

# Limit Spaces and Transfinite Types

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## Abstract

We give a characterisation of an extension of the Kleene-Kreisel continuous functionals to objects of transfinite types using limit spaces of transfinite types.

## 1 Introduction

The theory of (Scott-Ershov) domains with totality has evolved in the last few years through a number of papers by Normann (and Kristiansen) [5, 9, 11] and Berger [1, 2]. One source of inspiration for this theory is Martin-Löf's type theory, and in [18] it is shown how the theory of domains with totality can be used to construct semantics for this type theory; recall that (in the usual formulations) Martin-Löf's type theory is indeed a theory of total objects. Other applications of the theory in computability theory and proof theory can be found in the papers of Normann and Berger.

Whereas Berger takes a general point of view, Normann has for the most part considered certain specific hierarchies of domains, where the notion of totality is derived from the natural notion of a total function as a function that takes total elements to total elements. Normann's hierarchies are closed under  $\Pi$ - and  $\Sigma$ -constructors, which are generalised function and product types. Thus these hierarchies admit the definition of a general notion of continuous functionals; we call these *continuous functionals of transfinite type* and consider this as a natural extension of the notion of continuous functionals of finite type as studied in e.g. [4, 8, 10].

The purpose of the present paper is to provide a topological characterisation of these functionals. For the continuous functionals of finite type a characterisation using limit spaces has been given by Scarpellini [15]. We

extend the typed hierarchy of limit spaces to the transfinite case, and show a correspondence with the continuous functionals of transfinite type.

A main tool is what we call a lifting theorem; several variants of this theorem is formulated and proved in Waagbø's thesis [17], see also [19]. For the most general hierarchy we want to consider, the lifting theorem fails for ordinary continuity in the most natural topology, but will hold for limit space continuity. A consequence is that there is no natural characterisation of the continuous functionals of transfinite type in standard topological terms. At the end of the paper we will show that there is a characterisation also in such terms for a restricted hierarchy, where the empty space is systematically avoided.

## 2 Limit Spaces

### 2.1 Basic definitions

We let a *limit space* be a set with a notion of convergency satisfying three properties. We have taken our definition from Kuratowski [6], where we use the following notation: In a sequence  $\{x_n\}_{n \leq \omega}$  we will identify 'no index' with 'index  $\omega$ '. In this context,  $x_\omega$ , or simply  $x$ , will be called *the alleged limit*.

**Definition 1** Let  $X$  be a set.

A *limit structure* on  $X$  is a relation  $x = \lim_{n \rightarrow \infty} x_n$  on the set of sequences  $\{x_n\}_{n \leq \omega}$  from  $X$  satisfying

1. If  $x = x_n$  for all but finitely many  $n$ , then  $x = \lim_{n \rightarrow \infty} x_n$
2. If  $x = \lim_{n \rightarrow \infty} x_n$  and  $\{x_{n_k}\}_{k \in \mathbb{N}}$  is a subsequence, then  $x = \lim_{k \rightarrow \infty} x_{n_k}$
3. If  $x \neq \lim_{n \rightarrow \infty} x_n$ , then there is a subsequence such that no further subsequence will have  $x$  as a limit

where we let  $x \neq \lim_{n \rightarrow \infty} x_n$  mean  $\neg(x = \lim_{n \rightarrow \infty} x_n)$ .

A *limit space* is a set with a limit structure.

**Remark 1** The convergent sequences of a topological space will form a limit structure. Conversely, the limit structure will generate a topology. There are however examples of limit structures that are not induced from any topology; in this paper we will actually consider such examples.

**Definition 2** Let  $X$  and  $Y$  be two limit-spaces:

a)  $F : X \rightarrow Y$  is *continuous* if

$$x = \lim_{n \rightarrow \infty} x_n \Rightarrow F(x) = \lim_{n \rightarrow \infty} F(x_n).$$

b) Let  $X \rightarrow Y$  be the set of continuous functions.

c) We organise  $X \rightarrow Y$  to a limit space by

$$F = \lim_{n \rightarrow \infty} F_n$$

if

$$F(x) = \lim_{n \rightarrow \infty} F_n(x_n)$$

whenever  $x = \lim_{n \rightarrow \infty} x_n$ .

