

# Basic Thermodynamics of the Atmosphere

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## Abstract

We consider a model for atmospheric circulation based on the Euler equations for a compressible gas. We consider two hydrostatic base solutions depending on height, one with constant temperature and one isentropic with constant temperature gradient or lapse rate. We argue that these solutions represent solutions with maximal and minimal turbulent dissipation with the observed real lapse rate somewhere in between. We find an atmospheric cyclic thermodynamic process of rising-expanding-cooling and descending-compressing-warming air, which is similar to that of an air conditioner or refrigerator. We seek solutions as perturbations of the hydrostatic base solutions with the perturbations satisfying a modified form of the incompressible Euler equations.

## 1 Compressible/Incompressible Euler as Climate Model

As a model of the atmosphere we consider the Euler equations for a compressible perfect gas occupying a volume  $\Omega$ : Find  $(\rho, u, T)$  with  $\rho$  density,  $u$  velocity and  $T$  temperature depending on  $x$  and  $t > 0$ , such that for  $x \in \Omega$  and  $t > 0$ :

$$\begin{aligned} D_u \rho + \rho \nabla \cdot u &= 0 \\ D_u m + m \nabla \cdot u + \nabla p + g \rho e_3 &= 0 \\ D_u T + RT \nabla \cdot u &= q \end{aligned} \tag{1}$$

where  $m = \rho u$  is momentum,  $p = R\rho T$  is pressure  $R = c_p - c_v$  with  $c_v$  and  $c_p$  specific heats under constant volume and pressure, and  $D_u v = \dot{v} + u \cdot \nabla v$  is the material time derivative with respect to the velocity  $u$  with  $\dot{v} = \frac{\partial v}{\partial t}$  the partial derivative with respect to time  $t$ ,  $e_3 = (0, 0, 1)$  is the upward direction,  $g$  gravitational acceleration and  $q$  is a heat source. For air  $c_p = 1$  and  $\frac{c_p}{c_v} = 1.4$ . The Euler equations are complemented by initial values for  $\rho$ ,  $m$  and  $T$  at  $t = 0$ , and the boundary condition  $u \cdot n = 0$  on the boundary of  $\Omega$  where  $n$  is normal to the boundary.

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## 2 The 2nd Law of Thermodynamics

We recall the 2nd Law of Thermodynamics as stated in [4]:

$$\dot{K} + \dot{P} = W - D, \quad \dot{E} = -W + D + Q, \quad (2)$$

where

$$\begin{aligned} K(t) &= \frac{1}{2} \int_{\Omega} \rho u \cdot u(x, t) dx, \\ P(t) &= \int_0^t \int_{\Omega} g \rho u(x, s) dx ds, \\ E(t) &= \int_{\Omega} c_v \rho T(x, t) dx, \\ W(t) &= \int_Q p \nabla \cdot u(x, t) dx, \end{aligned} \quad (3)$$

is momentary total kinetic/potential/internal energy and work, respectively, and  $D(t) \geq 0$  is rate of turbulent dissipation and  $Q(t)$  rate of supplied heat.

Thermodynamics essentially concerns transformations between heat energy and kinetic/potential energy and the 2nd Law sets the following limits for these transformations:

- heat energy can be transformed to kinetic/potential energy only under expansion with  $W > 0$ ,
- turbulent dissipation transforms kinetic energy into heat energy, but not vice versa (because  $D(t) \geq 0$ ).

## 3 Basic Stability

Assuming  $Q = 0$ , we obtain adding the equations of the above 2nd Law that the total energy is conserved:

$$\dot{K} + \dot{P} + \dot{E} = 0, \quad (4)$$

which can be viewed as a form of basic stability.

