $B(a, r/2)$ is an element of the Σ^0_1 family $\mathcal{D}_n$. Also it is clear that the family $\mathcal{D}$ fulfills (a) and (b) of 5.13(2). $\square$

Let us say that a compact complete separable metric space X is *nice* if for all sufficiently small $\delta > 0$ and all $x \in X$, the open ball

$$B(x, \delta) \ = \ \{\, y \in X \ : \ d(x, y) < \delta \,\}$$

is connected. Such a δ is called a *modulus of niceness* for X. Compact spaces like [0,1], $[0, 1]^n$, $[0, 1/4] \cup [1/2, 1]$ are nice; the Cantor space is an example of compact space which is not nice (in fact it is totally disconnected).

Remark 5.15. We notice that the meaning of 5.13(2)(b) is that there exists a known number n such that $(a, r/n) \in \mathcal{D}$ (in our case we fixed $n = 2$). Therefore we can see that in nice spaces with modulus of niceness δ, we can give an equivalent reformulation of the concept of weak locatedness saying that a closed set C is weakly located if there exists a Σ^0_1 collection $\mathcal{D}$ of pairs (a, r), $a \in A$, $r \in \mathbb{Q}^+$ such that

(a) $(a, r) \in \mathcal{D} \implies B(a, r) \cap C = \emptyset$.

(b)$'$ $B(a, r) \cap C = \emptyset \implies (a, r/2^k) \in \mathcal{D}$ where k is the least such that $r/2^k < \delta$.

Indeed, to show that the riformulation is equivalent to the original definition, it is possible to prove the analogous of corollary 5.13 considering in 5.13(2) the property (b)$'$ in place of (b). On one hand, we repeat the proof of (1) $\Longrightarrow$ (2) in corollary 5.13. On the other hand, the only modification is that in the right hand side of (6) n must be such that $2^{-n} < \delta$.

The proof of lemma 5.14 can be repeated for *nice* spaces using remark 5.15 where δ is the the modulus of niceness. In fact if a ball with enough small radius (less than a fixed positive number) must be connected, then we can say that if it is included in the union of disjoint open sets, then it must be included in one of them. Therefore the following corollary holds.

Corollary 5.16 (RCA_0)**.** *Let X be a nice space. Let $\langle C_n : n \in \mathbb{N}\rangle$ be a sequence of closed weakly located sets such that their complements $U_n = X \setminus C_n$, $n \in \mathbb{N}$, are pairwise disjoint open sets. Moreover assume that there exists $\langle \mathcal{D}_n : n \in \mathbb{N}\rangle$, the sequence of Σ^0_1 families which witness the weak locatedness of the C_n's. Then $C = \bigcap_{n\in\mathbb{N}} C_n$ is a closed weakly located set.*

Remark 5.17. The hypothesis of niceness in corollary 5.16 cannot be dropped. Indeed, in the Cantor space every closed set C is of the form $C = \bigcap_{n\in\mathbb{N}} C_n$ as in corollary 5.16, provably in RCA_0. Hence it is enough to show that in the Cantor space there is a closed set which is not weakly located; this follows from remark 5.9.

6. Tietze Extension Theorem

In this section we study applications of the results obtained in the previous sections. In particular we focus on some versions of Tietze's extension theorem for compact metric spaces. For comparison we remark that the following theorem is already known (see [17, II.7.5] or [2, 1.32, page 46]).

Theorem 6.1 (RCA_0)**.** *If C is a closed set in a complete separable metric space $\widehat{A}$ and $f : C \to [a, b]$ is a continuous function, there exists a continuous function $F : \widehat{A} \to [a, b]$ such that $F \restriction C = f$, i.e. $F(x) = f(x)$ for every $x \in C$.*

To state our main result of this section (theorem 6.4) we need the following definition.

Definition 6.2 (RCA_0)**.** Let $\widehat{A}, \widehat{B}$ be complete separable metric spaces, $f : \widehat{A} \to \widehat{B}$ a uniformly continuous partial function with modulus of uniform continuity $h : \mathbb{N} \to \mathbb{N}$. We say that a code Φ for f is *uniform* if whenever $(a, r)\Phi(b, s)$ and $(a', r')\Phi(b', s')$

$$d(a, a') < 2^{-h(n)} \implies d(b, b') < 2^{-n} + s + s'$$

Definition 6.2 above describes a natural property for total functions, as the following lemma shows.

Lemma 6.3 (RCA_0)**.** *Let $X = \widehat{A}$ and $Y = \widehat{B}$ be complete separable metric spaces and let $f : X \to Y$ be a total uniformly continuous function with modulus of uniform continuity h. Then the code Φ for f is uniform.*

Proof. Let (a, r, b, s), $(a', b', r', s') \in \Phi$ be such that $d(a, a') < 2^{-h(n)}$. Since f is total, it is defined at a and a' and it assumes values $f(a)$ and $f(a')$. Using the uniform continuity of the function,

$$d(b, b') \leq d(b, f(a)) + d(f(a), f(a')) + d(f(a'), b') < s + 2^{-n} + s'$$

and therefore the proof is complete. □

We are now ready to present the main result of this section: a version of the strong Tietze theorem provable in RCA_0. The proof will follow the lines of the usual one in topology, which uses Urysohn's lemma. However to carry out that proof in RCA_0 much more work is needed. Indeed, we cannot use in RCA_0 a $\mathcal{C}(X)$-version of Urysohn's lemma (about the separation of two disjoint closed sets by a uniformly continuous function with modulus of uniform continuity) since such a version implies WKL_0 (see e.g. [7] or [14]) and therefore it is not available in RCA_0.

Theorem 6.4 (RCA_0)**.** *Let $X = \widehat{A}$ be a compact complete separable metric space, $C \subseteq X$ a weakly located closed subset, $f : C \to \mathbb{R}$ a uniformly continuous function with modulus of uniform continuity and uniform code. Then there exists $F \in \mathcal{C}(X)$ such that $F \restriction C = f$.*

In order to prove theorem 6.4, we need the following preliminary results.