**Remark 2** It is easy to verify that this defines a limit structure. If  $X$  and  $Y$  are metric spaces, this definition says that  $F$  is the pointwise limit of the equicontinuous family  $\{F_n\}_{n \in \mathbb{N}}$ .

$\mathbb{N}$  is a limit space with the ‘almost constant’ sequences as the only convergent ones. Then every finite type  $\sigma$  can be interpreted as a limit space  $L(\sigma)$  following the definition above.

**Proposition 1 ( Scarpellini [15])**

*Let  $Ct(\sigma)$  be the Kleene-Kreisel continuous functionals of type  $\sigma$ .*

*Then  $L(\sigma) = Ct(\sigma)$  as sets.*

Our main theorem will be a generalisation of this characterisation.

## 2.2 Parameterisations

**Definition 3** Let  $X$  be a limit space and  $F$  a map from  $X$  to the class of limit spaces.

a) A *limit structure* on  $(X, F)$  is a limit structure on the set  $\{(x, y) \mid x \in X \wedge y \in F(x)\}$  such that

1.  $(x, y) = \lim_{n \rightarrow \infty} (x_n, y_n) \Rightarrow x = \lim_{n \rightarrow \infty} x_n$ .

2. If  $x = x_n$  for almost all  $n$ , then  $(x, y) = \lim_{n \rightarrow \infty} (x_n, y_n)$  if and only if  $y = \lim_{n \rightarrow \infty} y_n$  in  $F(x)$ .

We then let  $\Sigma(X, F)$  be this set of ordered pairs with the limit structure.

- b) A *limit parameterisation* is a parameterisation of limit spaces with a limit structure. We will only use the term *parameterisation* when this cannot cause any confusion.
- c) A *parameterisation of parameterisations* is a tripple  $(X, F, G)$  such that  $(X, F)$  is a parameterisation and  $(\Sigma(X, F), G)$  is a parameterisation.

This definition is inspired from Berger [2].

We make the trivial observation

**Lemma 1** *If  $(X, F, G)$  is a parameterisation of parameterisations and  $x \in X$ , then  $(F(x), \lambda y \in F(x).G(x, y))$  is a parameterisation.*

We may view dependent products and dependent sums as operators from the class of parameterisations to the class of limit spaces, but we may also view them as operators from the class of parameterisations of parameterisations to the class of parameterisations:

**Definition 4 a)** Let  $(X, F)$  be a parameterisation. We let  $\Pi(X, F)$  be the set of functions  $f$  defined on  $X$  such that

1.  $f(x) \in F(x)$  for all  $x \in X$ .
2. If  $x = \lim_{n \rightarrow \infty} x_n$  then  $(x, f(x)) = \lim_{n \rightarrow \infty} (x_n, f(x_n))$ .

We let  $f = \lim_{n \rightarrow \infty} f_n$  if  $(x, f(x)) = \lim_{n \rightarrow \infty} (x_n, f_n(x_n))$  whenever  $x = \lim_{n \rightarrow \infty} x_n$ .

- b) If  $(X, F, G)$  is a parameterisation of parameterisation, we define  $(X, \Sigma(F, G))$  by

$$\Sigma(F, G)(x) = \Sigma(F(x), \lambda y \in F(x).G(x, y))$$

with the obvious limit structure.

- c) If  $(X, F, G)$  is a parameterisation of parameterisations, we define  $(X, \Pi(F, G))$  by

1.  $\Pi(F, G)(x) = \Pi(F(x), \lambda y \in F(x).G(x, y))$
2.  $(x, f) = \lim_{n \rightarrow \infty} (x_n, f_n)$  if  $x = \lim_{n \rightarrow \infty} x_n$  and for all subsequences  $\{x_{n_k}\}_{k \in \mathbb{N}}$ , all  $y \in F(x)$  and all sequences  $\{y_{n_k}\}_{k \in \mathbb{N}}$  with  $y_{n_k} \in F(x_{n_k})$  we have

$$(x, y) = \lim_{k \rightarrow \infty} (x_{n_k}, y_{n_k}) \Rightarrow (x, y, f(y)) = \lim_{k \rightarrow \infty} (x_{n_k}, y_{n_k}, f_{n_k}(y_{n_k})).$$