## 4 Basic Isothermal and Isentropic Solutions

We identify the following hydrostatic equilibrium base solutions:

$$\begin{aligned} \bar{u} = 0, \quad \bar{T} = 288(K), \quad \bar{\rho} = \alpha \exp(-gx_3), \quad \bar{p} = R 288 \alpha \exp(-gx_3), \\ \bar{u} = 0, \quad \bar{T} = 288 - gx_3, \quad \bar{\rho} = \alpha (288 - gx_3)^{\frac{1}{\gamma}}, \quad \bar{p} = R \alpha (288 - gx_3)^{\frac{1}{\gamma} + 1}, \end{aligned} \quad (5)$$

where  $\gamma = \frac{R}{c_p}$  ( $= 0.4$ ) and thus  $R(\frac{1}{\gamma} + 1) = c_p = 1$ , and  $\alpha$  denotes a positive constant to be determined by data. The first solution has constant temperature and exponential

drop of density and pressure. The second solution is isentropic with  $D = 0$  and  $Q = 0$  in the 2nd Law:

$$\dot{E} + W = 0, \quad (6)$$

or in conventional notation

$$c_v dT + p dV = 0, \quad (7)$$

which combined with hydrostatic balance  $\frac{\partial p}{\partial x_3} = -g\rho$  and the differentiated form  $p dV + V dp = R dT$  of the gas law, gives

$$(c_v + R) \frac{\partial T}{\partial x_3} = -g. \quad (8)$$

The isentropic (dry adiabatic) lapse rate is thus  $\approx -10$  C/km, since  $c_v + R = c_p = 1$ .

We summarize the properties of the above base solutions (with  $Q = 0$ ):

- isothermal: maximal turbulent dissipation:  $D = W$ ,
- isentropic: minimal turbulent dissipation:  $D = 0$ .

We find real solutions between these extreme cases, with roughly  $D = \frac{W}{2}$  and  $\bar{\rho} \sim (288 - gx_3)^5$ ,  $\bar{p} \sim (288 - gx_3)^6$ , with a quicker drop with height than for the isentropic solution with  $\bar{\rho} \sim (288 - gx_3)^{2.5}$  and  $\bar{p} \sim (288 - gx_3)^{3.5}$ , or turned the other way, with a smaller lapse rate ( $\approx -6$  C/km).

The lapse rate can also be diminished by evaporation close to the Earth surface and condensation higher up (moist adiabatic lapse rate), with  $Q$  representing latent heat ( $Q < 0$  in evaporation and  $Q > 0$  in condensation).

## 5 Basic Data

Observation gives the following data:

- $Q \approx 250$  Watts (per m2) from insolation,
- $\dot{P} \approx 0.01 \times 0.65 \times 10000 \times g = 650$  Watts,
- average vertical velocity = 0.01 m/s,
- average density = 0.65 kg/m3,
- average thickness of main part of atmosphere (troposphere) = 10000 m,
- $\dot{E} \approx 0.01 \times 0.65 \times 10000 \times 6 = 400$  Watts,
- lapse rate  $\approx -6$  C/km,

which are compatible with  $\dot{P} = W - D = 650$  and  $\dot{E} = -W + D + Q = -400$  Watts.

With an isentropic lapse rate of 10 C/km we would have  $dE = -650$  Watts, and we can thus view the input of 250 Watts as being spent on turbulent dissipation, effectively reducing the temperature drop with increasing height. In this case  $D = 250 \times 10^{-4}$

which is compatible with the scaling  $D \sim U^2$  of turbulent dissipation with  $U \approx 0.1$  as mean velocity.

In this argument with do not account for the effect of latent heat included in the term  $Q$ , which in evaporation with  $Q > 0$  together with turbulent dissipation  $D > 0$  adds heat energy (and thus may decrease the lapse rate).

## 6 Basic Thermodynamics

We have formulated a basic thermodynamic model of an atmosphere acting in a cyclic thermodynamic convective process of an ascending/expanding/cooling and descending/compressing/warming flow of air, which is driven by insolation spent on maintaining the convection under turbulent dissipation. This model is compatible with observation without any presence of so-called greenhouse gases, and thus suggests that global climate is mainly determined by thermodynamics and not by greenhouse gases.