Lemma 6.5 (RCA_0)**.** *If $f, g \in \mathcal{C}(X)$ then $\min\{f, g\}$, $\max\{f, g\} \in \mathcal{C}(X)$.*

Proof. Use the relations

$$\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$$

$$\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$$

and notice that sum, difference and absolute value of elements of $\mathcal{C}(X)$ is an element of $\mathcal{C}(X)$. □

Lemma 6.6 (RCA_0)**.** *Let $\sum_{k=0}^{\infty} \alpha_k$ be a convergent series of nonnegative real numbers $\alpha_k \geq 0$. Let $\langle f_k : k \in \mathbb{N} \rangle$ be a sequence of elements of $\mathcal{C}(X)$ such that $|f_k(x)| \leq \alpha_k$ for all $k \in \mathbb{N}$ and $x \in X$. Then $f = \sum_{k=0}^{\infty} f_k$ is an element of $\mathcal{C}(X)$.*

Proof. By [2, theorem 1.27 page 36] we know that f is coded as a continuous function. Therefore it is enough to prove that f has modulus of uniform continuity.

Let

$$g_m(x) = \sum_{k=0}^{m} f_k(x)$$

For all $m \in \mathbb{N}$, $g_m \in \mathcal{C}(X)$ and hence it can be viewed as a uniformly continuous function with modulus of uniform continuity h_m. For all $x \in X$

$$|f(x) - g_m(x)| = \sum_{k=m+1}^{\infty} |f_k(x)| \leq \sum_{k=m+1}^{\infty} \alpha_k$$

Since $\sum_{k=0}^{\infty} \alpha_k$ is a convergent series, for all n there exists an index $i(n)$ such that

$$\sum_{k=i(n)+1}^{\infty} \alpha_k < 2^{-n-2}.$$

Hence, by recusion we can define a function $n \mapsto i(n)$. Let us define $h : \mathbb{N} \to \mathbb{N}$ as follows:

$$h(n) = h_{i(n)}(n+1).$$

We now verify that h is modulus of uniform continuity for f. For all $x, y \in X$ such that $d(x,y) < 2^{-h(n)}$; then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - g_{i(n)}(x)| + |g_{i(n)}(x) - g_{i(n)}(y)| + |g_{i(n)}(y) - f(y)| \\ &\leq 2^{-n-2} + 2^{-n-1} + 2^{-n-2} = 2^{-n} \end{aligned}$$

Therefore the proof is complete. □

Notice that lemma 6.6 strengthens [17, lemma II.6.5].

Lemma 6.7 (RCA_0)**.** *Let $X = \widehat{A}$ be a compact complete separable metric space, $C \subseteq X$ a closed subset, $f : C \to \mathbb{R}$ a uniformly continuous partial function with modulus of uniform continuity h and uniform code Φ. Then f is bounded.*

Proof. Let $\langle B_{h(1),m} : m \leq i_{h(1)} \rangle$ be the $h(1)$-net. Within this proof B_m denotes the ball $B_{h(1),m}$. Consider the following Σ^0_1 formula:

$$\varphi(m) : \ \exists (a,r) \Phi (b,s) \ (a,r) < B_m. \tag{7}$$

By bounded Σ^0_1 comprehension, there exists a finite set $I \subseteq \{m : m \leq i_{h(1)}\}$ such that $m \in I$ if and only if $\varphi(m)$. Hence the sequence $\langle B_m : m \in I \rangle$ is such that $\bigcup_{m \in I} B_m \supseteq C$. Let m be such that $\varphi(m)$ and let $(a_m, r_m)\Phi(b_m, s_m)$ be the first witness of (7). Every point $x \in C$ is included in some ball B_m of the $h(1)$-net; since the B_m's, $m \in I$, cover

C and $\neg\varphi(m)$ may hold just for balls of the $h(1)$-net disjoint from C, it follows that $\varphi(m)$. Moreover, by hypothesis, f is defined at x. Using the properties of the code for a continuous function, $\exists(a,r,b,s) \in \Phi$ such that $x \in (a,r)$, $(a,r) < B_m$ and $s < 1$. In particular $f(x) \in (b,s)$ and $d(a,a_m) < 2^{-h(1)+1} \le 2^{-h(0)}$. The uniformity of the code implies that

$$|b - b_m| < 1 + s_m + s \le 2 + s_m.$$

Hence

$$|f(x) - b_m| \le |f(x) - b| + |b - b_m| < s + 1 + s_m \le 2 + s_m$$

Therefore for all $x \in C$

$$\begin{aligned} |f(x)| &\le |f(x) - b_m| + |b_m| \\ &\le 1 + \max\{s_m + |b_m| : m \in I\} \end{aligned}$$

□

For the proof of theorem 6.4 we shall need to repeatedly and uniformly apply the following lemma which is an *ad hoc* version of Urysohn's lemma.

Lemma 6.8 (RCA_0)**.** *Let X be a compact complete separable metric space and let $C \subseteq X$ be closed and weakly located. Let $g : C \to [-c, c]$, $c > 0$ be a uniformly continuous function with modulus of uniform continuity h and uniform code Φ. Let*

$$C_0 = g^{-1}\left(\left[-c, -\frac{1}{3}c\right]\right)$$

and

$$C_1 = g^{-1}\left(\left[\frac{1}{3}c, c\right]\right).$$

We can effectively find $G \in \mathcal{C}(X)$ with values in $[0,1]$ such that for all $i < 2$ if $x_i \in C_i$ then $G(x_i) = i$.