**Remark 3** Part c),2. of this definition may seem unnecessarily complicated since we demand diagonal convergency for any subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  and sequence  $\{y_{n_k}\}_{k \in \mathbb{N}}$ , but it is not. We have to foresee the possibility that some  $F(x_n) = \emptyset$ . In this case there will be no sequence  $\{y_n\}_{n \in \mathbb{N}}$  with  $y_n \in F(x_n)$  for all  $n$ . A careless definition will in this situation have the convergency of all sequences  $\{(x_n, f_n)\}_{n \in \mathbb{N}}$  to  $(x, f)$ , as long as  $x = \lim_{n \rightarrow \infty} x_n$ , as a consequence. We don't want this. We would actually not be able to verify axiom 2 for limit structures if we tried to take the easy way out here.

If we let  $G$  be the constant parameterisation  $G(x, y) = X$ , we may view  $(X, F, G)$  as a parameterisation of parameterisations in the obvious way. We let  $(X, F \rightarrow X)$  denote the parameterisation  $(X, \Pi(F, G))$ .

**Definition 5** Let  $(X, F)$  be a parameterisation.

We let  $par(X, F) = \Sigma(X, F \rightarrow X)$

As the terminology indicates,  $par(X, F)$  will represent a set of parameterisations. We give the precise definition, including the limit structure, below:

**Definition 6** If  $(x, f) \in par(X, F)$ ,  $(x, f)$  will induce a parameterisation  $(F(x), F \circ f)$  where  $(y, z) = \lim_{n \rightarrow \infty} (y_n, z_n)$  if  $(x, y) = \lim_{n \rightarrow \infty} (x, y_n)$  and  $(f(y), z) = \lim_{n \rightarrow \infty} (f(y_n), z_n)$ , both limits in the sense of the parameterisation  $(X, F)$ .

### 2.3 The transfinite hierarchy of limit spaces

If  $X_1, \dots, X_n$  are limit spaces, we may form the disjoint union

$$X_1 \oplus \dots \oplus X_n = (\{1\} \times X_1) \cup \dots \cup (\{n\} \times X_n)$$

with the obvious limit structure.

We let  $X_0 = \{*\}$  be the one point limit space. We also consider  $\emptyset$  and  $\mathbb{N}$  as limit spaces.

**Theorem 1** *There is a minimal solution to the following equations for a parameterisation  $(T, L)$ :*

1.  $T = X_0 \oplus X_0 \oplus \text{par}(T, L) \oplus \text{par}(T, L)$ .
2.  $L(1, *) = \emptyset$ .
3.  $L(2, *) = \mathbb{N}$
4.  $L(3, (t, F)) = \Sigma(L(t), L \circ F)$ .
5.  $L(4, (t, F)) = \Pi(L(t), L \circ F)$ .

*Proof*

We prove this theorem by constructing  $T$  by transfinite recursion over the ordinals. Let  $T_\alpha$  be the part of  $T$  constructed at level  $\alpha$ .

For each  $t \in T_\alpha$  we will define  $L(t)$  independently of  $\alpha$ .

Each  $T_\alpha$  will be equipped with a limit structure, but we will permit sequences from  $T_\alpha$  with an alledged limit in  $T_\alpha$  that does not converge to the limit in the sense of  $T_\alpha$  to converge in the sense of  $T_\beta$  for some  $\beta > \alpha$ .

Actually, the five clauses are clauses for an inductive definition of a parameterisation. This gives us the following construction of  $(T_\alpha, L_\alpha)$ , where  $L_\alpha$  obeys points 2. - 5. in the statement of the theorem.

