A DISCOURSE ON THREE COMBINATORIAL DISEASES

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Frank Harary in his article [2] was the first to speak of combinatorial diseases (actually, he spoke of graphical diseases). His original diseases were the four-colour disease, the hamiltonian disease and the reconstruction disease. Shortly thereafter, Read and Corneil [9] have discussed the graph isomorphism disease, and somewhat later, Huang, Kotzig and Rosa [4] spoke of a labelling disease.

When does a combinatorial problem become a disease? Certainly the extreme ease of formulating the problem has something to do with it: most identified "diseases" are understandable to undergraduates or even to good high school students. They are highly contagious, and so they attract the attention of not only professional mathematicians but also of scores of layman mathematicians.

In what follows I would like to discuss briefly three of my favourite diseases: the labelling disease, the Erdös-Faber-Lovász disease, and Buratti's problem.

1. The labelling disease

The origins of the labelling disease go back to the Smolenice conference in 1963 where Gerhard Ringel [10] proposed the following problem, known today as the *Ringel's conjecture.*

Ringel's conjecture (RC). The complete graph K_{2n+1} with $2n + 1$ vertices can be decomposed into $2n + 1$ subgraphs, each isomorphic to a given tree with n edges.

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Shortly thereafter, graph labellings were introduced as means intended to attack the Ringel's conjecture. Of these, the graceful labelling gained most prominence.

A *graceful labelling* of a graph $G = (V, E)$ with $|E| = n$ is a $1-1$ mapping $\phi : V \to \{0, 1, \dots, n\}$ such that the induced edge-labelling $\bar{\phi} : E \to \{1, 2, \dots, n\}$ given by $\bar{\phi}(\{u, v\}) = |\phi(u) - \phi(v)|, \{u, v\} \in E$, is onto.

Graceful labelling was originally called β-labelling; the name "graceful" is due to S. Golomb.

Among the related labellings defined at the same time (cf. [11]) were ρ-labelling, σ-labelling, and α-labelling.

Given a graph $G = (V, E)$ with n edges and a mapping $\phi : V \to N$ (the set of nonnegative integers), consider the following conditions:

(a) $\phi(V) \subseteq \{0, 1, \dots, n\}$

(b) $\phi(V) \subseteq \{0, 1, \dots, 2n\}$

(c) $\phi(E) = \{1, 2, \dots, n\}$

(d) $\phi(E) = \{x_1, x_2, \dots, x_n\}$ where $x_i = i$ or $x_i = 2n + 1 - i$

(e) there exists x such that either $\phi(u) < x < \phi(v)$ or $\phi(v) < x < \phi(u)$ whenever $\{u, v\} \in E$.

Then α-labelling satisfies (a),(c) and (e), β-labelling (=graceful) satisfies (a) and (c), σ-labelling satisfies (b) and (c), and ρ-labelling satisfies (b) and (d).

The hierarchy of these labellings is apparent, i.e. α-labelling is the "strongest" and ρ-labelling the "weakest".

Among the conjectures formulated after the Ringel's conjecture appeared in print, the most prominent ones are:

Kotzig's conjecture (KC). The complete graph K_{2n+1} can be *cyclically* decomposed into $2n + 1$ subgraphs isomorphic to a given tree with n edges.

Graceful tree conjecture (GTC). Every tree has a graceful labelling.

The ρ-labelling conjecture (ρC). Every tree has a ρ-labelling.

The relationship of labellings to decompositions is exemplified by the following theorem.

Theorem. *The complete graph K_{2n+1} can be cyclically decomposed into $2n+1$ subgraphs isomorphic to a graph G with n edges if and only if G has a ρ-labelling.*

Thus, obviously, GTC implies KC which is equivalent to ρC which in turn implies RC.

Among the labellings mentioned, only α-labelling has the stronger property, that, namely, if a graph G with n edges has an α-labelling then there exists a decomposition of K_{2kn+1} into copies of G, for all $k = 1, 2, \ldots$. Consequently, effort has been put into defining and investigating several types of "intermediate" labellings which, while somewhat weaker than α-labellings, still achieve as much as far as decompositions are concerned. Authors that need to be mentioned in this connection include Roberto Frucht, Charles Vanden Eynden, Saad El-Zanati, Dalibor Fronček, Jozef Širáň, and others.

There are several clases of trees that have been shown to possess either graceful labellings or some of the 'weaker' labellings (see, e.g., [1] for an up-to-date survey).

Of course, there is wide belief that all of the above mentioned conjectures, in particular, the RC and the GTC are true. But there are some differences, as far as the current state of affairs is concerned. For example, it has been proved that GTC holds for trees of diameter up to 5 while, on the other hand, RC holds for any tree of diameter up to 7. Similarly, it has been shown that any tree with ≤ 27 vertices has a graceful labelling but RC holds for any tree with ≤ 55 vertices, etc.

Perhaps the strongest result concerning labellings of the defined type is due to Kotzig [6] who showed in 1973 the following.

Kotzig's Theorem. . *Let T be an arbitrary tree and let $e \in E(T)$ be an arbitrary edge of T. Let $T_i, i = 1, 2, \ldots$ be the tree obtained from T by replacing e by a path with i edges. Then the set $\mathcal{T} = \{T_i : i = 1, 2, \ldots\}$ contains at most a finite number of trees without an α-labelling.*

But the cure of the labelling disease appears to be nowhere in sight.

2. The Erdös-Faber-Lovász disease

The original formulation of this innocently looking problem which originated at a party in 1972, is as follows.