Proof. We start the proof giving some definitions and notations. Let q be such that $2^{-q} < 1/192\, c$, let $\ell = h(q) + 2$, let $\langle B_{\ell,m} : m \le i_\ell \rangle$ be the ℓ-net of closed balls, and let $B'_{\ell,m} = B(x_{\ell,m}, 2^{-\ell+1})$, for $m \le i_\ell$. Lemma 6.8 follows from the following

Claim 1 (RCA_0)**.** *In the hypothesis of lemma 6.8, there exists $J \subseteq \{m : m \le i_\ell\}$ such that:*

$$C_1 \subseteq \bigcup_{m \in J} B_{\ell,m} \quad \text{and} \quad C_0 \cap \bigcup_{m \in J} B'_{\ell,m} = \emptyset.$$

Assuming claim 1 the proof of lemma 6.8 is completed as follows. For every $m \in J$ let us define the basic functions

$$b_{\ell,m}(x) = \begin{cases} 1 & \text{if } x \in B_{\ell,m} \\ \frac{2^{-\ell+1}-d(x_{\ell,m},x)}{2^{-\ell}} & \text{if } x \in B'_{\ell,m} \setminus B_{\ell,m} \\ 0 & \text{if } x \notin B'_{\ell,m} \end{cases}$$

Define, forall $x \in \widehat{A}$,

$$G(x) = \max\{b_{\ell,m}(x) : m \in J\}.$$

$G \in \mathcal{C}(X)$ (see theorem 6.5) and since $C_1 \subseteq \bigcup_{m\in J} B_{\ell,m}$, for all $x \in C_1$ $G(x) = 1$. Since $\bigcup_{m\in J} B'_{\ell,m}$ is disjoint from C_0, for all $x \in C_0$ $G(x) = 0$. Therefore, assuming clim 1, the proof of lemma 6.8 is complete.

Proof of claim 1: First step. We show that there are three open sets U_0, U_1, U_2 such that:

(1) $U_0 \cup U_1 \cup U_2 \supseteq C$.
(2) $d(C_0, U_1 \cup U_2) \geq 2^{-h(q)+2}$.
(3) $d(C_1, U_0 \cup U_2) \geq 2^{-h(q)+2}$.

Notice that we shall prove (2) and (3) in a comparative sense, without assuming the existence of the code for d as continuous function.

Let

$$U_0 = \{(a,r) \in A{\times}\mathbb{Q}^+ \ : \ \exists b,s\ (a,r)\Phi(b,s) \wedge s < \frac{1}{96}c \wedge r < 2^{-\ell} \wedge b < -\frac{1}{4}c\}$$

$$U_1 = \{(a,r) \in A{\times}\mathbb{Q}^+ \ : \ \exists b,s\ (a,r)\Phi(b,s) \wedge s < \frac{1}{96}c \wedge r < 2^{-\ell} \wedge b > \frac{1}{4}c\}$$

$$U_2 = \{(a,r) \in A{\times}\mathbb{Q}^+ \ : \ \exists b,s\ (a,r)\Phi(b,s) \wedge s < \frac{1}{96}c \wedge r < 2^{-\ell} \wedge |b| < \frac{7}{24}c\}$$

be codes for open sets (indeed the formulas defining U_0,U_1, U_2 are Σ^0_1 (cf. lemma 2.3)).

We prove (1). Since g is defined at every point of C, using the properties of the code, for every $x \in C$ there exists $(a,r)\Phi(b,s)$ such that $d(a,x) < r < 2^{-\ell}$, and $s < 1/96\ c$. Therefore U_0, U_1, U_2 cover C and (1) is proved.

We prove (2). Let $x \in C_0$. Since g is defined at x, there exists $(a,r)\Phi(b,s)$ such that $x \in (a,r)$, $r < 2^{-\ell}$ and $s < 1/96\ c$; moreover notice in particular that $b < -1/3\ c$. Let $(a',r')\Phi(b',s')$ be such that $(a',r') \in U_1 \cup U_2$. By contradiction, let $d(a,a') < 2^{-h(q)+2}$. Since in the definition of U_i for $i < 3$ we have $r < 2^{-\ell}$, since the code is uniform and since $2^{-h(q)} < 2^{-h(q)+2} \leq 2^{-h(q-2)}$ (use monotonicity of h), we get:

$$|b - b'| < 2^{-q+2} + s + s' \leq \frac{1}{48}c + \frac{1}{96}c + \frac{1}{96}c = \frac{1}{24}c.$$

But on the other hand, since $b < -1/3\ c$ and $b' > -7/24\ c$, we have

$$|b - b'| > \frac{1}{24}.$$

Therefore we have got a contradiction and hence (2) follows.

We prove (3). Reason as in (2).

Second step. We use the hypothesis that C is weakly located. Let $\mathcal{D}$ be as in corollary 5.13(2). For all $m \leq i_\ell$ we prove that at least one of the following properties holds:

- $\varphi_0(\ell, m)$: $\exists (a, r) \in U_0\ (a, r) < B'_{\ell,m}$.
- $\varphi_1(\ell, m)$: $\exists (a, r) \in U_1\ (a, r) < B'_{\ell,m}$.
- $\varphi_2(\ell, m)$: $\exists (a, r) \in U_2\ (a, r) < B'_{\ell,m}$.
- $\varphi_3(\ell, m)$: $B_{\ell,m} \in \mathcal{D}$.

First notice that if $\varphi_3(\ell, m)$ fails, then $B_{\ell,m}$ intersect C; since the codes for U_0, U_1, U_2 are given in terms of the code for the (uniformly) continuous function g, they contain balls of radius arbitrarily small (in particular smaller than $2^{-\ell}$). Therefore, if x is a point in $B_{\ell,m} \cap C$, there exists $(a, r)\Phi(b, s)$ such that $x \in B(a, r)$ and $(a, r) < B_{\ell,m}$. Thus, since (1) holds, we have shown that for some $j < 4\ \varphi_j(\ell, m)$. Moreover $\varphi_j(\ell, m)$, $j < 4$ are Σ^0_1 formulas; using lemma 2.13 we get four finite sets of indices $I_0(\ell), I_1(\ell), I_2(\ell), I_3(\ell)$ such that:

- $\{m : m \leq i_\ell\} = I_0(\ell) \cup I_1(\ell) \cup I_2(\ell) \cup I_3(\ell)$
- $\forall j < 4\ m \in I_j(\ell) \implies \varphi_j(\ell, m)$ holds.

Third step: we prove that:

(α): $\varphi_0(\ell, m) \vee \varphi_2(\ell, m) \vee \varphi_3(\ell, m) \implies B_{\ell,m} \cap C_1 = \emptyset$.

(β): $\varphi_1(\ell, m) \implies B'_{\ell,m} \cap C_0 = \emptyset$.