1.  $T_0 = X_0 \oplus X_0 \oplus \emptyset \oplus \emptyset$ .
2.  $T_{\alpha+1} = X_0 \oplus X_0 \oplus \text{par}(T_\alpha, L_\alpha) \oplus \text{par}(T_\alpha, L_\alpha)$ .
3. If  $\alpha$  is a limit ordinal, we let  $T_\alpha = \bigcup_{\beta < \alpha} T_\beta$  and  $L_\alpha = \bigcup_{\beta < \alpha} L_\beta$ . We let the convergent sequences of the union structures be those that are bounded in rank below some  $\beta < \alpha$  and converge in the sense of  $T_\beta$  or  $(T_\beta, L_\beta)$ .

In our definition of the new limit parameterisation, we use convergency negatively. This will, however, only be convergency within one of the limit spaces  $L(t)$ , and as soon as  $t \in T_\alpha$ , the limit structure of  $L(t)$  stays fixed. It follows that our construction is positive, we enlarge the sets of points and we enlarge the sets of convergent sequences.

It is an easy task to prove that the limit space axioms are preserved under unions of increasing families of limit spaces. If a sequence diverges because it has unbounded rank, there will be a subsequence such that any further subsequence also will have unbounded rank.

### 3 Domains with totality

In Normann [9] we constructed a transfinite hierarchy of domains with totality based on the flat domain of natural numbers and closed under dependent products. Density of the total objects in each of the domains was established. In Kristiansen and Normann [5] this construction and theorem was extended to also cover dependent sums. Berger [2, 3] gave a smooth reformulation of these constructions and extended them to cover universe operators relative to certain constructors, dependent sums, products and the universe operators themselves are in this class of constructors. He extended the density theorem to all these hierarchies. In Normann [11] we included the singleton domain with no total objects in the hierarchy and closed it under dependent sums and products. For this new base type, the density property trivially fails, so it fails as a general property of this extended hierarchy. Using terminology inspired from [2, 3] we will indicate how this hierarchy is constructed and show that it is essentially equivalent to the limit space hierarchy constructed in the previous section. In this proof, the lifting theorems of Waagbø [17, 19] are instrumental.

#### 3.1 The underlying domains

We will let a *domain* be an algebraic domain, or a Scott-Ershov domain as defined in Stoltenberg-Hansen & al. [16]. This means that we have a partially ordered set of *compacts* where all finite bounded sets have least upper bounds, and that this set is a subordering of the domain consisting exactly of the least upper bounds of directed sets of compacts. If  $D$  is a domain we let  $\sqsubseteq$  denote the ordering.

Following Stoltenberg-Hansen & al. [16] the class of domains is organised to a category  $DOM$  where  $\eta : D \rightarrow E$  is a morphism if there is a map  $\eta^- : E \rightarrow D$  such that

$$\eta^- \circ \eta = id_D \wedge \eta \circ \eta^- \sqsubseteq id_E.$$

$\eta^-$  is uniquely determined from  $\eta$  in this case.

Berger [2, 3] use the more restricted class of *good embeddings*, and actually all embeddings we will construct will be good, but we will not make any use of this fact here.

Following Palmgren and Stoltenberg-Hansen [14] we let a functor be *continuous* if it commutes with direct limits.

A *parameterisation* of domains will be a pair  $(X, F)$  where  $X$  is a domain (considered as a category with one morphism  $i_{x,y}$  exactly when  $x \sqsubseteq y$ ) and  $F$  is a continuous functor from  $X$  to  $DOM$ .

Given a parameterisation  $(X, F)$  we may form the set

$$\Sigma(X, F) = \{(x, y) \mid x \in X \wedge y \in F(x)\}$$

with the canonical ordering. This construction is due to Palmgren and Stoltenberg-Hansen [14], and they also proved that  $\Sigma(X, F)$  is a domain.

Now we define a parameterisation of parameterisations to be a triple  $(X, F, G)$  in complete analogy with the definition in the limit space case. The classes of parameterisations and of parameterisations of parameterisations are organised to categories in a natural way. This was first considered by Palmgren (unpublished) and is elaborated in Berger [2, 3].