In reality, turbulent dissipation gives strict inequality in the 2nd Law (7). The observed lapse rate of  $-6$  can thus partly be seen as an effect of turbulent dissipation, with another major effect coming from evaporation and condensation.

The isentropic lapse rate can be seen as being established by a cyclic thermodynamic process with hot light air rising under expansion/cooling and cool air descending under compression/warming, combined with evaporation/condensation. The atmosphere thus acts like an air conditioner or refrigerator transporting heat from the Earth surface (received by insolation) to the top of the atmosphere from where it is radiated to into space, by a cyclic thermodynamic process of expansion/cooling and compression/warming with efficiency boosted by evaporation/condensation. The thermodynamics of a refrigerator is driven by a compressor, which in the case on an atmosphere is taken over by gravitation causing compression of descending air.

## 7 Joule's Experiment

The basic thermodynamic process of cooling under expansion was experimentally studied by Joule letting a high pressure-density-temperature gas expand from equilibrium in one chamber into another chamber and measuring the temperature difference/gap in the chambers at the new equilibrium. In this process the gas gains kinetic energy by cooling and then comes to rest by turbulent dissipation causing warming. The resulting temperature gap depends on the dynamics of the expansion process, which with maximal turbulent dissipation (or maximal entropy increase) results in zero gap, while isentropic expansion without turbulent dissipation gives maximal gap. In a real process the gap is somewhere in between these extremes as shown in computational simulation in Chapter 166 of [5].

The experience from the Joule experiment suggests to view the real lapse as determined by the amount of turbulent dissipation between the limits of isentropic zero dissipation with maximal lapse rate and maximal dissipation with zero lapse rate (combined with the effect of evaporation/dissipation decreasing the lapse rate).

## 8 An Incompressible Model

We seek a solution to (1) on the form  $(\bar{\rho} + \rho, \bar{u} + u, \bar{T} + T)$ , where  $(\bar{\rho}, \bar{u}, \bar{T})$  is a basic solution of the form given above (only depending on  $x_3$ ) with  $\bar{u} = 0$  and  $\nabla \bar{p} = -g\bar{\rho}e_3$  expressing hydrostatic balance. We assume that the velocity perturbation  $u$  satisfies  $\nabla \cdot u = 0$  motivated by the fact that the Mach number of atmospheric air flow is small. Inserting this Ansatz into (1), we obtain the following modified form of the incompressible Euler equations: Find  $(\rho, u, p, T)$  such that

$$\begin{aligned} D_u \rho + u \cdot \nabla \bar{\rho} &= 0 \\ D_u((\bar{\rho} + \rho)u) + \nabla p + g\rho &= 0, \\ \nabla \cdot u &= 0, \\ D_u T + u \cdot \nabla \bar{T} &= q. \end{aligned} \tag{9}$$

where  $q$  is a heat source modelling radiative forcing and evaporation/condensation.

In this model, the heat source is decoupled from the mass and momentum equations, which is unphysical. This can be corrected by a source term in the mass equation, or by replacing  $\rho$  in the momentum equation assuming that  $\rho\bar{T} + \bar{\rho}T \approx 0$  expressing that the dynamic pressure  $p$  does not interact with the internal energy, which leads to the following alternative formulation used in the computations presented below:

$$\begin{aligned} D_u \rho + u \cdot \nabla \bar{\rho} &= 0 \\ D_u((\bar{\rho} + \rho)u) + \nabla p - g\frac{\bar{\rho}}{\bar{T}}T &= 0, \\ \nabla \cdot u &= 0, \\ D_u T + u \cdot \nabla \bar{T} &= q. \end{aligned} \tag{10}$$

## 9 Marginal Stability of Incompressible Perturbations

We consider an (infinitesimal) perturbation  $(\rho, u, T)$  with  $\nabla \cdot u = 0$  of a base solution  $(\bar{\rho}(x_3), 0, \bar{T}(x_3))$  satisfying the linearized Euler equations

$$\begin{aligned} \dot{\rho} + u_3 \frac{\partial \bar{\rho}}{\partial x_3} &= 0 \\ \bar{\rho}\dot{u} + R\nabla(\rho\bar{T} + \bar{\rho}T) + g\rho e_3 &= 0 \end{aligned} \tag{11}$$