We prove (α): If $\varphi_0(\ell, m)$, let (a, r) be a witness for $\varphi_0(\ell, m)$ and let $x \in C_1$. We have, using (3), $d(x, x_{\ell,m}) \geq d(x, a) - d(a, x_{\ell,m}) \geq 2^{-h(q)+2} - 2^{-h(q)-1} > 2^{-h(q)}$ and therefore $x \notin B_{\ell,m}$. If $\varphi_2(\ell, m)$ we reason analogously. If $\varphi_3(\ell, m)$, the ball $B_{\ell,m}$ does not intersect C and hence, for $i < 2$, it does not intersect C_i.

We prove (β): assume $\varphi_1(\ell, m)$ and let $x \in C_0$. Let $(a, r) < B'_{\ell,m}$ be a witness for $\varphi_1(\ell, m)$. Then we have: $d(x, x_{\ell,m}) \geq d(x, a) - d(a, x_{\ell,m}) \geq 2^{-h(q)+2} - 2^{-h(q)-1} > 2^{-h(q)+1}$ and therefore $x \notin B'_{\ell,m}$.

Fourth step: let us define $J = I_1(\ell)$. Properties (α) and (β) imply that

$$C_1 \subseteq \bigcup_{m \in J} B_{\ell,m} \quad \text{and} \quad C_0 \cap \bigcup_{m \in J} B'_{\ell,m} = \emptyset.$$

And the proof of claim 1 (and hence of lemma 6.8) is complete. □

We are now able to prove theorem 6.4 which is a version of the strong Tietze theorem (cf. section 1).

Proof. Since by theorem 6.7 the range of f is bounded, we may assume $f : C \to [-1, 1]$ for some. Let $c_n = (2/3)^n$. We define, by recursion, sequences $\langle F_n : n \in \mathbb{N}\rangle$ and $\langle f_n : n \in \mathbb{N}\rangle$ where $F_n \in \mathcal{C}(X)$ and $f_n : C \to [-c_n, c_n]$, for $n \in \mathbb{N}$. Let $f_0 = f$. Given f_n, let

$$C_0 = f_n^{-1}\left(\left[-c_n, -\frac{1}{3}c_n\right]\right)$$

and

$$C_1 = f_n^{-1}\left(\left[\frac{1}{3}c_n, c_n\right]\right).$$

By lemma 6.8 we can effectively find $G_n \in \mathcal{C}(X)$ such that $G_n \restriction C_i = i$. Define for $n \in \mathbb{N}$

$$F_n(x) = \frac{2}{3}\left(G_n(x) - \frac{1}{2}\right) c_n$$

and let $f_{n+1} = f_n - F_n$. Notice that $f_{n+1} = f - \sum_{k=0}^{n} F_k$. Then

- $|F_n(x)| \leq 1/3\ c_n$ for all $x \in X$.
- $|f_{n+1}(x)| \leq c_{n+1}$ for all $x \in C$.

Hence F_n's fulfill the hypothesis of lemma 6.6 and therefore the series $\sum_{n\in\mathbb{N}} F_n$ converges uniformly and the sum F of the series is an element of $\mathcal{C}(X)$. Since every F_n is total, F is a total function and $F \restriction C = f$. □

Theorem 6.9 strengthens Brown's theorem [2, 1.35 page 51].

Theorem 6.9 (RCA_0)**.** *The following are equivalent:*

(1) ACA_0.
(2) *Let X be a compact complete separable metric space, let $\overline{C}$ be a separably closed subset of X and let $f : \overline{C} \to \mathbb{R}$ be a continuous function. Then there exists a continuous function F such that $F \restriction \overline{C} = f$.*
(3) *Special case of (2) with $X = [0, 1]$.*

Proof. (1) $\Longrightarrow$ (2): Since in ACA_0 every separably closed set in a compact space is closed (see theorem 4.2), the conclusion follows from Brown's result 6.1 [2].

(2) $\Longrightarrow$ (3): Trivial.

(3) $\Longrightarrow$ (1): First we prove that (2) implies the following statement which is equivalent to WKL_0: "Let $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ be one-to-one functions such that $\forall n \in \mathbb{N}\ \forall m \in \mathbb{N}\ g_0(n) \neq g_1(m)$. Then

$\exists Y\ \forall m(g_0(m) \in Y \ \wedge\ g_1(m) \notin Y)$." (see [17, 16]). Let $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ be one-to-one functions such that $\forall n \in \mathbb{N}\ \forall m \in \mathbb{N}\ g_0(n) \neq g_1(m)$; we define in [0,1] the separably closed set

$$\overline{C} = \left\langle 2^{-g_0(n)} : n \in \mathbb{N} \right\rangle \cup \left\langle 2^{-g_1(n)} : n \in \mathbb{N} \right\rangle \cup \{0\}.$$

If $x \in \overline{C}$ then $x = 2^{-p}$ where p is equal to $g_i(m)$ for some $i < 2$ and for some $m \in \mathbb{N}$. Define the function f on $\overline{C}$ such that

$$f(2^{-p}) = \begin{cases} 2^{-p} & \text{if } \exists m\ g_0(m) = p \\ -2^{-p} & \text{if } \exists m\ g_1(m) = p \end{cases}$$

and $f(0) = 0$. f can be coded as a uniformly continuous function with modulus of uniform continuity. By (3) it can be extended to a continuous function F on [0,1]. Consider

$$Y_0 = \{p : F(2^{-p}) = 2^{-p}\} \qquad Y_1 = \{p : F(2^{-p}) = -2^{-p}\}$$

Since

$$p \in Y_0 \longleftrightarrow \forall (a,r)\Phi(b,s)\ (d(a, 2^{-p}) < r \longrightarrow d(b, 2^{-p}) \leq s),$$

it follows that Y_0 is Π^0_1. Analogously, we prove that Y_1 is Π^0_1. Hence we have two Π^0_1 and in RCA_0 we can define Y which separates the ranges of g_0 and g_1 (Π^0_1-separation: see [17]).