At this level of generality Palmgren and Stoltenberg-Hansen also defined dependent products  $\Pi(X, F)$  of parameterisations. The constructions of dependent sums and of dependent products are functorial in the sense that they define continuous functors from the category of parameterisations of parameterisations to the category of parameterisations. See [2, 3] for details. As a consequence of all this we obtain

**Theorem 2** *The following set of equations have a minimal domain parameterisation  $(S, I)$  as a solution:*

1.  $S = \{*\}_\perp \oplus \{*\}_\perp \oplus \Sigma(S, I \rightarrow S) \oplus \Sigma(S, I \rightarrow S)$ .
2.  $I(1, *) = \{\perp\}$ .
3.  $I(2, *) = \mathbb{N}_\perp$ .
4.  $I(3, (s, F)) = \Sigma(I(s), I \circ F)$ .
5.  $I(4, (s, F)) = \Pi(I(s), I \circ F)$ .

**Remark 4** The essentials of this construction can be found in a number of papers. The constructions in [5] and in [11] are quite explicit. Berger [2, 3] introduce the terminology used here. Waagbø [17, 19] also views this construction as the limit of a continuous procedure.

### 3.2 The well founded hierarchy

From now on we will use the following notation for the elements of  $S$ , and for the elements of the limit space  $T$  of the previous section:

1.  $O$  for  $(1, *)$
2.  $N$  for  $(2, *)$
3.  $(\sigma, s, F)$  for  $(3, (s, F))$
4.  $(\pi, s, F)$  for  $(4, (s, F))$ .

In the flat domain  $\mathbb{N}_\perp$  of natural numbers, we will consider the genuine natural numbers as the *total* objects, while  $\perp$  denotes ‘the undefined’.

**Definition 7** By transfinite induction we may isolate the well founded elements  $S_{\text{wf}}$  of  $S$  and the total elements  $\bar{I}(s)$  for  $s \in S_{\text{wf}}$ :

1.  $O \in S_{\text{wf}}$ .  $\bar{I}(O) = \emptyset$ .
2.  $N \in S_{\text{wf}}$ .  $\bar{I}(N) = \mathbb{N}$ .
3. If  $t = (\sigma, s, F) \in S$ ,  $s \in S_{\text{wf}}$  and  $F(x) \in S_{\text{wf}}$  for all  $x \in \bar{I}(s)$ , then  $t \in S_{\text{wf}}$ .  
 $(x, y) \in \bar{I}(t)$  if  $x \in \bar{I}(s)$  and  $y \in \bar{I}(F(x))$ .
4. If  $t = (\pi, s, F) \in S$ ,  $s \in S_{\text{wf}}$  and  $F(x) \in S_{\text{wf}}$  for all  $x \in \bar{I}(s)$ , then  $t \in S_{\text{wf}}$ .  
 $f \in \bar{I}(t)$  if  $f(x) \in \bar{I}(F(x))$  whenever  $x \in \bar{I}(s)$ .

We let  $\sqcap$  denote the greatest lower bound. The following was proved in Normann [11]:

**Theorem 3** *There are equivalence relations  $\sim$  on  $S_{\text{wf}}$  and  $\approx$  on  $\Sigma(S_{\text{wf}}, \bar{I})$  satisfying*

1.  $s \sim O$  if and only if  $s = O$ ,  $s \sim N$  if and only if  $s = N$  and  $(s, y) \approx (N, n)$  if and only if  $(x, y) = (N, n)$ .

2.  $t \sim s = (\sigma, s', F)$  if and only if  $t$  is of the form  $(\sigma, t', G)$ ,  $t' \sim s'$  and whenever  $(t', x) \approx (s', y)$ , then  $F(x) \sim G(y)$ .  
Moreover  $(t, (y_1, y_2)) \approx (s, (x_1, x_2))$  if and only if  $(t', y_1) \approx (s', x_1)$  and  $(G(y_1), y_2) \approx (F(x_1), x_2)$ .
3.  $t \sim s = (\pi, s', F)$  if and only if  $t$  is of the form  $(\pi, t', G)$ ,  $t' \sim s'$  and whenever  $(t', x) \approx (s', y)$ , then  $F(x) \sim G(y)$ .  
Moreover  $(t, g) \approx (s, f)$  if and only if  $(G(y), g(y)) \approx (F(x), f(x))$  whenever  $(t', y) \approx (s', x)$ .