Given n sets $A_1, \ldots, A_n$, $|A_i| = n$, $|A_i \cap A_j| \leq 1$, $i, j \in \{1, \ldots, n\}, i \neq j$, can one colour the elements of $\cup A_i$ with n colours so that for each i, no two elements of A_i get the same colour?

There are several equivalent formulations of this problem [5], [12]. By dualizing, one obtains the following formulation.

Does every linear hypergraph (partial linear space, partial pairwise balanced design, etc.) with n vertices have chromatic index at most n?

Kahn [5] has shown that for sufficiently large n, the chromatic index $\chi'(H)$ of a linear hypergraph H satisfies $\chi'(H) \leq n + o(n)$.

A few results on special cases of the problem were obtained by Hindman, Füredi and others. Very recently, for example, Sánchez-Arroyo [12] has established the truth of (an equivalent version of) the conjecture for dense hypergraphs.

And yet, the asymptotics of Kahn notwithstanding, the problem remains unsettled even for Steiner triple systems. Pippenger and Spencer [8] have shown already in 1989 that for a Steiner triple system(STS) S, $\chi'(S) \leq \frac{n}{2} + o(n)$ for sufficiently large n. It is well known that for STSs, the chromatic index χ' is $\geq \frac{n-1}{2}$ if $n \equiv 3 \ (mod\ 6)$, and is $\geq \frac{n+1}{2}$ if $n \equiv 1 \ (mod\ 6)$, and that the equality is attainable for all admissible $n \neq 7, 13$.

Examples of STSs exist whose chromatic index exceeds the minimum possible by 1 or by 2. But apart from the Fano plane, no example of an STS is known whose chromatic index exceeds the minimum possible by more than 2! It is annoying that even in this particular case – where it is even conceivable that χ' never exceeds (minimum possible value plus two) for all STSs of order greater than 7 –, noone has been able to prove the validity of the Erdös-Faber-Lovász conjecture (except asymptotically, of course). All that has been proved is that the Erdös-Faber-Lovász conjecture holds for cyclic STSs, indeed, for any cyclic Steiner 2-design.

3. Marco Buratti's problem

Marco Buratti formulated the following problem (which I hesitate to describe as a disease yet but which has in my opinion all the makings of a full-blown epidemic).

Let $p = 2n+1$ be a prime, let L be any list (=multiset) of $2n$ elements, each from the set $\{1, 2, \ldots, n\}$. Does there exist a hamiltonian path H in K_p, with $V(K_p) = Z_p$, such that the set of edge-lengths of H comprises L?

Here, the length of an edge $\{x, y\} \in E(K_p)$, or the distance from the vertex x to vertex y, is th usual cyclic distance $d(x, y) = min(|x-y|, p - |x-y|)$.

The hamiltonian path $(x_0, x_1, \ldots, x_{2n})$, $x_i \in Z_p$, is a *realization* of L if $\{d(x_i, x_{i+1}) : i = 0, 1, \ldots, 2n-1\} = L$.

Let $\mathcal{L}_p$ be the set of all lists (of $p-1 = 2n$ elements). For $L \in \mathcal{L}_p$, we write $L = 1^{a_1} 2^{a_2} \ldots n^{a_n}$ or $L = [a_1, a_2, \ldots, a_n]$ (where $\sum_{i=1}^{n} a_i = 2n$). Clearly, $|\mathcal{L}_p| = \binom{3n-1}{n-1}$. Define the graph G_p as follows: $V(G_p) = \mathcal{L}_p$; if $L, L' \in \mathcal{L}_p$, $L = [a_1, a_2, \ldots, a_n]$, $L' = [a_1', a_2', \ldots, a_n']$ then $\{L, L'\} \in E(G_p)$ iff $\delta(L, L') = \frac{1}{2} \sum_{i=1}^{n} |a_i - a_i'|$.

Given a hamiltonian path $H = (x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_{2n-1}, x_{2n})$, replacing the edge $\{x_i, x_{i+1}\}$ with the edge
(1) $\{x_0, x_n\}$, or
(2) $\{x_0, x_{i+1}\}$, or
(3) $\{x_i, x_{2n}\}$
results again in a hamiltonian path. A hamiltonian path H' obtained from H by either (1) or (2) or (3) is said to be obtained from H by an α-transformation. The graph A_p has as its vertices all $(2n+1)!$ realizations of the lists $L = [a_1, \ldots, a_n]$. Two vertices of A_p are adjacent if one is obtained from the other by an α-transformation.

Theorem. . *The graph A_p is connected.*

The graph $A_p{}^*$, the reduced graph of A_p, has as its vertex set $\mathcal{L}_p$, and is obtained from A_p by contracting to a single vertex all realizations of a given list L (while suppressing loops and/or multiple edges). Buratti's conjecture holds for a given prime p precisely when $A_p{}^*$ is a spanning subgraph of G_p.

We were able to derive some conditions that the lexicographically smallest list without a realization must satisfy [3] but at present this is not strong enough to conclude that no suhc list exists.

Some sufficient conditions for the existence of a realization of a list $L = [a_1, \ldots, a_n]$ are given in the following theorems.

Theorem. *Let L be such that $a_1 \geq 2$, $a_i \leq 2$ for $i = 2, 3, \ldots, n$. Then L has a realization.*

Theorem. *Let L be such that $\sum_{i=2}^{n} \leq 4$. Then L has a realization.*

Theorem. *Let for some $i.j$, $a_i > 0$, $A_j > 0$, and $A_k = 0$ for all $k \neq, i, j$. Then L has a realization.*

Buratti's conjecture has been verified for all primes $p \leq 23$ [7]. But it remains widely open.

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