Now, working in WKL_0, we prove that (3) implies ACA_0. Let $\langle a_n : n \in \mathbb{N}\rangle$ be an increasing sequence of reals in [0,1] without supremum. Let us consider in $[0, 1]$ the two separably closed sets:

$$\overline{C_0} = \langle a_n : n \in \mathbb{N}\rangle \qquad \overline{C_1} = \langle b_n : n \in \mathbb{N}\rangle$$

where $b_n = (a_n + a_{n+1})/2$. Since the a_n's have no supremum, also the b_n's have no supremum and therefore $\overline{C_0}$ and $\overline{C_1}$ are disjoint. We can define on them a code Φ for the continuous function f which assumes value i on $\overline{C_i}$, $i < 2$. If f, which is defined on $\overline{C} = \overline{C_0} \cup \overline{C_1}$, could be extended to a continuous function on the whole space, since we work in WKL_0, the extension should be uniformly continuous. We show that this cannot be the case. Since a_n's and b_n's are bounded, for all $m \in \mathbb{N}$ there exists n such that $|a_n - b_n| < 2^{-m}$. Hence, in a comparative sense, the distance between $\overline{C_0}$ and $\overline{C_1}$ is null. However the values assumed by f at a_n's and at b_n's have distance 1. Therefore f is not uniformly continuous. □

Before giving another version of Tietze's extension theorem (theorem 6.14), we state the following result (cf. [17, theorem IV 1.8 and VIII.2.5]) which will be used in the proof of that theorem. Also, we will introduce the definition of $\mathcal{C}(X, h)$.

Lemma 6.10 (WKL_0)**.** *Let X be a compact space and let ψ be a Π^0_1 formula. Assume that for any fixed $n \in \mathbb{N}$ and for any finite sequence $\langle x_0, \dots, x_{n-1}\rangle$ of points of X there exists $x \in X$ such that $\psi(\langle x_0, \dots, x_{n-1}\rangle, x)$. Then there exists a sequence of points $x_n \in X$, $n \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $\psi(\langle x_0, \dots, x_{n-1}\rangle, x_n)$.*

Definition 6.11 (RCA_0)**.** Let X be a compact complete separable metric space. For all $h : \mathbb{N} \to \mathbb{N}$ we define

$$\mathcal{C}(X,h) = \{f \in \mathcal{C}(X) : \forall x\ (|f(x)| \leq 1) \\ \wedge\ \ \forall x \forall y\ (d(x,y) < 2^{-h(n)} \longrightarrow |f(x) - f(y)| \leq 2^{-n})\}$$

Notice that

$$\mathcal{C}(X) = \bigcup_{m \in \mathbb{N}} \bigcup_{h \in \mathbb{N}^{\mathbb{N}}} m \cdot C(X,h).$$

Lemma 6.12 (RCA_0)**.** *Let X be a compact complete separable metric space. Then $\mathcal{C}(X,h)$ is a compact (and in particular closed) subset of $\mathcal{C}(X)$.*

Proof. Since $\mathcal{C}(X,h)$ is defined by a Π^0_1 property, it follows that $\mathcal{C}(X,h)$ is closed. Let $\langle\langle B_{n,m} : m \leq i_n\rangle : n \in \mathbb{N}\rangle$ be the net and let us consider $\left\langle\left\langle B_{h(n),m} : m \leq i_{h(n)}\right\rangle : n \in \mathbb{N}\right\rangle$. Since any function $f \in \mathcal{C}(X,h)$ has modulus of uniform continuity h, the values of f in $B_{h(n),m}$, for every $m \leq i_{h(n)}$, differ less than 2^{-n}. For every $n \in \mathbb{N}$ and for every $m \leq i_h(n)$, let $B'_{h(n),m} = B(x_{h(n),m}, 2^{-h(n)+1})$ and let $p_{h(n),m}$ be basic functions which assume values between 0 and 1 and assume 0 out of $B'_{h(n),m}$ and 1 in the closed ball $B_{h(n),m}$. Hence there exists a natural number $j(m) \in \{0, \dots, 2^n\}$ such that

$$|f(x) - j(m)2^{-n}p_{h(n),m}(x)| \leq 2^{-n} \quad \forall x \in B_{h(n),m}.$$

To define the witnesses of the compactness of $\mathcal{C}(X,h)$, fix n and set

$$J = \{j \mid j : \{0, \dots, i_{h(n)}\} \to \{0, \dots, 2^n\}\}.$$

J is a finite set of functions. Let

$$f_{n,j}(x) = \max_{m \leq i_n}(j(m)2^{-n}p_{h(n),m}(x)).$$

It is straightforward to verify that $\langle\langle f_{n,j} : j \in J\rangle : n \in \mathbb{N}\rangle$ witnesses the compactness of $\mathcal{C}(X,h)$. □

Lemma 6.13 (RCA_0)**.** *Let X be a compact complete separable metric space and let $\overline{C} \subseteq X$ be a separably closed set. Let $g : \overline{C} \to [-c,c]$, $c > 0$ be a uniformly continuous function with modulus of uniform continuity h. Let*

$$C_0 = g^{-1}\left(\left[-c, -\frac{1}{3}c\right]\right)$$

and

$$C_1 = g^{-1}\left(\left[\frac{1}{3}c, c\right]\right).$$

Then C_0 and C_1 are coded as separably closed sets.

Let $h'(j) = j + h(q) + 2$ where q is such that $2^{-q} < 1/192\, c$. Let

$$K = \{G \in \mathcal{C}(X, h') \ : \ \forall i < 2 \ G \restriction C_i = i\}.$$

Then K is a nonempty closed set in $\mathcal{C}(X, h')$.

Proof. Let $\ell = h(q) + 2$, let $\langle B_{\ell,m} : m \leq i_\ell \rangle$ be the ℓ-net of closed balls and let Φ be a code for g as continuous function.

Let us define

$$U_0 = \left\{(a, r) \in A \times \mathbb{Q}^+ \ : \ \exists b, s \ (a, r)\Phi(b, s) \ \wedge \ s < \frac{1}{96}c \ \wedge \ r < 2^{-\ell} \ \wedge \ b < -\frac{1}{4}c\right\}$$

$$U_1 = \left\{(a, r) \in A \times \mathbb{Q}^+ \ : \ \exists b, s \ (a, r)\Phi(b, s) \ \wedge \ s < \frac{1}{96}c \ \wedge \ r < 2^{-\ell} \ \wedge \ b > \frac{1}{4}c\right\}.$$

Notice that U_i, $i < 2$, are codes for open sets. The same argument used in the first step of the proof of claim 1 proves, in a comparative sense, that

(1) $d(C_0, U_1) \geq 2^{-h(q)+2}$.
(2) $d(C_1, U_0) \geq 2^{-h(q)+2}$.