The coupling between  $\rho$  and  $u$  is marginally stable if  $\frac{\partial \bar{\rho}}{\partial x_3} < 0$ , that is if the base density  $\bar{\rho}(x_3)$  decreases with height in the sense the perturbation develops into an oscillating solution without exponential decay or growth. This follows by combining the first equation multiplied by  $\rho$  and the second by  $u$  so as to make the coupling terms cancel.

## 10 Stability of General Perturbations

We next consider the question of stability of static solution to general perturbations. We first observe that a perturbation of finite size changing the density profile into increasing

with height will cause unstable overturning with rising light air. Heating changes the density and thus sufficient heating will lead to unstable overturning.

We now check the stability of static solutions between the extremes of isothermal with zero lapse rate and isentropic with maximal lapse rate of 10 K/km. We consider a perturbation in the form of a parcel of air which is heated into rising by bouyancy. If the parcel upon rising expands and cools quicker than the surrounding basic state, then it will gain heat and the process will accelerate. In other words, an isentropic perturbation of a non-isentropic state will cause instability. But if the perturbation develops turbulence, then it will cool less quickly, possibly less quickly than the surrounding and thus give off heat and stabilize.

This indicates that the isentropic state is stable, but it may not be achievable in reality since it would require dynamics without turbulence. On the other hand, the isothermal state may be viewed as the result of maximally turbulent perturbation process.

A static solution with an intermediate lapse rate is thus unstable to isentropic perturbations but stable to perturbations with smaller lapse rates.

Note that there is no heat transport (and thus no input of heat) in the static solutions under consideration, because there is no convection, conduction or radiation. In reality the atmosphere transports heat from the Earth surface to the top of the atmosphere in turbulent convective motion which generates an intermediate lapse rate.

## 11 A Simple Radiation Model

Let  $E(x_3)$  be the radiation from an atmospheric layer at height  $x_3$ . Subdivide the atmosphere into horisontal layers of width  $h$  identified by  $h(j-1) < x_3 < hj$ . Balance of incoming and outgoing radiation can, assuming full absorption/emission, be expressed as

$$E(x_3 - h) - 2E(x_3) + E(x_3 + h) = 0 \quad (12)$$

Assuming that  $E = cT^4$  according to Stefan-Boltzmann's Radiation Law, this leads to the following differential equation

$$-TT'' = 3(T')^2 \quad (13)$$

with  $T' = \frac{dT}{dx_3}$ . This equation effectively priviliges a linear temperature profile with  $T'' \approx 0$ , while the slope or lapse rate  $T'$  still is to be determined. The basic problem at hand can be seen as heat conduction modeled by

$$-\epsilon T''(x_3) = 0 \quad \text{for } 0 < x_3 < 1, \quad T(1) = 0, \quad -\epsilon T'(0) = Q \quad (14)$$

with  $\epsilon$  a coefficient of heat conduction and a Neumann condition at  $x_3 = 0$ , effectively defining  $T'(x_3) = T'(0) = -\frac{Q}{\epsilon}$ . In this model the lapse rate of the atmosphere is determined principally by the heat exchange ocean-atmosphere at the ocean surface.

## 12 G2 Computational Results

We solve the system (10) using the G2 finite element method [1] starting from the above adiabatic base solution with forcing from heating at  $x_3 = 0$  and cooling at  $x_3 = 1$ , with the objective of determining the temperature drop from  $x_3 = 0$  to  $x_3 = 1$ . We find buoyancy driven turbulent solutions, which will be presented shortly. Related results for thermohaline ocean circulation are presented in [2] and [3].

### References

- [1] J. Hoffman and C. Johnson, *Computational Turbulent Incompressible Flow*, Springer, 2007.
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