$\sim$  and  $\approx$  correspond to extensional equality, and that every function-like total object of this hierarchy is extensional. In the proof we extend a method of proof due to Longo and Moggi [7] to transfinite types.

**Definition 8** By transfinite recursion we now construct the *set of types*  $T_D$ , the interpretation  $Tp(t)$  for each  $t \in T_D$  as the image of the collapsing function  $\rho : S_{\text{wf}} \rightarrow T_D$  and the collapsing functions  $\rho_s : \bar{I}(s) \rightarrow Tp(\rho(s))$  as follows:

- $\rho(0) = 0$ ,  $\rho(N) = N$  and  $\rho_N(n) = n$ .
- Let  $\tau$  be  $\sigma$  or  $\pi$ .  
 $\rho(\tau, s, F) =$   
 $(\tau, \rho(s), \lambda x \in Tp(\rho(s)).\rho(F(\rho_s^{-1}(x))))$ .
- $\rho_{(\sigma, s, F)}(x, y) = (\rho_s(x), \rho_{F(x)}(y))$ .
- $\rho_{(\pi, s, F)}(f) =$   
 $\lambda x \in \rho(s).\rho_{F(\rho_s^{-1}(x))}(f(\rho_s^{-1}(x)))$ .

**Remark 5** We have used the notation  $\rho_s^{-1}$  even if the  $\rho$ 's are not invertible functions. We can do this because it does not matter which element in the inverse image we use, since  $\rho$  only identifies equivalent objects, and all parameterisations and functions will respect these equivalence relations. There is in general no way to find a continuous selection function for  $\rho_s^-$ .

**Theorem 4 (The Main Theorem)**

$T_D$  is the set of elements in the limit structure  $T$ .

Moreover, for all  $t \in T$  we have that  $Tp(t)$  is the set of elements in  $L(t)$ .

In the next section we will prove the Main Theorem by adding limit structures to  $T_D$  and  $Tp(t)$  and prove that we actually get  $T$  resp.  $L(t)$ .

### 3.3 The Lifting Theorems

**Definition 9** Let  $t \in T_D$  and let  $\{t_n\}_{n \in \mathbb{N}}$  be a sequence from  $T_D$ .

We call  $t$  a *strong limit* of  $\{t_n\}_{n \in \mathbb{N}}$  if there is a sequence  $\{s_n\}_{n \in \mathbb{N}}$  from  $S_{\text{wf}}$  converging to  $s \in S_{\text{wf}}$  such that  $\rho(s) = t$  and  $\rho(s_n) = t_n$  for all  $n \in \mathbb{N}$ .

In this case we call the sequence  $\{s_n\}_{s \leq \omega}$  a *lifting* of the sequence  $\{t_n\}_{n \leq \omega}$ .

We will use the analogue definition of strong limits and liftings for sequences from  $Tp(\rho(s))$  and sequences from  $\Sigma(T_D, Tp)$ .

$T_D$  and each  $Tp(t)$  will have topologies inherited as quotient topologies of the underlying domains. In Normann [12] it is shown that in general there are more sequences that will converge in the topological sense than in the strong limit sense, i.e. there are convergent sequences in  $T_D$  than cannot be lifted. This shows that we cannot characterise this hierarchy of topological spaces in any natural way, interpreting function spaces as the set of continuous functions. However, as a part of the proof of the Main Theorem we will show that strong limit is indeed the notion of convergency we defined in the limit space approach.

A domain  $X$  is *separable* if the set of compacts is countable. All domains considered in the present paper are separable. We consider a subset  $\bar{X}$  of a separable domain to be a *totality* if

$$x \in \bar{X} \wedge x \sqsubseteq y \in X \Rightarrow y \in \bar{X}.$$

The following theorem is proved by Waagbø in [17, 19].

#### Theorem 5

- a) Let  $(X, \bar{X})$  be a separable domain with totality and let  $F : \bar{X} \rightarrow T_D$  be such that  $F$  maps convergent sequences with limits in  $\bar{X}$  to sequences with strong limits in  $T_D$ .