Let $B'_{\ell,m} = \overline{B}(x_{\ell,m}, 2^{-\ell+1})$, for $m \leq i_\ell$. Consider the following Σ^0_1 formulas.

- $\varphi_0(m) : \ \exists (a, r) \in U_0 \ (a, r) < B'_{\ell,m}$.
- $\varphi_1(m) : \ \exists (a, r) \in U_1 \ (a, r) < B'_{\ell,m}$.

Using bounded Σ^0_1 comprehension there exist two finite sets I_0 and I_1 such that

- $I_0, \ I_1 \subseteq \{m : m \leq i_\ell\}$.
- $\forall j < 2 \ m \in I_j(\ell) \implies \varphi_j(\ell, m)$ holds.

Hence, analogously as in claim 1, we have

- (α): $\varphi_0(m) \implies B_{\ell,m} \cap C_1 = \emptyset$.
- (β): $\varphi_1(m) \implies B'_{\ell,m} \cap C_0 = \emptyset$.

Therefore, if we define $J = I_1$, we have

$$C_1 \subseteq \bigcup_{m \in J} B_{\ell,m} \quad \text{and} \quad C_0 \cap \bigcup_{m \in J} B'_{\ell,m} = \emptyset.$$

Let us consider, for every $m \in J$, the basic functions

$$b_{\ell,m}(x) = \begin{cases} 1 & \text{if } x \in B_{\ell,m} \\ \frac{2^{-\ell+1} - d(x_{\ell,m}, x)}{2^{-\ell}} & \text{if } x \in B'_{\ell,m} \setminus B_{\ell,m} \\ 0 & \text{if } x \notin B'_{\ell,m} \end{cases}$$

and define

$$G(x) = \max\{b_{\ell,m}(x) : m \in J\} \qquad \forall x \in \widehat{A}.$$

$G \in \mathcal{C}(X)$ (theorem 6.5) and since $C_1 \subseteq \bigcup_{m\in J} B_{\ell,m}$, for $x \in C_1$, $G(x) = 1$. Since $\bigcup_{m\in J} B'_{\ell,m}$ is disjoint from C_0, for $x \in C_0$, $G(x) = 0$. Moreover the modulus of uniform continuity for the function G is $h' : \mathbb{N} \to \mathbb{N}$ defined as $h(j) = j + \ell = j + h(q) + 2$. Therefore K is nonempty.

To complete the proof it remains to prove that K is closed. We show that its complement is open. Let $G \in \mathcal{C}(X, h') \setminus K$. We may assume that there exists $x_0 \in C_0$ such that $G(x_0) > \varepsilon > 0$. Let us consider the open set $V = \{F \in \mathcal{C}(X, h') : \|G - F\| < \varepsilon/2\}$. We prove that $V \cap K = \emptyset$. Indeed for all $F \in V$ we have

$$|F(x_0)| \geq |G(x_0)| - |F(x_0) - G(x_0)| > \varepsilon - \varepsilon/2 > 0$$

and hence $F \notin K$. □

Theorem 6.14 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0.
(2) *Let X be a compact complete separable metric space, let $\overline{C}$ be a separably closed subset of X and let $f : \overline{C} \to \mathbb{R}$ be a uniformly continuous function with modulus of uniform continuity h. Then there exists an element $F \in \mathcal{C}(X)$ such that $F \restriction \overline{C} = f$.*
(3) *Special case of (2) with $X = [0, 1]$.*

Proof. (1) $\Longrightarrow$ (2). We imitate the general lines of the proof of theorem 6.4. Since by theorem 6.7 the range of f is bounded, we may assume $f : \overline{C} \to [-1, 1]$. Let $c_n = (2/3)^n$ and let q_n be (the least) such that $2^{-q_n} < 1/192\, c_n$. Let $f_0 = f$. Given f_n, let

$$C_0 = f_n^{-1}\left(\left[-c_n, -\frac{1}{3}c_n\right]\right)$$

and

$$C_1 = f_n^{-1}\left(\left[\frac{1}{3}c_n, c_n\right]\right).$$

Applying lemma 6.13 to $f_0 = f$, the set

$$K_0 = \{G \in \mathcal{C}(X, h'_0) : \forall i < 2\ G \restriction C_i = i\}$$

where $h'_0(j) = j + h(q_0) + 2$, is a nonempty and closed set in $\mathcal{C}(X, h'_0)$. Hence we choose a function $G_0 \in K_0 \subseteq \mathcal{C}(X, h'_0)$. Let us define $F_0 = 2/3\ \ (G_0(x) - 1/2)\, c_0$. Let $f_1 = f_0 - F_n$. Applying lemma 6.13 to f_1, the set $K_1 \subseteq \mathcal{C}(X, h_1)$, where $h'_1(j) = j + h(q_1) + 2$, is nonempty and closed. Hence we select $G_1 \in K_1$ to define $F_2 \in K_1$ and we set $f_2 = f_1 - F_2$. Given $f_n = f_{n-1} - F_{n-1}$ the same argument proves that

$K_n \subseteq \mathcal{C}(X, h'_n)$, where where $h'_n(j) = j + h(q_n) + 2$, is nonempty and closed. Hence we select $G_n \in K_n$ to define F_n and f_{n+1}.

Notice that we can define in advance in RCA_0 the sequence $\langle h'_n : n \in \mathbb{N} \rangle$ of moduli of uniform continuity and the sequence of compact spaces $\mathcal{C}(X, h'_n),\ n \in \mathbb{N}$.

The situation described above uses the "dependent choice principle" as stated in lemma 6.10. Indeed we can think of K_n's as closed subsets of the space $Y = \Pi_{n\in\mathbb{N}}\mathcal{C}(X, h'_n)$ which is compact (use lemma 6.12 and [17, III.2.5]). Therefore, using lemma 6.10, in WKL_0 we are able to give a sequence $\langle F_n : n \in \mathbb{N} \rangle$ of elements of $\mathcal{C}(X)$ such that $F = \sum_{n\in\mathbb{N}} F_n$ extends f (for details cf. proof of theorem 6.4).