Then there is a total, continuous  $\hat{F} : X \rightarrow S$  such that for all  $x \in \bar{X}$ ,

$$F(x) = \rho(\hat{F}(x)).$$

- b) The same as a) for  $F : \bar{X} \rightarrow \Sigma(T_D, Tp)$ .

We will call  $\hat{F}$  a *lifting* of  $F$ .

## 4 Proof of the Main Theorem

We have not yet established that  $(T_D, Tp)$  with strong limits satisfies the axioms of limit spaces. Nevertheless, we may consider this as a *structure of limits*, i.e. as a set where some of the sequences are claimed to be convergent to some of the points. It is indeed trivial to verify the two first properties of limit structures, while the third actually will be a consequence of our characterisation. Now the paparameterisation  $(T, L)$  is the fixpoint of an operator  $\Gamma$  on limit parameterisations. This operator  $\Gamma$  may be applied on the parameterisation  $(T_D, Tp)$  with strong limits as well.

**Lemma 2**  $\Gamma(T_D, Tp) \subseteq (T_D, Tp)$  as a set parameterisation.

*Proof*

$\Gamma(T_D, Tp)$  is a parameterisation  $(U, V)$  where

$$U = X_0 \oplus X_0 \oplus \text{par}(T_D, Tp) \oplus \text{par}(T_D, Tp).$$

Let us focus on  $(t, F) \in \text{par}(T_D, Tp)$ .

By Definition 5 then,  $t \in T_D$  and

$F : Tp(t) \rightarrow T_D$  will map a strongly convergent sequence in  $Tp(t)$  to a strongly convergent sequence in  $T_D$ .

Let  $s \in S_{\text{wf}}$  with  $\rho(s) = t$ , and let  $G : \bar{I}(s) \rightarrow T_D$  be defined by

$G(x) = F(\rho_s(x))$ . Then  $G$  will map a convergent sequence to a strongly convergent sequence, and by the lifting theorem,  $G$  has a lifting  $\hat{G}$ .

Then  $\rho(\sigma, s, \hat{G}) = (\sigma, t, F)$  and  $\rho(\pi, s, \hat{G}) = (\pi, s, G)$

We show that  $V(\pi, t, F) \subseteq Tp(\pi, t, F)$  by a similar argument. For the other parameters  $t'$ , the equality  $V(t') = Tp(t')$  is trivial.

This ends the proof of the lemma.

**Lemma 3**  $(T_D, Tp) \subseteq \Gamma(T_D, Tp)$  as a set parameterisation.

*Proof*

This is trivial, every object that is inherited from a continuous domain object will map sequences with liftings to sequences with liftings. We do not give the details.

**Lemma 4**  $(T_D, Tp)$  seen as a parameterisation with a structure of limits is a fix-point of  $\Gamma$ .

*Proof*

By induction on the alledged limit, we show that  $t = \lim_{n \rightarrow \infty} t_n$  in the sense of  $\Gamma(T_D, Tp)$  if and only if  $\{t_n\}_{n \leq \omega}$  has a lifting to a convergent sequence. That a sequence with lifting converges in the sense of the limit space construction is as easy as Lemma 3.

For the other direction we consider the case of parameterisations; the rest of the arguments needed for the full proof follow the same or simpler ideas.

Let  $(t, F) = \lim_{n \rightarrow \infty} (t_n, F_n)$  in the sense of the limit space definition of  $par(T_D, Tp)$  where  $(T_D, Tp)$  is considered with strong limits. In particular  $t = \lim_{n \rightarrow \infty} t_n$ , and by the induction hypothesis there is a lifting  $\{s_n\}_{n \leq \omega}$ .  $\omega + 1$  as a topological space can be represented as the total elements  $\bar{\Omega}$  of a separable domain  $\Omega$  quite easily.

One standard way to do so is to introduce two compacts  $n$  and  $n_c$  for each number  $n \in \mathbb{N}$ . We let the ordering of the  $n$ 's be the standard ordering,  $n < m_c \Leftrightarrow n \leq m$  and we let the  $n_c$ 's be completely unordered.