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). Repeating the first part of the proof of (3) $\Longrightarrow$ (1) in theorem 6.9, we get (2) implies WKL_0 over RCA_0 and the proof is complete. □

At the present moment some questions remain open and these are our conjectures:

Conjecture 6.15 (RCA_0)**.** *We conjecture that the following are equivalent:*

1. WKL_0.
2. *Let X be a compact complete separable metric space, let C be a closed subset of X and let $f : C \to \mathbb{R}$ be a uniformly continuous function with modulus of uniform continuity. Then there exists an element $F \in \mathcal{C}(X)$ such that $F \restriction C = f$.*
3. *Same as (2) with "closed" replaced by "closed and separably closed".*
4. *Special case of (2) with $X = [0, 1]$.*
5. *Special case of (3) with $X = [0, 1]$.*

Using the fact that in RCA_0 (cf. theorem 6.1 and results in [2]) there exists a continuous extension which, in WKL_0, is uniformly continuous, it is immediate to prove (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) in conjecture 6.15. Notice that (2) and (3) are versions of what we called the strong Tietze theorem in section 1.

On the other hand, to prove that (5) $\Longrightarrow$ (1) is not so easy. To discuss more in detail this problem we recall some definitions and notations of recursion theory (for more details see [12] and [18]). Let $p \in \mathbb{Q}[x]$ be a polynomial with rational coefficients; the code for p is given by $\sharp(p)$. The code for $f \in \mathcal{C}[0, 1] \cap \mathrm{REC}$ is given by a sequence $\langle p_n : n \in \mathbb{N} \rangle$ of polynomials $p_n \in \mathbb{Q}[x]$ such that $\|f - p_n\| < 2^{-2n-2}$ (where $\|\ \|$ is the usual sup norm), and the function which associates $n \mapsto \sharp(p_n)$ is a (partial) recursive function in the variable n coded by its Gödel number. We assume that $\sharp : \mathbb{Q}[x] \to \mathbb{N}$ is one-to-one and onto. Let φ_e be a partial recursive function; we say that $\varphi_{e,s}(x) = y$ if $x, y, e < s$ and y is the output of $\varphi_e(x)$ in less than s steps of the Turing program P_e.

Lemma 6.16. *Let* REC *be the model of recursive sets. The strong Tietze theorem for closed and separably closed sets in* $[0,1]$ *(theorem 6.15(5)) fails in* REC.

Proof. To build the desired recursive counterexample, let us consider an enumeration $\langle \varphi_e : e \in \mathbb{N} \rangle$ of all the partial recursive functions. Let

$$s_e = \text{least natural number } s \text{ (if it exists) such that } \varphi_{e,s}(e) \downarrow$$

(i.e. s_e is the first step at which $\varphi_{e,s_e}(e)$ converges). Let $\langle (a_k, b_k) : k \in \mathbb{N} \rangle$ be a covering of the recursive reals of [0,1] with no finite subcovering. Also we may assume that for all $k \in \mathbb{N}$ $-2^{-2} < a_k < b_k < 1 + 2^{-2}$. Let us define for all $e \in \mathbb{N}$ the interval

$$I_e = \left[\frac{1}{2^{2e+1}},\ \frac{1}{2^{2e}} \right].$$

Using the linear transformation $x \mapsto (x+1)/2^{2e+1}$ we transfer to I_e the covering of [0,1] which we denote by $\langle (a_{e,k}, b_{e,k}) : k \in \mathbb{N} \rangle$.

If $\varphi_{e,s}(e) \downarrow$, define

$$J_e = I_e \setminus \bigcup_{k=0}^{s_e} (a_{e,k},\ b_{e,k})$$

If $\varphi_{e,s}(e) \downarrow$ then there exists a polynomial $p \in \mathbb{Q}[x]$ such that $\varphi_e(e) = \sharp(p)$. We define $f(x)$ on J_e as follows:

$$f(x) = \begin{cases} x & \text{if } \exists\, x_0 \in J_e \text{ such that } |p(x_0) - x_0| \geq \dfrac{1}{2^{2e+1}}\,, \\ -x & \text{if } \forall\, x_0 \in J_e \ \ |p(x_0) - x_0| < \dfrac{1}{2^{2e+1}}\,. \end{cases}$$

The domain of f is

$$\operatorname{dom}(f) = \bigcup_{\{e : \varphi_{e,s}(e)\downarrow\}} J_e \cup \{0\}.$$

Since the property which defines f is recursive, it is possible to give a code for f as recursive and uniformly continuous function with modulus of uniform continuity and uniform code.

Assume that there exists $F \in \mathcal{C}[0,1] \cap \text{REC}$ which extends f. F is coded by a recursive function φ_e such that $\varphi_e(n) = \sharp(p_n)$ where

$\langle p_n : n \in \mathbb{N}\rangle$ is a sequence of polynomials with rational coefficients such that

$$\|p_n - F\| < \frac{1}{2^{2n+2}}. \tag{8}$$

We prove that this leads to a contradiction. In fact, let $x_0 \in J_e$ and assume first that $f(x_0) = F(x_0) = x_0$. Then, by definition of f, there exists a polynomial $p \in \mathbb{Q}[x]$ such that $\varphi_e(e) = \sharp(p)$ and $|p(x_0) - x_0| \geq 2^{-2e-1}$. But also we have $\varphi_e(e) = \sharp(p_e)$ and therefore $|p_e(x_0) - x_0| > 2^{-2e-1}$, contradicting (8).

Let $x_0 \in J_e$. If $f(x_0) = F(x_0) = -x_0$ by definition of f, there exists a polynomial $p \in \mathbb{Q}[x]$ such that $\varphi_e(e) = \sharp(p)$ and $|p(x_0) - x_0| < 2^{-2e-1}$. Hence $-2^{-2e-1} + 2x_0 < p(x_0) + x_0$ and (since $x_0 \in J_e$), we have $2^{-2e-1} \leq -2^{-2e-1} + 2x_0$. But also $\varphi_e(e) = \sharp(p_e)$ and therefore $2^{-2e-1} < p_e(x_0) + x_0$, contradicting (8).

Therefore F cannot coincide with any recursive function in $\mathcal{C}(X)$ and hence we get a contradiction. Thus f has no extension in $\mathcal{C}(X) \cap$ REC. □

Lemma 6.16 implies that if we drop the hypothesis of weak locatedness, theorem 6.4 no longer holds in RCA_0. Indeed it is possible to give a recursive counterexample to theorem 6.4.

Moreover, examining carefully the proof of lemma 6.16, we are able to prove something more. Using the usual notations for recursion theory (see e.g. [12]), we define the DNR axiom, which can be stated as follows:

$$\forall A\ \exists f : \mathbb{N} \to \mathbb{N}\ f \in \mathrm{DNR}^{\mathrm{A}}$$

where we say that f is a $\mathrm{DNR}^{\mathrm{A}}$ function if

$$\forall e\ f(e) \neq \varphi_e^A(e).$$

Lemma 6.17 (RCA_0). *The strong Tietze theorem for closed and separably closed sets in* $[0,1]$ *(theorem 6.15(5)) implies the* DNR *axiom.*

Proof. For simplicity, assume $A = \emptyset$. The result for arbitrary A is routinely obtained by relativization.

We repeat the first part of the proof of lemma 6.16 to define f. Now, assume that there exists $F \in \mathcal{C}[0,1]$ which extends f. Since we are working in [0,1], F is coded by a sequence of polynomials with rational coefficients $\langle p_n : n \in \mathbb{N}\rangle$ such that $\|p_n - F\| < 2^{-2n-2}$ (for more details see [17] and [2]). Let us define the recursive function $g : \mathbb{N} \to \mathbb{N}$ as

$$g(n) = \sharp(p_n).$$

We claim that $g \in \mathrm{DNR}$, i.e. $\forall e\ g(e) \neq \varphi_e(e)$. In fact, if there exists e such that $g(e) = \varphi_e(e)$, let $x \in J_e$. The same argument as in 6.16 leads to a contradiction. □

A question naturally arises: what is the strength of the DNR axiom in the context of subsystems of second order arithmetic? Yu and Simpson [20] introduced a subsystem of second order arithmetic known as WWKL_0, consisting of RCA_0 plus the following axiom: *if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then*

$$\lim_{n\to\infty} \frac{|\{\,\sigma \in T \mid \text{length}(\sigma) = n\}|}{2^n} \;=\; 0\,. \tag{9}$$

This axiom is known as Weak Weak König's Lemma (WWKL). It is a weaker axiom than Weak König's Lemma (WKL), which reads as follows: *if T is a subtree of $2^{<\mathbb{N}}$ with no infinite path, then T is finite.* We present the following lemma which gives a partial answer to our question.

Lemma 6.18. *The* DNR *axiom can be proved in* WWKL_0.

Proof. We shall prove in WWKL_0 that there exists a DNR function. The DNR axiom is obtained similarly by relativization.

We briefly recall some notations and definitions from recursion theory which can be coded in RCA_0. Let $A \subseteq \omega$; $\overline{A}$ denotes the complement of A. $W_e = \text{dom}\varphi_e$. $W_{e,s} = \text{dom}\varphi_{e,s}$. An infinite set A is *effectively immune* if there is a recursive function p such that $\forall e\ (W_e \subseteq A \;\longrightarrow\; |W_e| < p(e))$.

Let us define

$$P = \{A \subseteq \mathbb{N} \;:\; \forall e \forall s\ (|W_{e,s}| \geq e{+}3 \;\longrightarrow\; (A{\cap}W_{e,s} \neq \emptyset \wedge \overline{A}{\cap}W_{e,s} \neq \emptyset))\}.$$

We prove that P is nonempty. Since subsets of $\mathbb{N}$ are identified with characteristic functions in $2^{\mathbb{N}}$, $P \subseteq 2^{\mathbb{N}}$; moreover P is described by a Π^0_1 formula. We equip $2^{\mathbb{N}}$ with the usual product measure μ. Following the argument in Jockusch's paper [9], we prove in WWKL_0 that the complement of P has measure at most $1/2$. Indeed, fixed any $e \in \mathbb{N}$, if $A \notin P$, then the measure of the class of such A's is at most 2^{-e-2}. Hence the measure of the complement of P is at most $\sum_{e\in\mathbb{N}} 2^{-e-2} = 1/2$. Now, let $T \subseteq 2^{<\mathbb{N}}$ be a recursive tree such that $P = \{A : A \text{ is a path through } T\}$. Since

$$\lim_{n\to\infty} \sum_{\sigma\in T,\ \text{lh}(\sigma)=n} 2^{-\,\text{lh}(\sigma)} = \mu(P) > 0$$

and since WWKL holds, there exists a path through T. Therefore P is nonempty.

We prove that P is a Π^0_1 class which contains only effectively bi-immune sets; actually, it is enough to prove that if $A \in P$ then A is effectively immune because $A \in P \longleftrightarrow \overline{A} \in P$. Assume, by contradiction, that A is not effectively immune. Then, for some e, $|W_e| \geq e + 3$ and $W_e \subseteq A$. Hence $A \notin P$.

To every effectively immune set A (which is infinite) we can associate a function which is in DNR. The following argument is due to Jockusch [10]. Let $g \leq_T A$ be such that

$$W_{g(e)} = \begin{cases} \text{the first } p(\varphi_e(e)) \text{ elements of A} & \text{if } \varphi_e(e) \text{ is defined,} \\ \emptyset & \text{if } \varphi_e(e) \text{ is undefined.} \end{cases}$$

We claim that g is a DNR function. If this is not the case, assume that $g(e) = \varphi_e(e)$; hence $W_{g(e)} = W_{\varphi_e(e)} \subseteq A$ and therefore $|W_{\varphi_e(e)}| < p(\varphi_e(e))$ which is a contradiction because $|W_{g(e)}| = p(\varphi_e(e))$. □

Lemma 6.18 allows us to interpret lemma 6.17 as a partial reversal. However, we do not know yet either if the DNR axiom implies WWKL_0 or if the strong Tietze theorem is provable in WWKL_0 or if statements 6.15(2)–(5) imply WWKL_0.

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