$\Omega$  will then be the set of ideals and  $\bar{\Omega}$  the set of maximal ideals.

From now on we let  $n, \omega$  denote the corresponding elements in  $\bar{\Omega}$ .

The convergent sequence  $\{s_n\}_{n \leq \omega}$  can be represented by a continuous function  $s^* : \Omega \rightarrow S$ .

Let

$$X = \Sigma(x \in \Omega)I(s^*(x))$$

with the canonical totality. Let  $(n, x) \in \bar{X}$  if  $n \leq \omega$  and  $x \in \bar{I}(s_n)$ .

Let  $G(n, x) = F_n(\rho(x))$ . By the lifting theorem,  $G$  has a lifting  $\hat{G}$ .

Then

$$(s_\omega, \lambda x. \hat{G}(\omega, x)) = \lim_{n \rightarrow \infty} (s_n, \lambda x. \hat{G}(n, x))$$

in the sense of domains, and we have produced a lifting of the sequence  $\{(t_n, F_n)\}_{n \leq \omega}$ .

This ends the proof of the lemma.

The three lemmas together show that  $(T_D, Tp)$  with the strong limit structure is a fix-point of  $\Gamma$ . It is now easy to show by induction on the rank of the elements of  $T_D$  that they will be in the least fix-point and that all strongly convergent sequences are convergent in the sense of the least fix-point. This concludes our proof of the main theorem.

**Remark 6** The sets  $T_D$  and  $Tp(t)$  have topologies inherited from the underlying domains. The Main Theorem actually provides us with characteri-

sations of those topologies in an indirect way:

$O \subseteq T_D$  is open if and only if for every convergent sequence  $t = \lim_{n \rightarrow \infty} t_n$  from  $T$ , if  $t \in O$ , then  $t_n \in O$  for almost every  $n$ . The topology on  $Tp(t)$  is characterised in a similar way. As remarked in the discussion just after Definition 9 this does not imply a direct characterisation of the topological spaces  $T_D$  and  $Tp(t)$ .

## 5 Omitting the empty type

In this section we will consider the special case where we will not include the empty limit space as a base space in the hierarchy. Let  $T^*$  be limit space of parameters in this case. Let  $L^*$  be the corresponding parameterisation of limit spaces. We may also define the restricted analogues  $S^*$ ,  $S_{\text{wf}}^*$  and  $I^*$  with totality  $\bar{I}^*$ . We then use the argument of the Main Theorem and obtain

**Corollary 1**  $T^* = T_D^*$  and  $L^* = Tp^*$ .

In this case, we may actually use the induced topology on  $T_D^*$  and the typed parameterisation. Waagbø [17, 19] showed that in this case a sequence will converge in the strong sense if and only if it converges with respect to the induced topology, and that every continuous function  $F : \bar{X} \rightarrow T_D^*$  has a lifting. The same of course holds for maps into the parameterisation and into the individual types. Since these topologies are again definable from the convergent sequences as the finest one accepting all these convergencies, we have a characterisation of the parameterisation  $(T^*, L^*)$  of topological spaces via characterising the convergent sequences of these topologies. In fact,  $T^* \subseteq T$  and  $L^*$  is  $L$  restricted to  $T^*$ .

The parameterisation  $(S_{\text{wf}}^*, \bar{I})$  has a rich fine structure, including effective density ([2, 3, 5, 9]). By Corollary 1 some of these properties will transfer to the limit space parameterisation  $(T^*, L^*)$ . However, we view the limit space construction as simpler than the domain approach, and the domain approach permits us to define concepts from recursion theory for these spaces, concepts that are not easily defined in the pure limit space approach. One consequence of our characterisations is that the limit spaces  $T$ ,  $T^*$  and the parameterisations are separable in the sense that there are countable dense subsets in the obvious sense.

There is an alternative characterisation of the parameterisation  $(T_D, Tp)$  using reduced products of finite type structures. This is shown in Normann, Palmgren and Stoltenberg-Hansen [13].

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